Electronic Journal of Probability

Vol. 16 (2011), Paper no. 19, pages 531–551.

Journal URL
http://www.math.washington.edu/~ejpecp/

Sharp and strict L^p-inequalities for Hilbert-space-valued orthogonal martingales*

Adam Osękowski†
http://www.mimuw.edu.pl/~ados
ados@mimuw.edu.pl

Abstract

The paper contains the proofs of sharp moment estimates for Hilbert-space valued martingales under the assumptions of differential subordination and orthogonality. The results generalize those obtained by Bañuelos and Wang. As an application, we sharpen an inequality for stochastic integrals with respect to Brownian motion.

Key words: Martingale, differential subordination, orthogonal martingales, moment inequality, stochastic integral, Brownian motion, best constants.

AMS 2000 Subject Classification: Primary 60G44; Secondary: 60G42.

Submitted to EJP on August 24, 2010, final version accepted February 6, 2011.

*Partially supported by MNiSW Grant N N201 397437.

†Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

1 Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, equipped with a right-continuous filtration $(\mathscr{F}_t)_{t\geq 0}$ such that $\mathscr{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ be two adapted martingales, which have right-continuous paths, with limits from the left. We assume that these processes take values in a separable Hilbert space $\mathscr{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and the norm $|\cdot|$. With no loss of generality we may assume that $\mathscr{H} = \ell^2$.

The martingales X and Y are said to be *orthogonal*, if for any i, $j \geq 1$ we have $[X^i, Y^j] \equiv 0$. Here X^i, Y^j stand for the i-th and j-th coordinates of X and Y, and for any real martingales M, N, the symbol $[M, N]$ denotes their quadratic covariance process (see e.g. [10] for details). The martingale Y is said to be *differentially subordinate* to X if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t (here $[X,X]_t = \sum_{n=1}^{\infty}[X^n, X^n]_t$). This notion was originally introduced by Burkholder [5] in the discrete-time setting and the above generalization is due to Wang [16] and Bañuelos and Wang [3]. It is well known that differential subordination implies many interesting martingale inequalities, which have numerous applications in various areas of mathematics. An excellent source of information in the discrete-time setting is the survey [8] by Burkholder. One can also find there a detailed description of the method which enables to obtain sharp versions of such estimates. By approximation and careful use of Itô's formula, these results can be extended to the continuous-time setting: see the paper by Wang [16]. For applications, consult e.g. [2], [3], [4], [6] and [12].

We shall only mention here the following famous result of Burkholder (see [5] for the discrete and [16] for the continuous-time version). For $1 \leq p \leq \infty$, we use the notation $||X||_p = \sup_{t\geq 0} ||X_t||_p$ for the p-th moment of a martingale X and we write p^* for the maximum of p and its harmonic conjugate $p/(p-1)$.

Theorem 1.1. *Assume that X and Y are $\mathscr{H}$-valued martingales such that Y is differentially subordinate to X. Then*

$$||Y||_p \leq (p^* - 1)||X||_p, \qquad 1 < p < \infty, \tag{1.1}$$

and the inequality is sharp. In addition, the inequality is strict provided $p \neq 2$ and $0 < ||X||_p < \infty$.

If $p \neq 2$ and we impose the orthogonality of X and Y, the constant above is no longer optimal. Bañuelos and Wang determined in [3] the best value under some additional assumptions on the dimension of the range of X and Y. Then the constant turns out to be the Pichorides-Cole constant $\cot(\pi/(2p^*))$ appearing in the sharp L^p-estimate for conjugate harmonic functions on the unit disc (cf. [11] and [14]). To be more specific, we have the following (cf. [3], [4]).

Theorem 1.2. *Let X, Y be two orthogonal martingales such that Y is differentially subordinate to X.*

(i) If Y is real valued (to be more precise, takes values in a one-dimensional subspace of $\mathscr{H}$), then

$$||Y||_p \leq \tan\left(\frac{\pi}{2p}\right)||X||_p, \qquad 1 < p \leq 2,$$

and the constant is the best possible. It is already the best possible if X is assumed to be real valued.

(ii) If X is real valued, then

$$||Y||_p \leq \cot\left(\frac{\pi}{2p}\right)||X||_p, \qquad 2 \leq p < \infty,$$

and the constant is the best possible. It is already the best possible if Y is assumed to be real valued. The inequalities are strict if $p \neq 2$ and $0 < ||X||_p < \infty$.

The motivation of the present paper comes from the question about the optimal constants in the estimates above in the case when there are no restrictions for the ranges of X and Y. Before we formulate the precise statement, let us first mention here a seemingly unrelated result, due to Davis [9]. Let w_p be the smallest positive zero of the confluent hypergeometric function of parameter p and let z_p be the largest positive zero of the parabolic cylinder function of parameter p (for the definitions of these objects, see [1] or Section 3 below).

Theorem 1.3. *Let B be a standard one-dimensional Brownian motion and let τ be an adapted stopping time. Then the following inequalities are sharp.*
(i) For $1 < p < \infty$ and $||\tau^{1/2}||_p < \infty$,

$$||\tau^{1/2}||_p \leq a_p||B_\tau||_p, \tag{1.2}$$

where $a_p = z_p^{-1}$ for $1 < p \leq 2$ and $a_p = w_p^{-1}$ for $2 \leq p < \infty$.
(ii) For $0 < p < \infty$,

$$||B_\tau||_p \leq A_p||\tau^{1/2}||_p, \tag{1.3}$$

where $A_p = w_p$ for $0 < p \leq 2$ and $A_p = z_p$ for $2 \leq p < \infty$.

The main result of the present paper can be stated as follows.

Theorem 1.4. *Let X, Y be two orthogonal martingales taking values in $\mathscr{H}$ such that Y is differentially subordinate to X. Then*

$$||Y||_p \leq C_p||X||_p, \qquad 1 < p < \infty, \tag{1.4}$$

where

$$C_p = \begin{cases} z_p^{-1} & \text{if } 1 < p \leq 2, \\ z_p & \text{if } 2 \leq p \leq 3, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } p \geq 3. \end{cases}$$

The constant C_p is the best possible. Furthermore, the inequality is strict if $p \neq 2$ and $0 < ||X||_p < \infty$.

Therefore we see that the constants C_p have quite surprising behavior. For $1 < p \leq 3$ they are the same as those in appropriate Davis' estimates (a_p for $1 < p \leq 2$ and A_p for $2 \leq p \leq 3$), while for $p \geq 3$ they are equal to Pichorides-Cole constants. In other words, comparing the above with Theorem 1.2, we see that the passage from real to $\mathscr{H}$-valued martingales affects the optimal constants if and only if $1 < p < 3$.

A few words about the proof and the organization of the paper. Our argumentation is based on Burkholder's technique, which reduces the problem of proving a given martingale inequality into that of finding an appropriate special function. This transference is described in the next section. The special function is constructed by means of confluent hypergeometric and parabolic cylinder functions, which are introduced and studied in Section 3. Then, in Section 4, we present the proof of our main result, Theorem 1.4. The final section of the paper is devoted to some applications to stochastic integrals.

2 On the method of proof

We start with describing the main tool which will be exploited to establish our result. We shall use the following notation: if $U : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ and $x, y, h \in \mathscr{H}$, then

$$\langle hU_{xx}(x,y), h\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{x_n x_m}(x,y) h_n h_m,$$

provided the partial derivatives exist. The term $\langle hU_{yy}(x,y), h\rangle$ is defined in a similar manner.

