Relaxation Schemes for Interacting Exclusions

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Abstract

We investigate the interaction of one-dimensional asymmetric exclusion processes of opposite speeds, where the exchange dynamics is combined with a creation-annihilation mechanism, and this asymmetric law is regularized by a nearest neighbor stirring of large intensity. The model admits hyperbolic (Euler) scaling, and we are interested in the hydrodynamic behavior of the system in a regime of shocks on the infinite line. This work is a continuation of a previous paper by Fritz and Nagy [FN06], where this question has been left open because of the lack of a suitable logarithmic Sobolev inequality. The problem is solved by extending the method of relaxation schemes to this stochastic model, the resulting a priory bound allows us to verify compensated compactness.

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1 Introduction and Main Result

A rigorous treatment of hyperbolic (Euler) scaling problems requires specific conditions because a direct compactness argument is not available; the celebrated Two Blocks Lemma, see Theorem 4.6 and Theorem 4.7 of [GPV88], can not be extended to macroscopic block averages. Just as in the case of parabolic energy inequalities, the diffusive (elliptic) component of the microscopic evolution does vanish in a hyperbolic scaling limit, thus we can not control spatial fluctuations. This problem is closely related to the formation of shock waves resulting in a breakdown of existence of classical solutions to the macroscopic equations. Assuming smoothness of the macroscopic solution, this difficulty can be avoided by means of the relative entropy method of [Yau91], *hydrodynamic limit* (HDL) for a large class of models can be derived in this way. In the case of *attractive systems* coupling and other specific techniques can be applied even in a regime of shocks, therefore advanced methods of PDE theory, as the entropy condition of S. Kruzkov play a crucial role in the proofs, see e.g. [Rez91], [KL99] and [Bah04] with some further references. Since the models of [FT04] and [FN06] are not attractive, a new tool, the stochastic theory of *compensated compactness* is used there to pass to the hydrodynamic limit; the microscopic *entropy flux* is evaluated by means of a *logarithmic Sobolev inequality* (LSI). Our present model is more difficult, these techniques alone are not sufficient to control the *mechanism of creation and annihilation*. The main purpose of this paper is to extend the PDE method of *relaxation schemes* to microscopic systems with a hyperbolic scaling, we are going to expose several versions of the argument.

1.1. Relaxation schemes: In case of non - attractive systems with general initial conditions the method of compensated compactness, cf. [Tar79] or [Ser00] and [Daf05] is an effective tool, its applications to microscopic systems are discussed in the papers [Fri01], [Fri04], [FT04], [FN06] and [Fri09]. However, compensated compactness alone is not sufficient in the present situation, it should be supplemented by another tool called the method of *relaxation schemes* in the PDE literature, see [Liu87], [CL93] and [CLL94] for the first results, [GT00] or [Daf05] for explanation and further discussions. The basic idea is not difficult: the single conservation law $\partial_t u + \partial_x f(u) = 0$ in one space dimension can be obtained as the *zero relaxation limit* of the linear system

$$\partial_t u_\varepsilon + \partial_x v_\varepsilon = 0, \quad \partial_t v_\varepsilon + \partial_x u_\varepsilon = \varepsilon^{-1} B(u_\varepsilon, v_\varepsilon)(f(u_\varepsilon) - v_\varepsilon) \tag{1.1}$$

with a nonlinear source (relaxation) term on its right hand side. We have $B > 0$, so unless $(f(u_\varepsilon) - v_\varepsilon) \to 0$, this term might explode if $0 < \varepsilon \to 0$; but the negative sign of $v_\varepsilon/\varepsilon$ is encouraging. To see a successful relaxation, we have to find a clever Liapunov function $h = h(u, v)$ such that $h'_v(u, v) B(u, v)(f(u) - v) \le -b(f(u) - v)^2$ with some $b > 0$. Then for classical solutions [1]

$$\partial_t h(u_\varepsilon, v_\varepsilon) + h'_u(u_\varepsilon, v_\varepsilon)\partial_x v_\varepsilon + h'_v(u_\varepsilon, v_\varepsilon)\partial_x u_\varepsilon \le -(b/\varepsilon)(f(u_\varepsilon) - v_\varepsilon)^2,$$

whence under suitable conditions on f and the initial data

$$\frac{b}{\varepsilon} \int_0^\infty \int_{-\infty}^\infty \left(f(u_\varepsilon(t,x)) - v_\varepsilon(t,x)\right)^2 dx\, dt \le C_0$$

follows, where C_0 does not depend on ε. Let us remark that the equation above is easily controlled if we have a function $J = J(u, v)$ such that $h'_u(u, v)\partial_x v + h'_v(u, v)\partial_x u = \partial_x J$, that is h is a *Lax entropy* for

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[1] Differentiation with respect to space and time is usually denoted by ∂_x and ∂_t, while h'_u and h'_v are the partial derivatives of h with respect to the state variables u and v.

(1.1) if $B = 0$, but there are other possibilities, too. Therefore it is reasonable to expect $v_\varepsilon \approx f(u_\varepsilon)$ in a mean square sense as $\varepsilon \to 0$, consequently the limit u of u_ε satisfies $\partial_t u + \partial_x f(u) = 0$ in a weak sense, see e.g. Sections 6.7 and 16.5 of [Daf05] for complete proofs which are technically much more complex than the presentation here.

The hyperbolic scaling limit of our model of *interacting exclusions with creation and annihilation* shall be understood as a microscopic version of the zero relaxation limit for the *LeRoux system*

$$
\begin{aligned}
\partial_t u_\varepsilon + \partial_x (\rho_\varepsilon - u_\varepsilon^2) &= 0, \\
\partial_t \rho_\varepsilon + \partial_x (u_\varepsilon - u_\varepsilon \rho_\varepsilon) &= \varepsilon^{-1} B(u_\varepsilon, \rho_\varepsilon)(F(u_\varepsilon) - \rho_\varepsilon),
\end{aligned}
\tag{1.2}
$$

where $x \in \mathbb{R}$, $u \in [-1, 1]$, $\rho \in [0, 1]$, $F(u) := (1/3)(4 - (4 - 3u^2)^{1/2}$ and $B \geq 1/2$, consequently the limiting equation for u reads as $\partial_t u + \partial_x (F(u) - u^2) = 0$. The first proof uses $h = (1/2)(F(u) - \rho)^2$ as our Liapunov function, the result obtained in this way can be improved by choosing h as a Lax entropy of the Leroux system.

1.2. The model: In view of our naive physical picture of electrophoresis, we consider ± 1 charges moving in an electric field on $\mathbb{Z}$ such that positive charges are jumping to the right at rate 1 if allowed (i.e. there is no particle on the next site), negative charges are jumping to the left at unit rates. The exclusion rule is in force: two or more particles (charges) can not coexist at the same site. However, when two opposite charges meet, then they either jump over each other at rate 2, or they are both annihilated at rate $\beta > 0$. To compensate annihilation, charges of opposite sign can be created at neighboring empty sites, again at rate β. Because of technical reasons, the process is regularized by a nearest neighbor stirring of intensity $\sigma > 0$, all elementary actions are independent of each other. The mathematical formulation of the model is summarized as follows.

The configuration space, Ω of our system is the set of sequences $\omega := (\omega_k \in \{0, 1, -1\} : k \in \mathbb{Z})$, i.e. ω_k is interpreted as the charge of the particle at site $k \in \mathbb{Z}$, $\omega_k = 0$ indicates an empty site, and $\eta_k := \omega_k^2$ denotes the occupation number. The process is composed of the following local operations. If $b = (k, k+1)$ is a bond of $\mathbb{Z}$, i.e. $b \in \mathbb{Z}^*$, then *stirring* $\omega \leftrightarrow \omega^b$ means that ω_k and ω_{k+1} are exchanged, the rest of the configuration is not altered. The action $\omega \leftrightarrow \omega^{b+}$ creates a couple of particles on the bond $b := (k, k+1)$ if it is empty:

$$
(\omega^{b+})_k = +1 \quad \text{and} \quad (\omega^{b+})_{k+1} = -1 \quad \text{if} \quad \omega_k = \omega_{k+1} = 0,
$$

other coordinates are not changed. Annihilation of a couple, $\omega \leftrightarrow \omega^{b\times}$ means that

$$
(\omega^{b\times})_k = (\omega^{b\times})_{k+1} = 0 \quad \text{if} \quad \omega_k = +1, \omega_{k+1} = -1 \, ; (\omega^{b\times})_j = \omega_j
$$

otherwise. The stochastic dynamics is then defined by the following formal generators, see [Lig85] on the construction of interacting particle systems. These operators are certainly defined for *finite functions,* i.e. for $\varphi : \Omega \mapsto \mathbb{R}$ depending only on a finite number of variables, and the set of finite functions is a *core* of the full generator. The *totally asymmetric process of interacting exclusions* (INTASEP) is generated by

$$
\mathcal{L}_o \varphi(\omega) := \sum_{b \in \mathbb{Z}^*} c_b(\omega) \left(\varphi(\omega^b) - \varphi(\omega) \right),
\tag{1.3}
$$

where $c_b(\omega) := (1/2)(\eta_k + \eta_{k+1} + \omega_k - \omega_{k+1})$ if $b = (k, k+1)$. This generator lets $\oplus$ particles jump to the right, $\ominus$ particles jump to the left at rate 1, if allowed, while a collision $\oplus\ominus \to \ominus\oplus$

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occurs at rate 2. Both particle number $\sum \eta_k$ and total charge $\sum \omega_k$ are preserved by INTASEP. The two-parameter family, $\{\lambda_{\rho,u} : 0 < \rho < 1, 0 \le |u| < \rho\}$ of translation invariant stationary product measures is characterized by $\lambda_{\rho,u}(\eta_k) = \rho$ and $\lambda_{\rho,u}(\omega_k) = u$; here and later on we use the short hand notation $\lambda(\varphi) \equiv \int \varphi \, d\lambda$. The degenerated stationary states $\lambda_{\rho,u}$ with $\rho = 0$, $\rho = 1$ or $|u| = \rho$ play no role in our calculations. The study of interacting exclusions and some related models goes back to the paper [TV03] by B. Tóth and B. Valkó, where HDL of INTASEP with hyperbolic scaling is derived in a smooth regime with periodic boundary conditions.

The *creation - annihilation process* (CRANNI) is generated by $\mathscr{L}_* := \mathscr{L}_o + \beta \mathscr{G}_*$, where $\beta > 0$ and

$$\mathscr{G}_* \varphi(\omega) := \sum_{b \in \mathbb{Z}^*} c_b^+(\omega) \left(\varphi(\omega^{b+}) - \varphi(\omega) \right) + \sum_{b \in \mathbb{Z}^*} c_b^\times(\omega)(\varphi(\omega^{b\times}) - \varphi(\omega)), \qquad (1.4)$$

where

$$c_b^+(\omega) := \mathbf{1}[\eta_k = 0, \eta_{k+1} = 0] = (1 - \eta_k)(1 - \eta_{k+1}),$$

$$c_b^\times(\omega) := \mathbf{1}[\omega_k = 1, \omega_{k+1} = -1] = (1/4)(\eta_k + \omega_k)(\eta_{k+1} - \omega_{k+1})$$

if $b = (k, k+1)$, and $\mathbf{1}[A]$ denotes the *indicator function* of the event $A \subset \Omega$. For any bond $(k, k+1) = b \in \mathbb{Z}^*$ the elementary action $\omega \longleftrightarrow \omega^{b*}$ is defined by $\omega^{b*} := \omega^{b+}$ if $(\omega_k, \omega_{k+1}) = (0, 0)$, $\omega^{b*} = \omega^{b\times}$ if $(\omega_k, \omega_{k+1}) = (+1, -1)$, while $\omega^{b*} = \omega$ otherwise. Since $c_b^+ + c_b^\times = 1$ if $\omega^{b*} \ne \omega$, we can rewrite $\mathscr{G}_*$ as

$$\mathscr{G}_* \varphi(\omega) = \sum_{b \in \mathbb{Z}^*} \left(\varphi(\omega^{b*}) - \varphi(\omega) \right).$$

Only total charge $\sum \omega_k$ is preserved by CRANNI, and within the class $\lambda_{\rho,u} : 0 \le |u| < \rho$ its stationary measures are characterized by the *principle of microscopic balance:* $\lambda_{\rho,u}[\omega_k = 1, \omega_{k+1} = -1] = \lambda_{\rho,u}[\omega_k = \omega_{k+1} = 0]$. This means that $C(\rho, u) = 0$, where

$$C(\rho, u) := (1 - \rho)^2 - (1/4)(\rho^2 - u^2) = (1/4) \left(3\rho^2 - 8\rho + u^2 + 4 \right) \qquad (1.5)$$
$$= (3/4) \left(\rho - F(u) \right) \left(\rho - F_*(u) \right),$$

$$F(u) := \frac{1}{3} \left(4 - \sqrt{4 - 3u^2} \right), \quad F_*(u) := \frac{1}{3} \left(4 + \sqrt{4 - 3u^2} \right). \qquad (1.6)$$

The smaller root, $F(u)$ is between $2/3$ and 1, while the second one is not allowed because $F_*(u) \ge 5/3$ for all $u \in [-1, 1]$. Consequently $\lambda_u^* := \lambda_{F(u),u}$ is a stationary measure of the process generated by $\mathscr{L}_*$ if $|u| < 1$, and $\lambda_u^*(\omega_k) = u$, while $\lambda_u^*(\eta_k) = \rho = F(u)$. Note that these measures are reversible with respect to $\mathscr{G}_*$, this fact shall be exploited several times in our computations.

Since we want to pass to HDL in a regime of shocks by means of the theory of compensated compactness, our process has to be regularized, e.g. by an overall *stirring* of large intensity, cf. [FT04] and [FN06]. The generator of the stirring process reads as

$$\mathscr{S} \varphi(\omega) := \sum_{b \in \mathbb{Z}^*} \left(\varphi(\omega^b) - \varphi(\omega) \right). \qquad (1.7)$$

This process is reversible with respect to any $\lambda_{\rho,u}$, and both $\sum \eta_k$ and $\sum \omega_k$ are preserved. HDL of the process generated by $\mathscr{L}_\sigma := \mathscr{L}_o + \sigma \mathscr{S}$ in a regime of shocks was determined in [FT04], here we are interested in the hyperbolic scaling limit of the creation - annihilation process generated by $\mathscr{L} := \mathscr{L}_o + \beta \mathscr{G}_* + \sigma \mathscr{S}$, see (1.3), (1.4) and (1.7) for definitions, where β and σ are positive

parameters to be specified later. The main goal of our paper is to develop a microscopic theory of relaxation schemes, by means of which the macroscopic behavior of this creation - annihilation process can be described.

Let us remark that in the paper [FN06] HDL of the process generated by $\mathscr{L}_\kappa := \mathscr{L}_o + \alpha \mathscr{G}_\kappa + \sigma \mathscr{S}$ was investigated. The *spin - flip generator* $\mathscr{G}_\kappa$ reads as

$$\mathscr{G}_\kappa \varphi(\omega) := \sum_{k \in \mathbb{Z}} (\eta_k - \kappa \omega_k) \left(\varphi(\omega^k) - \varphi(\omega) \right), \tag{1.8}$$

where $\kappa \in (-1, 1)$ is a constant, and $\omega \longleftrightarrow \omega^k$ means that $(\omega^k)_k = -\omega_k$, while $(\omega^k)_j = \omega_j$ otherwise. Since $\mathscr{G}_\kappa$ violates conservation of total charge, and

$$(1 - \kappa)\lambda_{\rho,u}[\omega_k = 1] = (1 + \kappa)\lambda_{\rho,u}[\omega_k = -1]$$

in equilibrium, i.e. $(1 - \kappa)(\rho + u)/2 = (1 + \kappa)(\rho - u)/2$, we have $u = \kappa\rho$. Therefore the family of stationary product measures is just $\{\lambda_\rho^\kappa := \lambda_{\rho,\kappa\rho} : 0 < \rho < 1\}$ such that $\lambda_\rho^\kappa(\eta_k) = \rho$ and $\lambda_\rho^\kappa(\omega_k) = \kappa\rho$. Although we can not improve results of [FN06] in that way, it might be interesting to see that this model also exhibits relaxation.

1.3. Currents: To understand the microscopic structure of our model, let us summarize some more information on the generators; $j^{\pi,?}$ below denotes the current of a conservative quantity π, which is induced by a generator $\mathscr{L}_?$. By direct computations we get $\mathscr{L}_o \omega_k = j_{k-1}^{\omega o} - j_k^{\omega o}$, where

$$j_k^{\omega o}(\omega) := \tfrac{1}{2} (\eta_k + \eta_{k+1} - 2\omega_k \omega_{k+1} + \omega_k \eta_{k+1} - \eta_k \omega_{k+1} + \omega_k - \omega_{k+1}), \tag{1.9}$$

whence $\mathfrak{J}^{uo}(\rho, u) := \lambda_{\rho,u}(j_k^{\omega o}) = \rho - u^2$ and $\lambda_u^*(j_k^{\omega o}) = f(u) := F(u) - u^2$.

Similarly, $\mathscr{L}_o \eta_k = j_{k-1}^{\eta o} - j_k^{\eta o}$, where

$$j_k^{\eta o}(\omega) := \frac{1}{2} (\omega_k + \omega_{k+1} - \omega_k \eta_{k+1} - \omega_{k+1}\eta_{k+1} + \eta_k - \eta_{k+1}), \tag{1.10}$$

whence $\mathfrak{J}^{\rho o}(\rho, u) := \lambda_{\rho,u}(j_k^{\eta o}) = u - u\rho$ and $\lambda_\rho^\kappa(j_k^{\eta o}) = \kappa(\rho - \rho^2)$.

