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Convergence of Rescaled Competing Species Processes to a Class of SPDEs

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Abstract

One can construct a sequence of rescaled perturbations of voter processes in dimension $d = 1$ whose approximate densities are tight. By combining both long-range models and fixed kernel models in the perturbations and considering the critical long-range case, results of Cox and Perkins (2005) are refined. As a special case we are able to consider rescaled Lotka-Volterra models with long-range dispersal and short-range competition. In the case of long-range interactions only, the approximate densities converge to continuous space time densities which solve a class of SPDEs (stochastic partial differential equations), namely the heat equation with a class of drifts, driven by Fisher-Wright noise. If the initial condition of the limiting SPDE is integrable, weak uniqueness of the limits follows. The results obtained extend the results of Mueller and Tribe (1995) for the voter model by including perturbations. In particular, spatial versions of the Lotka-Volterra model as introduced in Neuhauser and Pacala (1999) are covered for parameters approaching one. Their model incorporates a fecundity parameter and models both intra- and interspecific competition.

Key words: Voter model, Lotka-Volterra model, spatial competition, stochastic partial differential equations, long-range limits.

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1 Introduction

We define a sequence of rescaled competing species models ξ_t^N in dimension $d = 1$, which can be described as perturbations of voter models. In the N^{th}-model the sites are indexed by $x \in N^{-1}\mathbb{Z}$. We label the state of site x at time t by $\xi_t^N(x)$ where $\xi_t^N(x) = 0$ if the site is occupied at time t by type 0 and $\xi_t^N(x) = 1$ if it is occupied by type 1.

In what follows we shall write $x \sim y$ if and only if $0 < |x - y| \leq N^{-1/2}$, i.e. if and only if x is a neighbour of y. Observe that each x has $2c(N)N^{1/2}$, $c(N) \overset{N\to\infty}{\to} 1$ neighbours.

The rates of change incorporate both long-range models and fixed kernel models with finite range. The long-range interaction takes into account the densities of the neighbours of x at long-range, i.e.

$$f_i^{(N)}(x,\xi) \equiv \frac{1}{2c(N)\sqrt{N}} \sum_{\substack{0<|y-x|\leq 1/\sqrt{N},\\ y\in\mathbb{Z}/N}} 1(\xi^N(y) = i), \quad i = 0,1$$

and the fixed kernel interaction considers

$$g_i^{(N)}(x,\xi) \equiv \sum_{y\in\mathbb{Z}/N} p(N(x-y))1(\xi^N(y) = i), \quad i = 0,1, \tag{1}$$

where $p(x)$ is a random walk kernel on $\mathbb{Z}$ of finite range, i.e. $0 \leq p(x) \leq 1$, $\sum_{x\in\mathbb{Z}} p(x) = 1$ and $p(x) = 0$ for all $|x| \geq C_p$. In what follows we shall often abbreviate $f_i^{(N)}(x,\xi)$ by $f_i^{(N)}$ and $g_i^{(N)}(x,\xi)$ by $g_i^{(N)}$ if the context is clear. Note in particular that $0 \leq f_i^{(N)}, g_i^{(N)} \leq 1$ and $f_0^{(N)} + f_1^{(N)} = g_0^{(N)} + g_1^{(N)} = 1$.

Now define the rates of change of our configurations. At site x in configuration $\xi^N \in \{0,1\}^{\mathbb{Z}/N}$ the coordinate $\xi^N(x)$ makes transitions

$$\begin{aligned} 0 \to 1 \text{ at rate } & N f_1^{(N)} + f_1^{(N)} \left\{ g_0^{(N)} G_0^{(N)}\left(f_1^{(N)}\right) + g_1^{(N)} H_0^{(N)}\left(f_1^{(N)}\right) \right\}, \\ 1 \to 0 \text{ at rate } & N f_0^{(N)} + f_0^{(N)} \left\{ g_0^{(N)} G_1^{(N)}\left(f_0^{(N)}\right) + g_1^{(N)} H_1^{(N)}\left(f_0^{(N)}\right) \right\}, \end{aligned} \tag{2}$$

where $G_i^{(N)}, H_i^{(N)}, i = 0,1$ are functions on $[0,1]$.

Every configuration ξ_t^N can be rewritten in terms of its corresponding measure. Indeed, introduce the following notation.

Notation 1.1. For $f, g : N^{-1}\mathbb{Z} \to \mathbb{R}$, we set $\langle f, g \rangle = \frac{1}{N}\sum_x f(x)g(x)$. Let ν be a measure on $N^{-1}\mathbb{Z}$. Then we set $\langle \nu, f \rangle = \int f \, d\nu$.

Now we can rewrite every configuration ξ_t^N in terms of its corresponding measure as follows. Let

$$\nu_t^N \equiv \frac{1}{N} \sum_x \delta_x 1(\xi_t^N(x) = 1), \tag{3}$$

then $\langle \xi_t^N, f \rangle = \langle \nu_t^N, f \rangle$.

We next define approximate densities $A(\xi_t^N)$ for the configurations ξ_t^N via

$$A(\xi_t^N)(x) = \frac{1}{2c(N)N^{1/2}} \sum_{y \sim x} \xi_t^N(y), \qquad x \in N^{-1}\mathbb{Z} \tag{4}$$

and note that $A(\xi_t^N)(x) = f_1^{(N)}\left(x, \xi_t^N\right)$. By linearly interpolating between sites we obtain approximate densities $A(\xi_t^N)(x)$ for all $x \in \mathbb{R}$.

Notation 1.2. Set $\mathscr{C}_1 \equiv \{f : \mathbb{R} \to [0,1] \text{ continuous}\}$ and let $\mathscr{C}_1$ be equipped with the topology of uniform convergence on compact sets.

We obtain that $t \mapsto A(\xi_t^N)$ is cadlag $\mathscr{C}_1$-valued, where we used that

$$0 \le A(\xi_t^N)(x) \le 1 \text{ for all } x \in N^{-1}\mathbb{Z}.$$

Definition 1.3. Let S be a Polish space and let $D(S)$ denote the space of cadlag paths from $\mathbb{R}_+$ to S with the Skorokhod topology. Following Definition VI.3.25 in Jacod and Shiryaev [8], we shall say that a collection of processes with paths in $D(S)$ is C-tight if and only if it is tight in $D(S)$ and all weak limit points are a.s. continuous. Recall that for Polish spaces, tightness and weak relative compactness are equivalent.

In what follows we shall investigate tightness of $\{A(\xi_\cdot^N) : N \ge 1\}$ in $D(\mathscr{C}_1)$ and tightness of $\{\nu_t^N : N \ge 1\}$ in $D(\mathscr{M}(\mathbb{R}))$, where $\mathscr{M}(\mathbb{R})$ is the space of Radon measures equipped with the vague topology ($\mathscr{M}(\mathbb{R})$ is indeed Polish, see Kallenberg [9], Theorem A2.3(i)). We next impose certain assumptions on the functions $G_i^{(N)}$ and $H_i^{(N)}$ in (2). These assumptions are rather technical and only become clear later in the proper context.

Definition 1.4. Let $\vec{\mathscr{P}}_0$ be the class of sequences $(f^{(N)} : N \in \mathbb{N})$ of real-valued functions on $[0,1]$, that can be expressed as power series $f^{(N)}(x) \equiv \sum_{m=0}^{\infty} \gamma^{(m+1,N)} x^m$, $x \in [0,1]$ with $\gamma^{(m+1,N)} \in \mathbb{R}$, $m \ge 0$ such that there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{N \ge N_0} \sum_{m=0}^{\infty} \left\{ \left(\gamma^{(m+1,N)}\right)^+ + (m+1)\left(\gamma^{(m+1,N)}\right)^- \right\} < \infty, \tag{5}$$

where $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$ for $a \in \mathbb{R}$.

Theorem 1.5. *Suppose that $A(\xi_0^N) \to u_0$ in $\mathscr{C}_1$. Let the transition rates of $\xi^N(x)$ be as in (2) with $(G_i^{(N)} : N \in \mathbb{N}), (H_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_0, i = 0,1$. Then $\left(A(\xi_t^N) : t \ge 0\right)$ are C-tight as cadlag $\mathscr{C}_1$-valued processes and the $\left(\nu_t^N : t \ge 0\right)$ are C-tight as cadlag Radon measure valued processes with the vague topology. If $\left(A(\xi_t^{N_k}), \nu_t^{N_k}\right)_{t \ge 0}$ converges to $(u_t, \nu_t)_{t \ge 0}$, then $\nu_t(dx) = u_t(x)dx$ for all $t \ge 0$.*

From now on we consider the special case of no fixed kernel interaction (also to be called no short-range competition in what follows, for reasons that become clear later) and investigate the limits of our tight sequences. Recall that $g_0^{(N)} + g_1^{(N)} = 1$. Hence, the special case can be obtained by choosing $G_i^{(N)} \equiv H_i^{(N)}, i = 0,1$ in (2).

Definition 1.6. Let $\vec{\mathscr{P}}_1 \subset \vec{\mathscr{P}}_0$ be the class of sequences $(f^{(N)} : N \in \mathbb{N})$ of real-valued functions on $[0,1]$ such that $f^{(N)}(x) = \sum_{m=0}^{\infty} \gamma^{(m+1,N)} x^m$ with

$$\gamma^{(m+1,N)} \overset{N\to\infty}{\to} \gamma^{(m+1)} \text{ for all } m \geq 0 \tag{6}$$

and

$$\lim_{N\to\infty} \sum_{m=0}^{\infty} \left\{ \left(\gamma^{(m+1,N)}\right)^+ + (m+1)\left(\gamma^{(m+1,N)}\right)^- \right\} = \sum_{m=0}^{\infty} \left\{ \left(\gamma^{(m+1)}\right)^+ + (m+1)\left(\gamma^{(m+1)}\right)^- \right\}. \tag{7}$$

Remark 1.7. *For $(f^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_1$ let $f(x) \equiv \lim_{N\to\infty} f^{(N)}(x)$. Then we have*

$$f(x) = \sum_{m=0}^{\infty} \gamma^{(m+1)} x^m, x \in [0,1].$$

Indeed, this holds by (7) and Royden [16], Proposition 11.18.