Theorem 2.1. *Let U be a continuous function on $\mathscr{H} \times \mathscr{H}$, bounded on bounded sets, of class C^1 on $E = \mathscr{H} \times \mathscr{H} \setminus (\{|x| = 0\} \cup \{|y| = 0\})$ and such that its first order derivative is bounded on bounded subsets of E not containing a neighborhood of $(0,0)$. Assume that U satisfies*

$$U(x+h, y) - U(x,y) - \langle U_x(x,y), h\rangle \leq 0 \tag{2.1}$$

for all $(x,y) \in E$ and $h \in \mathscr{H}$. Moreover, assume that U is C^2 on S_i, $i \geq 1$, where S_i is a sequence of open connected subsets of E such that the union of the closure of S_i is $\mathscr{H} \times \mathscr{H}$. Suppose for each $i \geq 1$, there exists a nonnegative measurable function c_i defined on S_i such that for all $(x,y) \in S_i$ and $h, k \in \mathscr{H}$,

$$\langle hU_{xx}(x,y), h\rangle + \langle kU_{yy}(x,y), k\rangle \leq -c_i(x,y)(|h|^2 - |k|^2). \tag{2.2}$$

Assume further that there exists a nondecreasing sequence $(M_n)_{n\geq 1}$ such that

$$\sup c_i(x,y) < M_n < \infty, \tag{2.3}$$

where the supremum is taken over all $(x,y) \in S_i$ such that $1/n^2 \leq |x|^2 + |y|^2 \leq n^2$ and all $i > 1$. Let X and Y be $\mathscr{H}$-valued orthogonal martingales such that Y is differentially subordinate to X. If $\sup_s |U(X_s, Y_s)|$ is integrable, then for any $0 \leq s \leq t$ we have

$$\mathbb{E}\left[U(X_t, Y_t)|\mathscr{F}_s\right] \leq U(X_s, Y_s) \quad \textit{almost surely}. \tag{2.4}$$

This is a slight modification of Proposition 1 from [4]. Essentially, the difference is that the inequality (2.1) is replaced there by a more restrictive condition.

The proof of Theorem 2.1 is based on approximation and Itô's formula. To be more specific, one reduces the problem to the finite-dimensional case $\mathscr{H} = \mathbb{R}^n$ and convolves U with a C^∞ function to get a smooth $\overline{U}$ on $\mathbb{R}^n \times \mathbb{R}^n$. Then one applies Itô's formula to $\overline{U}(X_t, Y_t)$, takes conditional expectation of both sides and uses the conditions (2.1) and (2.2) to control the finite-variation terms. Since similar argumentation appears in so many places (see e.g. Proposition 1 in [4], Lemma 1.1 in [3], Theorem 2.1 in [13], Lemma 3 in [16] ...), we have decided not to include the details here.

Fix $p \in (1, \infty)$ and let us now sketch the proof of (1.4). Obviously, we may and do assume that $||X||_p < \infty$; otherwise there is nothing to prove. By Burkholder's inequality (1.1), we obtain that $||Y||_p$ is also finite. In consequence, all we need is to show that

$$\mathbb{E}|Y_t|^p \leq C_p^p \mathbb{E}|X_t|^p \qquad \text{for all } t \geq 0.$$

Here the inequality (2.4) comes into play: if we manage to find a function U_p as in Theorem 2.1, satisfying the majorization

$$U_p(x,y) \geq |y|^p - C_p^p |x|^p \qquad \text{for all } x,\ y \in \mathscr{H}, \tag{2.5}$$

and the condition

$$U_p(x,y) \leq 0 \qquad \text{for all } x,\ y \in \mathscr{H} \text{ satisfying } |y| \leq |x|, \tag{2.6}$$

then we will be done. Indeed, we have $|Y_0| \leq |X_0|$ by the differential subordination, so the properties of U_p imply

$$\mathbb{E}(|Y_t|^p - C_p^p |X_t|^p) \leq \mathbb{E} U_p(X_t, Y_t) \leq \mathbb{E} U_p(X_0, Y_0) \leq 0.$$

We search for U_p in the class of functions of the form

$$U_p(x,y) = V_p(|x|,|y|), \tag{2.7}$$

where $V_p : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfies

$$|V_p(x,y)| \leq c_p(x^p + y^p) \tag{2.8}$$

for all x, $y \geq 0$ and some c_p depending only on p. The latter condition immediately implies the integrability of $\sup_s |U_p(X_s, Y_s)|$, by means of Doob's inequality. Let us rephrase the requirements (2.1), (2.2), (2.5) and (2.6) in terms of the function V_p. We start from the latter condition: the inequality (2.6) reads

$$V_p(x,y) \leq 0 \qquad \text{for all } x \geq y \geq 0. \tag{2.9}$$

The majorization (2.5) takes the form

$$V_p(x,y) \geq y^p - C_p^p x^p \qquad \text{for all } x,\ y \geq 0. \tag{2.10}$$

The condition (2.1) can be rewritten as follows: for all $(x,\ y) \in \bigcup_i S_i$ and $h \in \mathscr{H}$,

$$\left[V_{pxx}(|x|,|y|) - \frac{V_{px}(|x|,|y|)}{|x|} \right] \langle x', h\rangle^2 + \frac{V_{px}(|x|,|y|)}{|x|} |h|^2 \leq 0, \tag{2.11}$$

where $x' = x/|x|$ for $x \neq 0$ and $x' = 0$ for $x = 0$. To see this, observe first that by continuity, we may restrict ourselves to linearly independent vectors x and h: $x + th \neq 0$ for all $t \in \mathbb{R}$. The estimate (2.1) is equivalent to saying that for any fixed x, y, $h \in \mathscr{H}$, the function $G_{x,y,h} : \mathbb{R} \to \mathbb{R}$ given by

$$G_{x,y,h}(t) = U_p(x + th, y) = V_p(|x + th|, |y|)$$

is concave. Thus, since this function is of class C^1, (2.1) will follow if we check that $G''_{x,y,h}(t) \leq 0$ for all those t, for which $(x + th, y) \in \bigcup_i S_i$. By the translation property $G_{x,y,h}(t + s) = G_{x+th,y,h}(s)$ valid for all s, $t \in \mathbb{R}$, we see that it suffices to check the latter inequality for $t = 0$ only. By (2.7), this is precisely (2.11).

Finally, the formula (2.7) transforms (2.2) into the following: for $(x,y) \in S_i$ with $|x||y| \neq 0$ and any h, $k \in \mathscr{H}$,

$$\begin{aligned} &\left[V_{pxx}(|x|,|y|) - \frac{V_{px}(|x|,|y|)}{|x|} \right] \langle x', h\rangle^2 \\ &+ \left[V_{pyy}(|x|,|y|) - \frac{V_{py}(|x|,|y|)}{|y|} \right] \langle y', k\rangle^2 \\ &+ \frac{V_{px}(|x|,|y|)|h|^2}{|x|} + \frac{V_{py}(|x|,|y|)|k|^2}{|y|} \leq -c_i(x,y)(|h|^2 - |k|^2). \end{aligned} \tag{2.12}$$

Summarizing, in order to establish (1.4), we need to construct a sufficiently smooth function V_p : $[0,\infty) \times [0,\infty) \to \mathbb{R}$, which satisfies (2.8), (2.9), (2.10), (2.11) and (2.12) and for which the corresponding functions c_i enjoy the bound (2.3).

3 Parabolic cylinder functions and their properties

In this section we introduce a family of special functions and present some of their properties, needed in our further considerations. Much more information on this subject can be found in [1].

We start with the definition of the *Kummer confluent hypergeometric* function $M(a,b,z)$. It is a solution of the differential equation

$$zy''(z) + (b-z)y'(z) - ay(z) = 0$$

and its explicit form is given by

$$M(a,b,z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \dots.$$

The *confluent hypergeometric function* M_p is defined by the formula

$$M_p(x) = M(-p/2, 1/2, x^2/2), \qquad x \in \mathbb{R}.$$

If p is an even positive integer: $p = 2n$, then M_p is a constant multiple of the Hermite polynomial of order $2n$ (where the constant depends on n).