The case of $\mathscr{S}$ is trivial: $\mathscr{S}\omega_k = \Delta_1 \omega_k$ and $\mathscr{S}\eta_k = \Delta_1 \eta_k$, where $\Delta_1 \xi_k := \xi_{k+1} + \xi_{k-1} - 2\xi_k$ for any sequence, ξ indexed by $\mathbb{Z}$, thus $j_k^{\omega s}(\omega) = \omega_k - \omega_{k+1}$ and $j_k^{\eta s}(\omega) = \eta_k - \eta_{k+1}$ are the associated currents. The spin - flip dynamics does not induce any current of η because $\mathscr{G}_\kappa \eta_k = 0$. Moreover, the identity $\mathscr{G}_\kappa \omega_k = \mathscr{G}_\kappa(\omega_k - \kappa\eta_k) = 2(\kappa\eta_k - \omega_k)$ indicates relaxation in the sense that $\omega_k \approx \kappa\eta_k$ in the scaling limit, see the heuristic explanation in Section 1.4 below.

The action of the creation - annihilation process is less transparent. We have $\mathscr{G}_* \omega_k = j_{k-1}^{\omega*} - j_k^{\omega*}$, where

$$j_k^{\omega*} := c_b^\times(\omega) - c_b^+(\omega), \quad b = (k, k+1), \tag{1.11}$$

thus $\mathfrak{J}^{u*}(\rho, u) := \lambda_{\rho,u}(j_k^{\omega*}) = -C(\rho, u)$, cf. (1.5). Since λ_u^* is reversible with respect to $\mathscr{G}_*$, the effect of $j^{\omega*}$ vanishes in the hydrodynamic limit; $\lambda_u^*(j_k^{\omega*}) = 0$. On the other hand,

$$\mathscr{G}_* \eta_k = c_b^+(\omega) - c_b^\times(\omega) + c_{b-}^+(\omega) - c_{b-}^\times(\omega), \tag{1.12}$$

where $b = (k, k+1)$ and $b- := (k-1, k)$, whence $\lambda_{\rho,u}(\mathscr{G}_* \eta_k) = 2C(\rho, u) = (3/2)(\rho - F(u))(\rho - F_*(u))$. Observe now that the second factor is negative because $F_*(u) \geq 5/3$, thus we have a good

reason to suspect that the terms $\mathscr{G}_* \eta_k$ give rise to relaxation: $\rho \approx F(u)$ in the scaling limit. This problem, however, is more difficult than the previous one because $\mathscr{G}_* \eta$ is not a linear function of ω and η, see the end of Section 1.4, and also Section 1.5 for some hints.

1.4. Macroscopic equations: Under hyperbolic scaling of space and time, at a level $\varepsilon \in (0, 1]$ of scaling the scaled densities read as $\rho_\varepsilon(t, x) := \eta_k(t/\varepsilon)$ and $u_\varepsilon(t, x) := \omega_k(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$; later on we shall redefine these empirical processes in terms of block averages. A formal application of the principle of local equilibrium suggests that in the case of interacting exclusions ρ_ε and u_ε converge in some sense to weak solutions of the following version [2] of the LeRoux system:

$$\partial_t \rho + \partial_x(u - u\rho) = 0 \quad \text{and} \quad \partial_t u + \partial_x(\rho - u^2) = 0, \tag{1.13}$$

for a correct derivation see [TV03] or [FT04] concerning $\mathscr{L}_o$ or $\mathscr{L}_\sigma = \mathscr{L}_0 + \sigma \mathscr{S}$, respectively. In the second case we had to assume that $\sigma = \sigma(\varepsilon) \to +\infty$ such that $\varepsilon\sigma(\varepsilon) \to 0$ and $\varepsilon\sigma^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$; the uniqueness of the limit is not known in a regime of shocks. Under the same assumptions on σ, in the paper [FN06] it was shown that HDL of the spin - flip dynamics generated by $\mathscr{L}_\kappa$ results in a Burgers equation:

$$\partial_t \rho + \kappa \, \partial_x(\rho - \rho^2) = 0, \tag{1.14}$$

and the limit is unique even in the regime of shocks. As we suggest in the next subsection, and demonstrate at the end of the paper, (1.14) can also be derived as the zero relaxation limit of the modified LeRoux system

$$\partial_t \tilde{\rho}_\varepsilon + \partial_x(\tilde{u}_\varepsilon - \tilde{u}_\varepsilon \tilde{\rho}_\varepsilon) = 0, \quad \partial_t \tilde{u}_\varepsilon + \partial_x(\tilde{\rho}_\varepsilon - \tilde{u}_\varepsilon^2) = (2\alpha/\varepsilon)(\kappa\tilde{\rho}_\varepsilon - \tilde{u}_\varepsilon) \tag{1.15}$$

allowing us to do the replacement $\tilde{u}_\varepsilon \approx \kappa\tilde{\rho}_\varepsilon$. Since $\mathscr{G}_\kappa \eta_k = 0$ and $\mathscr{G}_\kappa \omega_k = 2(\kappa\eta_k - \omega_k)$, this heuristic picture reveals quite well the microscopic structure of the spin - flip dynamics, note that the vanishing terms, $\varepsilon\sigma(\varepsilon) \partial_x^2 \tilde{\rho}_\varepsilon$ and $\varepsilon\sigma(\varepsilon) \partial_x^2 \tilde{u}_\varepsilon$ have been omitted.

In view of the previous subsection, see (1.9), (1.11) and (1.12) in particular, a formal calculation yields

$$\partial_t u = \partial_x \left(u^2 - F(u) \right) = \partial_x \left(u^2 + \tfrac{1}{3} \sqrt{4 - 3u^2} \right) \tag{1.16}$$

as the macroscopic equation for the creation - annihilation process generated by $\mathscr{L} := \mathscr{L}_o + \beta \mathscr{G}_*$. Assuming smoothness of the macroscopic solution, it would not be difficult to formulate and prove the statement in a rigorous manner. Let us remark that the flux $F(u) - u^2$ is neither convex nor concave, thus the structure of shocks developed by this equation is rather complex. It is important that the graph of $F(u) - u^2$ does not contain any linear segment, thus the uniqueness theorem of [CR00] applies. The relaxation scheme for (1.16) is less convincing than that of (1.15), it can be written as

$$\partial_t \tilde{u}_\varepsilon + \partial_x(\tilde{\rho}_\varepsilon - \tilde{u}_\varepsilon^2) + \beta \, \partial_x \mathfrak{J}^{u*}(\tilde{\rho}_\varepsilon, \tilde{u}_\varepsilon) = 0,$$
$$\partial_t \tilde{\rho}_\varepsilon + \partial_x(\tilde{u}_\varepsilon - \tilde{u}_\varepsilon \tilde{\rho}_\varepsilon) = (2\beta/\varepsilon) \, C(\tilde{\rho}_\varepsilon, \tilde{u}_\varepsilon), \tag{1.17}$$

where $C(\rho, u) = B(u, \rho)(F(u) - \rho)$ with $B(u, \rho) = (3/4)(F_*(u) - \rho) \geq 1/2$, see (1.5). As it has been stressed already in Section 1.1, in order to demonstrate relaxation $\tilde{\rho}_\varepsilon \approx F(\tilde{u}_\varepsilon)$, we have to find an effective Liapunov function, possibly a *Lax entropy* for (1.13). Since $(\rho - F(u))C(\rho, u) \leq$

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[2] The standard form, $\partial_t v + \partial_x(\pi v) = 0$ and $\partial_t \pi + \partial_x(v + \pi^2) = 0$ of the LeRoux system is obtained by substituting $v = 1 - \rho$ and $\pi = -u$ into (1.13), cf. [Ser00].

$-(1/2)(\rho - F(u))^2$, even $H := (1/2)(\rho - F(u))^2$ seems to be an effective Liapunov function, see Section 1.1, and (1.2) in particular.

1.5. Thermodynamic entropy: Conservation laws play a fundamental role in the study of nonlinear hyperbolic systems, see [Ser00] or [Daf05] with references on the original work by Peter D. Lax. The function $S = S(\rho, u)$ of the state variables $\rho, u \in \mathbb{R}$ is a *Lax entropy* with *flux* $\Phi = \Phi(\rho, u)$ for a *system of two conservation laws,* as the LeRoux system, if $\partial_t S + \partial_x \Phi = 0$ along classical solutions. Such a couple (S, Φ) is called a *Lax entropy - flux pair*; in case of (1.13) these are characterized by

$$
\begin{aligned}
\Phi'_\rho(\rho, u) &= S'_u(\rho, u) - u S'_\rho(\rho, u), \\
\Phi'_u(\rho, u) &= (1 - \rho) S'_\rho(\rho, u) - 2u S'_u(\rho, u).
\end{aligned}
\tag{1.18}
$$

Differentiating the right hand sides above with respect to u and v, a linear wave equation

$$
S''_{u,u}(u, v) = (1 - \rho) S''_{\rho, \rho}(u, v) - u S''_{u, \rho}(u, v)
\tag{1.19}
$$

is obtained for S, which admits a rich class of solutions, cf. [Ser00].

An interesting example, the relative entropy of $\lambda_{\rho, u}$ and $\lambda_{v, \pi}$ at one site is defined as

$$
\begin{aligned}
S(\rho, u) :&= \sum_{s=0, \pm 1} \lambda_{\rho, u}[\omega_k = s] \log \frac{\lambda_{\rho, u}[\omega_k = s]}{\lambda_{v, \pi}[\omega_k = s]} \\
&= \frac{\rho + u}{2} \log \frac{\rho + u}{v + \pi} + \frac{\rho - u}{2} \log \frac{\rho - u}{v - \pi} + (1 - \rho) \log \frac{1 - \rho}{1 - v}.
\end{aligned}
\tag{1.20}
$$

By a direct computation

$$
\begin{aligned}
S'_\rho(\rho, u) &= \frac{1}{2} \log \frac{\rho + u}{v + \pi} + \frac{1}{2} \log \frac{\rho - u}{v - \pi} - \log \frac{1 - \rho}{1 - v}, \\
S'_u(\rho, u) &= \frac{1}{2} \log \frac{\rho + u}{v + \pi} - \frac{1}{2} \log \frac{\rho - u}{v - \pi},
\end{aligned}
\tag{1.21}
$$

therefore the thermodynamic entropy satisfies (1.19), consequently it is really a Lax entropy for (1.13).

Let us demonstrate at an intuitive level that thermodynamic entropy might be an effective Liapunov function for our relaxation schemes. For this purpose the parameters v and π should be specified such that $S'_u(\rho, \kappa\rho) = 0$ in case of spin - flips, while $S'_\rho(F(u), u) = 0$ for creation and annihilation.

Choosing $v = 1/2$ and $\pi = \kappa/2$ in the definition of S, we see that

$$
S'_u(\rho, u)(\kappa\rho - u) \leq -2(u - \kappa\rho)^2,
$$

consequently the spin - flip dynamics exhibits relaxation to the Burgers equation. Although this problem has been solved already in [FN06] with another method, some remarks on this approach are added at the end of the paper.

To understand relaxation of creation and annihilation, set e.g. $\pi = 0$ and $v = F(0) = 2/3$ in the definition of S, then

$$
S'_\rho(\rho, u) = \frac{1}{2} \log \frac{\rho^2 - u^2}{4} - \frac{1}{2} \log(1 - \rho)^2,
$$

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whence by a direct computation

$$S'_\rho(\rho,u)C(\rho,u) \le -(1/4)C^2(\rho,u) \le -(1/16)(\rho - F(u))^2\,,$$

which is the necessary bound for relaxation to (1.16).

Unfortunately, our probabilistic calculations presuppose that the underlying entropy has bounded first and second derivatives, therefore we have to look for something else. In Section 3 we show that, although it is not a Lax entropy, $H(\rho,u) := (1/2)\big(\rho - F(u)\big)^2$ is an effective Liapunov function even for the microscopic system of creation and annihilation, while $H_\kappa(\rho,u) := (1/2)(u - \kappa\rho)^2$ applies in the case of spin flips. Computations are quite simple in both cases, but the results are not as good as possible. In case of the creation - annihilation process generated by $\mathcal{L} = \mathcal{L}_* + \sigma\mathcal{S}$, this statement can be improved a bit by means of a clever Lax entropy - flux pair, see Sections 1.6 and 5.6. The optimal result on the spin - flip dynamics is that of [FN06], cf. Section 5.7.

1.6. Main result: It is well known that (1.16) develops shocks in a finite time, and uniqueness of its *weak solutions* breaks down at the same time, thus we must be careful with definitions. A measurable $u : \mathbb{R}^2_+ \mapsto [-1,1]$ is a weak solution to (1.16) with initial value $u_0(x) = u(0,x)$ if

$$\int_0^\infty \int_{-\infty}^\infty (\psi'_t(t,x)u(t,x) + \psi'_x(t,x)(F(u(t,x)) - u^2(t,x)))\,dx\,dt$$
$$+ \int_{-\infty}^\infty \psi(0,x)u_0(x)\,dx = 0 \tag{1.22}$$

for all $\psi \in C^1_c(\mathbb{R}^2)$. Here and below a subscript "c" refers to compactly supported functions, $\mathbb{R}_+ := [0,+\infty)$, $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}$, and $C^1_{\mathrm{co}}(\mathbb{R}^2_+)$ is the space of continuously differentiable $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ with compact support in the interior of $\mathbb{R}^2_+$. The notion of Lax entropy plays a fundamental role in the study of weak solutions. A couple (h,J), $h,J \in C^1(\mathbb{R})$ is a Lax entropy - flux pair for (1.16) if $J'(u) = (F'(u) - 2u)h'(u)$, that is $\partial_t h(u) + \partial_x J(u) = 0$ along classical solutions; (h,J) is a *convex entropy - flux pair* if h is convex. A locally integrable $u(t,x)$ is a *weak entropy solution to* (1.16) *with initial data* u_0 if (1.22) holds true for $\psi \in C^1_c(\mathbb{R}^2)$, and

$$\int_0^\infty \int_{-\infty}^\infty \left(\psi'_t(t,x)h(u) + \psi'_x(t,x)J(u)\right)\,dx\,dt \ge 0 \tag{1.23}$$

for all convex entropy - flux pairs (h,J) and compactly supported $0 \le \psi \in C^1_{\mathrm{co}}(\mathbb{R}^2_+)$. The scaled density field of charge ω is defined as

$$U_\varepsilon(\psi,\omega) := \int_0^\infty \int_{-\infty}^\infty \psi(t,x)u_\varepsilon(t,x)\,dx\,dt \tag{1.24}$$

for $\psi \in C_c(\mathbb{R}^2)$, where $u_\varepsilon(t,x) = \omega_k(t/\varepsilon)$ if $|\varepsilon k - x| < \varepsilon/2$.

The initial conditions are specified in terms of a family $\mu_{\varepsilon,0}$ of probability measures, we are assuming that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^\infty \varphi(x)u_\varepsilon(0,x)\,dx = \int_{-\infty}^\infty \varphi(x)u_0(x)\,dx \tag{1.25}$$

in probability for all $\varphi \in C_c(\mathbb{R})$, where $-1 \le u_0 \le 1$ is a given measurable function. We are considering the process generated by $\mathcal{L} = \mathcal{L}_o + \beta(\varepsilon)\mathcal{G}_* + \sigma(\varepsilon)\mathcal{S}$, in its simplest form our main result can be stated as follows.

Theorem 1.1. *Suppose* (1.25) *and specify* $\sigma = \sigma(\varepsilon)$ *and* $\beta = \beta(\varepsilon)$ *such that* $\varepsilon\sigma(\varepsilon) \to 0$ *while* $\varepsilon\sigma^2(\varepsilon) \to +\infty$ *as* $0 < \varepsilon \to 0$, *finally* $\varepsilon\sigma^2(\varepsilon)\beta^{-4}(\varepsilon) \to +\infty$ *and* $\varepsilon\sigma^2(\varepsilon)\beta^2(\varepsilon) \to +\infty$ *as* $\varepsilon \to 0$. *Then*

$$\lim_{\varepsilon \to 0} U_\varepsilon(\psi) = \int_0^\infty \int_{-\infty}^\infty \psi(t,x)u(t,x)\,dx\,dt$$

in probability for all $\psi \in C_c(\mathbb{R}^2)$; *this* $u(t,x)$ *is the uniquely specified weak entropy solution to* (1.16) *with initial value* u_0.

Some remarks: The empirical process u_ε shall be redefined in terms of block averages, in that case we get convergence in the strong local topology of $L^1(\mathbb{R})$, see Theorem 5.1.

It is quite natural to fix the value of $\beta > 0$ because it is a parameter of the basic model, our conditions mean that $\beta(\varepsilon)$ can not be too small or too large. We can not improve the upper bound (growth condition) of β, it is needed to control $\beta j_k^{\omega*}$, and $\beta \mathcal{G}_* \eta_k$ in particular. However, the lower asymptotic bound $\varepsilon\sigma^2\beta^2 \to \infty$ of β can be relaxed to $\sigma(\varepsilon)\beta(\varepsilon) \to +\infty$ by using a Lax entropy of (1.13) instead of the trivial Liapunov function $H = (1/2)(\rho - F(u))^2$, see Theorem 5.2 in the last section.

The relaxation of the spin - flip dynamics to the Burgers equation is discussed in Sections 5.4. In this case $H_\kappa = (1/2)(u - \kappa\rho)^2$ is a nice Liapunov function, but the result is weaker that that of [FN06]. We are sorry to tell that entropy is not really helpful here because, unless $\kappa = 0$, we can not construct the required Lax entropy - flux pair of the LeRoux system.

2 An Outline of the Proof

As far as possible we follow the argument of [FN06], which is based on the stochastic theory of compensated compactness, while the necessary a priori bounds follow from the logarithmic Sobolev inequalities we do have for $\mathcal{S}$ and $\mathcal{G}_\kappa$. In fact, in [FN06] the second LSI is used to replace block averages of ω_k with those of $\kappa\eta_k$; in the present case this second step should consist in a replacement of block averages $\bar{\eta}_{l,k}$ of η with $F(\bar{\omega}_{l,k})$, where $\bar{\omega}_{l,k}$ denotes the corresponding block average of ω. However, we do not have any effective LSI involving $\mathcal{G}_*$; the required replacement will be carried out by exploiting relaxation of the microscopic system.

