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Parameter-dependent optimal stopping problems for one-dimensional diffusions*

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Abstract

We consider a class of optimal stopping problems for a regular one-dimensional diffusion whose payoff depends on a linear parameter. As shown in [Bank and Föllmer(2003)] problems of this type may allow for a universal stopping signal that characterizes optimal stopping times for any given parameter via a level-crossing principle of some auxiliary process. For regular one-dimensional diffusions, we provide an explicit construction of this signal in terms of the Laplace transform of level passage times. Explicit solutions are available under certain concavity conditions on the reward function. In general, the construction of the signal at a given point boils down to finding the infimum of an auxiliary function of one real variable. Moreover, we show that monotonicity of the stopping signal corresponds to monotone and concave (in a suitably generalized sense) reward functions. As an application, we show how to extend the construction of Gittins indices of [Karatzas(1984)] from monotone reward functions to arbitrary functions.

Key words: Optimal stopping, Gittins index, multi-armed bandit problems, American options, universal stopping signal.

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1 Introduction

In this paper, we study optimal stopping problems of one-dimensional regular diffusions that depend on a parameter. A well-known example is the perpetual American put option written on an underlying process X with strike k whose discounted payoff at time t is given by $e^{-rt}(k - X_t)^+$. More generally, with $\mathscr{S}$ denoting the set of stopping times, one can consider the optimal stopping problem with value function

$$V(x) \triangleq \sup_{\tau \in \mathscr{S}} E_x \left[e^{-r\tau}(u(X_\tau) - k) \right], \quad k \in \mathbb{R},$$

where $r > 0$ is a discount rate and the reward function is $h_k(\cdot) \triangleq u(\cdot) - k$. A general characterization of the value function in terms of excessive functions was proved by [Dynkin(1963)]. Moreover, efficient methods for the calculation of the value function are available if X is a one-dimensional regular diffusion, see e.g. [Dayanik and Karatzas(2003)]. This calculation is reduced to finding the smallest majorant of the reward function h_k which exhibits a suitably generalized form of concavity. Optimal stopping times can then be identified as the first entrance time of the underlying diffusion into the set where the value function and the reward function coincide. A purely probabilistic approach to find optimal stopping times is to compute the Snell envelope of the payoff process; see e.g. [Shiryayev(1978)], [El Karoui(1981)] or [Karatzas(1988)]. In any case, if the reward function depends on a parameter k as in the case of an American put option, both approaches outlined above have to be repeated anew for each choice of the parameter. By contrast, [Bank and Föllmer(2003)] show how to construct a universal stopping signal from the optional solution K to the representation problem

$$e^{-rT} u(X_T) = E \left[\int_T^\infty r e^{-rt} \sup_{T \le s \le t} K_s \, dt \,\middle|\, \mathscr{F}_T \right].$$

Optimal stopping times can be obtained from the solution K to the representation problem via a level-crossing principle: it is optimal to exercise a put with strike k as soon as K passes the threshold k. While explicit formulas for the signal process K have been found in special cases, this paper provides an explicit construction of universal stopping signals for general one-dimensional diffusions and reward functions. Specifically, the computation of the optimal stopping signal at a given point is reduced to finding the infimum of an auxiliary function of one variable that can be computed explicitly from the Laplace transforms of level passage times for the underlying diffusion. Moreover, we show that the infimum can be determined in closed form if the reward function satisfies certain concavity conditions. We also prove that monotone stopping signals correspond to monotone and concave (again in a suitably generalized sense) reward functions and provide explicit solutions in that case. Finally, we illustrate our method by applying them to optimal stopping problems for American options and to the computation of Gittins indices, extending results of [Karatzas(1984)].

This paper is organized as follows: in Section 2 we outline those results theory of optimal stopping for one-dimensional regular diffusions which are needed for our approach. In Section 3 we introduce a class of optimal stopping problems depending on a parameter k in a linear way and discuss how to compute universal stopping signals in that case. Our results are illustrated by some examples. We close by pointing out the connection of our results to stochastic representation problems.

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2 Optimal stopping of one-dimensional diffusions

In this section, we outline part of the theory of optimal stopping for one-dimensional diffusions with random discounting using the paper of [Dayanik(2008)] as a key reference; see also [Dayanik and Karatzas(2003)], [Peskir and Shiryaev(2006)], [Dynkin(1963)], [Fakeev(1971)], [Shiryayev(1978)] and the references therein. We consider a one-dimensional diffusion X specified by the SDE

$$dX_t = \sigma(X_t)\,dW_t + \mu(X_t)\,dt, \qquad X_0 = x \in \mathscr{J}, \tag{1}$$

where $\mathscr{J} = (a,b)$ with $-\infty \le a < b \le \infty$ and W is a Brownian motion.

Assumption 1. *The functions $\mu, \sigma : \mathscr{J} \longrightarrow \mathbb{R}$ are Borel-measurable and satisfy the following conditions:*

$$\sigma^2(x) > 0, \quad \textit{for all } x \in \mathscr{J},$$

and

$$\int_{\bar{a}}^{\bar{b}} \frac{1+|\mu(s)|}{\sigma^2(s)}\,ds < \infty, \quad \textit{for all } a < \bar{a} < \bar{b} < b.$$

Also, σ^2 is locally bounded, i.e.

$$\sup_{s\in[\bar{a},\bar{b}]} \sigma^2(s) < \infty, \quad \textit{for all } a < \bar{a} < \bar{b} < b.$$

Our assumptions are sufficient for the SDE (1) to have a weak solution $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\ge 0}, \{P_x : x \in \mathscr{J}\}, W, X)$ up to an explosion time that is unique in the sense of probability law; see [Karatzas and Shreve(1991), Section 5.5 C]. Define $T_y \triangleq \inf\{t \ge 0 : X_t = y\}$ the first hitting time of the point $y \in \mathscr{J}$. Assumption 1 implies that X is regular, i.e. $P_x\left[T_y < \infty\right] > 0$ for all $x \in \mathscr{J}, y \in \mathscr{J}$ (see [Dayanik and Karatzas(2003)]). Throughout this paper, we consider only natural boundaries:

Assumption 2. *The boundaries of $\mathscr{J}$ are natural for X, i.e.*

$$P_x\left[T_a < \infty\right] = P_x\left[T_b < \infty\right] = 0 \textit{ for any } x \in \mathscr{J}.$$

Next, we introduce our discounting process $(R_t)_{t\ge 0}$ given by

$$R_t = \int_0^t r(X_s)\,ds, \quad t \ge 0.$$

where we impose the following conditions on the function r:

Assumption 3. *The function $r : \mathscr{J} \to (0,\infty)$ is Borel-measurable and locally bounded. Moreover, there is $r_0 > 0$ such that $r(x) \ge r_0$ for all $x \in \mathscr{J}$.*

In particular, R is a continuous additive functional of X, i.e. R is a.s. nonnegative, continuous, vanishes at zero and has the additivity property

$$R_{s+t} = R_s + R_t \circ \theta_s, \quad s,t \geq 0,$$

where θ_s is the usual shift operator: $X_t \circ \theta_s = X_{t+s}$. For an account on additive functionals, see, e.g., Chapter X in [Revuz and Yor(1991)]. Moreover, we have $R_t \to \infty$ as $t \to \infty$. Sometimes we write $R(t)$ instead of R_t for notational convenience.

Let us now recall how to compute the Laplace transforms of $R(T_y)$ for $a < y < b$. To this end, consider the ODE

$$\mathscr{A} w(x) \triangleq \frac{1}{2}\sigma^2(x) w''(x) + \mu(x) w'(x) = r(x) w(x), \quad x \in \mathscr{J}, \tag{2}$$

where $\mathscr{A}$ is the infinitesimal generator of X. This ODE has two linearly independent, positive solutions. These are uniquely determined up to multiplication by constants, if we require one of them to be increasing and the other decreasing. We denote the increasing solution by ψ and the decreasing one by φ. They are continuously differentiable and satisfy

$$\lim_{x \downarrow a} \varphi(x) = \lim_{x \uparrow b} \psi(x) = \infty.$$

We refer to [Johnson and Zervos(2007)] for these results; see also [Dayanik and Karatzas(2003)] and [Itô and McKean, Jr.(1974)] for the case $r(x) \equiv r > 0$.

Lemma 1. *([Itô and McKean, Jr.(1974)],...) If $x, y \in \mathscr{J}$, then*

$$E_x\left[e^{-R_{T_y}}\right] = \begin{cases} \psi(x)/\psi(y), & x \leq y \\ \varphi(x)/\varphi(y), & x > y. \end{cases}$$

Remark 2. If $r(x) \equiv r$, the assumption that the boundaries of $\mathscr{J}$ are natural (Assumption 2) is equivalent to

$$\varphi(a+) = \psi(b-) = \infty \quad \text{and} \quad \psi(a+) = \varphi(b-) = 0,$$

see e.g. [Dayanik and Karatzas(2003), Itô and McKean, Jr.(1974)].