The *parabolic cylinder* functions (also known as Whittaker's functions) are closely related to the confluent hypergeometric functions. They are solutions of the differential equation

$$y''(x) + (ax^2 + bx + c)y(x) = 0.$$

We will be particularly interested in the special case

$$y''(x) - \left(\frac{1}{4}x^2 - p - \frac{1}{2}\right)y(x) = 0. \tag{3.1}$$

There are two linearly independent solutions of this equation, given by

$$y_1(x) = e^{-x^2/4} M\left(-\frac{p}{2}, \frac{1}{2}, \frac{x^2}{2}\right)$$

and

$$y_2(x) = xe^{-x^2/4} M\left(-\frac{p}{2} + \frac{1}{2}, \frac{3}{2}, \frac{x^2}{2}\right).$$

The parabolic cylinder function D_p is defined by

$$D_p(x) = A_1 y_1(x) + A_2 x y_2(x),$$

where

$$A_1 = \frac{2^{p/2}}{\sqrt{\pi}}\cos(p\pi/2)\Gamma((1+p)/2) \quad \text{and} \quad A_2 = \frac{2^{(1+p)/2}}{\sqrt{\pi}}\sin(p\pi/2)\Gamma(1+p/2). \tag{3.2}$$

Throughout the paper we will use the notation

$$\phi_p(s) = e^{s^2/4} D_p(s)$$

and z_p will stand for the largest positive root of D_p. If D_p has no positive roots, we set $z_p = 0$. Later on, we will need the following properties of ϕ_p.

Lemma 3.1. *Let p be a fixed number.*

(i) For all $s \in \mathbb{R}$ we have

$$p\phi_p(s) - s\phi_p'(s) + \phi_p''(s) = 0 \tag{3.3}$$

and

$$\phi_p'(s) = p\phi_{p-1}(s). \tag{3.4}$$

(ii) We have the asymptotics

$$\phi_p(s) = s^p \left(1 - \frac{p(p-1)}{2s^2} + \frac{p(p-1)(p-2)(p-3)}{8s^4} + o(s^{-5})\right) \quad \text{as } s \to \infty. \tag{3.5}$$

(iii) If $p \le 1$, then ϕ_p is strictly positive on $(0, \infty)$.

(iv) We have $z_p = 0$ for $p \le 1$ and $z_p > z_{p-1}$ for $p > 1$.

(v) For $p > 1$ we have

$$\phi_p(s) \ge 0, \quad \phi_p'(s) > 0 \quad \text{and} \quad \phi_p''(s) > 0 \qquad \text{on } [z_p, \infty). \tag{3.6}$$

Furthermore, if $1 < p \le 2$, then $\phi_p'''(s) \le 0$ on $[z_p, \infty)$, while for $p \ge 2$ we have $\phi_p'''(s) \ge 0$ on $[z_p, \infty)$; the inequalities are strict unless $p = 2$.

Proof. (i) This follows immediately from (3.1) and the definition of ϕ_p.

(ii) See 19.6.1 and 19.8.1 in [1].

(iii) If $p = 1$, then the assertion is clear, since $h_1(s) = s$. If $p < 1$ then, by (i) and 19.5.3 in [1], we have

$$\phi_p'(s) = p\phi_{p-1}(s) = \frac{p}{\Gamma(1-p)} \int_0^\infty u^{-p} \exp(-su - u^2/2) \mathrm{d}u > 0.$$

It suffices to use $\phi_p(0) = A_1 > 0$ (see (3.2)) to obtain the claim.

(iv) The first part is an immediate consequence of (iii). To prove the second, we use induction on $\lceil p \rceil$. When $1 < p \le 2$, we have $\phi_p(0) = A_1 < 0$ (see (3.2)) and, by (ii), $\phi_p(s) \to \infty$ as $s \to \infty$, so the claim follows from the Darboux property. To carry out the induction step, take $p > 2$ and write (3.3) in the form

$$\phi_p(z_{p-1}) = pz_{p-1}\phi_{p-1}(z_{p-1}) - p\phi_{p-1}'(z_{p-1}).$$

But, by the hypothesis, $z_{p-1} > 0$: this implies that z_{p-1} is the largest root of ϕ_{p-1}. Therefore, by asymptotics (3.5), we obtain $\phi_{p-1}'(z_{p-1}) \ge 0$. Plugging this above yields $\phi_p(z_{p-1}) \le 0$, so, again by (3.5), we have $z_{p-1} \le z_p$. However, the inequality is strict, since otherwise, by (i), we would have $\phi_{p-n}(z_p) = 0$ for all integers n. This would contradict (iii).

(v) This follows immediately from (iii), (iv) and the equalities $\phi_p' = p\phi_{p-1}$, $\phi_p'' = p(p-1)\phi_{p-2}$ and $\phi_p''' = p(p-1)(p-2)\phi_{p-3}$. □

The further property of ϕ_p is described in the following.

Lemma 3.2. *For $p > 1$, let $F_p : [0,\infty) \to \mathbb{R}$ be given by*

$$F_p(s) = p(p-2)\phi_p(s) - (2p-3)s\phi_p'(s) + s^2\phi_p''(s). \tag{3.7}$$

(i) If $0 < p \le 1$, then F_p is nonpositive.

(ii) If $1 < p \le 2$, then F_p is nonnegative.

(iii) If $2 \le p \le 3$, then F_p is nonpositive.

Proof. (i) By (3.3), we have

$$F_p(s) = p(p-2-s^2)\phi_p(s) + (s^2 - 2p + 3)s\phi_p'(s). \tag{3.8}$$

A little calculation gives that for $s > 0$,

$$F_p'(s) = -2ps\phi_p(s) + (p^2 - 4p + 3 - (p-3)s^2)\phi_p'(s) + (s^3 - (2p-3)s)\phi_p''(s),$$

which, by (3.3), can be rephrased in the form

$$F_p'(s) = ps(-s^2 + 2p - 5)\phi_p(s) + (s^4 - 3(p-2)s^2 + p^2 - 4p + 3)\phi_p'(s). \tag{3.9}$$

After lengthy but simple manipulations, this can be written as

$$F_p'(s) = F_p(s)\frac{s^4 - 3(p-2)s^2 + p^2 - 4p + 3}{s(s^2 - 2p + 3)} + \frac{p(p-1)(p-2)(3-p)\phi_p(s)}{s(s^2 - 2p + 3)}. \tag{3.10}$$

The second term above is nonnegative for $0 < p \le 1$ (see Lemma 3.1 (iii)). Furthermore,

$$\frac{s^4 - 3(p-2)s^2 + p^2 - 4p + 3}{s(s^2 - 2p + 3)} - s = \frac{(3-p)(s^2 - p + 1)}{s(s^2 - 2p + 3)} \ge 0.$$

Now suppose that $F_p(s_0) > 0$ for some $s_0 \ge 0$. Then, by (3.10) and the above estimates, $F_p'(s) \ge F_p(s)s$ for $s > s_0$, which yields $F_p(s) \ge F_p(s_0)\exp((s^2 - s_0^2)/2)$ for $s \ge s_0$. However, by (3.5), the function F_p has polynomial growth. A contradiction, which finishes the proof of (i).

(ii) It can be easily verified that we have $F_p'(s) = pF_{p-1}(s)$ for all p and s. In consequence, by the previous part, we have that F_p is nonincreasing and it suffices to prove that $\lim_{s\to\infty} F_p(s) \ge 0$. In fact, the limit is equal to 0, which can be justified using (3.8) and (3.5): F_p is of order at most s^{p+2} as $s \to \infty$, and one easily checks that the coefficients at s^p and s^{p+2} vanish.