Theorem 1.8. *Consider the special case with no short-range competition. Under the assumptions of Theorem 1.5 we have for $(G_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_1, i = 0,1$ that the limit points of $A(\xi_t^N)$ are continuous $\mathscr{C}_1$-valued processes u_t which solve*

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + (1-u)u\left\{G_0(u) - G_1(1-u)\right\} + \sqrt{2u(1-u)}\dot{W} \tag{8}$$

with initial condition u_0. If we assume additionally $\langle u_0, 1\rangle < \infty$, then u_t is the unique in law $[0,1]$-valued solution to the above SPDE.

1.1 Literature review

In [13], Mueller and Tribe show that the approximate densities of type 1 of rescaled biased voter processes converge to continuous space time densities which solve the heat equation with drift, driven by Fisher-Wright noise. This model and result are covered by our following example.

Example 1.9. *Choose $G_0^{(N)}(x) = H_0^{(N)}(x) \equiv \theta \in \mathbb{R}$ and $G_1^{(N)}(x) = H_1^{(N)}(x) \equiv 0$ in (2). Using that $g_0^{(N)} + g_1^{(N)} = 1$ by definition, we obtain a sequence of rescaled biased voter models with rates of change*

$$\begin{aligned} 0 \to 1 \text{ at rate } c(x,\xi) &= N\left(1 + \frac{\theta}{N}\right) f_1^{(N)}(x,\xi), \\ 1 \to 0 \text{ at rate } c(x,\xi) &= N f_0^{(N)}(x,\xi). \end{aligned}$$

For $\theta > 0$ we thus have a slight favour for type 1 and for $\theta < 0$ we have a slight favour for type 0. Theorem 1.5 and Theorem 1.8 yield that the sequence of approximate densities $A(\xi_t^N)$ is tight and every limit point solves (8) with $G_0(u) = \theta$, $G_1(1-u) = 0$ and initial condition u_0. Uniqueness in law holds for initial conditions of finite mass.

For $\theta \geq 0$, the above result, except for the part on weak uniqueness, coincides (up to scaling) with Theorem 2 of [13].

Example 1.10. *For $i = 0, 1$ choose*

$$a_{i(1-i)}^{(N)} \equiv 1 + \frac{\theta_i^{(N)}}{N} \text{ with } \theta_i^{(N)} \overset{N\to\infty}{\to} \theta_i. \tag{9}$$

Let $G_i^{(N)}(x) = H_i^{(N)}(x) \equiv \theta_{i(1-i)}^{(N)} x$ and observe that $G_i^{(N)}(x) = H_i^{(N)}(x) \overset{N\to\infty}{\to} \theta_i x \equiv G_i(x) = H_i(x)$. We obtain a sequence of rescaled Lotka-Volterra models with rates of change

$$\begin{aligned} 0 \to 1 \text{ at rate } & N f_1^{(N)} + \theta_0^{(N)} \left(f_1^{(N)}\right)^2 \\ &= N f_1^{(N)} + N(a_{01}^{(N)} - 1)\left(f_1^{(N)}\right)^2 = N f_1^{(N)} \left(f_0^{(N)} + a_{01}^{(N)} f_1^{(N)}\right), \\ 1 \to 0 \text{ at rate } & N f_0^{(N)} + \theta_1^{(N)} \left(f_0^{(N)}\right)^2 = N f_0^{(N)} \left(f_1^{(N)} + a_{10}^{(N)} f_0^{(N)}\right) \end{aligned} \tag{10}$$

instead, where we used that $f_0^{(N)} + f_1^{(N)} = 1$ by definition. The interpretation of $a_{01}^{(N)}$ and $a_{10}^{(N)}$ will become clear once we introduce spatial versions of the Lotka-Volterra model with finite range in (12) (choose $\lambda = 1$). Observe in particular that if we choose $a_{01}^{(N)}, a_{10}^{(N)}$ close to 1 as above, the Lotka-Volterra model can be seen as a small perturbation of the voter model.

Theorem 1.5 and Theorem 1.8 yield that the sequence of approximate densities $A(\xi_t^N)$ is tight and every limit point solves

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + (1-u)u\left\{\theta_0 u - \theta_1(1-u)\right\} + \sqrt{2u(1-u)}\dot{W} \tag{11}$$

with initial condition u_0. Uniqueness in law holds if $< u_0, 1 >< \infty$.

If we choose $\theta_0 = -\theta_1 > 0$ in (11) we obtain the Kolmogorov-Petrovskii-Piscuinov (KPP) equation driven by Fisher-Wright noise. This SPDE has already been investigated in Mueller and Sowers [12] in detail, where the existence of travelling waves was shown for θ_0 big enough.

In Cox and Perkins [5] it was shown that stochastic spatial Lotka-Volterra models as in (10), satisfying (9) and suitably rescaled in space and time, converge weakly to super-Brownian motion with linear drift. [5] extended the main results of Cox, Durrett and Perkins [4], which proved similar results for long-range voter models. Both papers treat the low density regime, i.e. where only a finite number of individuals of type 1 is present. Instead of investigating limits for approximate densities as we do, both papers define measure-valued processes X_t^N by

$$X_t^N = \frac{1}{N'} \sum_{x \in \mathbb{Z}/(M_N\sqrt{N})} \xi_t^N(x)\delta_x,$$

i.e. they assign mass $1/N'$, $N' = N'(N)$ to each individual of type 1 and consider weak limits in the space of finite Borel measures on $\mathbb{R}$. In particular, they establish the tightness of the sequence of measures and the uniqueness of the martingale problem, solved by any limit point.

Note that both papers use a different scaling in comparison to [13]. Using the notation in [4], for $d = 1$ they take $N' = N$ and the space is scaled by $M_N\sqrt{N}$ with $M_N/\sqrt{N} \to \infty$ (see for instance Theorem 1.1 of [4] for $d = 1$) in the long-range setup. According to this notation, [13] used $M_N = \sqrt{N}$, which is at the threshold of the results in [4], but not included. By letting $M_N = \sqrt{N}$ in our setup non-linear terms arise in our limiting SPDE. Also note the brief discussion of the case where $M_N/\sqrt{N} \to 0$ in $d = 1$ before (H3) in [4].

Next recall spatial versions of the Lotka-Volterra model with finite range as introduced in Neuhauser and Pacala [14] (they considered $\xi(x) \in \{1,2\}$ instead of $\{0,1\}$). They use transition rates

$$0 \to 1 \text{ at rate } c(x,\xi) = \frac{\lambda f_1(x,\xi)}{\lambda f_1(x,\xi) + f_0(x,\xi)} \left(f_0(x,\xi) + \alpha_{01} f_1(x,\xi)\right), \tag{12}$$

$$1 \to 0 \text{ at rate } c(x,\xi) = \frac{f_0(x,\xi)}{\lambda f_1(x,\xi) + f_0(x,\xi)} \left(f_1(x,\xi) + \alpha_{10} f_0(x,\xi)\right),$$

where $\alpha_{01}, \alpha_{10} \geq 0, \lambda > 0$. Here $f_i(x,\xi) = \frac{1}{|\mathcal{N}|} \sum_{y \in x + \mathcal{N}} 1(\xi(y) = i)$, $i = 0,1$ with the set of neighbours of 0 being $\mathcal{N} = \{y : 0 < |y| \leq R\}$ with $R \geq 1$.

We can think of R as the finite interaction range of the model. [14] use this model to obtain results on the parameter regions for coexistence, founder control and spatial segregation of types 0 and 1 in the context of a model that incorporates short-range interactions and dispersal. As a conclusion they obtain that the short-range interactions alter the predictions of the mean-field model.

Following [14] we can interpret the rates as follows. The second multiplicative factor of the rate governs the density-dependent mortality of a particle, the first factor represents the strength of the instantaneous replacement by a particle of opposite type. The mortality of type 0 consists of two parts, f_0 describes the effect of intraspecific competition, $\alpha_{01} f_1$ the effect of interspecific competition. [14] assume that the intraspecific competition is the same for both species. The replacement of a particle of opposite type is regulated by the fecundity parameter λ. The first factors of both rates of change added together yield 1. Thus they can be seen as weighted densities of the two species. If $\lambda > 1$, species 1 has a higher fecundity than species 0.