It will be convenient to introduce the increasing functions F and G defined by

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)}, \quad G(x) \triangleq -\frac{\varphi(x)}{\psi(x)}, \quad x \in \mathscr{J}.$$

Lemma 3. *[Dayanik(2008), Lemma 2.3] For every $x \in [y,z] \subset \mathscr{J}$, we have*

$$\begin{aligned} E_x[e^{-R_{T_y}} 1_{\{T_y < T_z\}}] &= \frac{\varphi(x)}{\varphi(y)} \cdot \frac{F(z) - F(x)}{F(z) - F(y)} = \frac{\psi(x)}{\psi(y)} \cdot \frac{G(z) - G(x)}{G(z) - G(y)}, \\ E_x[e^{-R_{T_z}} 1_{\{T_y > T_z\}}] &= \frac{\varphi(x)}{\varphi(z)} \cdot \frac{F(x) - F(y)}{F(z) - F(y)} = \frac{\psi(x)}{\psi(z)} \cdot \frac{G(x) - G(y)}{G(z) - G(y)}. \end{aligned}$$

Following [Dayanik and Karatzas(2003)], we use the following generalized notion of concavity.

Definition 4. Let $f : \mathscr{J} \to \mathbb{R}$ be a strictly monotone function. Then $U : \mathscr{J} \to \mathbb{R}$ is called f-concave if

$$U(x) \geq U(y)\frac{f(z)-f(x)}{f(z)-f(y)} + U(z)\frac{f(x)-f(y)}{f(z)-f(y)}, \quad x \in [y,z] \subseteq \mathscr{J}.$$

We speak of a strictly f-concave function if the inequality above holds in the strict sense whenever $y < x < z$.

Clearly, if $f(x) = x$, then a f-concave function is concave in the usual sense. Some facts about f-concave functions can be found in [Dynkin(1965)].

Remark 5. Note that by Lemma 3, the function $1/\varphi$ is strictly F-concave whereas $1/\psi$ is strictly G-concave on $\mathscr{J}$.

With these concepts at hand, we may now recall some very useful results of [Dayanik(2008)] that show how to solve optimal stopping problems in our setting; see [Dayanik and Karatzas(2003)] for these results in case of a constant discount rate. Let h be a Borel function such that $E_x\left[e^{-R_\tau}h(X_\tau)\right]$ is well defined for every $\mathbb{F}$-stopping time τ and $x \in \mathscr{J}$. Moreover, we assume that h is bounded on every compact subset of $\mathscr{J}$. By convention, we set $h(X_\tau) = 0$ on $\{\tau = \infty\}$. Denote by

$$V(x) \triangleq \sup_{\tau \in \mathscr{S}} E_x\left[e^{-R_\tau}h(X_\tau)\right], \quad x \in \mathscr{J}, \tag{3}$$

the value function of the optimal stopping problem with reward function h and discount rate $r(\cdot)$, where the supremum is taken over the set $\mathscr{S}$ of $\mathbb{F}$-stopping times.

Proposition 6. *[Dayanik(2008), Prop. 3.1 - 3.4, 3.10]*

1. *The value function V is either finite or infinite everywhere on $\mathscr{J}$. Moreover, $V \equiv \infty$ on $\mathscr{J}$ if and only if at least one of the limits*

$$\ell_a \triangleq \limsup_{x \downarrow a} \frac{h^+(x)}{\varphi(x)} \text{ and } \ell_b \triangleq \limsup_{x \uparrow b} \frac{h^+(x)}{\psi(x)}$$

   is infinite ($h^+(x) \triangleq \max\{0, h(x)\}$).

2. *If h is continuous on $\mathscr{J}$ and $\ell_a = \ell_b = 0$, the stopping time*

$$\tau^\star \triangleq \inf\{t \geq 0 : X_t \in \Gamma\}$$

   where $\Gamma \triangleq \{x \in \mathscr{J} : V(x) = h(x)\}$ is optimal, i.e.

$$V(x) = E_x\left[e^{-R_{\tau^\star}}h(X_{\tau^\star})\right].$$

3. *If ℓ_a and ℓ_b are both finite, the value function V is continuous on $\mathscr{J}$. Moreover, V is the smallest nonnegative majorant of h on $\mathscr{J}$ such that V/φ is F-concave (equivalently, V/ψ is G-concave) on $\mathscr{J}$.*

4. *With $W : F(\mathscr{J}) \to \mathbb{R}$ denoting the smallest nonnegative concave majorant of the function*

$$H(y) \triangleq (h/\varphi) \circ F^{-1}(y), \quad y \in F(\mathscr{J}),$$

   the value function V is given by

$$V(x) = \varphi(x)W(F(x)), \quad x \in \mathscr{J}.$$

3 Parameter-dependent optimal stopping

In many applications, the reward function h of an optimal stopping problem depends on some parameter k, i.e.

$$\sup_{\tau \in \mathscr{S}} E_x \left[e^{-R_\tau} h_k(X_\tau) \right], \quad x \in \mathscr{J}, k \in \mathbb{R}.$$

A very prominent example is the American put option with strike k, i.e. $h_k(x) = (k-x)^+$. In the previous section, we have seen how to compute the corresponding value function for a fixed value of k by finding the smallest majorant of the reward function h_k which is concave in some suitably generalized sense. An optimal stopping region Γ_k is then given by the set where the value function and the reward function coincide (cf. Proposition 6). If one has to determine optimal stopping rules corresponding to many different values of k, this approach might be quite tiresome, particularly if the structure of the stopping region is complex. In such a case, it would be desirable to have a universal stopping signal γ that characterizes the optimal stopping regions for any given parameter k, for instance as the level set $\Gamma_k = \{x : \gamma(x) \geq k\}$. In the sequel, we describe a method to compute such a universal stopping signal for reward functions of the form $h_k(x) = u(x) - k$, i.e. for the optimal stopping problem

$$V_k(x) \triangleq \sup_{\tau \in \mathscr{S}} E_x \left[e^{-R_\tau} (u(X_\tau) - k) \right], \quad x \in \mathscr{J}, k \in \mathbb{R}, \tag{4}$$

where u satisfies the following assumption:

Assumption 4. *The function $u \colon \mathscr{J} \to \mathbb{R}$ is continuously differentiable and*

$$\limsup_{x \downarrow a} \frac{u^+(x)}{\varphi(x)} < \infty \text{ and } \limsup_{x \uparrow b} \frac{u^+(x)}{\psi(x)} < \infty.$$

Remark 7. In view of Proposition 6, Assumption 4 ensures that the value function V_k of (4) is finite for any k.

3.1 Computation of optimal stopping signals

Let us discuss how to compute a universal stopping signal yielding an optimal stopping time for any k. To this end, let

$$\Gamma_k \triangleq \{x : V_k(x) = u(x) - k\}$$

denote the optimal stopping region (cf. Proposition 6).

Assumption 5. *For any k, the stopping time*

$$\tau_k \triangleq \inf\{t : X_t \in \Gamma_k\}$$

is optimal, i.e. $V_k(x) = E_x \left[e^{-R_{\tau_k}} (u(X_{\tau_k}) - k) \right]$.

A sufficient condition for optimality of the stopping times τ_k was given in Proposition 6. Moreover, granted there is an optimal stopping time, τ_k is optimal as well (see e.g. [El Karoui(1981)]).

Remark 8. Recall that we use the convention $h(X_\tau) = 0$ on the set $\{\tau = \infty\}$ for any stopping time τ. The existence of an optimal stopping time τ_k requires that the supremum in (4) is attained.

We aim to derive a function γ on $\mathscr{J}$ such that the optimal stopping regions Γ_k can be written as level sets $\Gamma_k = \{x : \gamma(x) \geq k\}$. In other words, knowledge of the function γ suffices to derive the optimal stopping regions for any parameter k. Our main result, Theorem 13 below, shows that $\gamma(x)$ can be found as the infimum over an auxiliary function $\eta^x(\cdot)$ of one real variable which is expressed in terms of the Laplace transforms φ and ψ of level passage times. The infimum can be computed explicitly under specific concavity conditions on the reward function u. In the sequel, we show how to obtain this result and give an explicit expression for η^x in (12).

Let $T_U \triangleq \inf\{t \geq 0 : X_t \notin U\}$ denote the first exit time from a measurable subset U of (a, b) and

$$\tilde{\mathscr{T}}(x) \triangleq \{T_U : U \subset (a, b) \text{ open}, x \in U\}$$

the class of all exit times from open neighborhoods of x. The following lemma shows that the stopping signal γ is the solution to a non-standard optimal stopping problem (cf. equation (25) in [Bank and Föllmer(2003)], also [El Karoui and Föllmer(2005)]).

Lemma 9. *For*

$$\gamma(x) \triangleq \inf_{T \in \tilde{\mathscr{T}}(x)} \frac{E_x\left[u(x) - e^{-R_T} u(X_T)\right]}{E_x\left[1 - e^{-R_T}\right]}, \quad x \in \mathscr{J}, \tag{5}$$

we have $\Gamma_k = \{x : \gamma(x) \geq k\}$.

Proof. For any k, we have $x \in \Gamma_k$ if and only if

$$E_x\left[e^{-R_T}(u(X_T) - k)\right] \leq u(x) - k \text{ for any } T \in \tilde{\mathscr{T}}(x).$$

To see this, note that if $x \in \Gamma_k$, then $V_k(x) = u(x) - k$ and the inequality above is true by definition of the value function. On the other hand, note that the sets Γ_k are closed in $\mathscr{J}$ since V_k is continuous for all k by Proposition 6. Thus, if $x \notin \Gamma_k$, $\tau_k = \inf\{t : X_t \in \Gamma_k\}$ is an exit time from an open neighborhood of x, i.e. $\tau_k \in \tilde{\mathscr{T}}(x)$. Since τ_k is optimal by Assumption 5, we get

$$u(x) - k < V_k(x) = E_x\left[e^{-R_{\tau_k}}(u(X_{\tau_k}) - k)\right].$$

Thus, for any k, we have

$$x \in \Gamma_k \iff \inf_{T \in \tilde{\mathscr{T}}(x)} \frac{E_x\left[u(x) - e^{-R_T} u(X_T)\right]}{E_x\left[1 - e^{-R_T}\right]} \geq k.$$

□

Let us now discuss how to compute the function γ of (5). The following lemma reduces the optimal stopping problem of the preceding lemma to finding the infimum of a function of two variables.