(iii) We proceed in the same manner as in the proof of (ii). The function F_p is nondecreasing and $\lim_{s\to\infty} F_p(s) = 0$, by means of (3.8) and (3.5) (this time one also has to check that the coefficient at s^{p-2} is equal to 0). □

Before we proceed to the contruction of the special functions V_p, let us mention here that the arguments presented in the proof of the above lemma (equations (3.8) and (3.9)) lead to some interesting bounds for the roots z_p, $1 \le p \le 3$. For example, if $1 \le p \le 2$, then, as we have shown, the function F_p is nonnegative: thus, putting $s = z_p$ in (3.8) and exploting (3.6) yields $z_p^2 \geq 2p - 3$ (which is nontrivial for $p > 3/2$). Furthermore, F_p is nonincreasing, so taking $s = z_p$ in (3.9) gives

$$z_p^4 - 3(p-2)z_p^2 + p^2 - 4p + 3 \leq 0,$$

which can be rewritten in the more explicit form

$$z_p^2 \leq \frac{3(p-2) + \sqrt{9(p-2)^2 - 4(p-1)(p-3)}}{2}.$$

Note that the bound is quite tight: we have equality for $p \in \{1, 2\}$. Similarly, in the case when $2 \leq p \leq 3$ we obtain the following estimates:

$$\frac{3(p-2) + \sqrt{9(p-2)^2 - 4(p-1)(p-3)}}{2} \leq z_p^2 \leq 2p - 3$$

and we have the (double) equality for $p \in \{2, 3\}$.

In particular, the above inequalities yield

Corollary 3.3. *We have $z_p^2 \leq p - 1$ for $1 < p \leq 2$ and $z_p^2 \geq p - 1$ for $2 \leq p \leq 3$.*

Furthermore, we get the following bound, which, clearly, is also valid for $p > 3$.

Corollary 3.4. *We have $C_p > 1$ for $p \neq 2$.*

4 Proof of Theorem 1.4

We turn to the proof of our main result. For the sake of convenience, we have decided to split it into five parts: the proof of (1.4) in the cases $1 < p \leq 2$, $2 < p < 3$, $p \geq 3$, the strictness and, finally, the sharpness of the estimate.

4.1 The case $1 < p \leq 2$

Let $V_p : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be given by

$$V_p(x, y) = \begin{cases} \alpha_p y^p \phi_p(x/y) & \text{if } y \leq z_p^{-1} x, \\ y^p - z_p^{-p} x^p & \text{if } y \geq z_p^{-1} x, \end{cases}$$

where

$$\alpha_p = -(z_p \phi_{p-1}(z_p))^{-1}. \tag{4.1}$$

Furthermore, let

$$S_1 = \{(x, y) \in \mathscr{H} \times \mathscr{H} : |y| > z_p^{-1}|x| > 0\}$$

and

$$S_2 = \{(x, y) \in \mathscr{H} \times \mathscr{H} : 0 < |y| < z_p^{-1}|x|\}.$$

We will check below in seven steps that the function V_p has all the properties described in Section 2.

1° Regularity. It is easy to see that V_p is continuous, of class C^1 on the set $(0,\infty)\times(0,\infty)$ and of class C^2 on the set $(0,\infty)\times(0,\infty)\setminus\{(x,y): y=z_p^{-1}x\}$. In consequence, the function U_p defined by (2.7) has the required smoothness. It can also be verified readily that the first order derivative of U_p is bounded on bounded sets not containing $0\in\mathscr{H}\times\mathscr{H}$.

2° The growth condition (2.8). This follows immediately from the asymptotics (3.5).

3° The inequality (2.9). Note that ϕ_p is increasing (by Lemma 3.1 (v)) and $z_p\leq 1$ (by Corollary 3.3). Thus, for $0<y\leq x$,

$$V_p(x,y)=\alpha_p y^p\phi_p(x/y)\leq \alpha_p y^p\phi_p(1)\leq 0.$$

4° The inequality (2.10). This is obvious for $y\geq z_p^{-1}x$, so we focus on the case $y<z_p^{-1}x$. Then the majorization is equivalent to

$$\alpha_p\phi_p(s)\leq 1-z_p^{-p}s^p \qquad \text{for } s>z_p.$$

Both sides are equal when $s=z_p$, so it suffices to establish an appropriate estimate for the derivatives: $\alpha_p\phi_p'(s)\leq -pz_p^{-p}s^{p-1}$ for $s>z_p$. We see that again both sides are equal when $s=z_p$; thus we will be done if we show that the function $s\mapsto \phi_p'(s)/s^{p-1}$ is nondecreasing on (z_p,∞). After differentiation, this is equivalent to

$$\phi_p''(s)s-(p-1)\phi_p'(s)\geq 0 \qquad \text{on } (z_p,\infty),$$

or, by Lemma 3.1 (i), $\phi_p'''(s)\leq 0$ for $s>z_p$. This is shown in the part (v) of that lemma.

Before we proceed, let us mention here a fact which will be exploited during the proof of the strictness. Namely, if $p\neq 2$, then the above reasoning gives that the majorization is strict on $\{(x,y): y<z_p^{-1}x\}$. Indeed, we have that ϕ_p''' is negative on $[z_p,\infty)$ (see Lemma 3.1 (v)).

5° The condition (2.11). If $z_p|y|>|x|>0$, then we have

$$\begin{aligned}\left[V_{pxx}(|x|,|y|)-\frac{V_{px}(|x|,|y|)}{|x|}\right]\langle x',h\rangle^2&=p(2-p)z_p^{-p}|x|^{p-2}\langle x',h\rangle^2\\&\leq p(2-p)z_p^{-p}|x|^{p-2}|h|^2\end{aligned}\tag{4.2}$$

and

$$\frac{V_{px}(|x|,|y|)}{|x|}|h|^2=-pz_p^{-p}|x|^{p-2}|h|^2,\tag{4.3}$$

so (2.11) is valid. When $|x|>z_p|y|>0$, we compute that

$$\begin{aligned}\left[V_{pxx}(|x|,|y|)-\frac{V_{px}(|x|,|y|)}{|x|}\right]\langle x',h\rangle^2&=\alpha_p|y|^{p-2}\left[\phi_p''(r)-r^{-1}\phi_p'(r)\right]\langle x',h\rangle^2\\&\leq \alpha_p|y|^{p-2}\left[\phi_p''(r)-r^{-1}\phi_p'(r)\right]|h|^2,\end{aligned}\tag{4.4}$$

where we have used the notation $r=|x|/|y|$. Furthermore,

$$\frac{V_{px}(|x|,|y|)}{|x|}|h|^2=\alpha_p|y|^{p-2}r^{-1}\phi_p'(r)\,|h|^2.\tag{4.5}$$

Adding this to (4.4) yields (2.11), since $\phi_p''(r) \geq 0$: see Lemma 3.1 (v).