2.1. Block averages: As it is more or less obligatory in the microscopic theory of hydrodynamics, first we rewrite the empirical process in terms of block averages. For $l \in \mathbb{N}$ and for any sequence ξ indexed by $\mathbb{Z}$ we define two sequences of moving averages, namely $\bar{\xi}_l = (\bar{\xi}_{l,k} : k \in \mathbb{Z})$ and $\hat{\xi}_l = (\hat{\xi}_{l,k} : k \in \mathbb{Z})$ such that

$$\bar{\xi}_{l,k} := \frac{1}{l}\sum_{j=0}^{l-1}\xi_{k+j} \quad \text{and} \quad \hat{\xi}_{l,k} := \frac{1}{l^2}\sum_{j=-l}^{l}||j| - l|\,\xi_{k+j}. \tag{2.1}$$

The usual arithmetic mean $\bar{\xi}_{l,k}$ is preferred in computing canonical expectations, the reason for using the "more smooth" averages $\hat{\xi}_{l,k}$ is rather technical. [3] For convenience the size $l = l(\varepsilon)$ of

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[3] Our a priori bounds are formulated in terms of block averages of type $\bar{\xi}_l$ when l is large. However, on the right hand side of the evolution equation we see discrete gradients as below, and in contrast to $l(\bar{\xi}_{l,k+1} - \bar{\xi}_{l,k})$, $l(\hat{\xi}_{l,k+1} - \hat{\xi}_{l,k}) = \bar{\xi}_{l,k+1} - \bar{\xi}_{l,k-l+1}$ is still a difference of large block averages if l is large. This simple fact is most relevant when we have to evaluate a Lax entropy, it was exploited in our earlier papers, too.

these blocks is chosen as the integer part of $(\sigma^2/\varepsilon)^{1/4}$, then our conditions on $\sigma = \sigma(\varepsilon)$ imply

$$\lim_{\varepsilon \to 0} \frac{l(\varepsilon)}{\sigma(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{l^2(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\varepsilon l^2(\varepsilon)}{\sigma(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\varepsilon l^3(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\sigma^2(\varepsilon)}{l^3(\varepsilon)} = 0, \tag{2.2}$$

while $\varepsilon l^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Because of technical reasons we have to assume that $l(\varepsilon)$ exceeds a certain threshold $l_0 \in \mathbb{N}$. For convenience we may, and do assume that $\varepsilon l^3(\varepsilon) \geq \sigma(\varepsilon) \geq l(\varepsilon) \geq l_0 \geq 1$.

Concerning $\beta = \beta(\varepsilon)$ we need

$$\lim_{\varepsilon \to 0} \frac{\beta(\varepsilon)l(\varepsilon)}{\sigma(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\varepsilon \beta^2(\varepsilon)l^2(\varepsilon)}{\sigma(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\beta(\varepsilon)l^2(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{1}{\beta(\varepsilon)\sigma(\varepsilon)} = 0 \tag{2.3}$$

and $\varepsilon\beta(\varepsilon)l^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$; the first and last relations are responsible for the conditions of Theorem 1.1. From now on the block size $l = l(\varepsilon)$ is specified as above, and these relations will frequently be used in the next coming computations, sometimes without any reference.

Our first statements on HDL will be formulated in terms of a modified empirical process $\hat{u}_\varepsilon$. It is defined as $\hat{u}_\varepsilon(t,x) := \hat{\omega}_{l,k}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$, and $\hat{\mathsf{P}}_\varepsilon$ denotes the distribution of $\hat{u}_\varepsilon$; several topologies can be introduced to study limit distributions of $\hat{\mathsf{P}}_\varepsilon$ as $\varepsilon \to 0$. The usual L^p norm of a measurable $\psi : \mathbb{R}_+^2$ is denoted by $\|\psi\|_p$, and $\langle \varphi, \psi \rangle$ is the scalar product in $L^2(\mathbb{R}_+^2)$. In general $\|\hat{u}_\varepsilon\|_1 = +\infty$, therefore we have to localize convergence by multiplying with a test function $\phi \in C^1_{co}(\mathbb{R}_+^2)$, say. The *local strong convergence* of ψ_n to ψ in L^p means that $\|\phi\psi_n - \phi\psi\|_p \to 0$ for all ϕ, while the *local weak convergence* $\psi_n \rightharpoonup \psi$ is defined by $\langle \phi, \psi_n \rangle \to \langle \phi, \psi \rangle$ whenever $\phi \in C^1_{co}(\mathbb{R}_+^2)$. The allowed class of test functions ϕ can be enlarged by means of the Banach - Steinhaus theorem. Since $|\hat{u}_\varepsilon(t,x)| \leq 1$ for all $(t,x) \in \mathbb{R}_+^2$, the set $\mathscr{U}$ of all realizations of the empirical process is relative compact in the local weak topology of $L^2(\mathbb{R}_+^2)$, therefore $\mathscr{U}$ is a separable metric space, and the family $\{\hat{\mathsf{P}}_\varepsilon : \varepsilon > 0\}$ is tight in this sense.

2.2. Measure - valued solutions: The notion of Young measure is a most convenient tool for the description of all limit distributions of our empirical process $\hat{u}_\varepsilon$, cf. [Tar79], [Daf05] or [Ser00]. Let Θ denote the set of measurable families, θ of probability measures $\theta = \{\theta_{t,x}(du) : (t,x) \in \mathbb{R}_+^2\}$ such that $\theta_{t,x}$ is a probability measure on $[-1,1]$ for each $(t,x) \in \mathbb{R}_+^2$, and $\theta_{t,x}(h)$ is a measurable function of (t,x) whenever $h : [-1,1] \mapsto \mathbb{R}$ is measurable and bounded; $\theta_{t,x}(h)$ denotes expectation of h with respect to $\theta_{t,x}$. We say that $\theta \in \Theta$ is a *measure - valued solution* to the macroscopic equation $\partial_t u + \partial_x f(u) = 0$ with initial value u_0 if

$$\int_0^\infty \int_{-\infty}^\infty \left(\psi'_t(t,x)\theta_{t,x}(u) + \psi'_x(t,x)\theta_{t,x}(f(u)) \right) dx\,dt$$

$$+ \int_{-\infty}^\infty \psi(0,x)u_0(x)\,dx = 0 \tag{2.4}$$

for all $\psi \in C^1_c(\mathbb{R}^2)$. A measurable function $u : \mathbb{R}_+^2 \mapsto [-1,1]$ is represented by a family $\theta \in \Theta$ of *Dirac measures* such that $\theta_{t,x}$ is concentrated on the actual value $u(t,x)$ of u; this θ is called the *Young representation* of u. Therefore any weak solution is a measure - solution. On the other hand, any $\theta \in \Theta$ can be identified as a locally finite measure m_θ by $dm_\theta := dt\,dx\,\theta_{t,x}(du)$ on $\mathbb{X} := \mathbb{R}_+^2 \times [-1,1]$; let $M_\theta(\mathbb{X})$ denote the set of such measures m_θ equipped with the associated

local weak topology. In view of the Young representation, our empirical process $\hat{u}_\varepsilon$ can be considered as a random element $\hat{m}_{\theta,\varepsilon}$ of $M_\theta(\mathbb{X})$; let $\hat{\mathsf{P}}_{\varepsilon,\theta}$ denote its distribution. The family $\{\hat{\mathsf{P}}_{\varepsilon,\theta} : \varepsilon > 0\}$ is obviously tight because the configuration space, Ω is compact; here and also later on $\hat{\mathscr{P}}_\theta$ denotes the set of weak limits $\hat{\mathsf{P}}_\theta := \lim_{n\to\infty} \hat{\mathsf{P}}_{\varepsilon(n),\theta}$ obtainable as $\varepsilon(n) \to 0$. In Section 4 we prove the following preliminary result.

Proposition 2.1. *Any limit distribution $\hat{\mathsf{P}}_\theta \in \hat{\mathscr{P}}_\theta$ of the Young representation of $\hat{u}_\varepsilon$ is concentrated on a set of measure - valued solutions.*

This is easy, but uniqueness of measure - valued solutions is rather problematic, see e.g. [Rez91] with further references. By means of the stochastic theory of compensated compactness, first we prove the *Dirac property* of the limiting Young measure, which means that limit distributions $\hat{\mathsf{P}}_\theta \in \hat{\mathscr{P}}_\theta$ of $\hat{\mathsf{P}}_{\varepsilon,\theta}$ are sitting on a set of measurable functions. Therefore we have convergence to a set of weak solutions, and uniqueness of weak solutions to a single conservation law is a well settled issue.

2.3. Entropy production: The microscopic version of entropy production $X_\varepsilon = \partial_t h + \partial_x J$ is defined as a distribution: for $\psi \in C_c^1(\mathbb{R}^2)$ and entropy - flux pairs (h, J) of (1.16) we introduce

$$X_\varepsilon(\psi, h) := - \int_0^\infty \int_{-\infty}^\infty \left(\psi_t'(t,x) h(\hat{u}_\varepsilon) + \psi_x'(t,x) J(\hat{u}_\varepsilon) \right) dx\, dt\,. \tag{2.5}$$

This follows by a formal integration by parts if $\psi \in C_c^1(\mathbb{R}_+^2)$ and $\psi(0,x) = 0 \;\forall\, x \in \mathbb{R}$. Calculating the stochastic differential of

$$H_\varepsilon(t, \psi, h) := \int_{-\infty}^\infty \psi(t,x) h(\hat{u}_\varepsilon(t,x))\, dx$$

we get a martingale $M_\varepsilon(t, \psi, h)$, see (4.2), such that

$$dH_\varepsilon = \int_{-\infty}^\infty \psi_t'(t,x) h(\hat{u}_\varepsilon)\, dx\, dt + \varepsilon^{-1} \mathscr{L} H_\varepsilon\, dt + dM_\varepsilon\,,$$

whence
$$X_\varepsilon(\psi, h) = H_\varepsilon(0, \psi, h) + L_\varepsilon(\psi, h) + J_\varepsilon(\psi, h) + M_\varepsilon(\infty, \psi, h) + N_\varepsilon(\psi, h)\,, \tag{2.6}$$

where L_ε, J_ε and M_ε are defined as follows. Since $\mathscr{L} = \mathscr{L}_o + \beta \mathscr{G}_* + \sigma \mathscr{S}$, we have $L_\varepsilon = L_\varepsilon^o + \beta L_\varepsilon^* + \sigma L_\varepsilon^s$ such that

$$L_\varepsilon^q(\psi, h) := \frac{1}{\varepsilon} \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\, \mathscr{A}_q h(\hat{u}_\varepsilon(t,x))\, dx\, dt\,,$$

$q \in \{o, *, s\}$ and $\mathscr{A}_o = \mathscr{L}_o$, $\mathscr{A}_* = \mathscr{G}_*$, $\mathscr{A}_s = \mathscr{S}$. Finally,

$$J_\varepsilon(\psi, h) := \frac{1}{\varepsilon} \int_0^\infty \int_{-\infty}^\infty \psi(t,x) \left(J(\hat{u}_\varepsilon(t,x)) - J(\hat{u}_\varepsilon(t,x-\varepsilon)) \right) dx\, dt\,, \tag{2.7}$$

while $N_\varepsilon(\psi, h)$ is a numerical error due to the lattice approximation of the space derivative, see (4.1).

Calculation of X_ε is quite easy when $h(u) \equiv u$ because $\mathscr{L} \omega_k$ is a difference of currents along adjacent bonds, thus rearranging sums by performing discrete integration by parts, the test function nicely

absorbs the factor ε^{-1} of $\mathscr{L}$. This is the way of proving convergence to the set of measure - valued solutions, see the proof of Proposition 2.1 in Section 4.5. Compensated compactness is applied then to show the Dirac property of these measure - valued solutions, it is based on a delicate evaluation of entropy production for some couples of entropy - flux pairs.

2.4. Compensated compactness: In view of the stochastic version of the Tartar - Murat theory of compensated compactness, cf. [Fri01], [Fri04] or [FT04], we have to find a decomposition $X_\varepsilon = Y_\varepsilon + Z_\varepsilon$ with the following properties. $Y_\varepsilon = Y_\varepsilon(\psi, h)$ and $Z_\varepsilon = Z_\varepsilon(\psi, h)$ are linear functionals of $\psi \in C_c^1(\mathbb{R}^2)$ such that

$$|Y_\varepsilon(\phi\psi, h)| \le A_\varepsilon(\phi)\|\psi\|_{+1} \quad \text{and} \quad \lim_{\varepsilon \to 0} \mathsf{E} A_\varepsilon(\phi) = 0, \tag{2.8}$$

$$|Z_\varepsilon(\phi\psi, h)| \le B_\varepsilon(\phi)\|\psi\| \quad \text{and} \quad \limsup_{\varepsilon \to 0} \mathsf{E} B_\varepsilon(\phi) < +\infty \tag{2.9}$$

for each $\phi \in C_{\text{co}}^2(\mathbb{R}_+^2)$, where $\|\psi\|$ denotes the uniform norm, and $\|\psi\|_{+1}$ is the H^{+1} norm of ψ, i.e. $\|\psi\|_{+1}^2 := \|\psi\|_2^2 + \|\psi_t'\|_2^2 + \|\psi_x'\|_2^2$. $A_\varepsilon(\phi)$ and $B_\varepsilon(\phi)$ are random variables depending only on ϕ and h. We say that *a random functional $\hat{X}_\varepsilon(\psi, h)$ is correctly decomposed* if $\hat{X}_\varepsilon = \hat{Y}_\varepsilon + \hat{Z}_\varepsilon$ as summarized above, i.e. $\hat{Y}_\varepsilon$ satisfies (2.8), while $\hat{Z}_\varepsilon$ satisfies (2.9). The stochastic version of the celebrated Div-Curl Lemma reads as follows.

Proposition 2.2. *Let (h_1, J_1) and (h_2, J_2) denote a couple of continuously differentiable entropy - flux pairs of (1.16), and suppose that both $X_\varepsilon(\psi, h_1)$ and $X_\varepsilon(\psi, h_2)$ are correctly decomposed. With probability one with respect to any limit distribution, $\hat{\mathsf{P}}_\theta \in \hat{\mathscr{P}}_\theta$ of the Young representation of $\hat{u}_\varepsilon$ we have*

$$\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1) \tag{2.10}$$

for almost every $(t, x) \in \mathbb{R}_+^2$.

Our main task now is the verification of conditions (2.8) and (2.9) above. Most terms on the right hand side of (2.6) will be split into further ones, and we shall show in Section 4 that each of them satisfies either (2.8) or (2.9). The martingale component and the numerical error both vanish, but in a regime of shocks $\sigma(\varepsilon)L_\varepsilon^s \nrightarrow 0$ as $\varepsilon \to 0$. The crucial part of the proof is to show that L_ε^o and J_ε cancel each other, while L_ε^* disappears when $\varepsilon \to 0$. The logarithmic Sobolev inequality for $\mathscr{S}$, and the relaxation mechanism induced by creation and annihilation are applied at these steps. The slightly sophisticated construction of the empirical process, $\hat{u}_\varepsilon$ is also relevant.

A slightly simplified version of the proof yields $\limsup_{\varepsilon \to 0} X_\varepsilon(\psi, h) \le 0$ in probability if h is convex and $\psi \ge 0$, whence by tightness of the family $\{\mathsf{P}_{\varepsilon,\theta} : \varepsilon > 0\}$ we get the Lax inequality in terms of the Young measure. With probability one with respect to any limit distribution $\hat{\mathsf{P}}_\theta \in \hat{\mathscr{P}}_\theta$ of $\hat{\mathsf{P}}_{\theta,\varepsilon}$ we have

$$\int_0^\infty \int_{-\infty}^\infty \left(\theta_{t,x}(h)\psi_t'(t,x) + \theta_{t,x}(J)\psi_x'(t,x) \right) dx\, dt \ge 0 \tag{2.11}$$

whenever (h, J) is a convex entropy - flux pair and $0 \le \psi \in C_{\text{co}}^1(\mathbb{R}_+^2)$.

The Dirac property of the Young measure follows from (2.10) in several situations. Since we are considering a single conservation law, to prove convergence of the empirical process $\hat{u}_\varepsilon$ to a set of weak solutions, it is sufficient to apply (2.10) to two entropy - flux pairs only. Let $h_1(u) :\equiv u$, $J_1(u) := F(u) - u^2$, and $h_2(u) := F(u) - u^2$ with the associated flux J_2; it is defined by $J_2(0) = 0$ and $J_2'(u) = J_1'^2(u)$. In view of Theorem 16.4.1 and Theorem 16.4.2 in [Daf05], the following statement holds true.

Proposition 2.3. *Suppose that the above couple of entropy - flux pairs satisfies* (2.10)*, then any limit distribution* $\hat{\mathsf{P}}_\theta \in \hat{\mathscr{P}}_\theta$ *of the Young measure is Dirac, i.e. it is concentrated on a set of measurable functions.*

As a direct consequence we obtain (1.22) from Proposition 2.1, while (2.11) turns into the weak entropy condition (1.23), which imply uniqueness of the limit by Main Theorem of [CR00].

The verification of the conditions of Proposition 2.2 is mainly based on inequalities involving relative entropy and the associated Dirichlet form; the basic ideas go back to [GPV88], see also [Var94] and [Yau97] with further references. These computations are supplemented with an extension of the method of relaxation schemes to microscopic systems, see Lemma 3.6 and Sections 5.4 and 5.6. Since we are going to treat hydrodynamic limit in infinite volume, we have to control entropy flux by means of its rate of production, cf. [Fri90], [Fri01] and [FN06] for some earlier results in this direction. This a priori bound allows us to apply the robust LSI and the relaxation scheme of the microscopic process, see Lemma 3.3, Lemma 3.4 and Lemma 3.6. Most technical details of this estimation procedure have been elaborated in [Fri01], [Fri04] and [FT04]; [FN06] is our basic reference. First we substitute the microscopic time derivative $\mathscr{L}_o h(\hat{u}_\varepsilon)$ with the spatial gradient of a *mesoscopic flux* depending on the block averages $\bar{\eta}_{l,k}$ and $\bar{\omega}_{l,k}$. The replacement of $\bar{\eta}_{l,k}$ with its empirical estimator $F(\bar{\omega}_{l,k})$ is based on the microscopic relaxation scheme, see Lemma 3.6 below, and also Section 5.6.