Example 1.11. *Choose the competition and fecundity parameters near one and consider the long-range case. Namely, the model at hand exhibits the following transition rates:*

$$0 \to 1 \textit{ at rate } N \left[\frac{\lambda^{(N)} f_1^{(N)}}{\lambda^{(N)} f_1^{(N)} + f_0^{(N)}} \left(f_0^{(N)} + a_{01}^{(N)} f_1^{(N)} \right) \right],$$

$$1 \to 0 \textit{ at rate } N \left[\frac{f_0^{(N)}}{\lambda^{(N)} f_1^{(N)} + f_0^{(N)}} \left(f_1^{(N)} + a_{10}^{(N)} f_0^{(N)} \right) \right].$$

We suppose that

$$\lambda^{(N)} \equiv 1 + \frac{\lambda'}{N},\ a_{01}^{(N)} \equiv 1 + \frac{a_{01}}{N},\ a_{10}^{(N)} \equiv 1 + \frac{a_{10}}{N}.$$

Using $f_0^{(N)} + f_1^{(N)} = 1$ *we can therefore rewrite the rates as*

$$0 \to 1 \textit{ at rate } \left(N + \lambda'\right) f_1^{(N)} \left(1 + \frac{a_{01}}{N} f_1^{(N)}\right) \sum_{n \geq 0} \left(-\frac{\lambda'}{N} f_1^{(N)}\right)^n, \tag{13}$$

$$\begin{aligned} 1 \to 0 \textit{ at rate } & N f_0^{(N)} \left(1 + \frac{a_{10}}{N} f_0^{(N)}\right) \sum_{n \geq 0} \left(-\frac{\lambda'}{N} f_1^{(N)}\right)^n \\ &= N f_0^{(N)} \left(1 + \frac{a_{10}}{N} f_0^{(N)}\right) \sum_{k \geq 0} \left(f_0^{(N)}\right)^k \left(\frac{\lambda'}{N}\right)^k \sum_{n \geq k} \binom{n}{k} \left(-\frac{\lambda'}{N}\right)^{n-k}. \end{aligned}$$

Here we used that $\left|f_i^{(N)}\right| \leq 1, i = 0,1$ *and that* $\left|\frac{\lambda'}{N}\right| \to 0$ *for* $N \to \infty$. *We can use the explicit calculations for a geometric series, in particular that we have* $\sum_{n\geq 0} n|q|^n < \infty$ *and* $\sum_{n\geq k} |q|^{n-k}\binom{n}{k} = \frac{1}{(1-|q|)^{k+1}}$ *for* $|q| < 1$ *to check that* $(G_i^{(N)} : N \in \mathbb{N}), (H_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_1, i = 0,1$. *Using Theorem 1.8 we obtain that the limit points of* $A(\xi_t^N)$ *are continuous* $\mathscr{C}_1$*-valued processes* u_t *which solve*

$$\begin{aligned}\frac{\partial u}{\partial t} =& \frac{\Delta u}{6} + (1-u)u\left\{\left(\lambda' + u\left(a_{01} - \lambda'\right)\right) - \left(-\lambda' + (1-u)\left(a_{10} + \lambda'\right)\right)\right\} + \sqrt{2u(1-u)}\dot{W} \\ =& \frac{\Delta u}{6} + (1-u)u\left\{\lambda' - a_{10} + u\left(a_{01} + a_{10}\right)\right\} + \sqrt{2u(1-u)}\dot{W}\end{aligned} \tag{14}$$

by rewriting the above rates (13) in the form (2) and taking the limit for $N \to \infty$. *For* $\langle u_0, 1\rangle < \infty$, u_t *is the unique weak* $[0,1]$*-valued solution to the above SPDE.*

Additionally, [4] and [5] consider fixed kernel models in dimensions $d \geq 2$ respectively $d \geq 3$. They set $g_i^{(N)}(x,\xi) = \sum_{y\in\mathbb{Z}^d/(M_N\sqrt{N})} p(x-y)1(\xi^N(y) = i)$, $i = 0,1$ (compare this to (1)) and choose $M_N = 1$ and a fixed random walk kernel q satisfying some additional conditions such that $p(x) = q(\sqrt{N}x)$ on $x \in \mathbb{Z}^d/(M_N\sqrt{N})$. In Cox and Perkins [6], the results of [5] for $d \geq 3$ are used to relate the limiting super-Brownian motions to questions of coexistence and survival of a rare type in the original Lotka-Volterra model.

Example 1.12. *Consider rescaled Lotka-Volterra models with long-range dispersal and short-range competition, i.e. where (10) gets generalized to*

$$\begin{aligned}&0 \to 1 \text{ at rate } N f_1^{(N)}\left(g_0^{(N)} + a_{01}^{(N)} g_1^{(N)}\right), \\ &1 \to 0 \text{ at rate } N f_0^{(N)}\left(g_1^{(N)} + a_{10}^{(N)} g_0^{(N)}\right).\end{aligned}$$

Here $f_i^{(N)}, i = 0,1$ *is the density corresponding to a long-range kernel* p^L *and* $g_i^{(N)}, i = 0,1$ *is the density corresponding to a fixed kernel* p^F *(also recall the interpretation of both multiplicative factors following equation (12)).*

We obtain for $a_{i(1-i)}^{(N)}, i = 0,1$ *as in (9) that* $H_0^{(N)}(x) \equiv \theta_0^{(N)}$, $G_1^{(N)}(x) \equiv \theta_1^{(N)}$ *and* $G_0^{(N)}(x) = H_1^{(N)}(x) \equiv 0$ *in (2). Under the assumption that the initial approximate densities* $A(\xi_0^N)$ *converge in* $\mathscr{C}_1$, *Theorem 1.5 yields tightness of the sequence of approximate densities* $A(\xi_t^N)$.

1.2 Discussion of results and future challenges

In the present paper we first prove tightness of the local densities for scaling limits of more general particle systems. The generalization includes two features.

Firstly, we extend the model in [13] to limits of small perturbations of the long-range voter model, including negative perturbations (recall Example 1.9 and that [13] assumed $\theta \geq 0$) and the setup from [14] (cf. Example 1.11). As the rates in [14] (see (12)) include taking ratios, we extend our perturbations to a set of power series (for extensions to polynomials of degree 2 recall (10)), thereby including certain analytic functions. Recall in particular from (9) that we shall allow the coefficients of the power series to depend on N.

Secondly, we combine both long-range interaction and fixed kernel interaction for the perturbations. As we see, the tightness results carry over (cf. Example 1.12).

Finally, in the case of long-range interactions only we show that the limit points are solutions of a SPDE similar to [13] but with a drift depending on the choice of our perturbation and small changes in constants due to simple differences in scale factors. Hence, we obtain a class of SPDEs that can be characterized as the limit of perturbations of the long-range voter model.

Example 1.13. *Let $G_i(x) = \sum_{m=0}^{\infty} \gamma_i^{(m+1)} x^m, i = 0, 1$ be two arbitrary power series with coefficients satisfying*

$$\sum_{i=0,1} \sum_{m=0}^{\infty} \left\{ \left(\gamma_i^{(m+1)}\right)^+ + (m+1)\left(\gamma_i^{(m+1)}\right)^- \right\} < \infty.$$

Set $G_i^{(N)}(x) \equiv G_i(x)$ for all $N \in \mathbb{N}$. Then $(G_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_1$ and Theorem 1.8 yields that a solution to (8) with $u_0 \in \mathscr{C}_1$ can be obtained as the limit point of a sequence of approximate densities $A(\xi_t^N)$, where $(\xi_t^N : N \in \mathbb{N})$ is a sequence of rescaled competing species models with rates of change

$$0 \to 1 \text{ at rate } N f_1^{(N)} + f_1^{(N)} G_0\left(f_1^{(N)}\right),$$
$$1 \to 0 \text{ at rate } N f_0^{(N)} + f_0^{(N)} G_1\left(f_0^{(N)}\right)$$

(recall (2) with $G_i^{(N)} \equiv H_i^{(N)}$) and initial configurations satisfying $A(\xi_0^N) \to u_0$ in $\mathscr{C}_1$. This includes in particular the case where $G_i(x), i = 0, 1$ are polynomials.

If the limiting initial condition u_0 satisfies $\int u_0(x)dx < \infty$, we can show the weak uniqueness of solutions to the limiting SPDE and therefore show weak convergence of the rescaled particle densities to this unique law.

To include more general perturbation functions we resolved to rewrite the transition rates (2) so that they involve non-negative contributions only (cf. (18)). A representation for the evolution in time of $\xi_t^N(x)$ is given in (23) and used to obtain a Green's function representation (37) (compare this to (2.9) in [13]). The right choice of Poisson processes in the graphical construction (cf. (22)) turned out to be crucial. For an example of how the proper choice of rates in (22) yields terms involving approximate densities $A(\xi_t^N)$ in the approximate semimartingale decomposition, see the third equality in (34). Also see (36) for an example of how error bounds can be obtained and why the additional fixed kernel interaction does not impact the result.

While dealing with non-constant functions $G_i^{(N)}, H_i^{(N)}, i = 0, 1$, certain cancelation tricks from [13] were not available anymore. Instead, techniques of [13] had to be modified and refined. See for instance the calculations (51) to (52), where only the leading term of the perturbation part of the transition rates in (18) effects the error bounds. Here we can also see best why the additional perturbations do not change the tightness result.

Finally, as a further extension to [13], we include results on weak uniqueness.

It would be of interest to see if the results can be extended to more general functions $G_i^{(N)}, H_i^{(N)}, i = 0, 1$ in (2). The techniques utilized in the present paper require the functions to be power series with coefficients satisfying (5). In short, we model each non-negative contribution to the rewritten transition rates (18) of the approximating processes ξ^N via independent families of i.i.d. Poisson processes (for more details see Subsection 3.2) and as a result obtain a graphical construction of ξ^N in (23). The non-negativity assumption makes it necessary to rewrite negative contributions in terms of positive contributions, which results in assumption (5).

When there exists a fixed kernel, the question of uniqueness of all limit points and of identifying the limit remains an open problem. Also, when we consider long-range interactions only with $\int u_0(x)dx = \infty$ the proof of weak uniqueness of solutions to the limiting SPDE remains open.

In Example 1.11 we apply our results to characterize the limits of spatial versions of the Lotka-Volterra model with competition and fecundity parameter near one (see (9)) in the case of long-range interactions only. We obtain a class of parameter-dependent SPDEs in the limit (see (14)). This opens up the possibility to interpret the limiting SPDEs and their behaviour via their approximating long-range particle systems and vice versa. For instance, a future challenge would be to use properties of the SPDE to obtain results on the approximating particle systems, following the ideas of [5] and [6].

A major question is how the change in the parameter-dependent drift, in particular, possible additional zeros, impacts the long-time behaviour of the solutions and if there exist phase transitions. The author conjectures that there are parameter regions that yield survival and others that yield extinction. Aronson and Weinberger [1] showed that the corresponding class of deterministic PDEs exhibits a diverse limiting behaviour.

1.3 Outline of the rest of the paper

In Section 2 we prove Theorem 1.5 and Theorem 1.8. The first part of the proof consists in rewriting the transition rates (2) of the rescaled models ξ_t^N. We then present results comparable to the results of Theorems 1.5 and 1.8 for a class of sequences of rescaled models with transition rates that include the transition rates of the rewritten system. The advantage of the new over the old model is, that the new model can be approached by the methods used in [13]. The proof of the results for the new model is given in Section 3.