6° *The condition* (2.12). If $z_p|y| > |x| > 0$, then

$$\left[V_{pyy}(|x|,|y|) - \frac{V_{py}(|x|,|y|)}{|y|}\right]\langle y',k\rangle^2 = p(p-2)|y|^{p-2}\langle y',k\rangle^2 \leq 0$$

and

$$\frac{V_{py}(|x|,|y|)}{|y|}|k|^2 = p|y|^{p-2}|k|^2.$$

Therefore, combining this with (4.2) and (4.3) we see that the left-hand side of (2.12) is not larger than

$$-p|y|^{p-2}(|h|^2 - |k|^2) - p[(p-1)z_p^{-p}|x|^{p-2} - |y|^{p-2}]|h|^2 \leq -p|y|^{p-2}(|h|^2 - |k|^2), \tag{4.6}$$

as needed. Here in the last passage we have used the estimates $z_p^{-1}|x| < |y|$ and $z_p^2 \leq p-1$ (see Corollary 3.3). On the other hand, if $0 < z_p|y| < |x|$, then recall the function F_p given by (3.7). A little calculation yields

$$\left[V_{pyy}(|x|,|y|) - \frac{V_{py}(|x|,|y|)}{|y|}\right]\langle y',k\rangle^2 = \alpha_p|y|^{p-2}F_p(r)\langle y',k\rangle^2,$$

which is nonpositive by means of Lemma 3.2 (here $r = |x|/|y|$, as before). Furthermore, by (3.3),

$$\frac{V_y(|x|,|y|)}{|y|} = \alpha_p|y|^{p-2}\left[p\phi_p(r) - r\phi_p'(r)\right]|k|^2 = -\alpha_p|y|^{p-2}\phi_p''(r)|k|^2,$$

which, combined with (4.4) and (4.5) implies that the left-hand side of (2.12) does not exceed

$$\alpha_p|y|^{p-2}\phi_p''(r)(|h|^2 - |k|^2).$$

7° *The bound* (2.3). In view of the above reasoning, we have to take $c_1(x,y) = p|y|^{p-2}$ and $c_2(x,y) = -\alpha_p|y|^{p-2}\phi_p''(r)$. It is evident that the estimate (2.3) is valid for this choice of c_i, $i = 1, 2$.

4.2 The case $2 < p < 3$

Let $V_p : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be given by

$$V_p(x,y) = \begin{cases} \alpha_p x^p \phi_p(y/x) & \text{if } y \geq z_p x, \\ y^p - z_p^p x^p & \text{if } y \leq z_p x, \end{cases}$$

where

$$\alpha_p = (z_p\phi_{p-1}(z_p))^{-1}. \tag{4.7}$$

Let

$$S_1 = \{(x,y) \in \mathscr{H} \times \mathscr{H} : 0 < |y| < z_p|x|\}$$

and

$$S_2 = \{(x,y) \in \mathscr{H} \times \mathscr{H} : |y| > z_p|x| > 0\}.$$

As in the previous case, we will verify that V_p enjoys the requirements listed in Section 2.

1° Regularity. Clearly, we have that V_p is continuous, of class C^1 on $(0,\infty)\times(0,\infty)$ and of class C^2 on $(0,\infty)\times(0,\infty)\setminus\{(x,y): y=z_px\}$. Hence U_p given by (2.7) has the necessary smoothness. In addition, it is easy to see that the first order derivative of U_p is bounded on bounded sets.

2° The growth condition (2.8). This is guaranteed by the asymptotics (3.5).

3° The inequality (2.9) This is obvious: $V_p(x,y)\leq x^p(1-z_p^p)\leq 0$ if $x\geq y\geq 0$.

4° The majorization (2.10). This can be established exactly in the same manner as in the previous case. In fact one can show that the majorization is strict provided $y>z_px$. The details are left to the reader.

5° The condition (2.11). Note that if $0<|y|<z_p|x|$, then

$$\left[V_{pxx}(|x|,|y|)-\frac{V_{px}(|x|,|y|)}{|x|}\right]\langle x',h\rangle^2=-p(p-2)z_p^p|x|^{p-2}\langle x',h\rangle^2\leq 0 \tag{4.8}$$

and

$$\frac{V_{px}(|x|,|y|)}{|x|}|h|^2=-pz_p^p|x|^{p-2}|h|^2\leq 0, \tag{4.9}$$

so (2.11) follows. Suppose then, that $|y|>z_p|x|>0$ and recall F_p introduced in Lemma 3.2. By means of this lemma, after some straightforward computations, one gets

$$\left[V_{pxx}(|x|,|y|)-\frac{V_{px}(|x|,|y|)}{|x|}\right]\langle x',h\rangle^2=\alpha_p|x|^{p-2}F_p(r)\langle x',h\rangle^2\leq 0, \tag{4.10}$$

where we have set $r=|y|/|x|$. Moreover, by (3.3),

$$\begin{aligned}\frac{V_{px}(|x|,|y|)}{|x|}|h|^2&=\alpha_p\left[p\phi_p(r)-r\phi_p'(r)\right]|x|^{p-2}|h|^2\\&=-\alpha_p\phi_p''(r)|x|^{p-2}|h|^2\end{aligned} \tag{4.11}$$

is nonpositive; this completes the proof of (2.11).

6° The condition (2.12). If $0<|y|<z_p|x|$, then

$$\left[V_{pyy}(|x|,|y|)-\frac{V_{py}(|x|,|y|)}{|y|}\right]\langle y',k\rangle^2=p(p-2)|y|^{p-2}\langle y',k\rangle^2\leq p(p-2)|y|^{p-2}|k|^2$$

and

$$\frac{V_{py}(|x|,|y|)}{|y|}|k|^2=p|y|^{p-2}|k|^2.$$

Therefore, combining this with (4.8) and (4.9) we see that the left-hand side of (2.12) can be bounded from above by

$$\begin{aligned}p(p-1)|y|^{p-2}|k|^2-pz_p^p|x|^{p-2}|h|^2&\leq p(p-1)z_p^{p-2}|x|^{p-2}|k|^2-pz_p^p|x|^{p-2}|h|^2\\&\leq -pz_p^p|x|^{p-2}(|h|^2-|k|^2).\end{aligned} \tag{4.12}$$

Here in the latter passage we have used Corollary 3.3. If $|y| > z_p|x| > 0$, then, again using the notation $r = |y|/|x|$,

$$\left[V_{pyy}(|x|,|y|) - \frac{V_{py}(|x|,|y|)}{|y|}\right]\langle y',k\rangle^2 = \alpha_p|x|^{p-2}\left[\phi_p''(r) - r^{-1}\phi_p'(r)\right]\langle y',k\rangle^2$$
$$\leq \alpha_p|x|^{p-2}\left[\phi_p''(r) - r^{-1}\phi_p'(r)\right]|k|^2$$

and

$$\frac{V_{py}(|x|,|y|)}{|y|}|k|^2 = \alpha_p|x|^{p-2}r^{-1}\phi_p'(r)|k|^2.$$

Combining this with (4.10) and (4.11), we get that the left-hand side of (2.12) does not exceed $-\alpha_p|x|^{p-2}\phi_p''(r)(|h|^2 - |k|^2)$.

7° The bound (2.3). By the above considerations, we are forced to take $c_1(x,y) = pz_p^p|x|^{p-2}$ and $c_2(x,y) = \alpha_p|x|^{p-2}\phi_p''(r)$ and it is clear that the condition is satisfied. This establishes (1.4) for $2 < p < 3$.

4.3 The case $p \geq 3$

Let $V_p : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as follows. First, set

$$V_p(x,y) = \alpha_p R^p \cos\left(p\left(\frac{\pi}{2} - \theta\right)\right),$$

if $\pi/2 - \pi/(2p) \leq \theta \leq \pi/2$. Here

$$\alpha_p = \frac{\cos^{p-1}\left(\frac{\pi}{2p}\right)}{\sin\frac{\pi}{2p}}$$

and we have used polar coordinates: $x = R\cos\theta$, $y = R\sin\theta$, with $\theta \in [0,\pi/2]$. For remaining values of θ, take

$$V_p(x,y) = y^p - \cot^p\left(\frac{\pi}{2p}\right)x^p.$$

In addition, let

$$S_1 = \{(x,y) \in \mathscr{H} \times \mathscr{H} : \pi/2 - \pi/(2p) < \theta < \pi/2\}$$

and

$$S_2 = \{(x,y) \in \mathscr{H} \times \mathscr{H} : 0 < \theta < \pi/2 - \pi/(2p)\},$$

with the similar convention as above: $|x| = R\cos\theta$ and $|y| = R\sin\theta$ for $\theta \in [0,\pi/2]$.