Lemma 10. *For any* $x \in \mathscr{J}$,

$$\gamma(x) = \inf_{a \leq y < x < z \leq b} f^x(y, z) \tag{6}$$

where

$$f^x(y,z) \triangleq \begin{cases} \dfrac{u(x)-u(z)\frac{\psi(x)}{\psi(z)}}{1-\frac{\psi(x)}{\psi(z)}} = \dfrac{\frac{u(x)}{\psi(x)} - \frac{u(z)}{\psi(z)}}{\frac{1}{\psi(x)} - \frac{1}{\psi(z)}}, & a = y < x < z < b, \\ \dfrac{\frac{u(x)}{\varphi(x)} - \frac{u(y)}{\varphi(y)}\frac{F(z)-F(x)}{F(z)-F(y)} - \frac{u(z)}{\varphi(z)}\frac{F(x)-F(y)}{F(z)-F(y)}}{\frac{1}{\varphi(x)} - \frac{1}{\varphi(y)}\frac{F(z)-F(x)}{F(z)-F(y)} - \frac{1}{\varphi(z)}\frac{F(x)-F(y)}{F(z)-F(y)}}, & a < y < x < z < b, \\ \dfrac{u(x)-u(y)\frac{\varphi(x)}{\varphi(y)}}{1-\frac{\varphi(x)}{\varphi(y)}} = \dfrac{\frac{u(x)}{\varphi(x)} - \frac{u(y)}{\varphi(y)}}{\frac{1}{\varphi(x)} - \frac{1}{\varphi(y)}}, & a < y < x < z = b, \\ u(x) & y = a, z = b. \end{cases}$$

Proof. First note that if $T \in \tilde{\mathscr{T}}(x)$, $T = T_y \wedge T_z$ P_x-a.s. for some $a \le y < x < z \le b$. Thus,

$$\gamma(x) = \inf_{a \le y < x < z \le b} \frac{E_x\left[u(x) - e^{-R_{T_y \wedge T_z}} u(X_{T_y \wedge T_z})\right]}{E_x\left[1 - e^{-R_{T_y \wedge T_z}}\right]}$$

Assumption 2 implies that $T_a = T_b = \infty$ a.s. The claim now follows from Lemma 1 and Lemma 3. □

Remark 11. According to Lemma 3, the functions φ and F may be replaced by ψ and G in the definition of f^x, i.e. for $a < y < x < z < b$,

$$f^x(y,z) = \frac{\frac{u(x)}{\psi(x)} - \frac{u(y)}{\psi(y)} \cdot \frac{G(z)-G(x)}{G(z)-G(y)} - \frac{u(z)}{\psi(z)} \cdot \frac{G(x)-G(y)}{G(z)-G(y)}}{\frac{1}{\psi(x)} - \frac{1}{\psi(y)} \cdot \frac{G(z)-G(x)}{G(z)-G(y)} - \frac{1}{\psi(z)} \cdot \frac{G(x)-G(y)}{G(z)-G(y)}}.$$

For simplicity of notation, let

$$\alpha_\varphi \triangleq \frac{u}{\varphi}, \quad \alpha_\psi \triangleq \frac{u}{\psi}, \quad \beta_\varphi \triangleq \frac{1}{\varphi}, \quad \beta_\psi \triangleq \frac{1}{\psi}.$$

For $a < y < x < z < b$, simple manipulations yield

$$f^x(y,z) = \frac{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - \frac{\alpha_\varphi(z)-\alpha_\varphi(x)}{F(z)-F(x)}}{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - \frac{\beta_\varphi(z)-\beta_\varphi(x)}{F(z)-F(x)}} = \frac{\frac{\alpha_\psi(x)-\alpha_\psi(y)}{G(x)-G(y)} - \frac{\alpha_\psi(z)-\alpha_\psi(x)}{G(z)-G(x)}}{\frac{\beta_\psi(x)-\beta_\psi(y)}{G(x)-G(y)} - \frac{\beta_\psi(z)-\beta_\psi(x)}{G(z)-G(x)}}. \tag{7}$$

We remark that $f^x(y,z) \to f^x(a,z)$ if $y \downarrow a$ iff $u(y)/\varphi(y) \to 0$ as $y \downarrow a$ and $f^x(y,z) \to f(y,b)$ if $z \uparrow b$ iff $u(z)/\psi(z) \to 0$ as $z \uparrow b$.

Let us discuss next how to compute the infimum of $f^x(y,z)$ over y and z for fixed x in order to find $\gamma(x)$ in (6). In view of the definition of f^x, it seems natural to ask what happens if one lets $y \uparrow x$ or $z \downarrow x$ in (7). One notices that a difference quotient similar to the definition of the usual derivative appears. Let us therefore recall the following definition.

Definition 12. Let $f: \mathscr{J} \to \mathbb{R}$ be a strictly monotone function. We say that $U: \mathscr{J} \to \mathbb{R}$ is f-differentiable at $x \in \mathscr{J}$ if

$$d_f U(x) \triangleq \lim_{y \to x} \frac{U(y) - U(x)}{f(y) - f(x)} \text{ exists.}$$

For $f(s) = s$, the definition above is just the usual derivative. Also, if U and f are differentiable at x, then $d_f U(x) = U'(x)/f'(x)$. In particular, under Assumption 4, α_φ and α_ψ are F- and G-differentiable. β_φ and β_ψ are always F- and G-differentiable.

From (7), we find that

$$\lim_{y \uparrow x} f^x(y,z) = \frac{d_F \alpha_\varphi(x) - \frac{\alpha_\varphi(z) - \alpha_\varphi(x)}{F(z) - F(x)}}{d_F \beta_\varphi(x) - \frac{\beta_\varphi(z) - \beta_\varphi(x)}{F(z) - F(x)}} = \frac{d_G \alpha_\psi(x) - \frac{\alpha_\psi(z) - \alpha_\psi(x)}{G(z) - G(x)}}{d_G \beta_\psi(x) - \frac{\beta_\psi(z) - \beta_\psi(x)}{G(z) - G(x)}}, \qquad x < z < b, \tag{8}$$

$$\lim_{z \downarrow x} f^x(y,z) = \frac{\frac{\alpha_\varphi(x) - \alpha_\varphi(y)}{F(x) - F(y)} - d_F \alpha_\varphi(x)}{\frac{\beta_\varphi(x) - \beta_\varphi(y)}{F(x) - F(y)} - d_F \beta_\varphi(x)} = \frac{\frac{\alpha_\psi(x) - \alpha_\psi(y)}{G(x) - G(y)} - d_G \alpha_\psi(x)}{\frac{\beta_\psi(x) - \beta_\psi(y)}{G(x) - G(y)} - d_G \beta_\psi(x)}, \qquad a < y < x. \tag{9}$$

Using l'Hôspital's rule, one computes

$$\lim_{y \uparrow x} f^x(y,b) = u(x) - u'(x)\frac{\varphi(x)}{\varphi'(x)} = \frac{d_F \alpha_\varphi(x)}{d_F \beta_\varphi(x)} \triangleq \kappa(x), \tag{10}$$

$$\lim_{z \downarrow x} f^x(a,z) = u(x) - u'(x)\frac{\psi(x)}{\psi'(x)} = \frac{d_G \alpha_\psi(x)}{d_G \beta_\psi(x)} \triangleq \rho(x). \tag{11}$$

We now define $\eta^x : [a,b] \setminus \{x\} \to \mathbb{R}$ by

$$\eta^x(y) \triangleq \begin{cases} \rho(x), & y = a, \\ \dfrac{d_F \alpha_\varphi(x) - \frac{\alpha_\varphi(y) - \alpha_\varphi(x)}{F(y) - F(x)}}{d_F \beta_\varphi(x) - \frac{\beta_\varphi(y) - \beta_\varphi(x)}{F(y) - F(x)}}, & x < y < b, \\ \dfrac{\frac{\alpha_\varphi(x) - \alpha_\varphi(y)}{F(x) - F(y)} - d_F \alpha_\varphi(x)}{\frac{\beta_\varphi(x) - \beta_\varphi(y)}{F(x) - F(y)} - d_F \beta_\varphi(x)}, & a < y < x, \\ \kappa(x), & y = b. \end{cases} \tag{12}$$

$\eta^x(\cdot)$ can also be written in terms of G, α_ψ and β_ψ instead of F, α_φ and β_φ in view of (8) and (9). In some proofs, it will be more convenient to use this alternative representation. Note that η^x may fail to be continuous at the boundaries a or b. In fact, we have $\lim_{y \downarrow a} \eta^x(y) = \rho(x)$ iff $\lim_{y \downarrow a} u(y)/\varphi(y) = 0$ and $\lim_{z \uparrow b} \eta^x(z) = \kappa(x)$ iff $\lim_{z \uparrow b} u(z)/\psi(z) = 0$.

We now show that the computation of the optimal stopping signal γ at a point $x \in \mathscr{J}$ amounts to finding the infimum of the function η^x of one real variable instead of the function f^x of two variables.