3   Entropy, Dirichlet Form, LSI and Relaxation

In this section we derive some fundamental estimates based on entropy and the associated Dirichlet forms. The parameters β, σ and l are almost arbitrary here, their dependence on the scaling parameter $\varepsilon > 0$ is not important. We only need $\beta > 0$, $\sigma \geq 1$ and $l \in \mathbb{N}$. We follow calculations of [FN06] with slight modifications.

3.1. Entropy and its temporal derivative: If μ and λ are probability measures on the same space, then *entropy of μ relative to λ* is defined by $\mathsf{S}[\mu|\lambda] := \mu(\log f)$ if $\mu \ll \lambda$ and $f := d\mu/d\lambda$. A frequently used entropy inequality, $\mu(\varphi) \leq \mathsf{S}[\mu|\lambda] + \log \lambda(e^\varphi)$ follows by convexity, and we have another definition of relative entropy:

$$\mathsf{S}[\mu|\lambda] := \sup_\varphi \left\{ \mu(\varphi) - \log \lambda(e^\varphi) : \lambda(e^\varphi) < +\infty \right\} ; \tag{3.1}$$

$\varphi = \log f$ is the condition of equality. Note that

$$f \log(g/f) = 2f \log \sqrt{g/f} \leq 2\sqrt{gf} - 2f = g - f - \left(\sqrt{g} - \sqrt{f} \right)^2 , \tag{3.2}$$

whence another useful inequality, $\mathsf{E}_\lambda(\sqrt{f} - 1)^2 \leq \mathsf{S}[\mu|\lambda]$ follows immediately.

Given a Markov generator $\mathscr{A}$, the *Donsker - Varadhan rate function* of large deviations is defined as

$$\mathsf{D}[\mu|\mathscr{A}] := -\inf \left\{ \int \frac{\mathscr{A}\psi}{\psi} \, d\mu : 0 < \psi \in \mathrm{Dom}(\mathscr{A}) \right\} ; \tag{3.3}$$

$\mathsf{D}[\mu|\mathscr{A}] = -\lambda(\sqrt{f} \mathscr{A} \sqrt{f})$ if $\mathscr{A}$ is self-adjoint in $L^2(\lambda)$ and $f = d\mu/d\lambda$. Of course, $\lambda(\varphi \mathscr{A} \varphi) = \lambda(\varphi \mathscr{G} \varphi)$ if $\mathscr{G}$ denotes the symmetric part of $\mathscr{A}$. In view of their variational characterizations (3.1)

and (3.3), both S and D are lower semi - continuous, convex functionals of μ, and the definitions and relations above extend to conditional distributions and densities, too.

As a reference measure we can choose any of the equilibrium product measures $\lambda = \lambda_u^*$ with $-1 < u < 1$ fixed, say $u = 0$. At a level $\varepsilon > 0$ of scaling, $\mu_{\varepsilon,t}$ denotes the evolved measure, $\mu_{\varepsilon,t,n}$ is the distribution of the variables $\{\omega_k : |k| \leq n\}$, and $f_n = f_{\varepsilon,t,n} := d\mu_{\varepsilon,t,n}/d\lambda$. Entropy in the box $\Lambda^n := [-n, n] \cap \mathbb{Z}$ is defined as

$$S_n(t) := \mathsf{S}[\mu_{\varepsilon,t,n}|\lambda] = \int \log f_{\varepsilon,t,n}\, d\mu_{\varepsilon,t}.$$

Local versions of the Dirichlet forms for $\mathscr{L}_o$, $\mathscr{S}$ and $\mathscr{G}_*$ at $\varphi = \sqrt{f_{\varepsilon,t,n}}$ can easily be computed; in the first line below $\tilde{c}_b(\omega) := (1/2)(\eta_k + \eta_{k+1})$ if $b = (k, k+1)$.

$$D_n^o(t) := \frac{1}{2} \sum_{b \subset \Lambda^n} \int \tilde{c}_b(\omega) \left(\sqrt{f_{\varepsilon,t,n}(\omega^b)} - \sqrt{f_{\varepsilon,t,n}(\omega)} \right)^2 \lambda(d\omega),$$

$$D_n^s(t) := \frac{1}{2} \sum_{b \subset \Lambda^n} \int \left(\sqrt{f_{\varepsilon,t,n}(\omega^b)} - \sqrt{f_{\varepsilon,t,n}(\omega)} \right)^2 \lambda(d\omega),$$

$$D_n^*(t) := \frac{1}{2} \sum_{b \subset \Lambda^n} \int \left(\sqrt{f_{\varepsilon,t,n}(\omega^{b*})} - \sqrt{f_{\varepsilon,t,n}(\omega)} \right)^2 \lambda(d\omega),$$

respectively. By convexity, S_n and any of D_n^q, $q \in \{o, *, s\}$ are nondecreasing sequences.

The Kolmogorov equation yields

$$\partial_t S_n(t) = \int (\partial_t + \mathscr{L}) \log f_{\varepsilon,t,n}(\omega)\, \mu_{\varepsilon,t}(d\omega)$$

$$= \int f_{\varepsilon,t,n+1}(\omega) \mathscr{L} \log f_{\varepsilon,t,n}(\omega)\, \lambda(d\omega)$$

$$= \beta \sum_{b \in \mathbb{Z}^*} \int f_{\varepsilon,t,n+1}(\omega) \log \frac{f_{\varepsilon,t,n}(\omega^{b*})}{f_{\varepsilon,t,n}(\omega)}\, \lambda(d\omega)$$

$$+ \sum_{b \in \mathbb{Z}^*} \int (c_b(\omega) + \sigma) f_{\varepsilon,t,n+1}(\omega) \log \frac{f_{\varepsilon,t,n}(\omega^b)}{f_{\varepsilon,t,n}(\omega)}\, \lambda(d\omega).$$

Taking into account inequality (3.2), it is easy to recover the local Dirichlet forms from the segments $-n \leq k < n$ of the corresponding sums, the rest gives the so called *boundary terms*.

3.2. Entropy flux: Our basic a priori bound on local entropy and Dirichlet forms is the content of

Lemma 3.1. *If $\sigma \geq 1$ and $\beta \leq \sigma$ then we have a universal constant C_0 such that*

$$S_n(t) + \beta \int_0^t D_n^*(\tau)\, d\tau + \sigma \int_0^t D_n(\tau)\, d\tau \leq C_0 \left(t + \sqrt{n^2 + \sigma t} \right)$$

for any initial distribution $\mu_{\varepsilon,0}$, $n \in \mathbb{N}$ and $t > 0$.

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Proof. We have to estimate the boundary terms of $\partial_t S_n$ by means of S_n and its associated Dirichlet forms D_n^q. Following the lines of the proof of Lemma 3.1 in [FN06], we arrive at a system

$$
\begin{aligned}
\partial_t S_n(t) + 2\beta D_n^*(t) + 2\sigma D_n(t) &\leq K_1 \left(S_{n+1}(t) - S_n(t) \right) \\
&\quad + \beta K_1 \sqrt{S_{n+1}(t) - S_n(t)} \sqrt{D_{n+1}^*(t) - D_n^*(t)} \\
&\quad + \sigma K_1 \sqrt{S_{n+1}(t) - S_n(t)} \sqrt{D_{n+1}(t) - D_n(t)},
\end{aligned}
\tag{3.4}
$$

of differential inequalities with some universal constant K_1. The cases of boundary term induced by $\mathscr{L}_o$ and $\mathscr{S}$ are the same as in [FN06], the asymmetric $\mathscr{L}_o$ yields the flux $S_{n+1} - S_n$ on the right hand side. When we estimate boundary terms induced by $\mathscr{S}$, we are exploiting its following properties. The jump rates are constant, and the elementary actions, $\omega \to \omega^b$ all preserve λ in a reversible way. Since $\mathscr{G}_*$ possesses all of these features, replacing ω^b with ω^{b*} in the computations concerning the boundary terms induced by $\mathscr{G}_*$, we get (3.4). This system can explicitly be solved by means of the following lemma, which is a simple generalization of Lemma 3 in [Fri90].

Lemma 3.2. *Suppose that $s_{n+1}(t) \geq s_n(t) \geq 0$, $u_{n+1}(t) \geq u_n(t) \geq 0$, $v_{n+1}(t) \geq v_n(t) \geq 0$ for $t \geq 0$ and $n \in \mathbb{N}$, moreover*

$$
\begin{aligned}
ds_n/dt + 2\beta u_n + 2\sigma v_n &\leq C(s_{n+1} - s_n) + \beta K \sqrt{(s_{n+1} - s_n)(u_{n+1} - u_n)} \\
&\quad + \sigma K \sqrt{(s_{n+1} - s_n)(v_{n+1} - v_n)},
\end{aligned}
$$

where $0 < \beta = \mathcal{O}(\sigma)$, $C, K > 0$ and $s_0 = u_0 = v_0 = 0$. Then we have a constant, M depending on C and K such that for all $t \geq 0$, $n \in \mathbb{N}$ we have

$$
s_n(t) + \beta \int_0^t u_n(s)\, ds + \sigma \int_0^t v_n(s)\, ds \leq \frac{M}{R} \sum_{m=0}^{+\infty} s_m(0) \exp\left(-\frac{m}{R} \right),
$$

where $R := M \left(t + (n^2 + \sigma t)^{1/2} \right)$.

Proof. Just as in [Fri90], a clever cutoff function $g_n(r)$ can be defined by

$$
g_n(r) = \int_{-\infty}^{\infty} g(x/r) g(n - x)\, dx
$$

for $n \in \mathbb{Z}_+$ and $r > 0$, where $g(x) = 1$ if $|x| \leq 1$, $g'(x) = -g(x)\operatorname{sign} x$ if $|x| \geq 3$ and $g'(x) = -(1/2)g(x)(x - \operatorname{sign} x)$ otherwise. It is easy to check that $0 < g_n(r) - g_{n+1}(r) \leq (2/r)g_{n+1}(r)$ if $r \geq 1$, moreover $g_n(r) - g_{n+1}(r) \leq 2\min\{g_n'(r), g_{n+1}'(r)\}$. Introduce now $s(t, r) := \xi(t, r)$ if $\xi_n(t) = s_n(t)$, $u(t, r) := \xi(t, r)$ if $\xi_n(t) = u_n(t)$ and $v(t, r) := \xi(t, r)$ if $\xi_n(t) = v_n(t)$, where

$$
\xi(t, r) = \sum_{n=0}^{+\infty} \left(g_n(r) - g_{n+1}(r) \right) \xi_n(t) = \sum_{n=0}^{+\infty} g_{n+1}(r) \left(\xi_{n+1}(t) - \xi_n(t) \right).
$$

Using βu_n and σv_n to estimate the corresponding square roots on the right hand side, we get

$$
\frac{\partial s(t, r)}{\partial t} + \beta u(t, r) + \sigma v(t, r) \leq M_1 \left(1 + \frac{\sigma}{r} \right) \frac{\partial s(t, r)}{\partial r}
$$

because $\beta = \mathcal{O}(\sigma)$, consequently

$$s(t, r(t)) + \beta \int_0^t u(s, r(s)) \, ds + \sigma \int_0^t v(s, r(s)) \, ds \leq s(0, r(0)),$$

provided that the decay of $r(t)$ is fast enough. More precisely, let r solve $r \, dr/dt + M_1 r + M_1 \sigma = 0$ with terminal condition $r(t) = n$. Since $r(0) = \mathcal{O}(t + (n^2 + \sigma t)^{1/2})$ in this case, and $g_n(r) \geq e^{-n/r}$ if $r \geq 1$, this completes the proof by a direct calculation. $\square$

Now we are in a position to complete the proof of Lemma 3.1. Since our reference measure λ_0^* is the uniform distribution on $\{0, 1, -1\}^{2n+1}$, $S_n(0) \leq (2n + 1) \log 3$. Therefore choosing $s_n = S_n$, $u_n = D_n^*$ and $v_n = D_n$ we see that (3.4) really implies Lemma 3.1. $\square$

This lemma is the fundamental a priori bound we need to materialize hydrodynamic limit in infinite volume. Since $\varepsilon\sigma \to 0$, $t \approx \tau/\varepsilon$ and $n \approx r/\varepsilon$ in its following consequences, $(r + \tau)/\varepsilon$ is the order of the bound, and $r + \tau \leq r\tau$ if $r, \tau \geq 1$. To simplify formulae, from now on we are assuming that $\varepsilon\sigma \leq 1$ and $\varepsilon l^3 \geq \sigma \geq l \geq 1$.

3.3. One block and two blocks estimates: The first replacement lemma for microscopic currents is based on the logarithmic Sobolev inequality for stirring $\mathscr{S}$. Given $\bar{\omega}_{l,k} = u$ and $\bar{\eta}_{l,k} = \rho$, let $\bar{\lambda}_{\rho,u}^{l,k}$ denote the conditional distributions of $\omega_k, \omega_{k+1}, ..., \omega_{k+l-1}$ with respect to λ. In view of Proposition 4 of [FT04], we have a universal constant, $\aleph$ such that

$$S[\bar{\mu} | \bar{\lambda}_{\rho,u}^{l,k}] \leq \aleph l^2 \sum_{b \subset [k, k+l)} \int \left(\sqrt{f(\omega^b)} - \sqrt{f(\omega)} \right)^2 \bar{\lambda}_{\rho,u}^{l,k}(d\omega) \qquad (3.5)$$

whenever $\bar{\mu} \ll \bar{\lambda}_{\rho,u}^{l,k}$ is a probability measure and $f := d\bar{\mu}/d\bar{\lambda}_{\rho,u}^{l,k}$. It is very important that $\aleph$ does not depend on ρ, u, l and f. This LSI allows us to estimate canonical expectations via the basic entropy inequality. The moment generating part is also a conditional expectation, consequently its bound should be independent of the conditions. Let us remark that Lemmas 3.3, 3.4 and 3.6 are consequences of the local entropy bound Lemma 3.1, therefore methods of [FN06] work also in case of the creation - annihilation process, and the results are valid for the spin - flip dynamics, too.

Lemma 3.3. *Let* $j(\omega_0, \omega_1)$ *denote a given function,* $\mathfrak{J}(\rho, u) := \lambda_{\rho,u}(\mathfrak{j})$ *and* $j_k(\omega) := j(\omega_k, \omega_{k+1})$. *We have a threshold* $l_0 \in \mathbb{N}$ *and a universal constant* C_1 *depending only on* C_0 *and* $\mathfrak{j}$ *such that*

$$\varepsilon^2 \sum_{|k| < r/\varepsilon} \int_0^{\tau/\varepsilon} \int \left(\bar{j}_{l-1,k} - \mathfrak{J}(\bar{\eta}_{l,k}, \bar{\omega}_{l,k}) \right)^2 d\mu_{\varepsilon,t} \, dt \leq C_1 \frac{\varepsilon r \tau l^2}{\sigma}$$

whenever $r, \tau \geq 1$ *and* $\varepsilon l^3 \geq \sigma \geq l \geq l_0$.

Proof. In view of Lemma 3.1 the statement is more or less a direct consequence of the first inequality of Proposition 1 in [FT04], where notation is slightly different from ours. Lemma 3.2 of [FN06] treats the case of $j_k = j_k^{\eta o}$, our problem is essentially the same. $\square$

This is a sharp form of the so called *One Block Lemma* of [GPV88], the explicit rate due to LSI is needed for the evaluation of microscopic currents in the expression of the Lax entropy production X_ε. The following comparison of block averages of type $\hat{\xi}$ and $\bar{\xi}$ follows also via LSI in much the same way as the previous lemma did.

Lemma 3.4. *We have a threshold $l_0 \in \mathbb{N}$ and a universal constant C_2 such that if $r, \tau \geq 1$ and $\varepsilon l^3 \geq \sigma \geq l \geq l_0$, then*

$$\varepsilon^2 \sum_{|k|<r/\varepsilon} \int_0^{\tau/\varepsilon} \xi_{l,k}^2 \, d\mu_{\varepsilon,t} \, dt \leq C_2 \frac{r\tau\varepsilon l^2}{\sigma}$$

where $\xi_{l,k} = \hat{\omega}_{l,k} - \bar{\omega}_{l,k}$, or $\xi_{l,k} = \hat{\eta}_{l,k} - \bar{\eta}_{l,k}$, or $\xi_{l,k} = \hat{\omega}_{l,k+l} - \hat{\omega}_{l,k}$.

The statement on block averages of η is a direct consequence of Lemma 3.3 in [FN06], the bound for $\xi_{l,k} = \hat{\omega}_{l,k} - \bar{\omega}_{l,k}$ can be proven in the same way. The argument works even if $\xi_{l,k} = \bar{\omega}_{l,k+m} - \bar{\omega}_{l,k}$, or $\xi_{l,k} = \bar{\eta}_{l,k+m} - \bar{\eta}_{l,k}$, but an integration by parts trick yields better results in these cases when m is large. By means of the Cauchy inequality the deviation of blocks of different size can also be estimated.

Lemma 3.5. *We have universal constants $l_0 \in \mathbb{N}$ and $C_3 < +\infty$ such that if $l_0 \leq l \leq m$ but $\varepsilon l^3 \geq \sigma \geq l$ and $r, \tau \geq 1$, then*

$$\varepsilon^2 \sum_{|k|<r/\varepsilon} \int_0^{\tau/\varepsilon} \int (\bar{\xi}_{m,k} - \bar{\xi}_{l,k})^2 \, d\mu_{\varepsilon,t} \, dt \leq C_3 \frac{r\tau\varepsilon m^2}{\sigma}$$

with $\xi = \omega$ or $\xi = \eta$.

For block averages of η the statement is proven in [FN06], see Lemma 3.4 and Section 6 there, the case of ω is the same. To estimate the deviation of consecutive block averages, set $m = 2l$. We see that $m = \delta\sqrt{\sigma/\varepsilon}$, $\delta \to 0$ is the maximal size of the bigger block, but $\varepsilon\sigma \to 0$, thus large microscopic block averages can not be replaced with small macroscopic ones: a strong compactness argument in not available.