2 Proof of Theorem 1.5 and Theorem 1.8

We prove both theorems together. Let $(G_i^{(N)} : N \in \mathbb{N}), (H_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_0, i = 0, 1$. Then we can write

$$G_i^{(N)}(x) \equiv \sum_{m=0}^{\infty} \alpha_i^{(m+1,N)} x^m \text{ and } H_i^{(N)}(x) \equiv \sum_{m=0}^{\infty} \beta_i^{(m+1,N)} x^m, \quad x \in [0,1] \tag{15}$$

with $i = 0, 1$ and $\alpha_j^{(m,N)}, \beta_j^{(m,N)} \in \mathbb{R}$ for all $j = 0, 1, m \in \mathbb{N}$, satisfying (5). We can now rewrite the rates of change (2) of $\xi^N(x)$ as

$$\begin{aligned} 0 \to 1 \text{ at rate } & N f_1^{(N)} + g_0^{(N)} \sum_{m=1}^{\infty} \alpha_0^{(m,N)} \left(f_1^{(N)}\right)^m + g_1^{(N)} \sum_{m=1}^{\infty} \beta_0^{(m,N)} \left(f_1^{(N)}\right)^m, \\ 1 \to 0 \text{ at rate } & N f_0^{(N)} + g_0^{(N)} \sum_{m=1}^{\infty} \alpha_1^{(m,N)} \left(f_0^{(N)}\right)^m + g_1^{(N)} \sum_{m=1}^{\infty} \beta_1^{(m,N)} \left(f_0^{(N)}\right)^m. \end{aligned} \tag{16}$$

Remark 2.1. *The above rates of change determine indeed a unique, $\{0,1\}^{\mathbb{Z}/N}$-valued Markov process ξ_t^N for $N \geq N_0$ with N_0 as in Definition 1.4. For a proof, see Appendix A.*

Following [13], we would like to model each term in (16) via independent families of i.i.d. Poisson processes. This technique is only applicable to non-negative contributions. As we allow the $\alpha_i^{(m,N)}, \beta_i^{(m,N)}$ to be negative, too, the first part of the proof consists in rewriting (16) with the help of $f_0^{(N)} + f_1^{(N)} = 1$ and $g_0^{(N)} + g_1^{(N)} = 1$ in a form, where all resulting coefficients are non-negative.

Lemma 2.2. *We can rewrite our transitions as follows.*

$$
\begin{aligned}
&0 \to 1 \text{ at rate} \qquad (17)\\
&\left(N - \theta^{(N)}\right) f_1^{(N)} + f_1^{(N)} \left\{ \sum_{i=0,1} a_i^{(N)} g_i^{(N)} + \sum_{m\geq 2, i,j=0,1} q_{ij}^{(0,m,N)} g_i^{(N)} f_j^{(N)} \left(f_1^{(N)}\right)^{m-2} \right\},\\
&1 \to 0 \text{ at rate}\\
&\left(N - \theta^{(N)}\right) f_0^{(N)} + f_0^{(N)} \left\{ \sum_{i=0,1} b_i^{(N)} g_i^{(N)} + \sum_{m\geq 2, i,j=0,1} q_{ij}^{(1,m,N)} g_i^{(N)} f_j^{(N)} \left(f_0^{(N)}\right)^{m-2} \right\},
\end{aligned}
$$

with corresponding $\theta^{(N)}, a_i^{(N)}, b_i^{(N)}, q_{ij}^{(k,m,N)} \in \mathbb{R}^+, i, j, k = 0, 1, m \geq 2.$

Proof. We shall drop the superscripts of $f_i^{(N)}, g_i^{(N)}, i = 0, 1$ in what follows to simplify notation. Suppose for instance $\alpha_0^{(m,N)} < 0$ for some $m \geq 1$ in (16). Using that

$$-x^m = (1-x)\sum_{l=1}^{m-1} x^l - x$$

and recalling that $1 - f_1 = f_0$ we obtain

$$g_0 \alpha_0^{(m,N)} f_1^m = g_0 \left\{ \left(-\alpha_0^{(m,N)}\right) f_0 \sum_{l=1}^{m-1} f_1^l + \alpha_0^{(m,N)} f_1 \right\}.$$

Finally, we can use $g_0 = 1 - g_1$ to obtain

$$g_0 \alpha_0^{(m,N)} f_1^m = g_0 \left(-\alpha_0^{(m,N)}\right) f_0 \sum_{l=1}^{m-1} f_1^l + g_1 \left(-\alpha_0^{(m,N)}\right) f_1 + \alpha_0^{(m,N)} f_1.$$

All terms on the r.h.s. but the last can be accommodated into an existing representation (17) as follows:

$$
\begin{aligned}
q_{00}^{(0,n,N)} &\to q_{00}^{(0,n,N)} + \left(-\alpha_0^{(m,N)}\right) \text{ for } 2 \leq n \leq m,\\
a_1^{(N)} &\to a_1^{(N)} + \left(-\alpha_0^{(m,N)}\right).
\end{aligned}
$$

Finally, we can assimilate the last term into the first part of the rate $0 \to 1$, i.e. we replace

$$\theta^{(N)} \to \theta^{(N)} - \alpha_0^{(m,N)}.$$

As we use the representation (17), a change in $\theta^{(N)}$ also impacts the rate $1 \to 0$ in its first term. Therefore we have to fix the rate $1 \to 0$ by adding a term of $(-\alpha_0^{(m,N)})f_0 = g_0 f_0(-\alpha_0^{(m,N)}) + g_1 f_0(-\alpha_0^{(m,N)})$ to the second and third term of the rate, i.e. by replacing

$$b_0^{(N)} \to b_0^{(N)} + \left(-\alpha_0^{(m,N)}\right), \qquad b_1^{(N)} \to b_1^{(N)} + \left(-\alpha_0^{(m,N)}\right).$$

The general case with multiple negative $\alpha's$ and/or $\beta's$ follows inductively. □

Remark 2.3. *The above construction yields the following non-negative coefficients:*

$$
\begin{aligned}
q_{00}^{(0,m,N)} &\equiv \sum_{n=m}^{\infty} \left(\alpha_0^{(n,N)}\right)^-, \qquad q_{10}^{(0,m,N)} \equiv \sum_{n=m}^{\infty} \left(\beta_0^{(n,N)}\right)^-, \\
q_{01}^{(0,m,N)} &\equiv \left(\alpha_0^{(m,N)}\right)^+, \qquad q_{11}^{(0,m,N)} \equiv \left(\beta_0^{(m,N)}\right)^+, \\
\theta^{(N)} &\equiv \sum_{j=0,1} \sum_{n=1}^{\infty} \left(\alpha_j^{(n,N)}\right)^- + \left(\beta_j^{(n,N)}\right)^-, \\
a_0^{(N)} &\equiv \left(\alpha_0^{(1,N)}\right)^+ + \sum_{n=1}^{\infty} \left(\beta_0^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\alpha_1^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\beta_1^{(n,N)}\right)^-, \\
a_1^{(N)} &\equiv \left(\beta_0^{(1,N)}\right)^+ + \sum_{n=1}^{\infty} \left(\alpha_0^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\alpha_1^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\beta_1^{(n,N)}\right)^-,
\end{aligned}
$$

$$
\begin{aligned}
q_{00}^{(1,m,N)} &\equiv \left(\alpha_1^{(m,N)}\right)^+, \qquad q_{10}^{(1,m,N)} \equiv \left(\beta_1^{(m,N)}\right)^+ \\
q_{01}^{(1,m,N)} &\equiv \sum_{n=m}^{\infty} \left(\alpha_1^{(n,N)}\right)^-, \qquad q_{11}^{(1,m,N)} \equiv \sum_{n=m}^{\infty} \left(\beta_1^{(n,N)}\right)^-, \\
b_0^{(N)} &\equiv \left(\alpha_1^{(1,N)}\right)^+ + \sum_{n=1}^{\infty} \left(\beta_1^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\alpha_0^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\beta_0^{(n,N)}\right)^-, \\
b_1^{(N)} &\equiv \left(\beta_1^{(1,N)}\right)^+ + \sum_{n=1}^{\infty} \left(\alpha_1^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\alpha_0^{(n,N)}\right)^- + \sum_{n=1}^{\infty} \left(\beta_0^{(n,N)}\right)^-.
\end{aligned}
$$

For $(G_i^{(N)} : N \in \mathbb{N}), (H_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_0, i = 0, 1$*, this implies in particular that there exists* $N_0 \in \mathbb{N}$ *such that*

$$\sup_{N \geq N_0} \sum_{i,j,k=0,1} \sum_{m \geq 2} q_{ij}^{(k,m,N)} < \infty.$$

Remark 2.4. *Observe that we can rewrite the transition rates in (17) such that* $a_i^{(N)} = b_i^{(N)} = 0$*,* $i = 0, 1$*, i.e.*

$$
\begin{aligned}
0 \to 1 \text{ at rate } & \left(N - \theta^{(N)}\right) f_1^{(N)} + f_1^{(N)} \sum_{m \geq 2, i,j=0,1} q_{ij}^{(0,m,N)} g_i^{(N)} f_j^{(N)} \left(f_1^{(N)}\right)^{m-2}, \qquad (18) \\
1 \to 0 \text{ at rate } & \left(N - \theta^{(N)}\right) f_0^{(N)} + f_0^{(N)} \sum_{m \geq 2, i,j=0,1} q_{ij}^{(1,m,N)} g_i^{(N)} f_j^{(N)} \left(f_0^{(N)}\right)^{m-2}.
\end{aligned}
$$

Indeed, using that $f_0^{(N)} + f_1^{(N)} = 1$, we can change for instance

$$a_0^{(N)} g_0^{(N)} + q_{00}^{(0,2,N)} g_0^{(N)} f_0^{(N)} \left(f_1^{(N)}\right)^0 + q_{01}^{(0,2,N)} g_0^{(N)} f_1^{(N)} \left(f_1^{(N)}\right)^0$$

with $a_0^{(N)}, q_{00}^{(0,2,N)}, q_{01}^{(0,2,N)}$ nonnegative into

$$\left(a_0^{(N)} + q_{00}^{(0,2,N)}\right) g_0^{(N)} f_0^{(N)} \left(f_1^{(N)}\right)^0 + \left(a_0^{(N)} + q_{01}^{(0,2,N)}\right) g_0^{(N)} f_1^{(N)} \left(f_1^{(N)}\right)^0,$$

where the new coefficients are nonnegative again.