Theorem 13. *Under Assumptions 1 - 5, for any $x \in \mathscr{J} = (a,b)$, it holds that*

$$\gamma(x) = \inf_{y \neq x} \eta^x(y),$$

where η^x is given by (12).

In order to prove Theorem 13, we need the following lemma.

Lemma 14. *For any $a \le y < x < z \le b$, it holds that*

$$f^x(y,z) \ge \min\{\eta^x(y), \eta^x(z)\}.$$

Proof. Suppose first that $a < y < x < z < b$. If $f^x(y,z) = 0$, then either $\eta^x(y) \le 0$ or $\eta^x(z) \le 0$. Indeed, if both expressions were positive, then since $f^x(y,z) = 0$, we would get (considering only the numerators of f^x in (7) and η^x in (12)) that

$$\begin{aligned} 0 &= \frac{\alpha_\varphi(x) - \alpha_\varphi(y)}{F(x) - F(y)} - \frac{\alpha_\varphi(z) - \alpha_\varphi(x)}{F(z) - F(x)} \\ &= \frac{\alpha_\varphi(x) - \alpha_\varphi(y)}{F(x) - F(y)} - d_F\alpha_\varphi(x) + d_F\alpha_\varphi(x) - \frac{\alpha_\varphi(z) - \alpha_\varphi(x)}{F(z) - F(x)} > 0, \end{aligned}$$

a contradiction.

If $f^x(y,z) > 0$, assume for the purpose of contradiction that the claim of the lemma is false, i.e.

$$\eta^x(z) = \frac{d_F\alpha_\varphi(x) - \frac{\alpha_\varphi(z)-\alpha_\varphi(x)}{F(z)-F(x)}}{d_F\beta_\varphi(x) - \frac{\beta_\varphi(z)-\beta_\varphi(x)}{F(z)-F(x)}} > \frac{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - \frac{\alpha_\varphi(z)-\alpha_\varphi(x)}{F(z)-F(x)}}{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - \frac{\beta_\varphi(z)-\beta_\varphi(x)}{F(z)-F(x)}} = f^x(y,z),$$

$$\eta^x(y) = \frac{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - d_F\alpha_\varphi(x)}{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - d_F\beta_\varphi(x)} > \frac{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - \frac{\alpha_\varphi(z)-\alpha_\varphi(x)}{F(z)-F(x)}}{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - \frac{\beta_\varphi(z)-\beta_\varphi(x)}{F(z)-F(x)}} = f^x(y,z).$$

Strict F-concavity of β_φ implies that all denominators are positive and the numerator of the right-hand side is also positive since $f^x(y,z) > 0$. Thus, we can rewrite the inequalities above as

$$\frac{d_F\alpha_\varphi(x) - \frac{\alpha_\varphi(z)-\alpha_\varphi(x)}{F(z)-F(x)}}{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - \frac{\alpha_\varphi(z)-\alpha_\varphi(x)}{F(z)-F(x)}} > \frac{d_F\beta_\varphi(x) - \frac{\beta_\varphi(z)-\beta_\varphi(x)}{F(z)-F(x)}}{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - \frac{\beta_\varphi(z)-\beta_\varphi(x)}{F(z)-F(x)}},$$

$$\frac{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - d_F\alpha_\varphi(x)}{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - \frac{\alpha_\varphi(z)-\alpha_\varphi(x)}{F(z)-F(x)}} > \frac{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - d_F\beta_\varphi(x)}{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - \frac{\beta_\varphi(z)-\beta_\varphi(x)}{F(z)-F(x)}}.$$

If we add both equations, we obtain $1 > 1$, which is absurd.

The case $f^x(y,z) < 0$ can be handled analogously.

The proof in the case $a < y < x, z = b$ is also along similar lines. Assume again that

$$\eta^x(b) = \kappa(x) = \frac{d_F\alpha_\varphi(x)}{d_F\beta_\varphi(x)} > \frac{\alpha_\varphi(x) - \alpha_\varphi(y)}{\beta_\varphi(x) - \beta_\varphi(y)} = f^x(y,b),$$

$$\eta^x(y) = \frac{\frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} - d_F\alpha_\varphi(x)}{\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)} - d_F\beta_\varphi(x)} > \frac{\alpha_\varphi(x) - \alpha_\varphi(y)}{\beta_\varphi(x) - \beta_\varphi(y)} = f^x(y,b).$$

If $f^x(y,b)=0$, then $\alpha_\varphi(x)=\alpha_\varphi(y)$, so we are led to the contradictory statement

$$d_F\alpha_\varphi(x)>0 \quad \text{and} \ -d_F\alpha_\varphi(x)>0.$$

If $f^x(y,b)\neq 0$, rearranging terms and adding the resulting equations as before, we also obtain a contradiction.

If $y=a$ and $z<b$, assume again that the claim of the lemma is false, i.e.

$$\eta^x(a)=\rho(x)=\frac{d_G\alpha_\psi(x)}{d_G\beta_\psi(x)}>\frac{\alpha_\psi(x)-\alpha_\psi(z)}{\beta_\psi(x)-\beta_\psi(z)}=f^x(a,z),$$

$$\eta^x(z)=\frac{d_G\alpha_\psi(x)-\frac{\alpha_\psi(z)-\alpha_\psi(x)}{G(z)-G(x)}}{d_G\beta_\psi(x)-\frac{\beta_\psi(z)-\beta_\psi(x)}{G(z)-G(x)}}>\frac{\alpha_\psi(x)-\alpha_\psi(z)}{\beta_\psi(x)-\beta_\psi(z)}=f^x(a,z).$$

Here we have used the representation of η^x involving G and ψ stated in (8). Since $\beta_\psi=1/\psi$ is G-concave and G is decreasing, this leads to a contradiction as before. Finally, if $y=a, z=b$, since $\psi'>0$ and $\varphi'<0$, clearly

$$\min\{\eta^x(a),\eta^x(b)\}=\min\left\{u(x)-u'(x)\frac{\psi(x)}{\psi'(x)},u(x)-u'(x)\frac{\varphi(x)}{\varphi'(x)}\right\}\leq u(x).$$

□

Proof. (Theorem 13). By Lemma 10 and Lemma 14, it is immediate that for all $x\in(a,b)$

$$\gamma(x)=\inf_{a\leq y<x<z\leq b}f^x(y,z)\geq\inf_{y\neq x}\eta^x(y).$$

To see the reverse inequality, note that for $z>x$,

$$\eta^x(z)=\lim_{u\uparrow x}f^x(u,z)\geq\inf_{a\leq u<x}f^x(u,z)\geq\gamma(x).$$

Similarly, for $y<x$,

$$\eta^x(y)=\lim_{u\downarrow x}f^x(y,u)\geq\inf_{x<u\leq b}f^x(y,u)\geq\gamma(x).$$

Thus,

$$\inf_{y\neq x}\eta^x(y)\geq\gamma(x).$$

□

3.2 Concave envelopes

According to Theorem 13, the computation of the universal stopping signal γ amounts to minimizing the function η^x given by (12) for every $x\in\mathscr{J}$. In the sequel, we show that under certain concavity conditions on u, the infimum of η^x is attained at the boundaries a or b of $\mathscr{J}$ which yields a closed form solution for $\gamma(x)$ since $\eta^x(a)=\kappa(x)$ (see (10)) and $\eta^x(b)=\rho(x)$ (see (11)). To this end, if $\varsigma\colon\mathscr{J}\to\mathbb{R}$ is a strictly monotone function, denote by $\widehat{u}^\varsigma$ the ς-concave envelope of u, i.e.

$$\widehat{u}^\varsigma(x)=\inf\{f(x):f\geq u \text{ and } f\ \ \varsigma\text{-concave}\}.$$

Provided that $\widehat{u}^{\varsigma}$ is finite, $\widehat{u}^{\varsigma}$ is itself ς-concave. We remark that $\widehat{u}^{\varsigma}(x) = \widehat{u \circ \varsigma^{-1}}(\varsigma(x))$ where $\widehat{u} = \widehat{u}^{id}$ refers to the usual concave envelope. With this concept at hand, we may now state the main result of this section.

Theorem 15. *Let $x \in \mathscr{J}$. Under Assumptions 1 - 5, we have*

$$\{\gamma = \rho\} = \{u = \widehat{u}^{\psi}\} \cap \{u' \geq 0\},$$

and

$$\{\gamma = \kappa\} = \{u = \widehat{u}^{\varphi}\} \cap \{u' \leq 0\}.$$

In other words, $\gamma(x) = \rho(x)$ (resp. $\gamma(x) = \kappa(x)$) iff the ψ-concave (resp. φ-concave) envelope of u coincides with the function u itself and $u'(x) \geq 0$ (resp. $u'(x) \leq 0$). Note that $\rho(x) = \kappa(x) = u(x)$ if $u'(x) = 0$. In order to prove Theorem 15, we need the following lemma.