3.4. The first relaxation inequality: To complete the evaluation of entropy production, we have to replace $\bar{\eta}_{l,k}$ with its empirical estimator $F(\bar{\omega}_{l,k})$.

Lemma 3.6. *For all $r, \tau \geq 1$ we have universal constants $l_0 \in \mathbb{N}$ and $C_4 < +\infty$ such that*

$$\varepsilon^2 \sum_{|k|<r/\varepsilon} \int_0^{\tau/\varepsilon} \int \left(\bar{\eta}_{l,k} - F(\bar{\omega}_{l,k}) \right)^2 d\mu_{\varepsilon,t} \, dt \leq C_4 \left(\frac{r\tau\sigma}{\beta l^2} + \frac{r\tau\varepsilon l^2}{\sigma} \right)$$

whenever $\tau, r \geq 1$ and $\varepsilon l^3 \geq \sigma \geq l \geq l_0$.

Proof. Let us consider the evolution of

$$\bar{H}(t, \psi) := \int_{-\infty}^{\infty} \psi(t, x) H(\bar{v}_\varepsilon(t, x)) \, dx$$

along the modified empirical process

$$\bar{v}_\varepsilon(t, x) = (\bar{\rho}_\varepsilon(t, x), \bar{u}_\varepsilon(t, x)) := \left(\bar{\eta}_{l,k}(t/\varepsilon), \bar{\omega}_{l,k}(t/\varepsilon) \right) \quad \text{if } |x - \varepsilon k| < \varepsilon/2,$$

where $0 \leq \psi \in C_c^1(\mathbb{R}^2)$, and $H(\rho, u) := (1/2) \left(\rho - F(u) \right)^2$ is our Liapunov function. Of course, the evolution of $\bar{v}_\varepsilon$ is governed by $\mathscr{L}$, thus the Kolmogorov equation yields

$$\bar{H}(t, \psi) = \bar{H}(0, \psi) + \bar{I}(t, \psi_t') + \frac{1}{\varepsilon} \int_0^t \mathscr{L}\bar{H}(s, \psi) \, ds + \bar{M}_\varepsilon(t, \psi),$$

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where $\bar{M}_\varepsilon(t,\psi)$ is a martingale and $I(t,\psi'_t) := \int_0^t \bar{H}(s,\psi'_t)\,ds$, whence

$$\mathsf{E}\bar{H}(\infty,\psi) = 0 = \mathsf{E}\bar{H}(0,\psi) + \mathsf{E}\bar{I}(\infty,\psi'_t) + \frac{1}{\varepsilon}\int_0^\infty \mathsf{E}\mathscr{L}\bar{H}(t,\psi)\,dt. \tag{3.6}$$

Here, and also later on, most calculations are done at the microscopic level. Since $\bar{v}_\varepsilon$ is a step function, the integral mean,

$$\psi_k(t) := \frac{1}{\varepsilon}\int_{\varepsilon k-\varepsilon/2}^{\varepsilon k+\varepsilon/2} \psi(\varepsilon t,x)\,dx$$

of ψ appears in such expressions. We write $V_k(t) := V(\bar{\eta}_{l,k}(t),\bar{\omega}_{l,k}(t))$ whenever V is a function of the empirical process; for instance $H'_{u,k}(t) = H'_u(\bar{v}_\varepsilon(\varepsilon t,\varepsilon k))$. In the forthcoming calculations notation and facts from Sections 1.2 and 1.3 may be used without any reference.

The first and second terms, $\mathsf{E}\bar{H}(0,\psi)$ and $\mathsf{E}\bar{I}(\infty,\psi'_t)$ on the right hand side of (3.6) are bounded; to evaluate the third one, let us consider its decomposition:

$$\frac{1}{\varepsilon}\int_0^\infty \mathscr{L}\bar{H}(t,\psi)\,dt = \bar{L}_\varepsilon^o(\psi) + \sigma\bar{L}_\varepsilon^s(\psi) + \beta\bar{L}_\varepsilon^*(\psi),$$

cf. $\mathscr{L} = \mathscr{L}_o + \beta\mathscr{G}_* + \sigma\mathscr{S}$, where

$$\bar{L}_\varepsilon^o(\psi) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)\mathscr{L}_oH\left(\bar{\eta}_{l,k}(t),\bar{\omega}_{l,k}(t)\right)\,dt,$$

$\bar{L}_\varepsilon^s(\psi)$ and $\bar{L}_\varepsilon^*(\psi)$ are defined analogously: we have to replace $\mathscr{L}_o$ by $\mathscr{S}$ or $\mathscr{G}_*$, respectively.

$$\mathscr{L}_oH(\bar{\eta}_{l,k},\bar{\omega}_{l,k}) = \sum_{b\in\mathbb{Z}^*}c_b(\omega)\left(H(\bar{\eta}_{l,k}^b,\bar{\omega}_{l,k}^b) - H\left(\bar{\eta}_{l,k},\bar{\omega}_{l,k}\right)\right);$$

$\bar{\omega}_{l,k}^b$ and $\bar{\eta}_{l,k}^b$ are block averages of the sequences $(\omega^b)_j$ and $(\eta^b)_j = (\omega^b)_j^2$, respectively. Next we expand $\mathscr{S}H$ by means of the Lagrange theorem. From

$$H(\bar{\eta}_{l,k}^b,\bar{\omega}_{l,k}^b) - H(\bar{\eta}_{l,k},\bar{\omega}_{l,k}) = H'_{\rho,k}(t)(\bar{\eta}_{l,k}^b - \bar{\eta}_{l,k}) + H'_{u,k}(t)(\bar{\omega}_{l,k}^b - \bar{\omega}_{l,k})$$
$$+ B_k(\bar{\eta}_{l,k}^b - \bar{\eta}_{l,k},\bar{\omega}_{l,k}^b - \bar{\omega}_{l,k}),$$

where B_k is a quadratic form with bounded coefficients, we get $\bar{L}_\varepsilon^s = R_\varepsilon^s + Q_\varepsilon^s$,

$$R_\varepsilon^s(\psi) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)\left(H'_{\rho,k}(t)\Delta_1\bar{\eta}_{l,k} + H'_{u,k}(t)\Delta_1\bar{\omega}_{l,k}\right)\,dt,$$

and $Q_\varepsilon^s(\psi)$ is a quadratic form of the differences $\bar{\eta}_{l,k}^b - \bar{\eta}_{l,k}$ and $\bar{\omega}_{l,k}^b - \bar{\omega}_{l,k}$. Remember now that

$$\mathscr{G}_*H(\bar{\eta}_{l,k},\bar{\omega}_{l,k}) = \sum_{b\in\mathbb{Z}^*}\left(H(\bar{\eta}_{l,k}^{b*},\bar{\omega}_{l,k}^{b*}) - H(\bar{\eta}_{l,k},\bar{\omega}_{l,k})\right)$$

and $H(\bar{\eta}_{l,k}^{b*},\bar{\omega}_{l,k}^{b*}) - H(\bar{\eta}_{l,k},\bar{\omega}_{l,k}) = H(\bar{\eta}_{l,k}^{b*},\bar{\omega}_{l,k}^{b*}) - H(\bar{\eta}_{l,k}^{b*},\bar{\omega}_{l,k}) + H(\bar{\eta}_{l,k}^{b*},\bar{\omega}_{l,k}) - H(\bar{\eta}_{l,k},\bar{\omega}_{l,k})$, where $\bar{\omega}_{l,k}^{b*}$ and $\bar{\eta}_{l,k}^{b*}$ are again block averages, those of ω^{b*} and η^{b*}, respectively. Moreover,

$$H(\bar{\eta}_{l,k}^{b*},\bar{\omega}_{l,k}) - H(\bar{\eta}_{l,k},\bar{\omega}_{l,k}) = H'_\rho(\bar{\eta}_{l,k},\bar{\omega}_{l,k})(\bar{\eta}_{l,k}^{b*} - \bar{\eta}_{l,k}) + (1/2)(\bar{\eta}_{l,k}^{b*} - \bar{\eta}_{l,k})^2$$

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and $H'_\rho(\rho, u) = \rho - F(u)$, finally from (1.12)

$$\mathscr{G}_* \bar\eta_{l,k} = \sum_{b \in \mathbb{Z}^*} (\bar\eta_{l,k}^{b*} - \bar\eta_{l,k}) = \frac{1}{l} \sum_{b \subset [k, k+l]} \left(c_b^+(\omega) - c_b^\times(\omega) + c_{b-}^+(\omega) - c_{b-}^\times(\omega) \right).$$

Having in mind also Lemma 3.3, we now decompose $\bar{L}_\varepsilon^*$ as $\bar{L}_\varepsilon^* = G_\varepsilon^u + Q_\varepsilon^\rho + \Gamma_\varepsilon^\rho + R_\varepsilon^\rho$, where

$$G_\varepsilon^u(\psi) := \sum_{k \in \mathbb{Z}} \varepsilon \sum_{b \in \mathbb{Z}^*} \int_0^\infty \psi_k(t) \left(H(\bar\eta_{l,k}^{b*}, \bar\omega_{l,k}^{b*}) - H(\bar\eta_{l,k}^{b*}, \bar\omega_{l,k}) \right),$$

$$\Gamma_\varepsilon^\rho(\psi) := \varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \psi_k(t) \left(\bar\eta_{l,k} - F(\bar\omega_{l,k}) \right) C_{l,k}^*(\omega) \, dt,$$

$$C_{l,k}^*(\omega) := \frac{2}{l-1} \sum_{b \subset [k, k+l)} \left(c_b^+(\omega) - c_b^\times(\omega) \right),$$

$$R_\varepsilon^\rho(\psi) := \varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \psi_k(t) \left(\bar\eta_{l,k} - F(\bar\omega_{l,k}) \right) \left(\mathscr{G}_* \bar\eta_{l,k} - C_{l,k}^*(\omega) \right) dt,$$

finally $Q_\varepsilon^\rho(\psi)$ is the contribution of the squared differences $(1/2)(\bar\eta_{l,k}^{b*} - \bar\eta_{l,k})^2$.

$\Gamma_\varepsilon^\rho(\psi)$ is the critical term here, it looks much larger than the others: its order seems to be $1/\varepsilon$. However, due to some cancelations we have

$$-\mathsf{E}\Gamma_\varepsilon^\rho(\psi) \leq K(\psi) \left(\frac{1+\beta}{\varepsilon\beta l} + \frac{\sigma}{\varepsilon\beta l^2} \right) \leq K(\psi) \left(\frac{1}{\varepsilon l} + \frac{2\sigma}{\varepsilon\beta l^2} \right), \qquad (3.7)$$

where $K(\psi) < +\infty$ is a constant depending only on ψ. To prove this, observe first that

$$\bar\omega_{l,k}^b - \bar\omega_{l,k} = \bar\eta_{l,k}^b - \bar\eta_{l,k} = \bar\omega_{l,k}^{b*} - \bar\omega_{l,k} = 0$$

unless $b = (k-1, k)$ or $b = (k+l-1, k+l)$, while any of $|\bar\omega_{l,k}^b - \bar\omega_{l,k}|$, $|\bar\eta_{l,k}^b - \bar\eta_{l,k}|$, $|\bar\omega_{l,k}^{b*} - \bar\omega_{l,k}|$ and $|\bar\eta_{l,k}^{b*} - \bar\eta_{l,k}|$ is bounded by $2/l$, finally $\bar\eta_{l,k}^{b*} - \bar\eta_{l,k} = 0$ if $b \subset (-\infty, k)$ or $b \subset [k+l, +\infty)$. The derivation of (3.7) reduces to these deterministic bounds by simple computations.

Indeed, $\bar{H}_\varepsilon(0, \psi)$ and $\bar{H}_\varepsilon(\infty, \psi'_t)$ are uniformly bounded, while $\bar{L}_\varepsilon^o(\psi) = \mathcal{O}(1/\varepsilon l)$, $G_\varepsilon^u(\psi) = \mathcal{O}(1/\varepsilon l)$, $Q_\varepsilon^s(\psi) = \mathcal{O}(\varepsilon^{-1}l^{-2})$, $Q_\varepsilon^\rho(\psi) = \mathcal{O}(1/\varepsilon l)$ and $R_\varepsilon^\rho(\psi) = \mathcal{O}(1/\varepsilon l)$. In the case of R_ε^s we do discrete integration by parts to get

$$R_\varepsilon^s(\psi) := -\varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \left(\nabla_1(\psi_k(t) H'_{\rho,k}(t)) \right) \nabla_1 \bar\eta_{l,k}(t)) \, dt$$

$$- \varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \left(\nabla_1(\psi_k(t) H'_{u,k}(t)) \right) \nabla_1 \bar\omega_{l,k}(t) \, dt,$$

where $\nabla_1 \xi_k = \xi_{k+1} - \xi_k$, whence $R_\varepsilon^s(\psi) = \mathcal{O}(1/\varepsilon l^2)$. Summarizing these computations we get (3.7) because $l \leq \sigma$ and $\beta \leq \sigma$.

Remember now again that

$$\lambda_{\rho,u}(C_{l,k}^*) = 2C(\rho, u) = (3/2)(\rho - F_*(u))(\rho - F(u)),$$

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and $\rho - F_*(u) \leq -2/3$. That is why we set

$$W_\varepsilon^\rho(\psi) := \frac{3\varepsilon}{2} \sum_{k \in \mathbb{Z}} \int_0^\infty \psi_k(t) \left(\bar{\eta}_{l,k} - F_*(\bar{\omega}_{l,k}) \right) \left(\bar{\eta}_{l,k} - F(\bar{\omega}_{l,k}) \right)^2 dt$$

$$\leq -\varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \psi_k(t) \left(\bar{\eta}_{l,k} - F(\bar{\omega}_{l,k}) \right)^2 dt \,,$$

and consider $2\Gamma_\varepsilon^\rho - W_\varepsilon^\rho = 2\Gamma_\varepsilon^\rho - 2W_\varepsilon^\rho + W_\varepsilon^\rho$. From

$$2\Gamma_\varepsilon^\rho - 2W_\varepsilon^\rho = 2\varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \psi_k(t) \left(\bar{\eta}_{l,k} - F(\bar{\omega}_{l,k}) \right) D_k(t) dt \,,$$

where $D_k := C_{l,k}^*(\omega) - 2C(\bar{\eta}_{l,k}, \bar{\omega}_{l,k})$ we obtain

$$2\Gamma_\varepsilon^\rho - W_\varepsilon^\rho \leq \varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \psi_k(t) D_k^2(t) dt \,.$$

Now we are in a position to apply Lemma 3.3 to conclude $2\mathsf{E}\Gamma_\varepsilon^\rho(\psi) - \mathsf{E}W_\varepsilon^\rho(\psi) \leq K'(\psi)(l^2/\sigma)$, whence by (3.7)

$$-\mathsf{E}W_\varepsilon^\rho(\psi) \leq \bar{K}(\psi) \left(\frac{1}{\varepsilon l} + \frac{2\sigma}{\varepsilon \beta l^2} + \frac{l^2}{\sigma} \right) \leq 2\bar{K}(\psi) \left(\frac{\sigma}{\varepsilon \beta l^2} + \frac{l^2}{\sigma} \right) \,,$$

where $K'(\psi)$ and $\bar{K}(\psi)$ depend only on ψ in a simple way. Choosing ψ such that $\psi(t, x) = 1$ if $0 \leq t \leq \tau$ and $|x| \leq 1 + r$, while $\psi(t, x) = 0$ if $t > \tau + 1$ and $|x| > 2 + r$, we obtain the statement of the lemma by a direct computation. $\qquad \square$

The full power of our tools has not been exploited in the proof of (3.7), it is possible to improve Lemma 3.6, thus also Theorem 1.1 a bit. Namely, the lower bound $\varepsilon \sigma^2(\varepsilon) \beta^2(\varepsilon) \to +\infty$ can be replaced with $\varepsilon \sigma^2(\varepsilon) \beta^3(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. We do not go into details because by means of a clever Lax entropy - flux pair a much better condition, $\sigma(\varepsilon) \beta(\varepsilon) \to +\infty$ can be proven, see Sections 5.5 and 5.6.

4　Estimation of Entropy Production

The main part of this section is devoted to the verification of the conditions of Proposition 2.2, first of all the components L_ε, M_ε, J_ε and N_ε of entropy production X_ε have to be evaluated. We are assuming that $h, J \in C^2(\mathbb{R})$ with bounded first and second derivatives; $J'(u) = (F'(u) - 2u)h'(u)$ is the relation of h and J. Our calculations are based on the a priori bounds of Section 3, the argument follows [FN06] with some modifications. Although h and J here are now functions of $\hat{\omega}$, considerable differences are coming only from the replacement of $\alpha \mathcal{G}_\kappa$ with $\beta \mathcal{G}_*$. Spin - flips are simple because $\mathcal{G}_\kappa \eta_k = 0$, and the linear expression, $\mathcal{G}_\kappa \omega_k = -2(\kappa \eta_k - \omega_k)$ is also easily controlled; $\mathcal{G}_* \eta_k$ and $\mathcal{G}_* \omega_k$ are more complicated, cf. Section 1.3. Nevertheless, the main lines of our computations are essentially the same as in [FN06], just Lemma 3.6 of this paper is used instead

of Lemma 3.5 of [FN06] at the final evaluation of entropy production. Among others, we have to show that due to reversibility of $\mathscr{G}_*$, the microscopic current $j^{\omega*}$ vanishes in the limit.