In the second part of this proof we shall now present results for rescaled competing species models ξ^N with transition rates as in (18). Tightness results for such models then immediately yield tightness results for the former models with rates of change as in (16). A bit more work is needed to translate convergence results of the latter model to obtain convergence results for the former model. The relevant part of the proof is given at the end of this section.

Moving on to models with transition rates as in (18), we introduce hypotheses directly on the $q_{ij}^{(k,m,N)}$ as the primary variables. Observe in particular that they will be assumed to be non-negative.

Hypothesis 2.5. *Assume that there exist non-negative $q_{ij}^{(k,m,N)}$, $i, j, k = 0, 1$ and $m \geq 2$ such that*

$$\sup_{N \geq N_0} \sum_{i,j,k=0,1} \sum_{m \geq 2} q_{ij}^{(k,m,N)} < \infty$$

for some $N_0 \in \mathbb{N}$.

Remark 2.6. *Recall the end of Remark 2.3. We can use the above condition as in Remark 2.1 to show that the rewritten transition rates can be used to determine a $\{0,1\}^{\mathbb{Z}/N}$-valued Markov process ξ_t^N for $N \geq N_0$.*

Hypothesis 2.7. *In the special case with no short-range competition, i.e. where we consider*

$$q_{00}^{(k,m,N)} = q_{10}^{(k,m,N)} \text{ and } q_{01}^{(k,m,N)} = q_{11}^{(k,m,N)} \tag{19}$$

in (18), we assume additionally to Hypothesis 2.5 that

$$\theta^{(N)} \overset{N\to\infty}{\to} \theta,$$
$$q_{0j}^{(k,m,N)} \overset{N\to\infty}{\to} q_{0j}^{(k,m)} \text{ for all } j,k = 0,1 \text{ and } m \geq 2$$

and

$$\lim_{N\to\infty} \sum_{j,k=0,1} \sum_{m\geq 2} q_{0j}^{(k,m,N)} = \sum_{j,k=0,1} \sum_{m \geq 2} q_{0j}^{(k,m)}. \tag{20}$$

Remark 2.8. *In the special case with no short-range competition, observe that if we assume that the $q_{0j}^{(k,m,N)}, j,k = 0,1, m \geq 2$ were obtained from $\alpha_j^{(m,N)}, j = 0,1, m \geq 1$ as described earlier in Remark 2.3 and Remark 2.4, then $G_i^{(N)} = H_i^{(N)}, i = 0,1$ implies (19). The additional assumption $(G_i^{(N)} : N \in \mathbb{N}), (H_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_1, i = 0,1$ implies Hypothesis 2.7. Indeed, use for instance [16], Proposition 11.18 together with Remark 2.3.*

Notation 2.9. For $k = 0, 1$ and $a \in \mathbb{R}$ we let

$$F_k(a) = \begin{cases} 1-a, & k=0, \\ a, & k=1. \end{cases}$$

We give the proof of the following result for rescaled competing species models ξ^N with rates of change as in (18) in Section 3.

Theorem 2.10. *Suppose that $A(\xi_0^N) \to u_0$ in $\mathscr{C}_1$. Let the transition rates of $\xi^N(x)$ be as in (18) and $q_{ij}^{(k,m,N)}$ satisfying Hypothesis 2.5. Then the $\left(A(\xi_t^N) : t \geq 0\right)$ are C-tight as cadlag $\mathscr{C}_1$-valued processes and the $\left(\nu_t^N : t \geq 0\right)$ are C-tight as cadlag Radon measure valued processes with the vague topology. If $\left(A\left(\xi_t^{N_k}\right), \nu_t^{N_k}\right)_{t\geq 0}$ converges to $(u_t, \nu_t)_{t\geq 0}$, then $\nu_t(dx) = u_t(x)dx$ for all $t \geq 0$.*
For the special case with no short-range competition we further have that if Hypothesis 2.7 holds, then the limit points of $A(\xi_t^N)$ are continuous $\mathscr{C}_1$-valued processes u_t which solve

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{6} + \sum_{k=0,1} (1-2k) \sum_{m\geq 2, j=0,1} q_{0j}^{(k,m)} F_j(u) \left(F_{1-k}(u)\right)^{m-1} F_k(u) + \sqrt{2u(1-u)}\dot{W} \tag{21}$$

with initial condition u_0. If we assume additionally $\langle u_0, 1\rangle < \infty$, then u_t is the unique in law $[0,1]$-valued solution to the above SPDE.

The claim of Theorem 1.5 now follows from the first part of Theorem 2.10. Indeed, rewrite (16) in the form (18) and use Remark 2.3.

Assume additionally that there is no short-range competition and recall Remark 2.8. The second part of Theorem 2.10 yields that the limit points of $A(\xi_t^N)$, with ξ_t^N being the system with transition rates as in (16), are continuous $\mathscr{C}_1$-valued processes u_t which solve (21). To obtain the claim of Theorem 1.8, it remains to show that every solution to (21) can be rewritten as a solution to (8). This follows from Corollary 2.11 below. Uniqueness in law of the former solution then implies uniqueness in law of the latter.

Corollary 2.11. *Under the assumptions of Theorem 1.8, the SPDE (21) may be rewritten as*

$$u_t = \frac{\Delta u}{6} + (1-u)u \sum_{m=0}^{\infty} \alpha_0^{(m+1)} u^m - u(1-u) \sum_{m=0}^{\infty} \alpha_1^{(m+1)} (1-u)^m + \sqrt{2u(1-u)}\dot{W}.$$

Proof. First recall Remark 1.7. Next use the definition of $F_k(a)$ and collect terms appropriately. Then recall how we rewrote the transition rates in Lemma 2.2 and Remark 2.4 to obtain (18) from (16). Now, analogously, rewrite (8) as (21). □

3 Proof of Theorem 2.10

3.1 Overview of the proof

The proofs in Subsections 3.2-3.7 are generalizations of the proofs in [13]. In [13], limits are considered for both the long-range contact process and the long-range voter process. Full details are given for the contact process. For the voter process, once the approximate martingale problem is derived, almost all of the remaining steps are left to the reader. Many arguments of our proof are similar to [13] but as they did not provide details for the long-range voter model and as additions and adaptations are needed due to our broader setup, we shall not omit the details.

In Subsection 3.2 we shall introduce a graphical construction for each approximating model ξ^N. This allows us to write out the time-evolution of our models. By integrating it against a test function and summing over $x \in \mathbb{Z}/N$ we finally obtain an approximate martingale problem for the N^{th}-process in Subsection 3.4. We defined the approximate density $A(\xi_t^N)(x)$ as the average density of particles of type 1 on $\mathbb{Z}/N$ in an interval centered at x of length $2/\sqrt{N}$ (recall (4)). By choosing a specific test function, the properties of which are under investigation at the beginning of Subsection 3.5, an approximate Green's function representation for the approximate densities $A(\xi_t^N)(\cdot)$ is derived towards the end of Subsection 3.5 and bounds on error-terms appearing in it are given. Making use of the Green's function representation, tightness of $A(\xi_t^N)(\cdot)$ is proven in Subsection 3.6. Here the main part of the proof consists in finding estimates on p^{th}-moment differences. In Subsection 3.7 the tightness of the approximate densities is used to show tightness of the measure corresponding to the sequence of configurations ξ_t^N. Finally, in the special case with no short-range competition, every limit is shown to solve a certain SPDE.

In Subsection 3.8 we additionally prove that this SPDE has a unique weak solution if $\int u_0(x)dx < \infty$. In this case, weak uniqueness of the limits of the sequence of approximate densities follows.

3.2 Graphical construction

Recall that the rates of change of the approximating processes ξ_t^N that we consider in Theorem 2.10 are given in (18). Note in particular that by Hypothesis 2.5 the coefficients $q_{ij}^{(k,m,N)}$ are non-negative. We shall first derive a graphical construction and evolution in time of our approximating processes ξ_t^N. The graphical construction uses independent families of i.i.d. Poisson processes:

$$\begin{aligned} &\Big(P_t(x;y): x,y \in N^{-1}\mathbb{Z}\Big) \text{ i.i.d. Poisson processes of rate } \frac{N-\theta^{(N)}}{2c(N)N^{1/2}}, \qquad (22)\\ &\text{and for } m \geq 2, i,j,k = 0,1,\\ &\Big(Q_t^{m,i,j,k}(x;y_1,\ldots,y_m;z): x,y_1,\ldots,y_m,z \in N^{-1}\mathbb{Z}\Big)\\ &\qquad \text{i.i.d. Poisson processes of rate } \frac{q_{ij}^{(k,m,N)}}{(2c(N))^m N^{m/2}} p(N(x-z)). \end{aligned}$$

Note that we suppress the dependence on N in the family of Poisson processes $P_t(x;y)$ and $Q_t^{m,i,j,k}(x;y_1,\ldots,y_m;z)$.

At a jump of $P_t(x;y)$ the voter at x adopts the opinion of the voter at y provided that y is a neighbour of x with opposite opinion.

At a jump of $Q_t^{m,i,j,k}(x;y_1,\ldots,y_m;z)$ the voter at x adopts the opinion $1-k$ provided that $y_1,\ldots,y_m$ are neighbours of x, y_1 has opinion j, all of $y_2\ldots,y_m$ have opinion $1-k$ and z has opinion i.