Lemma 16. *Let $\varsigma \colon \mathscr{J} \to \mathbb{R}$ be differentiable and strictly monotonic.*

1. *Then u is ς-concave iff for all $x, y \in \mathscr{J}$*

$$u(x) + d_{\varsigma}u(x)(\varsigma(y) - \varsigma(x)) \geq u(y).$$

2. *Let $x \in \mathscr{J}$. Then $x \in \{u = \widehat{u}^{\varsigma}\}$ if and only if*

$$u(y) \leq u(x) + d_{\varsigma}u(x)(\varsigma(y) - \varsigma(x)) \text{ for all } y \in \mathscr{J}. \tag{13}$$

Proof. 1. Note that u is ς-concave on $\mathscr{J}$ if and only if $u \circ \varsigma^{-1}$ is concave on $\varsigma(\mathscr{J})$. This is equivalent to

$$u(\varsigma^{-1}(x)) + \left(\frac{d}{dx}u(\varsigma^{-1}(x))\right) \cdot (y - x) \geq u(\varsigma^{-1}(y)) \quad \text{for all } x, y \in \varsigma(\mathscr{J}).$$

The claim follows upon noting that for all $x \in \varsigma(\mathscr{J})$

$$\frac{d}{dx}u(\varsigma^{-1}(x)) = \frac{u'(\varsigma^{-1}(x))}{\varsigma'(\varsigma^{-1}(x))} = \left(d_{\varsigma}u\right)(\varsigma^{-1}(x)).$$

2. Assume that $x \in \{u = \widehat{u}^{\varsigma}\}$. By definition of the ς-concave envelope, the differentiability of u and ς and the first part of the lemma, it must hold for all y that

$$u(y) \leq \widehat{u}^{\varsigma}(y) \leq \widehat{u}^{\varsigma}(x) + d_{\varsigma}\widehat{u}^{\varsigma}(x)(\varsigma(y) - \varsigma(x)) = u(x) + d_{\varsigma}u(x)(\varsigma(y) - \varsigma(x)).$$

On the other hand, the ς-concave function $f(y) \triangleq u(x) + d_{\varsigma}u(x)(\varsigma(y) - \varsigma(x))$ satisfies $f(x) = u(x)$ and $f \geq u$ if (13) holds. This implies $u(x) = \widehat{u}^{\varsigma}(x)$. □

Proof. (Theorem 15) Let $x, y \in \mathscr{J}$. For $y < x$, $\eta^{x}(y) \geq \kappa(x)$ is equivalent to

$$\frac{\alpha_{\varphi}(x) - \alpha_{\varphi}(y)}{F(x) - F(y)} - d_F\alpha_{\varphi}(x) \geq \kappa(x) \cdot \left(\frac{\beta_{\varphi}(x) - \beta_{\varphi}(y)}{F(x) - F(y)} - d_F\beta_{\varphi}(x)\right).$$

We have used the fact that $\beta_\varphi = 1/\varphi$ is F-concave (Remark 5), F is increasing and part 1 of Lemma 16. Since $\kappa(x) = d_F\alpha_\varphi(x)/d_F\beta_\varphi(x)$, we may write

$$\eta^x(y) \geq \kappa(x) \Longleftrightarrow \frac{\alpha_\varphi(x)-\alpha_\varphi(y)}{F(x)-F(y)} \geq \kappa(x)\cdot\frac{\beta_\varphi(x)-\beta_\varphi(y)}{F(x)-F(y)}.$$

Some algebra shows that this is further equivalent to

$$u(x)+d_\varphi u(x)(\varphi(y)-\varphi(x)) \geq u(y).$$

If $y > x$, a similar calculation leads to the same result showing that $\eta^x(y) \geq \kappa(x) = \eta^x(b)$ for all $y \in \mathscr{J}$ if and only if

$$u(x)+d_\varphi u(x)(\varphi(y)-\varphi(x)) \geq u(y) \quad \text{for all } y \in \mathscr{J}. \tag{14}$$

In view of part 2 of Lemma 16, we deduce from (14) that $\kappa(x) \leq \eta^x(y)$ for all $y \in \mathscr{J}$ if and only if $x \in \{u = \widehat{u}^\varphi\}$. Now $\kappa(x) \leq \eta^x(a) = \rho(x)$ if and only if

$$u(x) - u'(x)\frac{\varphi(x)}{\varphi'(x)} \leq u(x) - u'(x)\frac{\psi(x)}{\psi'(x)}$$

which is equivalent to $u'(x) \leq 0$. This proves the first assertion of the theorem.

The proof of the second claim is along the same lines. For $a < y < x$, $\eta^x(y) \geq \rho(x)$ is equivalent to

$$\frac{\alpha_\psi(x)-\alpha_\psi(y)}{G(x)-G(y)} - d_G\alpha_\psi(x) \geq \rho(x)\cdot\left(\frac{\beta_\psi(x)-\beta_\psi(y)}{G(x)-G(y)} - d_G\beta_\psi(x)\right).$$

Since $\rho = d_G\alpha_\psi/d_G\beta_\psi$, $\eta^x(y) \geq \rho(x)$ for $y < x$ is further equivalent to

$$\alpha_\psi(x)-\alpha_\psi(y) \geq \rho(x)(\beta_\psi(x)-\beta_\psi(y)).$$

One can show that $\eta^x(y) \geq \rho(x)$ for all $y \in \mathscr{J}$ if and only if

$$u(x)+d_\psi u(x)(\psi(y)-\psi(x)) \geq u(y) \quad \text{for all } y \in \mathscr{J}. \tag{15}$$

Since $\rho(x) \leq \kappa(x) = \eta^x(a)$ iff $u'(x) \geq 0$, the proof is complete in view of (15) and Lemma 16. □

We can now prove that a monotone stopping signal γ is equal either to κ or to ρ. Thus, by Theorem 15, a nondecreasing (resp. nonincreasing) signal corresponds to a nondecreasing (resp. nonincreasing) reward function u that is ψ-concave (resp. φ-concave). In order to prove this claim, we first show that γ is upper semi-continuous.

Lemma 17. *The function γ is upper semi-continuous on $\mathscr{J}$.*

Proof. For fixed k, $V_k(\cdot)$ is continuous on $\mathscr{J}$. Moreover, for fixed $x \in \mathscr{J}$, $k \mapsto V_k(x)$ is Lipschitz continuous with Lipschitz constant 1 which in turn yields that $(k,x) \mapsto V_k(x)$ is continuous. Therefore, the hypograph $\{(x,k) : \gamma(x) \geq k\} = \{(x,k) : V_k(x) = u(x) - k\}$ (Lemma 9) is closed or equivalently, γ is upper semi-continuous. □

Corollary 18. *Under Assumptions 1 - 5, γ is nonincreasing on $\mathcal{J}$ if and only if $\gamma = \kappa$ and γ is nondecreasing on $\mathcal{J}$ if and only if $\gamma = \rho$.*

Proof. Assume that γ is nonincreasing. By Lemma 17, γ must be left-continuous. Let $x \in \mathcal{J}$ and set $k = \gamma(x)$.

If $\{\gamma > k\} = \emptyset$, we have that $\gamma(y) = k$ for all $y \in (a, x]$ and

$$V_l(y) = \begin{cases} u(y) - l, & l \leq k = \gamma(y), \\ 0, & l > \gamma(y). \end{cases}$$

Since $k \mapsto V_k(y)$ is continuous for any $y \in \mathcal{J}$, we deduce that $u(y) - k = 0$ for all $y \in (a, x]$. In particular, $u'(x) = 0$ and therefore, $\kappa(x) = u(x) = k = \gamma(x)$.

Assume next that $\{\gamma > k\} \neq \emptyset$. If $\{\gamma = k\} = \{x\}$, then for $y > x$, we have due to Lemma 1 that

$$\begin{aligned} V_k(y) &= E_y\left[e^{-R_{T_x}}(u(X_{T_x}) - k)\right] = \frac{\varphi(y)}{\varphi(x)}(u(x) - k) \\ &= \sup_{v<y} E_y\left[e^{-R_{T_v}}(u(X_{T_v}) - k)\right] = \sup_{v<y} \frac{\varphi(y)}{\varphi(v)}(u(v) - k). \end{aligned}$$

Set $g(v) \triangleq \varphi(y)/\varphi(v)(u(v) - k)$ for $v \leq y$. Note that $g(y) = u(y) - k < V_k(y)$ since $y \notin \Gamma_k$. Thus, g attains its maximum at the interior point x and x must satisfy $g'(x) = 0$ which is equivalent to $\kappa(x) = k$. By definition, also $\gamma(x) = k$.

Next, consider the case $\{\gamma = k\} = (x_1, x_2]$ where $a < x_1 < x_2 < b$ and $x \in (x_1, x_2]$. For all $y \in (x_1, x_2]$, we have that $V_k(y) = u(y) - k$ since $\gamma(y) = k$ amounts to stopping immediately. Moreover, we claim that

$$\hat{\tau}_k \triangleq \inf\{t : \gamma(X_t) > k\} = T_{x_1}$$

is also an optimal stopping time for the parameter k if the diffusion is started at $y \in (x_1, x_2]$. Indeed, since $\{\gamma > k\} \neq \emptyset$, there is $\hat{a}$ such that $\gamma(\hat{a}) > k$. Set $\epsilon = \gamma(\hat{a}) - \gamma(x) > 0$. Then $\tau_{k+\epsilon/n} \downarrow \hat{\tau}_k$ P_y-a.s. for any $y \in \mathcal{J}$ and $X_{\tau_{k+\epsilon/n}} \in [\hat{a}, x)$ P_y-a.s. for any $y > \hat{a}$. Applying Fatou, we obtain

$$\begin{aligned} V_k(y) &= \limsup_{n\to\infty} V_{k+\epsilon/n}(y) = \limsup_{n\to\infty} E_y\left[e^{-R(\tau_{k+\epsilon/n})}(u(X_{\tau_{k+\epsilon/n}}) - (k + \frac{\epsilon}{n}))\right] \\ &\leq E_y\left[e^{-R(\hat{\tau}_k)}(u(X_{\hat{\tau}_k}) - k)\right] \leq V_k(y), \end{aligned}$$

which proves the claim that $\hat{\tau}_k$ is optimal.