The a priori bounds (2.8) and (2.9) we need for compensated compactness are localized by a smooth function $\phi \in C^2_{co}(\mathbb{R}^2_+)$ of compact support, thus $H_\varepsilon(0, \phi\psi, h) = 0$, see (2.6) also for the basic decomposition of $X_\varepsilon(\psi, h)$. Since $\hat{u}_\varepsilon(t, x)$ is a step function of $x \in \mathbb{R}$, the integral mean,

$$\psi_k(t) := \frac{1}{\varepsilon} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + \varepsilon/2} \varphi(\varepsilon t, x) \, dx$$

of $\varphi(t, x) := \phi(t, x)\psi(t, x)$ appears quite frequently in our equations. Finally, $\nabla_\varepsilon \varphi(x) := \varepsilon^{-1}(\varphi(x + \varepsilon) - \varphi(x))$ for functions, while in the case of sequences we write $\nabla_l \xi_k := l^{-1}(\xi_{k+l} - \xi_k)$, $\nabla_l^* \xi_k := l^{-1}(\xi_{k-l} - \xi_k)$, and $\Delta_l \xi_k := -\nabla_l^* \nabla_l \xi_k$. Note that ∇_l^* is the adjoint of ∇_l in $\ell^2(\mathbb{Z})$, $\nabla_1 \hat{\xi}_{l,k} = \nabla_l \bar{\xi}_{l,k+1-l}$ and $\nabla_1^* \hat{\xi}_{l,k} = \nabla_l^* \bar{\xi}_{l,k}$. For $\nabla_1 \psi_k$ we have an identity:

$$\nabla_1 \psi_k(t) = \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} (\varepsilon - |x|) \varphi'_x(\varepsilon t, \varepsilon k + x + \varepsilon/2) \, dx,$$

where $\varphi = \phi\psi$, whence by the Schwarz inequality

$$\left(\nabla_1 \psi_k(t)\right)^2 \le \frac{2\varepsilon}{3} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + 3\varepsilon/2} \varphi'^2_x(\varepsilon t, x) \, dx.$$

A similar bound of $(\nabla_l \psi_k)^2$ follows easily because $\nabla_l \psi_k = \nabla_1 \bar{\psi}_{l,k}$, thus

$$\left(\nabla_l \psi_k(t)\right)^2 \le \frac{1}{l} \sum_{j=k}^{k+l-1} \left(\psi_{j+1}(t) - \psi_j(t)\right)^2.$$

Such estimates are frequently used in the following calculations to obtain bounds in terms of $\|\psi\|_{+1}$. From now on we are assuming that the parameters $\sigma(\varepsilon)$, $\beta(\varepsilon)$ and $l(\varepsilon)$ of our problem are specified as in Theorem 1.1 and before (2.2).

4.1. The numerical error: This is the easiest case, by a direct calculation

$$N_\varepsilon(\phi\psi, h) = \varepsilon \sum_{k \in \mathbb{Z}} \int_0^\infty \left(\nabla_1 \psi_k(t) - \varepsilon \nabla_\varepsilon \varphi(\varepsilon t, \varepsilon k - \varepsilon/2)\right) J_k(t) \, dt. \tag{4.1}$$

Since J' is bounded, without any modification of the argument of the proof of Lemma 4.1 of [FN06], we obtain that the numerical error, N_ε satisfies (2.8) with a vanishing bound.

4.2. The martingale: We estimate the H^{-1} norm of $M_\varepsilon(\infty, \psi, h)$ as follows. The stochastic differential $dh(\hat{u}_\varepsilon) = \varepsilon^{-1}\mathscr{L}h(\hat{u}_\varepsilon) + dm_\varepsilon$ defines a martingale $m_\varepsilon = m_\varepsilon(t, x)$ for each $x \in \mathbb{R}$ such that $m_\varepsilon(t, x) = m_\varepsilon(t, \varepsilon k)$ if $|x - \varepsilon k| < \varepsilon/2$ and

$$M_\varepsilon(t, \phi\psi, h) = \int_{-\infty}^\infty \int_0^t \psi(s, x) \phi(s, x) m_\varepsilon(ds, x) \, dx. \tag{4.2}$$

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The martingale m_ε is identified by the intensity q_ε of its quadratic variation:

$$q_\varepsilon(t,x) := \frac{1}{\varepsilon}\left(\mathscr{L}h^2(\hat{u}_\varepsilon) - 2h(\hat{u}_\varepsilon)\mathscr{L}h(\hat{u}_\varepsilon)\right)$$

$$= \frac{1}{\varepsilon}\sum_{b\in\mathbb{Z}^*}(c_b(\omega) + \sigma(\varepsilon))\left(h(\hat{\omega}_{l,k}^b) - h(\hat{\omega}_{l,k})\right)^2$$

$$+ \frac{\beta(\varepsilon)}{\varepsilon}\sum_{b\in\mathbb{Z}^*}\left(h(\hat{\omega}_{l,k}^{b*}) - h(\hat{\omega}_{l,k})\right)^2$$

if $|x - \varepsilon k| < \varepsilon/2$, where $\hat{\omega}_{l,k}^{b*}$ denotes the block average of ω^{b*}.

Lemma 4.1. $M_\varepsilon(\infty, \phi\psi, h)$ *satisfies* (2.8) *with a vanishing bound.*

Proof. Let $\dot{m}_\varepsilon(t,x)$ denote the time derivative of m_ε in the H^{-1} sense, we have to show that $\mathsf{E}\|\varphi\,\dot{m}_\varepsilon\|_{-1}^2 \to 0$ as $\varepsilon \to 0$. Since $\phi\dot{m}_\varepsilon = \partial_t(\phi m_\varepsilon) - \phi_t' m_\varepsilon$ in H^{-1}, we have

$$|M_\varepsilon| \leq \|\psi\|_{+1}\|\phi\dot{m}_\varepsilon\|_{-1} \leq \|\psi\|_{+1}\left(\|\phi m_\varepsilon\|_2 + \|\phi_t' m_\varepsilon\|_2\right),$$

consequently we have to estimate

$$\mathsf{E}\,m_\varepsilon^2(t,x) = \int_0^t \mathsf{E}\,q_\varepsilon(\tau,x)\,d\tau.$$

However, $|\hat{\omega}_{l,k}^b - \hat{\omega}_{l,k}| \leq 2/l^2$ and $|\hat{\omega}_{l,k}^{b*} - h(\hat{\omega}_{l,k})| \leq 1/l^2$, thus independently of the configuration we have $q_\varepsilon(t,x) = \mathcal{O}((\sigma + \beta)/\varepsilon l^3(\varepsilon))$, which completes the proof. □

4.3. The microscopic current: The starting point of the estimation of L_ε is an identity,

$$\mathscr{L}h(\hat{\omega}_{l,k}) = h'(\hat{\omega}_{l,k})\mathscr{L}\hat{\omega}_{l,k} + \frac{1}{2}\sum_{b\in\mathbb{Z}^*}h''(\tilde{\omega}_k^b)(c_b(\omega) + \sigma(\varepsilon))(\hat{\omega}_{l,k}^b - \hat{\omega}_{l,k})^2$$

$$+ \frac{\beta(\varepsilon)}{2}\sum_{b\in\mathbb{Z}^*}h''(\tilde{\omega}_k^{b*})(\hat{\omega}_{l,k}^{b*} - \hat{\omega}_{l,k})^2, \tag{4.3}$$

where $\tilde{\omega}_k^b$ and $\tilde{\omega}_k^{b*}$ are intermediate values. The contribution of the quadratic remainders vanishes in the space of measures in an obvious way, cf. (2.9), because $\mathcal{O}(l^{-2})$ is the order of both differences, and $\sigma/\varepsilon l^3 \to 0$.

Therefore we are facing with the resultant of

$$\mathscr{L}\hat{\omega}_{l,k} = \nabla_l^* \bar{j}_{l,k}^{\omega o} + \beta(\varepsilon)\nabla_l^* \bar{j}_{l,k}^{\omega *} + \sigma(\varepsilon)\nabla_l^* \bar{\omega}_{l,k},$$

see Section 1.3 for the definition of currents. Let us consider first the easy case of

$$L_\varepsilon^{\omega s}(\phi\psi, h) := \varepsilon\sigma(\varepsilon)\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)\Delta_1\hat{\omega}_{l,k}(t)\,dt, \tag{4.4}$$

it is the contribution of $\sigma \mathscr{S}$. We have $L_\varepsilon^{\omega s} = Y_\varepsilon^{\omega s} + Z_\varepsilon^{\omega s}$, where

$$Y_\varepsilon^{\omega s}(\phi\psi, h) := -\varepsilon\sigma(\varepsilon)\sum_{k\in\mathbb{Z}}\int_0^\infty (\nabla_1\psi_k(t))h'_k(t)\nabla_1\hat{\omega}_{l,k}(t)\,dt,$$

$$Z_\varepsilon^{\omega s}(\phi\psi, h) := -\varepsilon\sigma(\varepsilon)\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_{k+1}(t)(\nabla_1 h'_k(t))\nabla_1\hat{\omega}_{l,k}(t)\,dt.$$

Since the entropic Dirichlet form of $\mathscr{S}$ also has a factor $\sigma(\varepsilon)$ in our fundamental a priori bound, Lemma 3.1, we have

Lemma 4.2. $Y_\varepsilon^{\omega s}$ *satisfies* (2.8) *with a vanishing bound, while* $Z_\varepsilon^{\omega s}$ *satisfies* (2.9). *The bound of* $Z_\varepsilon^{\omega s}$ *does not vanish, and* $Z_\varepsilon^{\omega s} \leq 0$ *if h is convex and* $\phi\psi \geq 0$.

Proof. It is exactly the same as that of the second part of Lemma 4.2 in [FN06]. First we separate the factors by means of the Cauchy - Schwarz inequality to let Lemma 3.5 work. For example, suppose that ϕ is supported in the rectangle $[0, \tau] \times [-r-1, r+1]$, then $|Y_\varepsilon^{\omega s}| \leq (\sigma/\varepsilon)\|h'\|\sqrt{\Psi_1 Q_1}$, where

$$\Psi_1 := \varepsilon^2\sum_{k\in\mathbb{Z}}\int_0^\infty (\nabla_1\psi_k(t))^2\,dt, \quad Q_1 := \varepsilon^2\sum_{|k|<r/\varepsilon}\int_0^{\tau/\varepsilon}(\nabla_1\hat{\omega}_{l,k})^2\,dt.$$

It is plain that $\Psi_1 \leq \varepsilon^2\|\phi\|^2\|\psi\|_{+1}^2$, while $Q_1 = \mathcal{O}(\varepsilon/\sigma)$ follows by Lemma 3.5 because $\nabla_1\hat{\omega}_{l,k} = \nabla_l\bar{\omega}_{l,k}$. This trick will be used several times in the next coming computations. $\qquad\square$

By means of the one and two blocks estimates the contributions of the microscopic currents $j^{\omega o}$ and $j^{\omega*}$ can be reduced as follows. Let

$$L_\varepsilon^{\omega o}(\phi\psi, h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h'_k(t)\nabla_l^*\bar{j}_{l,k}^{\omega o}(\omega(t))\,dt, \tag{4.5}$$

$$L_\varepsilon^{\omega*}(\phi\psi, h) := \varepsilon\beta(\varepsilon)\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h'_k(t)\nabla_l^*\bar{j}_{l,k}^{\omega*}(\omega(t))\,dt, \tag{4.6}$$

and introduce their *mesoscopic* counterparts:

$$V_\varepsilon^{\omega o}(\phi\psi, h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h'_k(t)\nabla_l^*\mathfrak{J}^{uo}(\bar{\eta}_{l,k}(t), \bar{\omega}_{l,k}(t))\,dt, \tag{4.7}$$

$$V_\varepsilon^{\omega*}(\phi\psi, h) := \varepsilon\beta(\varepsilon)\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h'_k(t)\nabla_l^*\mathfrak{J}^{u*}(\bar{\eta}_{l,k}(t), \bar{\omega}_{l,k}(t))\,dt, \tag{4.8}$$

where $\mathfrak{J}^{uo}(\rho, u) = \lambda_{\rho,u}(j_k^{\omega o})$ and $\mathfrak{J}^{u*}(\rho, u) = \lambda_{\rho,u}(j_k^{\omega*})$, see Section 1.3. Now we split the corresponding differences by doing discrete integration by parts such that $L_\varepsilon^{\omega o} - V^{\omega o} = Y_\varepsilon^{\omega o} + Z^{\omega o}$ and $L_\varepsilon^{\omega*} - V^{\omega*} = Y_\varepsilon^{\omega*} + Z^{\omega*}$, where

$$Y_\varepsilon^{\omega o}(\phi\psi, h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty (\nabla_l\psi_k)h'_k(t)(\bar{j}_{l,k}^{\omega o} - \mathfrak{J}^{uo}(\bar{\eta}_{l,k}, \bar{\omega}_{l,k}))\,dt,$$

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$$Z_\varepsilon^{\omega o}(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_{k+l}(t)(\nabla_l h_k')(\bar{\jmath}_{l,k}^{\omega o} - \mathfrak{J}^{uo}(\bar{\eta}_{l,k},\bar{\omega}_{l,k}))\,dt\,,$$

$$Y_\varepsilon^{\omega*}(\phi\psi,h) := \varepsilon\beta(\varepsilon)\sum_{k\in\mathbb{Z}}\int_0^\infty (\nabla_l\psi_k)h_k'(t)(\bar{\jmath}_{l,k}^{\omega*} - \mathfrak{J}^{u*}(\bar{\eta}_{l,k},\bar{\omega}_{l,k}))\,dt\,,$$

$$Z_\varepsilon^{\omega*}(\phi\psi,h) := \varepsilon\beta(\varepsilon)\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_{k+l}(t)(\nabla_l h_k')(\bar{\jmath}_{l,k}^{\omega*} - \mathfrak{J}^{u*}(\bar{\eta}_{l,k},\bar{\omega}_{l,k}))\,dt\,.$$

The following bounds are more or less direct consequences of Lemma 3.3 and Lemma 3.4 or 3.5; β can not be too large here.

Lemma 4.3. $Y_\varepsilon^{\omega o}$ and $Y_\varepsilon^{\omega*}$ satisfy (2.8) while $Z_\varepsilon^{\omega o}$ and $Z_\varepsilon^{\omega*}$ satisfy (2.9); all bounds vanish as $\varepsilon\to 0$.

Proof. It follows the argument of the first part of Lemma 4.2 in [FN06]. First we separate $\bar{\jmath}_{l,k}^{\omega o} - \mathfrak{J}^{uo}(\bar{\eta}_{l,k},\bar{\omega}_{l,k})$ and $\bar{\jmath}_{l,k}^{\omega*} - \mathfrak{J}^{u*}(\bar{\eta}_{l,k},\bar{\omega}_{l,k})$ by means of the Cauchy inequality from their factors, then we can use Lemma 3.3 and Lemma 3.4 or Lemma 3.5 in both cases. The procedure is terminated by the elementary computation of $\nabla_l\psi_k$. We get

$$\mathsf{E}|Y_\varepsilon^{\omega o}| = \|\psi\|_{+1}\,\mathscr{O}_\phi\left(l\,\sqrt{\varepsilon/\sigma}\right)\,,\quad \mathsf{E}|Y_\varepsilon^{\omega*}| = \|\psi\|_{+1}\,\mathscr{O}_\phi\left(\beta l\,\sqrt{\varepsilon/\sigma}\right)$$

and $\mathsf{E}|Z_\varepsilon^{\omega o}| = \|\psi\|\,\mathscr{O}_\phi(l/\sigma)$, while $\mathsf{E}|Z_\varepsilon^{\omega*}| = \|\psi\|\,\mathscr{O}_\phi(\beta l/\sigma)$, which complete the proof as $\beta l\sqrt{\varepsilon/\sigma}\to 0$ and $\beta l/\sigma\to 0$ as $\varepsilon\to 0$, cf. (2.3). $\qquad\square$

So far we have replaced the dominant parts of the microscopic currents of $\hat{\omega}_{k,l}$ with their canonical expectations, when $\bar{\eta}_{k,l}$ and $\bar{\omega}_{l,k}$ are given. The crucial step of the whole proof follows right now, it is the replacement of $\bar{\eta}_{l,k}$ with $F(\bar{\omega}_{l,k})$.

4.4. Relaxation in action: As we have indicated above, the last step of the evaluation of entropy production consists in a comparison of J_ε and $V_\varepsilon^{\omega o}$, see (2.7) and (4.7). Indeed, as total charge is preserved by the creation - annihilation mechanism, we expect that $V_\varepsilon^{\omega*}$ vanishes in the limit, while J_ε and $V_\varepsilon^{\omega o}$ cancel each other. By a direct calculation we get

$$J_\varepsilon(\phi\psi,h) = -\varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)\nabla_1^* J_k(t)\,dt\,, \tag{4.9}$$

where $J_k(t) = J(\hat{\omega}_{l,k}(t))$, $h_k' = h'(\hat{\omega}_{l,k}(t))$ and $J'(u) = h'(u)f'(u)$ with $f(u) := F(u) - u^2$, see (1.6) for the definition of F. We are going to replace $J_{k-1} - J_k$ with $h_k'\nabla_l^* f(\bar{\omega}_{l,k})$. We have

$$\begin{aligned}
J_{k-1} - J_k - h_k'\nabla^* f(\bar{\omega}_{l,k}) &= h_k' f_k'\nabla_l^*\bar{\omega}_{l,k} - h_k' f'(\tilde{\omega}_{l,k})\nabla_l^*\bar{\omega}_{l,k}\\
&= h_k' f''(\tilde{\omega}_{l,k}')(\hat{\omega}_{l,k} - \tilde{\omega}_{l,k})\nabla_l^*\bar{\omega}_{l,k}
\end{aligned}$$

with some intermediate values $\tilde{\omega}_{l,k}$ and $\tilde{\omega}_{l,k}'$ such that the quadratic remainders on the right hand side can be neglected. More precisely, contribution of these remainders to J_ε satisfies (2.9) with a vanishing bound; this follows by Lemma 3.4 and Lemma 3.5 in the usual way.