In the proof of (1.4) we will need the following auxiliary fact.

Lemma 4.1. *If $\beta \in (0,\pi/(2p)]$, then*

$$\frac{\cot^{p-1}\beta}{\cot((p-1)\beta)} \geq \cot^p\left(\frac{\pi}{2p}\right). \tag{4.13}$$

Furthermore, the inequality is strict if $\beta \neq \pi/(2p)$.

Proof. Both sides are equal for $\beta = \pi/(2p)$, so it suffices to show that the left-hand side is decreasing as a function of $\beta \in (0, \pi/(2p))$. If we calculate the derivative, we see that this is equivalent to the inequality $\sin(2\beta) < \sin(2(p-1)\beta)$. But this follows immediately from the bounds $0 < 2\beta < 2(p-1)\beta < \pi - 2\beta$. □

Furthermore, due to the complexity of the calculations, it is convenient to gather the bounds for the partial derivatives of V_p in a separate lemma.

Lemma 4.2. *(i) For any x, $y > 0$ with $y > \cot(\pi/(2p))x$ we have*

$$V_{pxx}(x,y)x \geq V_{px}(x,y) \tag{4.14}$$

and

$$V_{pyy}(x,y)y \geq V_{py}(x,y) \tag{4.15}$$

(ii) For any x, $y > 0$ with $y < \cot(\pi/(2p))x$ we have

$$V_{pxx}(x,y)x \leq V_{px}(x,y), \tag{4.16}$$

$$V_{pyy}(x,y)y \geq V_{py}(x,y) \tag{4.17}$$

and

$$xV_{pyy}(x,y) + V_{px}(x,y) \leq 0. \tag{4.18}$$

Proof. (i) We derive that

$$V_{px}(x,y) = \alpha_p p R^{p-1} \sin\left((p-1)\left(\frac{\pi}{2} - \theta\right)\right)$$

and

$$V_{pxx}(x,y) = -\alpha_p p(p-1)R^{p-2} \cos\left((p-2)\left(\frac{\pi}{2} - \theta\right)\right), \tag{4.19}$$

so (4.14) is equivalent to

$$-(p-1)\cos\left((p-2)\left(\frac{\pi}{2} - \theta\right)\right)\sin\left(\frac{\pi}{2} - \theta\right) + \sin\left((p-1)\left(\frac{\pi}{2} - \theta\right)\right) \geq 0,$$

or

$$-(p-2)\cos\left((p-2)\left(\frac{\pi}{2} - \theta\right)\right)\sin\left(\frac{\pi}{2} - \theta\right) + \sin\left((p-2)\left(\frac{\pi}{2} - \theta\right)\right)\cos\left(\frac{\pi}{2} - \theta\right) \geq 0$$

for $\theta \in (\pi/2 - \pi/(2p), \pi/2)$. This holds true, since both sides are equal for $\theta = \pi/2$ and the derivative of the left-hand side equals

$$(3-p)(p-1)\sin\left((p-2)\left(\frac{\pi}{2} - \theta\right)\right)\sin\left(\frac{\pi}{2} - \theta\right) \leq 0.$$

To prove (4.15), we carry out similar calculations and transform the estimate into the equivalent form

$$(p-2)\cos\left((p-2)\left(\frac{\pi}{2} - \theta\right)\right)\cos\left(\frac{\pi}{2} - \theta\right) + \sin\left((p-2)\left(\frac{\pi}{2} - \theta\right)\right)\sin\left(\frac{\pi}{2} - \theta\right) \geq 0,$$

for $\theta \in (\pi/2-\pi/(2p), \pi/2)$. Denoting the left-hand side by $F(\theta)$, we easily check that

$$F\left(\frac{\pi}{2}-\frac{\pi}{2p}\right)=(p-3)\cos\left((p-2)\frac{\pi}{2p}\right)\cos\frac{\pi}{2p}+\cos\left((p-3)\frac{\pi}{2p}\right)\geq 0$$

and

$$F'(\theta)=(p-1)(p-3)\sin\left((p-2)\left(\frac{\pi}{2}-\theta\right)\right)\sin\theta\geq 0.$$

This yields the claim.

(ii) If $y<\cot(\pi/(2p))x$, then

$$xV_{pxx}(x,y)-V_{px}(x,y)=-p(p-2)\cot^p\left(\frac{\pi}{2p}\right)x^{p-1}\leq 0$$

and

$$yV_{pyy}(x,y)-V_{py}(x,y)=p(p-2)y^{p-1}\geq 0.$$

Furthermore,

$$\begin{aligned}xV_{pyy}(x,y)+V_{px}(x,y)&=px\left[(p-1)y^{p-2}-\cot^p\left(\frac{\pi}{2p}\right)x^{p-2}\right]\\&\leq pxy^{p-2}\left(p-1-\cot^2\frac{\pi}{2p}\right)=pxy^{p-2}\left[p-\left(\sin\frac{\pi}{2p}\right)^{-2}\right],\end{aligned}$$

so it suffices to show that the expression in the square brackets is nonpositive. This follows immediately from

$$\sin^2\frac{\pi}{2p}\leq\left(\frac{\pi}{2p}\right)^2=\frac{1}{p}\cdot\frac{\pi^2}{4p}\leq\frac{1}{p}$$

and we are done. □

Now we are ready for the proof of (1.4). As in the previous cases, we verify that V_p has the properties studied in Section 2.

1° Regularity. It is easily checked that V_p is continuous, of class C^1 on $(0,\infty)\times(0,\infty)$ and of class C^2 outside $\{(x,y):\theta=\pi/2-\pi/(2p)\}$. This guarantees the appropriate smoothness of U_p given by (2.7). Furthermore, it is clear that the first order derivative of U_p is bounded on bounded sets.

2° The growth condition (2.8). This is clear from the very formula for the function V_p.

3° The condition (2.9). This is obvious: for $x\geq y\geq 0$ we have

$$V_p(x,y)=y^p-\cot^p\left(\frac{\pi}{2p}\right)x^p\leq x^p\left(1-\cot^p\left(\frac{\pi}{2p}\right)\right)\leq 0.$$

4° The majorization (2.10). Clearly, it suffices to show this inequality on the set $\{(x,y):\theta>\pi/2-\pi/(2p)\}$, where it can be rewritten in the form

$$\sin^p\theta-\cot^p\left(\frac{\pi}{2p}\right)\cos^p\theta\leq\frac{\cos^{p-1}\left(\frac{\pi}{2p}\right)}{\sin\frac{\pi}{2p}}\cos\left(p\left(\frac{\pi}{2}-\theta\right)\right)$$

or, after substitution $\beta = \pi/2 - \theta \in [0, \pi/(2p))$,

$$\frac{\cos^p \beta - \cot^p(\pi/(2p)) \sin^p \beta}{\cos(p\beta)} \leq \frac{\cos^{p-1}\left(\frac{\pi}{2p}\right)}{\sin \frac{\pi}{2p}}.$$

Since both sides become equal when we let $\beta \to \pi/(2p)$, it suffices to show that the left-hand side, as a function of β, is nondecreasing on $(0, \pi/(2p))$. Differentiating, we see that this is equivalent to (4.13). In fact, we have that the majorization is strict on $\{(x, y) : \theta > \pi/2 - \pi/(2p)\}$ (since (4.13) is strict for $\beta < \pi/(2p)$).