Thus, for any $y \in (x_1, x_2]$, we have that

$$V_k(y) = u(y) - k = E_y\left[e^{-R(T_{x_1})}(u(X_{T_{x_1}}) - k)\right] = \frac{\varphi(y)}{\varphi(x_1)}(u(x_1) - k),$$

showing that u is an affine transformation of φ on $(x_1, x_2]$, i.e. $u(y) = c \cdot \varphi(y) + k$ for some constant $c = c(x_1) > 0$. Now

$$\kappa(x) = u(x) - u'(x)\frac{\varphi(x)}{\varphi'(x)} = k = \gamma(x).$$

The same reasoning applies in the remaining case $\{\gamma = k\} = (x_1, b)$.

On the other hand, if $\gamma = \kappa$ on $\mathscr{J}$, u must be nonincreasing and φ-concave on $\mathscr{J}$ due to Theorem 15. By Lemma 16, we have for all $x, y \in \mathscr{J}$ that

$$u(x) + d_\varphi u(x)(\varphi(y) - \varphi(x)) \geq u(y)$$

which we may rewrite as

$$\kappa(x) = u(x) - d_\varphi u(x)\varphi(x) \geq u(y) - d_\varphi u(x)\varphi(y). \tag{16}$$

Note that $d_\varphi u(x) = (u \circ \varphi^{-1})'(\varphi(x))$. Since $u \circ \varphi^{-1}$ is concave, its derivative is nonincreasing showing that $d_\varphi u(\cdot)$ is nondecreasing. Thus, if $x \leq y$, we see from (16) that $\kappa(x) \geq \kappa(y)$. The proof of the second assertion is similar. □

Let us further investigate the structure of the stopping signal γ if the reward function u is monotone. For instance, if u is nonincreasing, we know from Theorem 15 that $\gamma(x) = \kappa(x)$ iff $u(x) = \widehat{u}^\varphi(x)$. Since the set $\{\widehat{u}^\varphi > u\}$ is open, it is the countable union of disjoint open intervals which we denote by $(l_n, r_n), n \in \mathbb{N}$. In this setting, we have the following result:

Proposition 19. *If u is nonincreasing on $\mathscr{J}$ and $[l_n, r_n] \subset \mathscr{J}$, then*

$$\gamma(l_n) = \kappa(l_n) = \kappa(r_n) = \gamma(r_n)$$

and $\gamma(x) < \kappa(l_n)$ for all $x \in (l_n, b_n)$.

Proof. By Theorem 15, we have $\gamma(l_n) = \kappa(l_n)$ and $\gamma(r_n) = \kappa(r_n)$. From the definition of the φ-concave envelope, we see that

$$d_\varphi u(l_n) = d_\varphi u(r_n), \quad \widehat{u}^\varphi(x) = u(l_n) + d_\varphi u(l_n)(\varphi(x) - \varphi(l_n)), \quad x \in [l_n, r_n].$$

This implies that $\kappa(l_n) = \kappa(r_n)$.

It remains to show that $\gamma(x) < \kappa(l_n)$ for all $x \in (l_n, b_n)$. Indeed, if $\gamma(x) \geq \kappa(l_n)$ or equivalently, $x \in \Gamma_{\kappa(l_n)}$ for some $x \in (l_n, b_n)$, then

$$\begin{aligned} V_{\kappa(l_n)}(x) = u(x) - \kappa(l_n) &\geq E_x\left[e^{-R(T_{l_n})}(u(X_{T_{l_n}}) - \kappa(l_n))\right] \\ &= \frac{\varphi(x)}{\varphi(l_n)}(u(l_n) - \kappa(l_n)) = d_\varphi u(l_n)\varphi(x). \end{aligned}$$

This is equivalent to $u(x) \geq u(l_n) + d_\varphi u(l_n)(\varphi(x) - \varphi(l_n)) = \widehat{u}^\varphi(x)$, which is impossible since $x \in \{\widehat{u}^\varphi > u\}$. □

In particular, the function $\overline{\kappa} : \mathscr{J} \to \mathbb{R}$ defined by

$$\overline{\kappa}(x) = \begin{cases} \kappa(x), & x \in \{u = \widehat{u}^\varphi\}, \\ \kappa(l_n), & x \in (l_n, r_n) \text{ for some } n, \end{cases}$$

defines a nonincreasing majorant of γ and coincides with the optimal stopping signal on $\{u = \widehat{u}^\varphi\}$. Note that if the diffusion starts at $x_0 \in \{u = \widehat{u}^\varphi\}$, optimal stopping times can be derived directly from $\overline{\kappa}$. Thus, the computation of γ on the intervals (l_n, r_n) as the infimum of η^x as suggested by Theorem 13 is not required in that case.

Of course, an analogous result holds if u is nondecreasing. Again, $\{\widehat{u}^\psi > u\}$ is the countable union of disjoint open intervals denoted by (l_n, r_n) and we have

Proposition 20. *If u is nondecreasing on $\mathscr{J}$ and $[l_n, r_n] \subset \mathscr{J}$, then*

$$\gamma(l_n) = \rho(l_n) = \rho(r_n) = \gamma(r_n)$$

and $\gamma(x) < \rho(l_n)$ for all $x \in (l_n, b_n)$.

4 Applications and illustrations

For simplicity, we illustrate our results for a constant discount rate $r > 0$ which implies that $R_t = rt$. Examples with random discounting can be found in [Dayanik(2008)] and [Beibel and Lerche(2000)].

4.1 Optimal stopping

Example 21. Let us consider the well-known example of a perpetual put option written on an underlying whose dynamics are given by a geometric Brownian motion with volatility $\sigma > 0$ and drift $\mu \in \mathbb{R}$, i.e. consider the diffusion

$$dX_t = \sigma X_t \, dB_t + \mu X_t \, dt$$

on $\mathscr{J} = (0, \infty)$. The functions φ and ψ are the solutions to the ODE (2) which is

$$\frac{1}{2}\sigma^2 x^2 w''(x) + \mu x w'(x) = r w(x), \quad x \in (0, \infty).$$

One finds that $\varphi(x) = x^\alpha$ and $\psi(x) = x^\beta$, $x \in \mathscr{J}$ where $\alpha < 0$ and $\beta > 0$ are the roots of $q(x) = \frac{1}{2}\sigma^2 x(x-1) + \mu x - r$.

Let us show how to compute the optimal stopping signal for the family of American put options whose payoff profile is given by $h_k(x) = (k - x)^+$ where $k > 0$. The corresponding optimal stopping problem is

$$V_k(x) = \sup_{\tau \in \mathscr{S}} E_x \left[e^{-r\tau} (k - X_\tau)^+ \right].$$

Clearly, the quantities ℓ_0 and ℓ_∞ defined in Proposition 6 are equal to zero for any k since h_k is bounded. Thus, the stopping times $\tau_k \triangleq \inf\{t : X_t \in \Gamma_k\}$ where $\Gamma_k = \{x : V_k(x) = (k - x)^+\}$ are optimal by Proposition 6 for any strike k. Note that $V_k > 0$ since X reaches the point $k/2$ with positive probability which in turn leads to a positive payoff. It follows that $V_k(x) = E_x \left[e^{-r\tau_k}(k - X_{\tau_k}) \right] = E_x \left[e^{-r\tau_k}(u(X_{\tau_k}) - (-k)) \right]$ for any $k > 0$, $u(x) = -x$ and $x \in \mathscr{J}$. Since u is decreasing and $u \circ \varphi^{-1}(x) = -x^{1/\alpha}$ is concave on $\mathscr{J}$, i.e u is φ-concave, we deduce from Theorem 15 that

$$\gamma(x) = \kappa(x) = u(x) - u'(x) \cdot \frac{\varphi(x)}{\varphi'(x)} = x \cdot (\frac{1}{\alpha} - 1).$$

Example 22. This example considers a reward function u that is not monotone. Nonetheless, u is φ- and ψ-concave and the optimal stopping signal can be computed explicitly in view of Theorem 15.

Assume that X is a standard Brownian motion and $r \leq 1$. The functions φ and ψ are given by

$$\varphi(x) = \exp(-\sqrt{r}x), \quad \psi(x) = \exp(\sqrt{r}x), \quad x \in \mathbb{R}.$$

Consider the reward function $u(x) = -\cosh(x)$. Then

$$u \circ \psi^{-1}(x) = u \circ \varphi^{-1}(x) = -\frac{1}{2}\left(x^{1/\sqrt{r}} + x^{-1/\sqrt{r}}\right), \quad x \in (0,\infty).$$

One checks that $u \circ \psi^{-1}$ and $u \circ \varphi^{-1}$ are concave if $r \leq 1$. Since $\{u' \geq 0\} = (-\infty, 0]$ and $\{u' \leq 0\} = [0,\infty)$, we deduce from Theorem 15 that

$$\gamma(x) = \begin{cases} \rho(x) = -\cosh(x) + \frac{\sinh(x)}{\sqrt{r}}, & x \leq 0 \\ \kappa(x) = -\cosh(x) - \frac{\sinh(x)}{\sqrt{r}}, & x > 0. \end{cases}$$

Note that $\gamma(-x) = \gamma(x)$ for all $x \in \mathbb{R}$ which is clear by symmetry.