Therefore

$$J_\varepsilon^{jo}(\phi\psi,h) := -\varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)\nabla_l^* f(\bar{\omega}_{l,k}(t))\,dt\,, \tag{4.10}$$

is the essential component of J_ε, we split $J^{jo} - V_\varepsilon^{\omega o}$ into two parts:

$$Y_\varepsilon^{jo}(\phi\psi, h) := \varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty (\nabla_l \psi_k) h_k'(t) \left(\mathfrak{J}^{uo}(\bar{\eta}_{l,k}, \bar{\omega}_{l,k}) - f(\bar{\omega}_{l,k}) \right) dt,$$

$$Z_\varepsilon^{jo}(\phi\psi, h) := \varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_{k+l}(t)(\nabla_l h_k') \left(\mathfrak{J}^{uo}(\bar{\eta}_{l,k}, \bar{\omega}_{l,k}) - f(\bar{\omega}_{l,k}) \right) dt,$$

i.e. $J^{jo} - V_\varepsilon^{\omega o} = Y_\varepsilon^{jo} + Z_\varepsilon^{jo}$. Similarly, $V^{\omega*} = Y_\varepsilon^{j*} + Z_\varepsilon^{j*}$, where

$$Y_\varepsilon^{j*}(\phi\psi, h) := \varepsilon\beta(\varepsilon) \sum_{k\in\mathbb{Z}} \int_0^\infty (\nabla_l \psi_k) h_k'(t) \mathfrak{J}^{u*}(\bar{\eta}_{l,k}, \bar{\omega}_{l,k}) dt,$$

$$Z_\varepsilon^{j*}(\phi\psi, h) := \varepsilon\beta(\varepsilon) \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_{k+l}(t)(\nabla_l h_k') \mathfrak{J}^{u*}(\bar{\eta}_{l,k}, \bar{\omega}_{l,k}) dt.$$

Since $\mathfrak{J}^{uo}(\rho, u) - f(u) = \rho - F(u)$ and $\mathfrak{J}^{u*}(\rho, u) = -(3/4)(\rho - F(u))(\rho - F_*(u))$, see Section 1.3, we are now in a position to apply Lemma 3.6.

Lemma 4.4. Y_ε^{jo} and Y^{j*} satisfy (2.8), while Z_ε^{jo} and Z^{j*} satisfy (2.9); all bounds do vanish.

Proof. In much the same way as in the proof of Lemma 4.3, by means of Lemma 3.6 we get

$$\mathsf{E}|Y_\varepsilon^{jo}| = \|\psi\|_{+1}\, \mathcal{O}_\phi \left(\sqrt{\sigma/\beta l^2 + \varepsilon l^2/\sigma} \right)$$

and

$$\mathsf{E}|Y_\varepsilon^{i*}| = \|\psi\|_{+1}\, \mathcal{O}_\phi \left(\sqrt{\beta\sigma/l^2 + \varepsilon\beta^2 l^2/\sigma} \right).$$

Finally, as in the case of Z_ε^ω, Lemma 3.4 and Lemma 3.6 imply

$$\mathsf{E}|Z_\varepsilon^{jo}| = \|\psi\|\, \mathcal{O}_\phi \left(\sqrt{(\varepsilon\beta l^2)^{-1} + l^2/\sigma^2} \right)$$

and

$$\mathsf{E}|Z_\varepsilon^{i*}| = \|\psi\|\, \mathcal{O}_\phi \left(\sqrt{\beta/\varepsilon l^2 + \beta^2 l^2/\sigma^2} \right).$$

Therefore we need

$$\lim_{\varepsilon\to 0} \frac{\sigma}{\beta l^2} = \lim_{\varepsilon\to 0} \frac{\beta\sigma}{l^2} = \lim_{\varepsilon\to 0} \frac{\varepsilon\beta^2 l^2}{\sigma} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon\beta l^2} = \lim_{\varepsilon\to 0} \frac{\beta}{\varepsilon l^2} = \lim_{\varepsilon\to 0} \frac{\beta^2 l^2}{\sigma^2} = 0,$$

which complete the proof as $l^2 \approx \sigma/\sqrt{\varepsilon}$, see also (2.2) and (2.3). $\qquad\square$

The results of this section can be summarized as follows. We have decomposed entropy production $X_\varepsilon(\psi, h)$ in a correct way, therefore Proposition 2.2 applies. Apart from Z_ε^s, all terms of the decomposition vanish, while $Z_\varepsilon^s(\psi, h) \le 0$ if h is convex and $\psi \ge 0$, thus $\limsup X_\varepsilon(\psi, h) \le 0$ in probability as $\varepsilon \to 0$ holds true in this case.

Proof of Theorem 1.1: First we prove Proposition 2.1. In fact we have to evaluate $X_\varepsilon(\psi, h)$ when $h(u) = u$, which is easy. Non - gradient analysis is not needed at all, and Z_ε^s is missing from the decomposition of X_ε. For $\psi \in C_c^1(\mathbb{R}^2)$ by Kolmogorov

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} \psi(t,x)\hat{u}_\varepsilon(t,x)\,dx = 0 = \int_{-\infty}^{\infty} \psi(0,x)\hat{u}_\varepsilon(t,x)\,dx + M_\varepsilon(\infty, \psi)$$
$$+ \int_0^\infty \int_{-\infty}^{\infty} \psi'_t(t,x)\hat{u}_\varepsilon(t,x)\,dx\,dt + \int_0^\infty \int_{-\infty}^{\infty} \left(\nabla_{l\varepsilon}\psi(t,x)\right) \bar{\jmath}_\varepsilon(t,x)\,dx\,dt,$$

where M_ε is the terminal value of a martingale, and $\bar{\jmath}_\varepsilon$ is a block average, $\bar{\jmath}_\varepsilon(t,x) := \bar{\jmath}_{l,k}^{\omega o}(t/\varepsilon) + \beta(\varepsilon)\bar{\jmath}_{l,k}^{\omega *}(t/\varepsilon) + \sigma(\varepsilon)\bar{\jmath}_{l,k}^{\omega s}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$, see Section 1.3. In view of Lemma 4.2, $\mathsf{E}M_\varepsilon^2 \to 0$ as $\varepsilon \to 0$, and the replacement of $\bar{\jmath}_\varepsilon$ with $f(\bar{u}_\varepsilon)$ is a consequence of Lemma 3.3 and Lemma 3.6; remember that $f(u) = F(u) - u^2$ is just the flux of (1.16). Finally, Lemma 3.4 implies that $\bar{u}_\varepsilon - \hat{u}_\varepsilon$ also vanish in the limit, thus we have $\mathsf{E}|X_\varepsilon(\psi, u)| \to 0$ as $\varepsilon \to 0$, which completes the proof because the distributions $\hat{\mathsf{P}}_{\varepsilon,\theta}$ of the Young measures form a tight family.

Now we are in a position to finish the proof of Theorem 1.1. Proposition 2.1 and Proposition 2.3 imply that any limit distribution, $\hat{\mathsf{P}}_\theta$ of the Young measures is concentrated on a set of weak solutions. On the other hand, $\limsup X_\varepsilon(\psi, h) \leq 0$ in probability if $\psi \geq 0$ and h is convex, thus first we get (2.11), whence Proposition 2.3 yields (1.23) almost surely with respect to any $\hat{\mathsf{P}}_\theta$. Therefore the uniqueness of the limiting solution follows by the Main Theorem of [CR00] on uniqueness of entropy solutions. Since the limit is deterministic, for $\psi \in C_c(\mathbb{R}^2)$ we have

$$\lim_{\varepsilon\to 0} \int_0^t \int_{-\infty}^{\infty} \psi(t,x)\hat{u}_\varepsilon(t,x)\,dx\,dt = \int_0^t \int_{-\infty}^{\infty} \psi(t,x)u(t,x)\,dx\,dt$$

in probability. The space integral on the left hand side is actually a sum, thus the block average can be transposed on ψ, which completes the proof of Theorem 1.1 because ψ is uniformly continuous.

5 Concluding Remarks

Here we summarize some improvements and explanations of our main result including further remarks on the method of relaxation schemes.

5.1. Strong convergence: The last step of the argument yields a stronger form of Theorem 1.1, cf. [FT04]. Let $\bar{u}_\varepsilon$ denote the empirical process of Section 3.4, $\bar{u}_\varepsilon(t,x) = \bar{\omega}_{l,k}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$, $l = l(\varepsilon)$ as in (2.2). This version is certainly more natural than $\hat{u}_\varepsilon$, which has been introduced because of technical reasons: $l\nabla_1^*\hat{\jmath}^{\omega o}$ is a difference of block averages, and $\bar{\jmath}^{\omega o}$ is well controlled by Lemma 3.3; $\nabla_1^*\bar{\jmath}^{\omega o}$ is more singular. The second statement of the following theorem is a consequence of Lemma 3.4.

Theorem 5.1. *Under conditions of Theorem 1.1, for $\tau, r > 0$ we have*

$$\lim_{\varepsilon\to 0} \mathsf{E} \int_0^\tau \int_{-r}^{r} |u(t,x) - \hat{u}_\varepsilon(t,x)|\,dx\,dt$$
$$= \lim_{\varepsilon\to 0} \mathsf{E} \int_0^\tau \int_{-r}^{r} |u(t,x) - \bar{u}_\varepsilon(t,x)|\,dx\,dt = 0,$$

where u denotes the unique entropy solution to (1.16) *with initial value* u_0.

5.2. Microscopic block averages: By means of Lemma 3.5 we can fill in the gap between large microscopic and small macroscopic block averages of the evolved configuration, see [FN06]. Let $\bar{u}_{\varepsilon,l}(t,x) := \bar{\omega}_{l,k}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$, where $l \in \mathbb{N}$ does not depend on ε, then for all $\tau, r > 0$ we have

$$\lim_{l\to\infty} \limsup_{\varepsilon\to0} \mathsf{E} \int_0^\tau \int_{-r}^r |\bar{u}_\varepsilon(t,x) - \bar{u}_{\varepsilon,l}(t,x)| \, dx \, dt = 0,$$

whence

$$\lim_{l\to\infty} \limsup_{\varepsilon\to0} \mathsf{E} \int_0^\tau \int_{-r}^r |u(t,x) - \bar{u}_{\varepsilon,l}(t,x)| \, dx \, dt = 0.$$

5.3. Measure - valued solutions: Convergence of the empirical process $\bar{u}_{\varepsilon,l}$ to a set of measure-valued solutions holds true under fairly general conditions. Let us consider CRANNI, that is the process generated by $\mathscr{L}^* = \mathscr{L}_o + \beta \mathscr{G}_*$ with a fixed value of $\beta > 0$. Let $\psi \in C_c^1(\mathbb{R}^2)$, then following the proof of Proposition 2.1 we write

$$\lim_{t\to\infty} \int_{-\infty}^\infty \psi(t,x)\bar{u}_{\varepsilon,l}(t,x)\,dx = 0 = \int_{-\infty}^\infty \psi(0,x)\bar{u}_{\varepsilon,l}(t,x)\,dx + \bar{M}_\varepsilon(\infty,\psi)$$
$$+ \int_0^\infty \int_{-\infty}^\infty \psi_t'(t,x)\bar{u}_{\varepsilon,l}(t,x)\,dx\,dt + \int_0^\infty \int_{-\infty}^\infty \left(\nabla_\varepsilon \psi(t,x)\right) \bar{\jmath}_{\varepsilon,l}(t,x)\,dx\,dt,$$

where $\bar{M}_\varepsilon$ is the terminal value of a martingale, and $\bar{\jmath}_{\varepsilon,l}$ is a block average, $\bar{\jmath}_{\varepsilon,l}(t,x) := \bar{\jmath}_{l,k}^{\omega o}(t/\varepsilon) + \beta(\varepsilon)\bar{\jmath}_{l,k}^{\omega*}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$; remember that $l \in \mathbb{N}$ does not depend on $\varepsilon > 0$. It is easy to show that $\mathsf{E}\bar{M}_\varepsilon^2 \to 0$ as $\varepsilon \to 0$, and the replacement of $\bar{\jmath}_{\varepsilon,l}$ with $f(\bar{u}_{\varepsilon,l})$ follows by the standard One - Block Lemma of [GPV88]. For $\tau, r > 0$ we have

$$\lim_{l\to\infty} \limsup_{\varepsilon\to0} \int_0^\tau \int_{-r}^r \mathsf{E}|\bar{\jmath}_{\varepsilon,l}(t,x) - f(\bar{u}_{\varepsilon,l}(t,x))| \, dx \, dt = 0$$

because the usual entropy argument shows that every translation invariant stationary measure is a superposition of product measures of type λ_u^*. Therefore all limit distributions of the Young representation of $\bar{u}_{\varepsilon,l}$ are concentrated on a set of measure - valued solutions, cf. (1.22); these limit distributions are obtained by sending $\varepsilon \to 0$ first, and $l \to +\infty$ next. The weak Lax inequality (2.11) can not be proven in this easy way, we are facing with a non - gradient problem there.

5.4. Relaxation of the spin flip model: Let $H_\kappa(\rho,u) := (1/2)(u - \kappa\rho)^2$, $0 \le \psi \in C_c^1(\mathbb{R}^2)$, and consider the evolution of

$$\bar{H}_\kappa(t,\psi) := \int_{-\infty}^\infty \psi(t,x)H_\kappa(\bar{v}_\varepsilon(t,x))\,dx,$$

where the empirical process $\bar{v}_\varepsilon$ is the same as in Section 3.4, but it is now generated by $\mathscr{L}_\kappa = \mathscr{L}_o + \alpha(\varepsilon)\mathscr{G}_\kappa + \sigma(\varepsilon)\mathscr{S}$. We follow the lines of the derivation of (3.7), the only difference consists in

the evaluation of the effect of $\mathscr{G}_\kappa$. Due to $\mathscr{G}_\kappa \eta_k = 0$, these computations are considerably simpler than those of Section 3.4. Denote $\bar{\omega}^j_{l,k}$ the block average of the sequence ω^j, then

$$
\begin{aligned}
\mathscr{G}_\kappa H(\bar{\eta}_{l,k}, \bar{\omega}_{l,k}) &= \frac{1}{2} \sum_{j \in \mathbb{Z}} (\eta_j - \kappa \omega_j) \left((\bar{\omega}^j_{l,k} - \kappa \bar{\eta}_{l,k})^2 - (\bar{\omega}_{l,k} - \kappa \bar{\eta}_{l,k})^2 \right) \\
&= \frac{1}{l} \sum_{j=k}^{k+l-1} (\kappa \eta_j - \omega_j)(\bar{\omega}^j_{l,k} + \bar{\omega}_{l,k} - 2\kappa \bar{\eta}_{l,k}) \\
&= -2(\bar{\omega}_{l,k} - \kappa \bar{\eta}_{l,k})^2 + (2/l)(\bar{\eta}_{l,k} - \kappa \bar{\omega}_{l,k})
\end{aligned}
$$

is an identity, whence by a direct calculation we obtain the final a priori bound:

$$
\varepsilon^2 \sum_{|k| < r/\varepsilon} \int_0^{\tau/\varepsilon} \int (\bar{\omega}_{l,k} - \kappa \bar{\eta}_{l,k})^2 \, d\mu_{\varepsilon,t} \, dt \le \bar{C}_4 \left(\frac{r\tau}{l} + \frac{r\tau\sigma}{\alpha l^2} \right).
$$

Let us remark that the second LSI in [FN06] yields a considerably better order, $\mathcal{O}(1/l + \varepsilon/\alpha)$, see Lemma 3.5 there. Our present bound is effective if $\alpha(\varepsilon) > 0$ is not too small, namely

$$
\lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\alpha(\varepsilon) l^2(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon \alpha(\varepsilon) l^2(\varepsilon) = +\infty.
$$

This means $\varepsilon \alpha(\varepsilon) \sigma^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$; the condition $\alpha(\varepsilon) \sigma(\varepsilon) \to +\infty$ of Theorem 1.1 in [FN06] has not been improved in this way.

5.5. Entropy - flux pairs of the LeRoux system: Regular entropy - flux pairs of (1.13) can easily be constructed because besides the trivial $S_a(\rho, u) := 1 - \rho + au - a^2$ and $\Phi_a(\rho, u) := -(a+u)S_a$, also $|S_a(\rho, u)|_+$ and $-(a+u)|S_a(\rho, u)|_+$ are convex entropy - flux pairs for all $a \in \mathbb{R}$, where $|S|_+ := \max\{S, 0\}$. Observe that $S_a(\rho, u) > 0$ means $w < a < z$, where z and w are the *Riemann invariants* of the LeRoux system:

$$
-1 \le w := \tfrac{u}{2} - \tfrac{1}{2}\sqrt{u^2 + 4 - 4\rho} \le 0 \le \tfrac{u}{2} + \tfrac{1}{2}\sqrt{u^2 + 4 - 4\rho} =: z \le 1;
$$

notice that $u = z + w$ and $\rho = 1 + zw$, moreover $S_z(\rho, u) = S_w(\rho, u) = 0$ in this case. Therefore

$$
S_v(\rho, u) = \int |S_a(\rho, u)|_+ \, v(da) = \int_w^z S_a(\rho, u) \, v(da)
$$

is again an entropy with flux

$$
\Phi_v(\rho, u) := -\int_w^z (a + u) S_a(\rho, u) \, v(da)
$$

whenever v is a finite signed measure on $[-1, 1]$. Suppose that v is absolutely continuous on the set $[-1, 0) \cup (0, 1]$, then

$$
\partial_\rho S_v(\rho, u) = -\int_w^z v(da) = -\left(v(\{0\}) + G^+(z) + G^-(w) \right),
$$

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where $G^+(a) = v((0, a))$ if $a \geq 0$, and $G^-(a) = v((a, 0))$ if $a < 0$.

To demonstrate relaxation to (1.16), we need

$$\partial_\rho S_v(\rho, u) C(\rho, u) \leq -b \left(\rho - F(u)\right)^2 \tag{5.1}$$

with some universal constant $b > 0$, which implies $\partial_\rho S_v(\rho, u) = 0$ if $\rho = F(u)$. In terms of the Riemann invariants this means

$$0 = C(\rho, u) = (1/4)(z^2 + w^2 + 3z^2 w^2 - 1),$$

whence $z^2 = (1 - w^2)/(1 + 3w^2)$. Therefore setting $G^+(z) = z^2$, $v(\{0\}) = -1$ and $G^-(w) = 1 - (1 - w^2)/(1 + 3w^2)$, we obtain

$$\partial_\rho S_v(\rho, u) = \frac{1 - z^2 - w^2 - 3z^2 w^2}{1 + 3w^2} = -\frac{4C(\rho, u)}{1 + 3w^2} \tag{5.2}$$

by a direct computation. However, $w^2 \leq 1$ and $|C(\rho, u)| \geq (1/2)|\rho - F(u)|$, consequently the above choice of v yields (5.1) with $b = 1/4$.