5° *The condition* (2.11). If $(x, y) \in \mathscr{H} \times \mathscr{H}$ satisfies $|y| > \cot(\pi/(2p))|x| > 0$, then by (4.14) and (4.19) we have that

$$\left[V_{pxx}(|x|,|y|) - \frac{V_{px}(|x|,|y|)}{|x|}\right] \langle x', h\rangle^2 + \frac{V_{px}(|x|,|y|)}{|x|}|h|^2 \leq V_{pxx}(|x|,|y|)|h|^2 \leq 0.$$

On the other hand, if $0 < |y| < \cot(\pi/(2p))|x|$, then, by (4.16),

$$\begin{aligned}\left[V_{pxx}(|x|,|y|) - \frac{V_{px}(|x|,|y|)}{|x|}\right] \langle x', h\rangle^2 + \frac{V_{px}(|x|,|y|)}{|x|}|h|^2 &\leq \frac{V_{px}(|x|,|y|)}{|x|}|h|^2 \\ &= -pC_p^p|x|^{p-2}|h|^2 \leq 0,\end{aligned}$$

as needed.

6° *The condition* (2.12). If $|y| > \cot(\pi/(2p))|x| > 0$, then by (4.14) and (4.15), the left-hand side of (2.12) does not exceed

$$V_{pxx}(|x|,|y|)|h|^2 + V_{pyy}(|x|,|y|)|k|^2 \leq V_{pxx}(|x|,|y|)(|h|^2 - |k|^2), \tag{4.20}$$

since $V_{pxx}(|x|,|y|) \leq 0$ and V_p is harmonic on $\{(x, y) : y > \cot(\pi/(2p)x > 0\}$. If we have $0 < |y| < \cot(\pi/(2p))|x|$, then by (4.16), (4.17) and (4.18) the left-hand side is not larger than

$$\frac{V_{px}(|x|,|y|)}{|x|}|h|^2 + V_{pyy}(|x|,|y|)|k|^2 \leq -V_{pyy}(|x|,|y|)(|h|^2 - |k|^2). \tag{4.21}$$

7° *The bound* (2.3). By (4.20) and (4.21), we take $c_1(x, y) = -V_{pxx}(|x|,|y|)$ and $c_2(x, y) = V_{pyy}(|x|,|y|)$. By (4.19) and the equality $V_{pyy}(|x|,|y|) = p(p-1)|y|^{p-2}$ it is clear that the estimate (2.3) is satisfied.

The proof of (1.4) is complete.

4.4 Strictness

Suppose that $p \neq 2$ and that X, Y are orthogonal $\mathscr{H}$-valued martingales such that $0 < ||X||_p < \infty$, Y is differentially subordinate to X and we have equality in (1.4). These conditions imply in particular that both martingales converge almost surely and in L^p. Denoting the corresponding limits by X_∞ and Y_∞, we may write $||Y_\infty||_p^p - C_p^p ||X_\infty||_p^p = 0$ and $\mathbb{E}U_p(X_\infty, Y_\infty) \leq 0$: to get the second estimate, we use (2.8), Doob's inequality and Lebesgue's dominated convergence theorem. Therefore, by (2.10), we have

$$||Y_\infty||_p^p - C_p^p ||X_\infty||_p^p = \mathbb{E}U_p(X_\infty, Y_\infty) = 0. \tag{4.22}$$

This implies

$$|Y_\infty| = C_p|X_\infty| \qquad \text{with probability 1.} \tag{4.23}$$

To see this, assume first that $1 < p < 2$. For this range of parameters p, the inequality (2.10) is strict if $|y| < C_p|x|$. This, by (4.22), implies $\mathbb{P}(|Y_\infty| \geq C_p|X_\infty|) = 1$ and, again by (4.22), yields (4.23). In the case $p > 2$ the reasoning is the same.

The condition (4.23) gives $U_p(X_\infty, Y_\infty) = 0$. Furthermore, by (2.4) and (2.8) we see that $(U_p(X_t, Y_t))_{t\geq 0}$ is a uniformly integrable supermartingale satisfying $U_p(X_0, Y_0) \leq 0$. This gives $\mathbb{P}(U_p(X_t, Y_t) = 0 \text{ for all } t \geq 0) = 1$. However, using the formulas for V_p in the cases $1 < p < 2$, $2 < p < 3$ and $p \geq 3$, we see that this is equivalent to

$$\mathbb{P}(|Y_t| = C_p|X_t| \text{ for all } t \geq 0) = 1. \tag{4.24}$$

For any $t \geq 0$ we have that

$$|X_0|^2 + \sum_{k=0}^{2^n-1} |X_{t(k+1)2^{-n}} - X_{tk2^{-n}}|^2 \xrightarrow{n\to\infty} [X,X]_t$$

in probability (see e.g. [10] for the proof in the real case; the reasoning presented there can be easily extended to the Hilbert-space setting). Using a similar statement for Y and combining this with (4.24) yields $C_p^2[X,X]_t = [Y,Y]_t$, which contradicts the differential subordination unless $C_p = 1$ or $[X,X] \equiv 0$. However, the first possibility occurs only for $p = 2$ (Corollary 3.4) and the second one implies $||X||_p = 0$; we have excluded both these possibilities at the beginning.

This completes the proof of the strictness.

4.5 Optimality of the constants

Clearly, it suffices to consider only the case $1 < p < 3$, since for $p \geq 3$ the constant C_p is the best possible even for real-valued processes (cf. [3], see also Section 5 below). Suppose first that $1 < p \leq 2$ and let $B = (B^{(1)}, B^{(2)})$ be a standard two-dimensional Brownian motion. Fix a positive integer n, a stopping time τ of B satisfying $\tau \in L^{p/2}$ and consider the martingales X, Y given by $X_t = (B^{(1)}_{\tau\wedge t}, 0, 0, \ldots)$ and

$$Y_t^{(n)} = (B^{(2)}_{\tau\wedge t\wedge 2^{-n}}, B^{(2)}_{\tau\wedge t\wedge(2\cdot 2^{-n})} - B^{(2)}_{\tau\wedge t\wedge 2^{-n}}, B^{(2)}_{\tau\wedge t\wedge(3\cdot 2^{-n})} - B^{(2)}_{\tau\wedge t\wedge(2\cdot 2^{-n})}, \ldots)$$

for $t \geq 0$. That is, the k-th coordinate of $Y_t^{(n)}$ is equal to the increment

$$B^{(2)}_{\tau\wedge t\wedge(k\cdot 2^{-n})} - B^{(2)}_{\tau\wedge t\wedge((k-1)\cdot 2^{-n})},$$

for $k = 1, 2, \ldots$. Obviously, X and $Y^{(n)}$ are orthogonal, since $B^{(1)}$ and $B^{(2)}$ are independent. In addition, $Y^{(n)}$ is differentially subordinate to X, because $[X,X]_t = [Y^{(n)}, Y^{(n)}]_t = \tau \wedge t$ for all $t \geq 0$. Therefore, we infer from the inequality (1.4) that for any $t \geq 0$,

$$||Y_t^{(n)}||_p \leq C_p||X_t||_p. \tag{4.25}$$

On the other hand, it is well known (see e.g. proof of Theorem 1.3 in [15]) that for any $t \geq 0$,

$$|Y_t^{(n)}|^2 = \sum_{k=1}^{\infty} |B^{(2)}_{\tau\wedge t\wedge(k\cdot 2^{-n})} - B^{(2)}_{\tau\wedge t\wedge((k-1)\cdot 2^{-n})}|^2 \xrightarrow{n\to\infty} [B^{(2)}, B^{(2)}]_{\tau\wedge t} = \tau \wedge t$$

in L^2. Therefore, letting $n \to \infty$ in (4.25) yields

$$||(\tau \wedge t)^{1/2}||_p \le C_p ||B^{(1)}_{\tau\wedge t}||_p \le C_p ||B^{(1)}_{\tau}||_p.$$

It suffices to take $t \to \infty$ and apply Davis' inequality (1.2) to get $C_p \ge z_p^{-1}$. The case $2 < p < 3$ is dealt with exactly in the same manner.