Example 23. Finally, we provide an example of a diffusion and a decreasing reward function u such that u is not φ-concave. The optimal stopping signal γ is computed numerically on the interval where u and its concave envelope do not coincide. To this end, consider a diffusion X specified by the SDE

$$dX_t = \sigma X_t \, dB_t + \mu \cdot \operatorname{sign}(X_t - c) X_t \, dt, \qquad X_0 = x_0 \in \mathscr{J} = (0,\infty)$$

for $\sigma, \mu, c > 0$. We would like to derive the optimal stopping signal related to the perpetual American put option, i.e. $u(x) = -x$. The functions ψ and φ must solve the ODE (2), i.e.

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 w''(x) - \mu x w'(x) = r w(x), \quad x \in (0,c), \\ \frac{1}{2}\sigma^2 x^2 w''(x) + \mu x w'(x) = r w(x), \quad x \in (c,\infty). \end{aligned}$$

One computes

$$\varphi(x) = \begin{cases} c^{\beta_1 - \alpha_1} \left(\frac{\beta_1 - \alpha_2}{\alpha_1 - \alpha_2}\right) \cdot x^{\alpha_1} + c^{\beta_1 - \alpha_2} \left(\frac{\alpha_1 - \beta_1}{\alpha_1 - \alpha_2}\right) \cdot x^{\alpha_2}, & x \in (0,c], \\ x^{\beta_1}, & x \in [c,\infty), \end{cases}$$

and

$$\psi(x) = \begin{cases} x^{\alpha_2}, & x \in (0,c], \\ c^{\alpha_2 - \beta_1} \left(\frac{\alpha_2 - \beta_2}{\beta_1 - \beta_2}\right) \cdot x^{\beta_1} + c^{\alpha_2 - \beta_2} \left(\frac{\beta_1 - \alpha_2}{\beta_1 - \beta_2}\right) \cdot x^{\beta_2}, & x \in (c,\infty), \end{cases}$$

where $\alpha_1 < 0, \alpha_2 > 0$ are the roots of $q_1(x) = \frac{1}{2}\sigma^2 x(x-1) - \mu x - r$ and $\beta_1 < 0, \beta_2 > 0$ are the roots of $q_2(x) = \frac{1}{2}\sigma^2 x(x-1) + \mu x - r$. In order to compute the universal stopping signal, we apply Theorem 15. Since u is decreasing, let us check whether u is φ-concave which is equivalent to convexity of φ. Clearly, the restriction of φ to $[c,\infty)$ is convex. One can show that φ'' has a root on $(0,c)$ iff

$$\frac{\sigma^2}{2} + \frac{r\sigma^2}{\mu} < \mu + \sqrt{(\frac{\sigma^2}{2} - \mu)^2 + 2r\sigma^2}.$$

Note that the right-hand side of the inequality above is increasing in μ. Thus, if the drift coefficient μ is sufficiently large compared to σ and r, φ changes its curvature and is not convex on $(0,\infty)$.

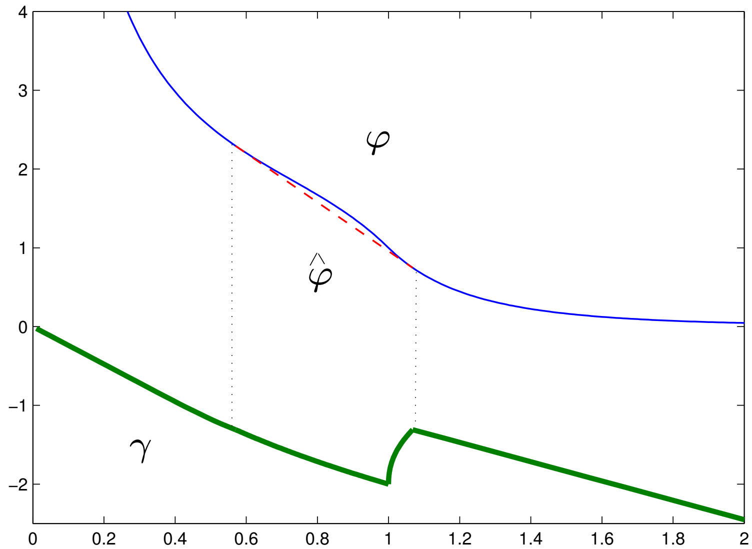

Figure 1: The function φ (solid thin line), its convex envelope $\hat{\varphi}$ (dashed line) and the universal stopping signal γ (bold line) for the parameters $r = 0.05$, $\sigma = 0.15$, $c = 1$.

For simplicity, assume that $c = 1$ and $r = \mu$. Then $u \circ \varphi^{-1}$ is not concave (or equivalently, φ is not convex) if $\sigma^2/2 < r$, see Figure 1.

It is now easy to determine the set $\{u = \widehat{u}^{\varphi}\}$. In view of Lemma 16, we find that $x \in \{u = \widehat{u}^{\varphi}\}$ if and only if

$$\varphi(y) \geq \varphi(x) + \varphi'(x)(y - x) \quad \text{for all } y.$$

The equation above has an obvious graphical interpretation: If (and only if) the tangent that touches the graph of φ at x remains below the graph of φ everywhere, then $x \in \{u = \widehat{u}^{\varphi}\}$; see Figure 1. We see that $\{\widehat{u}^{\varphi} > u\} = (\hat{a}, \hat{b})$ for unique $0 < \hat{a} < c < \hat{b}$. According to Theorem 15, we have $\gamma(x) = \kappa(x)$ for $x \notin (\hat{a}, \hat{b})$ whereas the value of $\gamma(x)$ can be computed numerically as the infimum over η^x given by (12) for $x \in (\hat{a}, \hat{b})$. The resulting stopping signal is shown in Figure 1. Note that not all the optimal stopping regions Γ_k are connected in this case.

4.2 Gittins indices

The approach described in this section can also be applied to the class of optimal stopping problems that appears in [Karatzas(1984)], namely

$$V_k(x) \triangleq \sup_{\tau \in \mathscr{S}} E_x \left[\int_0^\tau e^{-rt} u(X_t)\, dt + k e^{-r\tau} \right]. \tag{17}$$

These stopping problems arise in the context of multi-armed bandit problems where one seeks to dynamically allocate limited resources among several independent projects in order to maximize the discounted reward generated by these projects. The evolution of each project is modeled by a stochastic process such as a one-dimensional diffusion or a Lévy process (see [Kaspi and Mandelbaum(1995)]) and $u(x)$ specifies the reward generated by allocating resources to a project in state x. Optimal allocation rules can be found by the projects' Gittins indices, see e.g. [Gittins(1979), Gittins and Jones(1979)]. An optimal strategy is to engage in the project whose index is maximal. Specifically, the Gittins index of a project with diffusion dynamics X as in [Karatzas(1984)] and reward function u can be derived from the solution to (17). This optimal stopping problem can be reduced to a problem of type (4). To this end, set $p(x) \triangleq E_x\left[\int_0^\infty e^{-rt}u(X_t)\,dt\right]$. It follows from the strong Markov property that $e^{-r\tau}p(X_\tau) = E_x\left[\int_\tau^\infty e^{-rt}u(X_t)\,dt|\mathscr{F}_\tau\right]$ P_x-a.s. Thus,

$$\begin{aligned} V_k(x) &= \sup_{\tau\in\mathscr{S}}\left(p(x) - E_x\left[\int_\tau^\infty e^{-rt}u(X_t)\,dt\right] + E_x\left[ke^{-r\tau}\right]\right) \\ &= p(x) + \sup_{\tau\in\mathscr{S}} E_x\left[e^{-r\tau}(-p(X_\tau)) + ke^{-r\tau}\right]. \end{aligned}$$

This is now an optimal stopping problem of type (4). Note that the function p is a particular solution to the non-homogeneous linear ODE $\mathscr{A}f - rf = u$ on $\mathscr{J}$ since $p = U_r u$ where U denotes the resolvent corresponding to the diffusion X.

If the state of each project is given by a one-dimensional regular diffusion, one can show (Theorem 4.1 in [Karatzas(1984)], also [Gittins and Glazebrook(1977)]) that the Gittins index M is given by

$$\begin{aligned} M(x) &= \sup_{\tau>0}\frac{p(x) - E_x\left[\int_\tau^\infty e^{-rt}u(X_t)\,dt\right]}{1 - E_x\left[e^{-r\tau}\right]} = \sup_{\tau>0}\frac{p(x) - E_x\left[e^{-r\tau}p(X_\tau)\right]}{1 - E_x\left[e^{-r\tau}\right]} \\ &= -\inf_{\tau>0}\frac{-p(x) - E_x\left[e^{-r\tau}(-p(X_\tau))\right]}{1 - E_x\left[e^{-r\tau}\right]}. \end{aligned}$$

This show that the Gittins index is basically the optimal stopping signal γ associated with the optimal stopping problem (4) for the reward function $-p$.

In [Karatzas(1984)], the function u is assumed to be increasing, continuously differentiable and bounded. Then p inherits the same properties. Note that adding a positive constant c to the reward function u only shifts M upwards by the amount c/r. Thus, we may assume without loss of generality that u and p are positive. Using that $(\mathscr{A} - r)p = u$ and $(\mathscr{A} - r)\varphi = 0$, one can check that $-p \circ \varphi^{-1}$ is strictly concave in that case. Therefore, we apply Theorem 15 to deduce that the optimal stopping signal is given by $\gamma(x) = -p(x) + p'(x)\cdot(\varphi(x)/\varphi'(x))$ recovering Lemma 3.1 of [Karatzas(1984)]. We may also use Theorem 15 to compute the Gittins index if u is not increasing.