5.6. The second relaxation inequality: Using S_v instead of the trivial Liapunov function $H = (1/2)(F(u) - \rho)^2$, we can improve Lemma 3.6 as follows.

Lemma 5.1. *There exist two universal constants $l_0 \in \mathbb{N}$ and $C_5 < +\infty$ such that*

$$\varepsilon^2 \sum_{|k| < r/\varepsilon} \int_0^{\tau/\varepsilon} \int \left(\bar{\eta}_{l,k} - F(\bar{\omega}_{l,k})\right)^2 d\mu_{\varepsilon,t} \, dt \leq C_5 \left(\frac{r\tau\varepsilon}{\beta} + \frac{r\tau\varepsilon l^2}{\sigma}\right)$$

whenever $r, \tau \geq 1$ and $\varepsilon l^3 \geq \sigma \geq l \geq l_0$.

Proof. Let us consider

$$X_\varepsilon^v(\psi) := -\int_0^\infty \int_{-\infty}^\infty \left(\psi'_t(t, x) S_v(\hat{v}_\varepsilon) + \psi'_x(t, x) \Phi_v(\hat{v}_\varepsilon)\right) dx \, dt \tag{5.3}$$

for $0 \leq \psi \in C_c^1(\mathbb{R}^2)$, where

$$\hat{v}_\varepsilon(t, x) = \left(\hat{\rho}_\varepsilon(t, x), \hat{u}_\varepsilon(t, x)\right) := \left(\hat{\eta}_{l,k}(t/\varepsilon), \hat{\omega}_{l,k}(t/\varepsilon)\right) \quad \text{if } |x - \varepsilon k| < \varepsilon/2;$$

the process is generated by $\mathscr{L} = \mathscr{L}_o + \beta(\varepsilon)\mathscr{G}_* + \sigma(\varepsilon)\mathscr{S}$. The evaluation of X_ε^v follows the lines of the proof of Lemma 3.6 in Section 3.4; considerably better bounds are obtained in the first part of the argument. Concerning the decomposition of entropy production and estimation techniques we refer to Section 2.3 and Section 4. However, just as in case of Lemma 3.6, it suffices to control expectations only, which simplifies the argument. The main steps are outlined below with a concrete emphasis on nontrivial differences. It is not really problematic that our empirical process $\hat{v}_\varepsilon$ consists of two components, cf. [FT04], the crucial issue is to control the effect of $\mathscr{G}_*$.

Due to formal conservation of entropy, the contribution of $\mathscr{L}_o$ is compensated by the mesoscopic flux $\Phi_v(\hat{v}_\varepsilon)$, the vanishing $\varepsilon l^2/\sigma$ is the order of this difference. Indeed, apart from some quadratic remainders we have

$$\mathscr{L}_0 S_v(\hat{v}_\varepsilon) \approx \partial_\rho S_v(\hat{v}_\varepsilon) \mathscr{L}_0 \hat{\rho}_\varepsilon + \partial_u S_v(\hat{v}_\varepsilon) \mathscr{L}_0 \hat{u}_\varepsilon,$$

where $\mathscr{L}_0\bar{\rho}_\varepsilon = \nabla_l^* \bar{\jmath}_{l,k}^{\eta o}$, $\mathscr{L}_0\hat{u}_\varepsilon = \nabla_l^* \bar{\jmath}_{l,k}^{\omega o}$, and $\mathscr{O}(\varepsilon^{-1}l^{-3}) \to 0$ is the total contribution of the remainders. The microscopic currents are replaced by their canonical expectations via Lemma 3.3 at a total cost of $\mathscr{O}(l/\sigma)$, that is $\bar{\jmath}_{l,j}^{\eta o} \approx \bar{\omega}_{l,k} - \bar{\omega}_{l,k}\bar{\eta}_{l,k}$ and $\bar{\jmath}_{l,j}^{\omega o} \approx \bar{\eta}_{l,k} - (\bar{\omega}_{l,k})^2$. In a similar way, by means of Lemma 3.4 we get

$$\Phi_\nu(\hat{\eta}_{l,k-1}, \hat{\omega}_{l,k-1}) - \Phi_\nu(\hat{\omega}_{l,k}, \hat{\eta}_{l,k})$$
$$\approx (\partial_\rho S_\nu - \bar{\eta}_{l,k}\partial_\rho S_\nu - 2\bar{\omega}_{l,k}\partial_u S_\nu)\nabla_l^* \bar{\omega}_{l,k} + (\partial_u S_\nu - \bar{\omega}_{l,k}\partial_\rho S_\nu)\nabla_l^* \bar{\eta}_{l,k} \,;$$

again $\mathscr{O}(l/\sigma)$ is the total order of our error terms. By Lemma 4.1 the numerical error coming from the substitution of ψ_x' by $\varepsilon^{-1}(\psi(t, x+\varepsilon) - \psi(t,\varepsilon))$ also vanishes, thus (1.18) implies the desired cancelation. Finally, again by Lemma 3.4 the contribution of $\sigma \mathscr{S} S_\nu$ is bounded, see also Lemma 4.2, while the martingale component in the evolution equation of S_ν has zero expectation, consequently $\mathscr{O}(1)$ is the total order of those terms of X_ε^ν which have been considered so far.

Therefore, just as in case of Lemma 3.6, we have to concentrate on the effect of $\mathscr{G}_*$. Apart from the usual quadratic remainders

$$\mathscr{G}_* S_\nu(\hat{\eta}_{l,k}, \hat{\omega}_{l,k}) \approx \partial_\rho S_\nu(\hat{\eta}_{l,k}, \hat{\omega}_{l,k})\mathscr{G}_*\hat{\eta}_{l,k} + \partial_u S_\nu(\hat{\eta}_{l,k}, \hat{\omega}_{l,k})\mathscr{G}_*\hat{\omega}_{l,k}\,,$$

and $\mathscr{G}_*\hat{\omega}_{l,k}$ is a difference of mesoscopic block averages, therefore

$$\Gamma_\varepsilon^\nu(\psi) := \varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_k(t)\partial_\rho S_\nu(\hat{\eta}_{l,k}, \hat{\omega}_{l,k})\,\hat{C}_{l,k}^{\nu*}(\omega)\,dt$$

is the critical term, where

$$\hat{C}_{l,k}^{\nu*}(\omega) := \frac{2}{l^2 - l} \sum_{m=1}^{l-1} \sum_{b\subset[k-m,k+m]} \left(c_b^\times(\omega) - c_b^+(\omega) \right)\,;$$

remember that $\hat{C}_{l,k}^{\nu*}/2$ is a mean value of arithmetic averages with weights $(2m-1)/(l^2-l)$. Since $|\mathscr{G}_*\hat{\eta}_{l,k} - \hat{C}_{l,k}^{\nu*}(\omega)| = \mathscr{O}(1/l)$, moreover $|\hat{\omega}_{l,k}^{b*} - \hat{\eta}_{l,k}| = \mathscr{O}(l^{-2})$ while $|\hat{\eta}_{l,k}^{b*} - \hat{\eta}_{l,k}| = \mathscr{O}(1/l)$, we obtain

$$-\,\mathrm{E}\Gamma_\varepsilon^\nu(\psi) \le K_\nu(\psi)\left(\frac{1}{\beta} + \frac{1}{\varepsilon l}\right) \tag{5.4}$$

by a direct computation, see the derivation of (3.7). Note that $\sigma/\beta\varepsilon l^2 \approx \varepsilon^{-1/2}/\beta$ is much bigger then the corresponding $1/\beta$ above.

Now we are in a position to continue the argument of Section 3.4. Having in mind (5.2), let us introduce

$$W_\varepsilon^\nu(\psi) := 2\varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_k(t)\partial_\rho S_\nu(\hat{\eta}_{l,k}, \hat{\omega}_{l,k})C(\hat{\eta}_{l,k}, \hat{\omega}_{l,k})\,dt$$
$$\le -\frac{\varepsilon}{2} \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_k(t)(F(\hat{\omega}_{l,k}) - \hat{\eta}_{l,k})^2\,dt$$

and $\hat{D}_k(t) := \hat{C}_{l,k}^{\nu*}(\omega) - 2C(\hat{\eta}_{l,k}, \hat{\omega}_{l,k})$, whence

$$2\Gamma_\varepsilon^\nu - W_\varepsilon^\nu = 2\Gamma_\varepsilon^\nu - 2W_\varepsilon^\nu + W_\varepsilon^\nu \le \frac{\varepsilon}{4} \sum_{k\in\mathbb{Z}} \int_0^\infty \psi_k(t)\hat{D}_k^2(t)\,dt\,.$$

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In order to estimate the right hand side via Lemma 3.3, first of all we have to get rid of the block averages of type $\hat{\xi}_{l,k}$; note that we have to control squared differences. Lemma 3.4 allows us to replace $C(\hat{\eta}_{l,k}, \hat{\omega}_{l,k})$ by $C_k := C(\bar{\eta}_{2l,k-l+1}, \bar{\omega}_{2l,k-l+1})$, while $C_m \approx C_l$ follows by Lemma 3.5 if $l_0 \le m < l$. On the other hand, due to Cauchy

$$D_k \le 4 \sum_{m=1}^{l-1} \frac{2m-1}{l^2-l} (\tilde{C}_m - C_m)^2,$$

where

$$\tilde{C}_m := \frac{1}{2m-1} \sum_{b \subset [k-m,k+m]} \left(c_b^\times(\omega) - c_b^+(\omega) \right) \approx C_m$$

is a consequence of Lemma 3.3 if $m \ge l_0$, therefore $2\Gamma_\varepsilon^\nu - W_\varepsilon^\nu = \mathcal{O}(l^2/\sigma)$; the constant does depend on l_0, too.

Since $-W_\varepsilon^\nu = 2\Gamma_\varepsilon^\nu - W_\varepsilon^\nu - 2\Gamma_\varepsilon^\nu$, (5.4) implies a preliminary version of Lemma 5.1: instead of $(F(\bar{\omega}_{l,k}) - \bar{\eta}_{l,k})^2$ we have $(F(\hat{\omega}_{l,k}) - \hat{\eta}_{l,k})^2$ in the sum on the left hand side of the second relaxation inequality. The required substitution is immediate, Lemma 3.4 completes the proof of this lemma.

$\square$

The second relaxation inequality allows us to improve the previous lower bound on $\beta = \beta(\varepsilon)$.

Theorem 5.2. *The conclusion of Theorem 1.1 holds true even if its condition $\varepsilon\sigma^2(\varepsilon)\beta^2(\varepsilon) \to +\infty$ is replaced with $\beta(\varepsilon)\sigma(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.*

Proof. Applying the above bound instead of Lemma 3.6, and following the proof of Lemma 4.4, we see that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\beta} = \lim_{\varepsilon \to 0} \frac{\varepsilon\beta^2 l^2}{\sigma} = \lim_{\varepsilon \to 0} \frac{1}{\beta\sigma} = \lim_{\varepsilon \to 0} \frac{\beta^2 l^2}{\sigma^2} = 0$$

are the requirements of an effective relaxation, which completes the proof as $l^2 \approx \sigma/\sqrt{\varepsilon}$. $\square$

Of course, Theorem 5.1 also holds true under the above condition on $\beta = \beta(\varepsilon)$ of Theorem 5.2.

5.7. Entropy for the spin - flip model: In view of Section 5.5, in case of the spin - flip dynamics

$$\partial_u S_v(\rho, u) = \int_w^z a\, v(da) = M(z) - M(w)$$

is the relevant derivative; $dM = a\, v(da)$. The condition $u = \kappa\rho$ of equilibrium means $z + w = \kappa + \kappa zw$, consequently it is tempting to set $M(a) := a^2$ if $a \ge 0$ and $M(a) := (\kappa - a)^2(1 - \kappa a)^{-2}$ if $a < 0$, then

$$\partial_u S_v(\rho, u) = \frac{(u - \kappa\rho)\sqrt{u^2 + 4 - 4\rho}}{1 - 2\kappa w + \kappa^2 w^2},$$

whence an extremely strong bound,

$$\varepsilon^2 \sum_{|k| < r/\varepsilon} \int_0^{\tau/\varepsilon} \int (1 - \bar{\eta}_{l,k})^2 (\bar{\omega}_{l,k} - \kappa\bar{\eta}_{l,k})^2 \, d\mu_{\varepsilon,t} \, dt \le \bar{C}_4' \left(r\tau\varepsilon + \frac{r\tau\varepsilon}{\alpha} \right)$$

would follow, see Lemma 4.3 in [FN06] and Section 5.4 here, provided that v and S_v are well defined. The bound above is effective if $\alpha(\varepsilon)\sigma(\varepsilon) \to +\infty$ as $\varepsilon \to 0$; this condition on α is the very same as that of [FN06]. Unfortunately, unless $\kappa = 0$, no version of our construction is correct: the underlying measure v can not be defined because the map $z = (\kappa - w)/(1 - \kappa w)$ does not interchange the ranges of z and w if $\kappa \neq 0$. The case of $\kappa = 0$ is trivial at the level of hyperbolic scaling, nevertheless it is interesting to note that relaxation via entropy may be just as strong as LSI.

References

[Bah04] C. Bahadoran. Blockage hydrodynamics of one-dimensional driven conservative systems. *Ann. Probab.* **32**:805–854 (2004). MR2039944

[CLL94] G.-Q. Chen, C.D. Levermore and T.-P. Liu: Hyperbolic conservation laws with stiff relaxation and entropy. Comm. Pure Appl. Math. **47**:787–830 (1994). MR1280989

[CL93] G.-Q. Chen and T.-P. Liu: Zero relaxation and dissipation limits for hyperbolic conservation laws. Comm. Pure Appl. Math. **46**:255–281 (1993). MR1213992

[CR00] G.Q. Chen and M. Rascle: Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws. Arch. Rational Mech. Anal. **153**:205–220 (2000). MR1771520

[Daf05] C. M. Dafermos: Hyperbolic Conservation Laws in Continuum Physics. Grundlehren der Mathematishen Wissenschaften **325**, Second Ed. Springer Verlag, Berlin 2005. MR2169977

[Fri90] J. Fritz: On the diffusive nature of entropy flow in infinite systems: Remarks to a paper by Guo, Papanicolau and Varadhan. Commun. Math. Phys. **133**:331–352 (1990). MR1090429

[Fri01] J. Fritz: An Introduction to the Theory of Hydrodynamic Limits. Lectures in Mathematical Sciences **18.** The University of Tokyo, ISSN 0919-8180, Tokyo 2001.

[Fri04] J. Fritz: Entropy pairs and compensated compactness for weakly asymmetric systems. Advanced Studies in Pure Mathematics**39**:143–171 (2004). MR2073333

[Fri09] J. Fritz: Microscopic theory of isothermal elasticity. To appear in Arch. Ration. Mech. Anal. (2011).

[FN06] J. Fritz and K. Nagy: On uniqueness of the Euler limit of one-component lattice gas models. ALEA Latin American Journal on Probability and Math. Stat. **1**:367–392 (2006). MR2285732

[FT04] J. Fritz and B. Tóth: Derivation of the Leroux system as the hydrodynamic limit of a two-component lattice gas. Commun. Math. Phys. **249**:1–27 (2004). MR2077251

[GT00] L. Gosse and A. Tzavaras: Convergence of relaxation schemes to the equations of elastodynamics. Mathematics of Computation **70**:555–577 (2000). MR1813140

[GPV] G.-C. Guo, G. Papanicolau and S.R.S. Varadhan: Nonlinear diffusion limit for a system with nearest neighbor interactions. Commun. Math. Phys. **118**:31–59 (1988). MR0954674

[KL99] C. Kipnis C. Landim: Scaling Limit of Interacting Particle Systems. Grundlehren der Mathematishen Wissenschaften **320.** Springer Verlag, Berlin 1999. MR1707314

[Lig85] T.M. Liggett: Interacting Particle Systems. Grundlehren der Mathematishen Wissenschaften **276.** Springer Verlag, New York 1985. MR0776231

[Liu87] T.P. Liu: Hyperbolic conservation laws with relaxation. Commun. Math. Phys. **108**:153–175 (1987). MR0872145

[LL] L. Hsiao and T.-P. Liu: Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping. Commun. Math. Phys. **143**:599–605 (1992). MR1145602

[OV91] S. Olla and S.R.S. Varadhan: Scaling limit for interacting Ornstein - Uhlenbeck processes. Commun. Math. Phys. **135**:355–378 (1991). MR1087388

[Rez91] F. Rezakhanlou: Hydrodynamic limit for attractive particle systems on $\mathbb{Z}^d$. Commun. Math. Phys. **140**:417–448 (1991). MR1130693

[Ser00] D. Serre: Systems of Conservation Laws. Vol. 1-2. Cambridge University Press, Cambridge 2000. MR1775057

[Tar79] L. Tartar: Compensated compactness and applications to partial differential equations. In: Nonlinear analysis and mechanics. Heriot-Watt Symposium, Vol. IV.:136–212, Pitman, London 1979. MR0584398

[TV03] B. Tóth and B. Valkó: Onsager relations and Eulerian hydrodynamic limit for systems with several conservation laws. J. Statist. Phys. **112**:497–521 (2003). MR1997260

[Var94] S.R.S. Varadhan: Nonlinear diffusion limit for a system with nearest neighbor interactions II. In: Asymptotic Problems in Probability Theory: Stochastic Models and Diffusion on Fractals. Elworthy, K. D. and Ikeda, N. Eds. Pitman Research Notes in Mathematics. **283**:75–128. Wiley, New York 1994. MR1354152

[Yau91] H.-T. Yau: Relative entropy and hydrodynamics of Ginzburg - Landau models. Lett. Math. Phys. **22**:63–80 (1991). MR1121850

[Yau97] H.-T. Yau: Logarithmic Sobolev inequality for generalized simple exclusion processes. Probab. Theory Relat. Fields **109**:507–538 (1997). MR1483598