This yields the following stochastic integral equation to describe the evolution in time of our approximating processes ξ_t^N:

$$\xi_t^N(x) = \xi_0^N(x) + \sum_{y\sim x}\int_0^t \left\{1\left(\xi_{s-}^N(x)=0\right)1\left(\xi_{s-}^N(y)=1\right) - 1\left(\xi_{s-}^N(x)=1\right)1\left(\xi_{s-}^N(y)=0\right)\right\} dP_s(x;y) \tag{23}$$

$$+\sum_{k=0,1}(1-2k)\sum_{m\geq 2,i,j=0,1}\sum_{y_1,\dots,y_m\sim x}\sum_{z}\int_0^t 1\left(\xi_{s-}^N(x)=k\right)1\left(\xi_{s-}^N(y_1)=j\right)$$
$$\times\prod_{l=2}^m 1\left(\xi_{s-}^N(y_l)=1-k\right)1\left(\xi_{s-}^N(z)=i\right)dQ_s^{m,i,j,k}(x;y_1,\dots,y_m;z)$$

for all $x\in N^{-1}\mathbb{Z}$.

An explanation of why (23) has a unique solution can be found at the beginning of Section 4.3 of the author's thesis, Kliem [10]. There it is further shown that the solution is the spin-flip system with rates $c(x,\xi^N)$ given by (18). In what follows we shall often drop the superscripts w.r.t. N to simplify notation.

3.3 Preliminary notation

In what follows we shall consider $e_\lambda(x)=\exp(\lambda|x|)$ for $\lambda\in\mathbb{R}$ and we let

$$\mathscr{C} = \{f:\mathbb{R}\to[0,\infty)\text{ continuous with }|f(x)e_\lambda(x)|\to 0\text{ as }|x|\to\infty\text{ for all }\lambda<0\}$$

be the set of non-negative continuous functions with slower than exponential growth. Define

$$\|f\|_\lambda = \sup_x |f(x)e_\lambda(x)|$$

and give $\mathscr{C}$ the topology generated by the norms $(\|\cdot\|_\lambda\colon \lambda<0)$.

Remark 3.1. *We work on the space $\mathscr{C}$ instead of $\mathscr{C}_1$ because in Subsection 3.5 we shall introduce functions $0\leq\psi_t^z(x)\leq CN^{1/2}$ and shall show in Lemma 3.9(b) that they converge in $\mathscr{C}$ to the Brownian transition density $p\left(\frac{t}{3},z-x\right)$. Finally, in Subsection 3.6 we shall derive estimates on p^{th}-moment differences of $\hat{A}(\xi_t)(z)\equiv A(\xi_t)(z)-\langle\xi_0,\psi_t^z\rangle$, where $A(\xi_0)\to u_0$ in $\mathscr{C}$ to finally establish the tightness claim for the sequence of approximate densities $A(\xi^N)(x)$.*

Notation 3.2. For $x\in N^{-1}\mathbb{Z}, f:N^{-1}\mathbb{Z}\to\mathbb{R}$ and $\delta>0$ we shall write

$$D(f,\delta)(x) = \sup\{|f(y)-f(x)|:|y-x|\leq\delta, y\in N^{-1}\mathbb{Z}\}, \tag{24}$$
$$\Delta(f)(x) = \frac{N-\theta^{(N)}}{2c(N)N^{1/2}}\sum_{y\sim x}(f(y)-f(x)),$$

where we suppress the dependence on N.

3.4 An approximate martingale problem

We now derive the approximate martingale problem. In short, the idea is to express the integral of ξ_t against time-dependent test functions as the sum of a martingale, average (drift) terms and fluctuation (error) terms.

Take a test function $\phi : [0,\infty) \times N^{-1}\mathbb{Z} \to \mathbb{R}$ with $t \mapsto \phi_t(x)$ continuously differentiable and satisfying

$$\int_0^T \langle |\phi_s| + \phi_s^2 + |\partial_s \phi_s|, 1\rangle ds < \infty \tag{25}$$

(this ensures that the following integration and summation are well-defined). We apply integration by parts to $\xi_t(x)\phi_t(x)$, sum over x and multiply by $\frac{1}{N}$, to obtain for $t \leq T$ (recall the definition of ν_t from (3) and that $\langle \xi_t, \phi\rangle = \langle \nu_t, \phi\rangle$)

$$\langle \nu_t, \phi_t\rangle = \langle \nu_0, \phi_0\rangle + \int_0^t \langle \nu_s, \partial_s \phi_s\rangle ds \tag{26}$$

$$+ \frac{1}{N}\sum_x \sum_{y\sim x} \int_0^t \xi_{s-}(y)\left(\phi_s(x) - \phi_s(y)\right) dP_s(x;y) \tag{27}$$

$$+ \frac{1}{N}\sum_x \sum_{y\sim x} \int_0^t \xi_{s-}(x)\phi_s(x)\left(dP_s(y;x) - dP_s(x;y)\right) \tag{28}$$

$$+ \sum_{k=0,1}(1-2k) \sum_{m\geq 2, i,j=0,1} \frac{1}{N}\sum_x \sum_{y_1,\ldots,y_m\sim x} \sum_z \int_0^t 1\left(\xi_{s-}(x) = k\right) 1\left(\xi_{s-}(y_1) = j\right)$$
$$\times \prod_{l=2}^m 1\left(\xi_{s-}(y_l) = 1-k\right) 1\left(\xi_{s-}(z) = i\right) \phi_s(x) dQ_s^{m,i,j,k}(x; y_1, \ldots, y_m; z). \tag{29}$$

The main ideas for analyzing terms (27) and (28) will become clear once we analyze term (29) in detail. The latter is the only term where calculations changed seriously compared to [13]. Hence, we shall only summarize the results for terms (27) and (28) in what follows.

We break term (27) into two parts, an average term and a fluctuation term and after proceeding as for term (3.1) in [13] we obtain

$$(27) = \int_0^t \langle \nu_{s-}, \Delta(\phi_s)\rangle ds + E_t^{(1)}(\phi),$$

where

$$E_t^{(1)}(\phi) \equiv \frac{1}{N}\sum_x \sum_{y\sim x} \int_0^t \xi_{s-}(y)\left(\phi_s(x) - \phi_s(y)\right)\left(dP_s(x;y) - d\langle P(x;y)\rangle_s\right).$$

We have suppressed the dependence on N in $E_t^{(1)}(\phi)$. $E_t^{(1)}(\phi)$ is a martingale (recall that if $N \sim$ Pois(λ), then $N_t - \lambda t$ is a martingale with quadratic variation $\langle N\rangle_t = \lambda t$) with predictable brackets process given by

$$d\left\langle E^{(1)}(\phi)\right\rangle_t \leq \left\| D\left(\phi_t, \frac{1}{\sqrt{N}}\right)\right\|_\lambda^2 \langle 1, e_{-2\lambda}\rangle dt. \tag{30}$$

Alternatively we also obtain the bound

$$d\left\langle E^{(1)}(\phi)\right\rangle_t \le 4 \parallel \phi_t \parallel_0 \langle \left|\phi_t\right|, 1\rangle dt \tag{31}$$

with $\parallel \phi_t \parallel_0 = \sup_x |\phi_t(x)|$.

The second term (28) is a martingale which we shall denote by $M_t^{(N)}(\phi)$ (in what follows we shall drop the superscripts w.r.t. N and write $M_t(\phi)$). It can be analyzed similarly as the martingale $Z_t(\phi)$ of (3.3) in [13]. We obtain in particular that

$$\langle M(\phi)\rangle_t = 2\frac{N-\theta^{(N)}}{N}\left\{\int_0^t \langle \xi_{s-}, \phi_s^2\rangle ds - \int_0^t \langle A\big(\xi_{s-}\phi_s\big), \xi_{s-}\phi_s\rangle ds\right\}. \tag{32}$$

Using that

$$\left|A\big(\xi_{s-}\phi_s\big)(x)\right| \equiv \left|\frac{1}{2c(N)N^{1/2}}\sum_{y\sim x}\xi_{s-}(y)\phi_s(y)\right| \le \sup_{y\sim x}|\phi_s(y)|$$

we can further dominate $\langle M(\phi)\rangle_t$ by

$$\langle M(\phi)\rangle_t \le C(\lambda)\int_0^t \left(\parallel \phi_s\parallel_\lambda^2 \langle 1, e_{-2\lambda}\rangle\right) \wedge \left(\parallel \phi_s \parallel_0 \langle \xi_{s-}, |\phi_s|\rangle\right) ds. \tag{33}$$