The proof of the Theorem 1.4 is complete.

Remark 4.3. Referee asked a very interesting question whether the constant C_p, $1 < p < 2$, is still the best possible when we restrict ourselves to Y taking values in $\mathbb{R}^2$. The answer is affirmative and can be obtained by a modification of the above example. Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional Brownian motion, let τ be an arbitrary stopping time of B with $\tau \in L^{p/2}$ and define $X_t = B^{(1)}_{\tau\wedge t}$ for $t \ge 0$. For a fixed positive integer n we introduce $Y^{(n)}$ using the following inductive procedure:

(i) $Y^{(n)}_0 = (0, 0)$,

(ii) for $k = 0, 1, 2, \ldots$ and $t \in (k\cdot 2^{-n}, (k+1)\cdot 2^{-n}]$, let

$$Y^{(n)}_t = Y^{(n)}_{\tau\wedge k\cdot 2^{-n}} + \xi_k \cdot (B^{(2)}_{\tau\wedge t} - B^{(2)}_{\tau\wedge k\cdot 2^{-n}}),$$

where ξ_k is a norm-one vector in $\mathbb{R}^2$, orthogonal to $Y^{(n)}_{\tau\wedge k\cdot 2^{-n}}$, not depending on t.

In other words, for $t \in (k\cdot 2^{-n}, (k+1)\cdot 2^{-n}]$, the increment $Y^{(n)}_t - Y^{(n)}_{k\cdot 2^{-n}}$ is orthogonal to $Y_{k\cdot 2^{-n}}$: this guarantees the equality

$$|Y^{(n)}_t|^2 = \sum_{\ell=1}^{\infty} |B^{(2)}_{\tau\wedge t\wedge(\ell\cdot 2^{-n})} - B^{(2)}_{\tau\wedge t\wedge((\ell-1)\cdot 2^{-n})}|^2.$$

The remainder of the proof is the same as above: we easily check that X and $Y^{(n)}$ are orthogonal and that $[X,X]_t = [Y^{(n)}, Y^{(n)}]_t = \tau \wedge t$. Letting $n \to \infty$ we obtain that $C_p \ge z_p^{-1}$ due to Davis' inequality (1.2).

A similar reasoning can be carried out to show that the constant C_p, $2 < p < 3$, is the best possible if we restrict ourselves to X taking values in $\mathbb{R}^2$.

5 Inequalities for stochastic integrals

We will apply the results from the previous section and obtain some sharp inequalities for stochastic integrals. Let $M = (M_t)_{t\ge 0}$ be an adapted martingale and let $H = (H_t)_{t\ge 0}$ and $K = (K_t)_{t\ge 0}$ be two predictable processes. Let $X = H \cdot M$, $Y = K \cdot M$ denote the stochastic integrals of H and K with respect to M. The following result is due to Burkholder (see Theorem 5.1 in [7]).

Theorem 5.1. *Suppose that one of the following two conditions is satisfied:*

(i) M is $\mathcal{H}$-valued, but H and K are scalar valued,

(ii) M is scalar valued, but H and K are $\mathcal{H}$-valued.
If $|K_t| \le |H_t|$ for every $t \ge 0$, then for $1 < p < \infty$,

$$||Y||_p \le (p^* - 1)||X||_p$$

and the constant $p^* - 1$ is the best possible.

In the case where $M = B$, a d-dimensional Brownian motion, one needs H and K to be real valued in order to apply the theorem above. Theorem 1 in [16] (which is the inequality (1.1) under continuous-time differential subordination) allows to strengthen this result to the following fact (cf. [3] and [16]). It is essential for the applications to Riesz and Beurling transforms, see e.g. [2], [3], [4] and [12].

Theorem 5.2. *Suppose that for every $t \geq 0$, $H_t : \mathbb{R}^d \to \mathscr{H}$ and $K_t : \mathbb{R}^d \to \mathscr{H}$ are (random) linear operators satisfying*

$$||K_t||_{HS} \leq ||H_t||_{HS}, \tag{5.1}$$

where $||\cdot||_{HS}$ stands for the Hilbert Schmidt norm. Let B be a Brownian motion in $\mathbb{R}^d$ and let $X = H \cdot B$ and $Y = K \cdot B$. Then for $1 < p < \infty$,

$$||Y||_p \leq (p^* - 1)||X||_p$$

and the constant $p^* - 1$ is the best possible.

Our contribution is the following version of the above theorem for orthogonal H and K. It may be regarded as a generalization the statement used by Bañuelos and Wang in their study of Riesz transforms (see Corollary 4.3 and Corollary 4.6 in [3]).

Theorem 5.3. *Let B be a Brownian motion in $\mathbb{R}^d$, $d \geq 2$, and let $X = H \cdot B$ and $Y = K \cdot B$ for some predictable H and K, with $H_t : \mathbb{R}^d \to \mathscr{H}$ and $K_t : \mathbb{R}^d \to \mathscr{H}$ for all $t \geq 0$. Suppose that for any $t \geq 0$ the linear operators H_t and K_t satisfy the domination* (5.1) *and the further orthogonality property*

$$H_t K_t^* = 0 \quad \text{as an operator on } \mathscr{H}.$$

Then for $1 < p < \infty$,

$$||Y||_p \leq C_p ||X||_p \tag{5.2}$$

and the constant is the best possible, even for $d = 2$.

Proof of Theorem 5.3. By standard localization, we may assume that X and Y are martingales. We have

$$[X,X]_t - [Y,Y]_t = \int_0^t ||H_s||_{HS}^2 - ||K_s||_{HS}^2 \mathrm{d}s,$$

so, by (5.1), Y is differentially subordinate to X. In addition, for any $i, j \geq 1$ we have, by the orthogonality of H and K,

$$[X^i, Y^j]_t = \int_0^t H_s^i (K_s^j)^* \mathrm{d}s = 0,$$

where $H_s = (H_s^1, H_s^2, \ldots)$ and $K_s = (K_s^1, K_s^2, \ldots)$. Thus, by (1.4), the estimate (5.2) follows. It is clear that C_p is optimal for $1 < p \leq 3$ and $d = 2$: the example considered in Subsection 4.5 can be rewritten in the above language of stochastic integrals. For $p > 3$, take two-dimensional Brownian motion B and an analytic function $f = u + iv$ on the unit disc D, satisfying $f(0) = 0$. Consider the stopping time $\tau = \inf\{t : B_t \notin D\}$ and the martingales $X = (u(B_{\tau\wedge t}))_{t\geq 0}$, $Y = (v(B_{\tau\wedge t}))_{t\geq 0}$ which, by Itô's formula, admit the representation

$$X_t = \int_0^t 1_{\{\tau \geq s\}} \nabla u(B_s) \mathrm{d}B_s, \quad \text{and} \quad Y_t = \int_0^t 1_{\{\tau \geq s\}} \nabla v(B_s) \mathrm{d}B_s$$

for all $t \geq 0$. By Cauchy-Riemann equations, the integrands satisfy the domination (5.1) as well as the orthogonality. It suffices to note that (5.2) reduces to Pichorides' inequality

$$||v||_p \leq \cot\left(\frac{\pi}{2p^*}\right)||u||_p,$$

which is known to be sharp (see [14]). This completes the proof. □

Acknowledgment

The author would like to thank anonymous referees for many helpful comments and suggestions, in particular for the question studied in Remark 4.3.

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