4.3 A stochastic representation problem

Let us now consider the stochastic representation problem

$$e^{-rT}u(X_T) = E_x\left[\int_T^\infty re^{-rt}\sup_{T\le s\le t} K_s\,dt\,\middle|\,\mathscr{F}_T\right], \qquad T\in\mathscr{S}, \tag{18}$$

i.e. for a given function u and a constant $r > 0$, find an optional process K such that the equation above holds. If the representation above holds for some K, then for any $k \in \mathbb{R}$ and any stopping time T, we have

$$E_x\left[e^{-rT}(u(X_T)-k)\right] = E_x\left[\int_T^\infty re^{-rt}\sup_{T\le s\le t}(K_s-k)\,dt\right]$$
$$\le E_x\left[\int_T^\infty re^{-rt}\sup_{T\le s\le t}(K_s-k)^+\,dt\right].$$

Moreover, if we choose $T^k \triangleq \inf\{t : K_t \ge k\}$, the inequality above becomes an equality provided that the process K has upper semicontinuous paths (ensuring that $K_{T^k} \ge k$ if T^k is finite). Thus, the process K characterizes optimal stopping times for any parameter k just as the process $(\gamma(X_t))_{t\ge 0}$ in the previous section. Hence, there is a close relation of optimal stopping and stochastic representation problems and we refer to [Bank and Föllmer(2003)] for an overview, see also [Bank and El Karoui(2004)].

Note that if (18) holds, then choosing $T=0$ yields

$$u(x) = E_x\left[\int_0^\infty re^{-rt}\sup_{0\le s\le t}K_s\,dt\right]. \tag{19}$$

On the other hand, if (19) is satisfied, then by the strong Markov property, it holds for any $T\in\mathscr{S}$ that

$$e^{-rT}u(X_T) = e^{-rT}E_x\left[\int_0^\infty re^{-rt}\sup_{0\le s\le t}K_{s+T}\,dt\,\middle|\,\mathscr{F}_T\right]$$
$$= E_x\left[\int_0^\infty re^{-r(t+T)}\sup_{T\le s\le t+T}K_s\,dt\,\middle|\,\mathscr{F}_T\right] = E_x\left[\int_T^\infty re^{-rt}\sup_{T\le s\le t}K_s\,dt\,\middle|\,\mathscr{F}_T\right].$$

Therefore, solving (18) is equivalent to solving (19). The latter problem was considered by [El Karoui and Föllmer(2005)] in a more general setting without discounting. One can proceed in an analogous fashion in order to prove that $(K_t)_{t\ge 0}$ defined by $K_t \triangleq \gamma(X_t)$ solves the representation problem (18) if the diffusion X satisfies some additional uniform integrability assumptions. Let us just briefly outline the main ideas.

Let $\mathscr{T}(x)$ denote the class of all exit times from relatively compact open neighborhoods of x. Recall that any $T\in\mathscr{T}(x)$ is P_x-a.s. finite [Revuz and Yor(1991), Prop. VII.3.1]. Using the notation of El-Karoui and Föllmer, we set

$$\underline{D}u(x) \triangleq \inf_{T\in\mathscr{T}(x)} \frac{u(x)-E_x\left[e^{-rT}u(X_T)\right]}{E_x\left[1-e^{-rT}\right]}, \quad \overline{G}f(x) \triangleq E_x\left[\int_0^\infty re^{-rt}\sup_{0\le s\le t}f(X_s)\,dt\right].$$

As in [El Karoui and Föllmer(2005)], we say that a measurable function $u\colon \mathscr{J}\to\mathbb{R}$ is of class (D) if the family $\{e^{-rT}u(X_T) : T\in\mathscr{T}(x)\}$ is uniformly integrable with respect to P_x for all $x\in\mathscr{J}$.

Assumption 6. *The function u is continuous on $\mathscr{J}$, of class (D) and for all $x\in\mathscr{J}$, it holds that*

$$\lim_{t\to\infty} e^{-rt}u(X_t) = 0 \quad P_x\text{-a.s.}$$

Lemma 24. *Under Assumption 6, it holds that $\gamma = \underline{D}u$ on $\mathscr{J}$ where γ is given by* (5).

Proof. The inequality $\gamma(x) \leq \underline{D}u(x)$ is clear since $\mathscr{T}(x) \subset \tilde{\mathscr{T}}(x)$. In order to prove the reverse inequality, denote by $(U_n)_n$ a sequence of relatively compact open sets increasing to $\mathscr{J}$, and denote by S^n the exit time from the set U_n. For $T \in \tilde{\mathscr{T}}(x)$, the stopping times $T^n \triangleq T \wedge S^n \in \mathscr{T}(x)$ increase to T, and so we have $E_x\left[1-e^{-rT^n}\right] \uparrow E_x\left[1-e^{-rT}\right]$.

Moreover, since u is of class (D) by Assumption 6, it follows that

$$\lim_{n\to\infty} E_x\left[e^{-rT^n}u(X_{T^n})\right] = E_x\left[e^{-rT}u(X_T)\right].$$

Therefore, for any $T \in \tilde{\mathscr{T}}(x)$,

$$\frac{E_x\left[u(x)-e^{-rT}u(X_T)\right]}{E_x\left[1-e^{-rT}\right]} = \lim_{n\to\infty}\frac{E_x\left[u(x)-e^{-rT^n}u(X_{T^n})\right]}{E_x\left[1-e^{-rT^n}\right]} \geq \underline{D}u(x),$$

proving that

$$\gamma(x) = \inf_{T\in\tilde{\mathscr{T}}(x)} \frac{E_x\left[u(x)-e^{-rT}u(X_T)\right]}{E_x\left[1-e^{-rT}\right]} \geq \underline{D}u(x).$$

□

We can now prove the following uniqueness result which corresponds to Theorem 3.1 of [El Karoui and Föllmer(2005)] in our modified setting.

Proposition 25. *Under Assumption 6, suppose that the function $f : \mathscr{J} \to \mathbb{R}$ is upper-semicontinuous and*

$$u(x) = \overline{G}f(x) = E_x\left[\int_0^\infty re^{-rt}\sup_{0\leq s\leq t} f(X_s)\,dt\right].$$

Then $f(x) = \underline{D}u(x)$.

Proof. By the strong Markov property, it holds for any $T \in \mathscr{T}(x)$ that

$$\begin{aligned}\overline{G}f(X_T) &= E_x\left[\int_0^\infty re^{-rt}\sup_{0\leq s\leq t} f(X_{s+T})\,dt\,\middle|\,\mathscr{F}_T\right]\\ &= e^{rT}E_x\left[\int_0^\infty re^{-r(t+T)}\sup_{T\leq s\leq t+T} f(X_s)\,dt\,\middle|\,\mathscr{F}_T\right]\\ &= e^{rT}E_x\left[\int_T^\infty re^{-rt}\sup_{T\leq s\leq t} f(X_s)\,dt\,\middle|\,\mathscr{F}_T\right].\end{aligned}$$

It follows that

$$\begin{aligned}&u(x) - E_x\left[e^{-rT}u(X_T)\right] = \overline{G}f(x) - E_x\left[e^{-rT}\overline{G}f(X_T)\right]\\ &= E_x\left[\int_0^T re^{-rt}\sup_{0\leq s\leq t} f(X_s)\,dt + \int_T^\infty re^{-rt}\left(\sup_{0\leq s\leq t} f(X_s) - \sup_{T\leq s\leq t} f(X_s)\right)dt\right]\\ &\geq E_x\left[\int_0^T re^{-rt}\sup_{0\leq s\leq t} f(X_s)\,dt\right] \geq f(x)E_x\left[\int_0^T re^{-rt}\,dt\right] = f(x)E_x\left[1-e^{-rT}\right].\end{aligned} \tag{20}$$

This implies that $\underline{D}u(x) \geq f(x)$.

To prove the reverse inequality, fix $\alpha > f(x)$ and denote by T^α the exit time from the set $\{f < \alpha\}$. Since $\limsup_{y\to x} f(y) \leq f(x)$ for all x by upper-semicontinuity of f, the set $\{f \geq \alpha\}$ is closed. Thus, T^α is indeed an exit time from an open neighborhood of x. For $t \geq T^\alpha$, it holds P_x-a.s. that

$$\sup_{0\leq s\leq t} f(X_s) = \sup_{T^\alpha\leq s\leq t} f(X_s)$$

and the first inequality in (20) becomes an equality for $T = T^\alpha$. Thus,

$$E_x\left[u(x) - e^{-rT}u(X_{T^\alpha})\right] = E_x\left[\int_0^{T^\alpha} re^{-rt}\sup_{0\leq s\leq t} f(X_s)\,dt\right] \leq \alpha E_x\left[1 - e^{-rT^\alpha}\right].$$

Now take a sequence $\alpha_n \downarrow f(x)$ to conclude that

$$\gamma(x) = \inf_{T\in\tilde{\mathcal{T}}(x)} \frac{u(x) - E_x\left[e^{-rT}u(X_T)\right]}{E_x\left[1 - e^{-rT}\right]} \leq f(x).$$

The proof is complete due to Lemma 24. □

In order to prove that the process $(\gamma(X_t))_{t\geq 0}$ does indeed solve (18) under Assumption 6 or equivalently, $u(x) = \overline{G}\underline{D}u(x)$ for $x \in \mathcal{J}$, one can proceed in the spirit of [El Karoui and Föllmer(2005)].

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