We break the third term (29) into two parts, an average term and a fluctuation term. Recall Notation 2.9 and observe that if we only consider $a \in \{0,1\}$ we have $F_k(a) = 1(a = k)$. We can now rewrite (29) to

$$\begin{aligned}
&\sum_{k=0,1}(1-2k)\sum_{m\ge 2, i,j=0,1}\frac{1}{N}\sum_x \sum_{y_1,\dots,y_m\sim x}\sum_z \int_0^t 1\big(\xi_{s-}(x)=k\big)\, 1\big(\xi_{s-}(y_1)=j\big) \qquad (34)\\
&\times \prod_{l=2}^m 1\big(\xi_{s-}(y_l)=1-k\big)\, 1\big(\xi_{s-}(z)=i\big)\, \phi_s(x)\frac{q_{ij}^{(k,m,N)}}{(2c(N))^m N^{m/2}} p(N(x-z))ds + E_t^{(3)}(\phi)\\
&= \sum_{k=0,1}(1-2k)\sum_{m\ge 2, i,j=0,1} q_{ij}^{(k,m,N)} \int_0^t \frac{1}{N}\sum_x \left(\frac{1}{2c(N)N^{1/2}}\sum_{y_1\sim x} 1\big(\xi_{s-}(y_1)=j\big)\right)\\
&\quad \times \prod_{l=2}^m \left(\frac{1}{2c(N)N^{1/2}}\sum_{y_l\sim x} 1\big(\xi_{s-}(y_l)=1-k\big)\right)\left(\sum_z p(N(x-z))1\big(\xi_{s-}(z)=i\big)\right)\\
&\quad \times 1\big(\xi_{s-}(x)=k\big)\, \phi_s(x)ds + E_t^{(3)}(\phi)\\
&= \sum_{k=0,1}(1-2k)\sum_{m\ge 2, i,j=0,1} q_{ij}^{(k,m,N)} \int_0^t \frac{1}{N}\sum_x F_j(A(\xi_{s-})(x))\\
&\quad \times \big(F_{1-k}(A(\xi_{s-})(x))\big)^{m-1} F_i((p^N * \xi_{s-})(x))1\big(\xi_{s-}(x)=k\big)\, \phi_s(x)ds + E_t^{(3)}(\phi)\\
&= \sum_{k=0,1}(1-2k)\sum_{m\ge 2, i,j=0,1} q_{ij}^{(k,m,N)} \int_0^t \langle \big(F_j \circ A(\xi_{s-})\big)\\
&\quad \times \big(F_{1-k}\circ A(\xi_{s-})\big)^{m-1}\left(F_i \circ (p^N * \xi_{s-})\right) 1\big(\xi_{s-}(\cdot)=k\big), \phi_s\rangle ds + E_t^{(3)}(\phi),
\end{aligned}$$

where for $x \in \mathbb{Z}/N$ we set

$$\left(p^N * f\right)(x) \equiv \sum_{z \in \mathbb{Z}/N} p(N(x-z)) f(z) \tag{35}$$

and

$$\begin{aligned} E_t^{(3)}(\phi) \equiv & \sum_{k=0,1} (1-2k) \sum_{m \geq 2, i,j=0,1} \frac{1}{N} \sum_x \sum_{y_1,\ldots,y_m \sim x} \sum_z \int_0^t 1\left(\xi_{s-}(x)=k\right) \\ & \times 1\left(\xi_{s-}(y_1)=j\right) \prod_{l=2}^m 1\left(\xi_{s-}(y_l)=1-k\right) 1\left(\xi_{s-}(z)=i\right) \phi_s(x) \\ & \times \left(dQ_s^{m,i,j,k}(x; y_1, \ldots, y_m; z) - \frac{q_{ij}^{(k,m,N)}}{(2c(N))^m N^{m/2}} p(N(x-z)) ds \right). \end{aligned}$$

We have suppressed the dependence on N in $E_t^{(3)}(\phi)$. Here, $E_t^{(3)}(\phi)$ is a martingale with predictable brackets process given by

$$\begin{aligned} \left\langle E^{(3)}(\phi) \right\rangle_t \leq & \sum_{m \geq 2, i,j,k=0,1} q_{ij}^{(k,m,N)} \frac{1}{N^2} \sum_x \prod_{l=0}^m \left(\sum_{y_l \sim x} \frac{1}{2c(N) N^{1/2}} \right) \left(\sum_z p(N(x-z)) \right) \int_0^t \phi_s^2(x) ds \\ \leq & \frac{1}{N} \sum_{m \geq 2, i,j,k=0,1} q_{ij}^{(k,m,N)} \int_0^t \|\phi_s\|_\lambda^2 \langle e_{-2\lambda}, 1 \rangle ds. \end{aligned} \tag{36}$$

Taking the above together we obtain the following approximate semimartingale decomposition from (26).

$$\begin{aligned} \langle \nu_t, \phi_t \rangle = & \langle \nu_0, \phi_0 \rangle + \int_0^t \langle \nu_s, \partial_s \phi_s \rangle ds + \int_0^t \langle \nu_{s-}, \Delta(\phi_s) \rangle ds + E_t^{(1)}(\phi) + M_t(\phi) + E_t^{(3)}(\phi) \\ & + \sum_{k=0,1} (1-2k) \sum_{m \geq 2, i,j=0,1} q_{ij}^{(k,m,N)} \int_0^t \langle \left(F_j \circ A(\xi_{s-}) \right) \\ & \times \left(F_{1-k} \circ A(\xi_{s-}) \right)^{m-1} \left(F_i \circ (p^N * \xi_{s-}) \right) 1\left(\xi_{s-}(\cdot)=k\right), \phi_s \rangle ds. \end{aligned} \tag{37}$$

Remark 3.3. *Note that this approximate semimartingale decomposition provides the link between our approximate densities and the limiting SPDE in (21) for the case with no short-range competition. Indeed, uniqueness of the limit u_t of $A(\xi_t^N)$ will be derived by proving that u_t solves the martingale problem associated with the SPDE (21).*

3.5 Green's function representation

Analogous to [13], define a test function

$$\psi_t^z(x) \geq 0 \text{ for } t \geq 0, x, z \in N^{-1}\mathbb{Z}$$

as the unique solution, satisfying (25) and such that

$$\frac{\partial}{\partial t}\psi_t^z = \Delta\psi_t^z, \qquad \psi_0^z(x) = \frac{N^{1/2}}{2c(N)}1(x \sim z) \tag{38}$$

with

$$\Delta\psi_t^z(x) = \frac{N-\theta^{(N)}}{2c(N)N^{1/2}}\sum_{y\sim x}(\psi_t^z(y) - \psi_t^z(x)) \tag{39}$$

as in (24). Note that ψ_0^z was chosen such that $\langle \nu_t, \psi_0^z\rangle = A(\xi_t)(z)$ and that we suppress the dependence on N.

Next observe that Δ is the generator of a random walk $X_t \in N^{-1}\mathbb{Z}$, jumping at rate $\frac{N-\theta^{(N)}}{2c(N)N^{1/2}}\left(2c(N)N^{1/2}\right) = N - \theta^{(N)} = (1+o(1))N$ with symmetric steps of variance $\frac{1}{N}\left(\frac{1}{3}+o(1)\right)$, where we used that $c(N) \stackrel{N\to\infty}{\to} 1$. Here $o(1)$ denotes some deterministic function that satisfies $o(1) \to 0$ for $N \to \infty$. Define

$$\bar{\psi}_t^z(x) = N\mathbb{P}(X_t = x | X_0 = z)$$

then

$$\langle \psi_0^z, \bar{\psi}_t^x\rangle = \frac{N^{1/2}}{2c(N)}\sum_{y\sim z}\mathbb{P}(X_t = y|X_0 = x) = \mathbb{E}_x\left[\psi_0^z(X_t)\right] = \psi_t^z(x). \tag{40}$$

As we shall see later in Lemma 3.9(b), when linearly interpolated, the functions $\psi_t^z(x)$ and $\bar{\psi}_t^z(x)$ converge to $p\left(\frac{t}{3}, z-x\right)$ (the proof follows), where

$$p(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} \text{ is the Brownian transition density.} \tag{41}$$

The next lemma gives some information on the test functions ψ and $\bar{\psi}$ from above. Later on, this will provide us with estimates necessary for establishing tightness.

Lemma 3.4. *There exists $N_0 < \infty$ such that for $N \geq N_0, T \geq 0, z \in N^{-1}\mathbb{Z}$, $\lambda \geq 0$,*

(a) $\langle \psi_t^z, 1\rangle = \langle \bar{\psi}_t^z, 1\rangle = 1$ *and* $\|\psi_t^z\|_0 \leq CN^{1/2}$ *for all* $t \geq 0$.

(b) $\langle e_\lambda, \psi_t^z + \bar{\psi}_t^z\rangle \leq C(\lambda, T)e_\lambda(z)$ *for all* $t \leq T$,

(c) $\|\psi_t^z\|_\lambda \leq C(\lambda, T)\left(N^{1/2} \wedge t^{-2/3}\right)e_\lambda(z)$ *for all* $t \leq T$,

(d) $\langle \left|\bar{\psi}_t^z - \bar{\psi}_s^z\right|, 1\rangle \leq 2N|t-s|$ *for all* $s, t \geq 0$.

If we further restrict ourselves to $N \geq N_0$, $N^{-3/4} \leq s < t \leq T, y, z \in N^{-1}\mathbb{Z}$, $|y-z| \leq 1$, then

(e) $\|\psi_t^z - \psi_t^y\|_\lambda \leq C(\lambda, T)e_\lambda(z)\left(|z-y|^{1/2}t^{-1} + N^{-1/2}t^{-3/2}\right)$,

(f) $\|\psi_t^z - \psi_s^z\|_\lambda \leq C(\lambda, T)e_\lambda(z)\left(|t-s|^{1/2}s^{-3/2} + N^{-1/2}s^{-3/2}\right)$,

(g) $\left|\left|D\left(\psi_t^z, N^{-1/2}\right)(\cdot)\right|\right|_\lambda \leq C(\lambda, T)e_\lambda(z)N^{-1/4}t^{-1}$.

Proof. First we shall derive an explicit description for the test functions ψ_t^z and $\bar{\psi}_t^z$. We proceed as at the beginning of Section 4 in [13] by using that Δ as in (39) is the generator of a random walk. Let $(Y_i)_{i=1,2,\dots}$ be i.i.d. and uniformly distributed on $(jN^{-1} : 0 < |j| \leq \sqrt{N})$. Set

$$\rho(t) = \mathbb{E}\left[e^{itY_1}\right] \text{ and } S_k = \sum_{i=1}^{k} Y_i. \tag{42}$$

Note that $E[Y_1^2] \equiv \frac{c_2(N)}{3N}$, where $c_2(N) \to 1$ for $N \to \infty$ ($c_2(N)$ corresponds to $c_3 = c_3(N)$ in [13]). Similarly, $E[Y_1^4] \equiv \frac{c_4(N)}{5N^2}$, where $c_4(N) \to 1$ for $N \to \infty$.
The relation between the test functions $\psi_t^z, \bar{\psi}_t^z$ and S_k is as follows.

$$\psi_t^z(x) = \mathbb{E}_x\left[\psi_0^z(X_t)\right] = \sum_{k=0}^{\infty} \frac{((N-\theta^{(N)})t)^k}{k!} e^{-((N-\theta^{(N)})t)} N\mathbb{P}(S_{k+1} = x - z), \tag{43}$$
$$\bar{\psi}_t^z(x) = N\mathbb{P}(X_t = x | X_0 = z) = \sum_{k=0}^{\infty} \frac{((N-\theta^{(N)})t)^k}{k!} e^{-((N-\theta^{(N)})t)} N\mathbb{P}(S_k = x - z).$$

Now we can start proving the above lemma.

(a) follows as in the proof of Lemma 3(a), [13], using that $\mathbb{P}(S_k = x) \leq CN^{-1/2}$ for all $x \in N^{-1}\mathbb{Z}, k \geq 1$.

(b) follows as in the proof of Lemma 3(b), [13], where we shall use the bound $\mathbb{E}[e^{\mu Y_1}] \leq \exp\left\{\mu^2/N\right\}$ for all $\mu \geq 0$ to obtain the claim. Indeed, as Y_1 is uniformly distributed on $(jN^{-1} : 0 < |j| \leq \sqrt{N})$, we have

$$\mathbb{E}[e^{\mu Y_1}] = \frac{1}{c(N)\sqrt{N}} \sum_{j=1}^{\lfloor\sqrt{N}\rfloor} \cosh(\mu j/N) \leq \frac{1}{c(N)\sqrt{N}} \sum_{j=1}^{\lfloor\sqrt{N}\rfloor} e^{\mu^2 j^2/N^2} \leq e^{\mu^2/N}.$$

(c) Following the proof of Lemma 3(c) in [13], one can show that for $k \in \mathbb{N}$ and $|x| \geq 1$, $\mathbb{P}(S_k = x) \leq \frac{1}{N}\mathbb{P}(S_k \geq |x| - 1)$, which we can use to obtain $\mathbb{P}(S_k = x) \leq \frac{1}{N} e^{-\mu(|x|-1)} \exp\left\{5k\mu^2 \frac{1}{N}\right\}$.
Substituting this bound into (43) gives for any $\mu \geq 0$

$$\psi_t^z(x) \leq C(\mu, T) \exp\{-\mu|x - z|\} \tag{44}$$

for all $t \leq T$ and $|x - z| \geq 1$.
From (43) we further have for N big enough (recall the notation $p(t, x)$ from (41))

$$\psi_t^z(x) \equiv \mathbb{E}\left[p\left(\frac{c_2(N)(\mathscr{P}_t + 1)}{3N}, x - z\right)\right] + E(N, t, x - z),$$

where $\mathscr{P}_t \sim \text{Pois}((N - \theta^{(N)})t)$. Using Corollary B.2 we get as in the proof of [13], Lemma 3(c),

$$|E(N, t, x)| \leq C\frac{1}{N}\left(1 + t^{-3/2}\right) \qquad \text{for } N^{-3/4} \leq t.$$

Here we used that for $\mathscr{P} \sim \text{Pois}(r), r > 0$ we have

$$\mathbb{E}[(\mathscr{P} + 1)^a] \leq C(a) r^a \text{ for all } a < 0.$$

(This is obviously true for $0 < r < 1$. For $r \geq 1$ fixed, prove the claim first for all $a \in \mathbb{Z}$. Then extend this result to general $a < 0$ by an application of Hölder's inequality.)

Using the trivial bound $p(t,x) \leq Ct^{-1/2}$ we get from the above

$$\psi_t^z(x) \leq C(T)t^{-2/3} \qquad \text{for } N^{-3/4} \leq t \leq T.$$

Finally, we obtain with (44) and part (a) that

$$\begin{aligned}\|\psi_t^z\|_\lambda \leq & \sup_{\{x:|x-z|\geq 1\}} \left\{C(\lambda,T)e^{-\lambda|x-z|}e^{\lambda|x|}\right\} \vee \sup_{\{x:|x-z|<1, N^{-3/4}\leq t\leq T\}} \left\{C(T)t^{-2/3}e^{\lambda|x|}\right\} \\ & \vee \sup_{\{x:|x-z|<1, 0\leq t\leq N^{-3/4}\}} \left\{CN^{1/2}e^{\lambda|x|}\right\} \\ \leq & C(\lambda,T)\left(N^{1/2}\wedge t^{-2/3}\right)e_\lambda(z) \qquad \text{for all } t\leq T.\end{aligned}$$

This proves part (c).

(d) follows along the lines of the proof of [13], Lemma 3(d).

(e) For the remaining parts (e)-(g) we fix $N^{-3/4} \leq s < t \leq T, y, z \in N^{-1}\mathbb{Z}$, $|y-z| \leq 1$. For part (e) we follow the reasoning of the proof of [13], Lemma 3(e). The only change occurs in the derivation of the last estimate. In summary, we find as in [13] that

$$\|\psi_t^z - \psi_t^y\|_0 \leq C(T)\left(|z-y|t^{-1} + N^{-1}t^{-3/2}\right). \tag{45}$$

Now recall (44) with $\mu = 2\lambda$ to get $\psi_t^z(x) + \psi_t^y(x) \leq C(\lambda,T)\exp\{-2\lambda|x-z|\}$ for $|x-z| \geq 1$, $|x-y| \geq 1$, $|y-z| \leq 1$ and thus in particular for $|x-z| \geq 2, |y-z| \leq 1$. This yields

$$\begin{aligned}\|\psi_t^z - \psi_t^y\|_\lambda \leq & \sup_{\{x:|x-z|<2\}} \|\psi_t^z - \psi_t^y\|_0\, e_\lambda(x) + \sup_{\{x:|x-z|\geq 2\}} \left\{C(\lambda,T)\,\|\psi_t^z - \psi_t^y\|_0^{1/2}\, e^{-\lambda|x-z|}e_\lambda(x)\right\} \\ \leq & C(\lambda,T)e_\lambda(z)\left(\|\psi_t^z - \psi_t^y\|_0 + \|\psi_t^z - \psi_t^y\|_0^{1/2}\right) \\ \leq & C(\lambda,T)e_\lambda(z)\left(|z-y|^{1/2}t^{-1} + N^{-1/2}t^{-3/2}\right).\end{aligned}$$

This proves (e).

(f) The proof of part (f) follows analogously to the proof of part (e), with changes as suggested in the proof of [13], Lemma 3(f).

(g) Finally, to prove part (g), use part (e), $\psi_t^z(y) = \psi_t^y(z)$ (see (43)) and the definition of

$$D\left(\psi_t^z, N^{-1/2}\right)(x) = \sup\left\{\left|\psi_t^z(y) - \psi_t^z(x)\right| : |x-y| \leq N^{-1/2}, y \in N^{-1}\mathbb{Z}\right\}$$

to get

$$\begin{aligned}\|D\left(\psi_t^z, N^{-1/2}\right)(\cdot)\|_\lambda \overset{(44)}{\leq} & C(\lambda) \sup_{\{x:|x-z|<2\}} \left\{\sup_{y:|x-y|\leq N^{-1/2}} \left\{\left|\psi_t^z(y) - \psi_t^z(x)\right|\right\} e^{\lambda|z|}\right\} \\ & + C(\lambda,T) \sup_{\{x:|x-z|\geq 2\}} \left\{\sup_{y:|x-y|\leq N^{-1/2}} \left\{\left|\psi_t^z(y) - \psi_t^z(x)\right|^{1/2}\right\} e^{-\lambda|x-z|}e^{\lambda|x|}\right\}.\end{aligned}$$

Next use that $\psi_t^a(b) = \psi_t^b(a)$ to get as a further upper bound

$$
\begin{aligned}
&C(\lambda) \sup_{\{x:|x-z|<2\}} \left\{ \sup_{y:|x-y|\leq N^{-1/2}} \left\{ \left| \psi_t^y(z) - \psi_t^x(z) \right| \right\} e^{\lambda|z|} \right\} \\
&\quad + C(\lambda, T) \sup_{\{x:|x-z|\geq 2\}} \left\{ \sup_{y:|x-y|\leq N^{-1/2}} \left\{ \left| \psi_t^y(z) - \psi_t^x(z) \right|^{1/2} \right\} e^{\lambda|z|} \right\} \\
&\overset{(45)}{\leq} C(\lambda, T) e_\lambda(z) N^{-1/4} t^{-1},
\end{aligned}
$$

where we used $N^{-3/4} < t \leq T$. This finishes the proof of (g) and it also finishes the proof of the lemma. □

The following corollary uses the results of Lemma 3.4 to obtain estimates that we shall need later.

Corollary 3.5. *There exists $N_0 < \infty$ such that for $N \geq N_0$, $0 \leq \delta \leq u \leq t \leq T$ and $y, z \in N^{-1}\mathbb{Z}$, $\lambda \geq 0$, we have*

(a) $\int_u^t \|\psi_{t-s}^z\|_\lambda \, ds \leq C(\lambda, T)(t-u)^{1/3} e_\lambda(z)$ *and* $\int_0^t \|\psi_{t-s}^z\|_\lambda^2 \, ds \leq C(\lambda, T) N^{1/4} e_{2\lambda}(z)$.

(b) For $|y - z| \leq 1$ and $\delta \leq t - N^{-3/4}$ we further have

$$
\sup_{0\leq s\leq\delta} \|\psi_{t-s}^z - \psi_{t-s}^y\|_\lambda \leq C(\lambda, T) e_\lambda(z) \left\{ |z-y|^{1/2}(t-\delta)^{-1} + N^{-1/2}(t-\delta)^{-3/2} \right\}.
$$

(c) We also have $\int_\delta^t \|\psi_{t-s}^z - \psi_{t-s}^y\|_\lambda \, ds \leq C(\lambda, T)\left(e_\lambda(z) + e_\lambda(y)\right)(t-\delta)^{1/3}$.

(d) For $N^{-3/4} \leq u - \delta$ we have

$$
\sup_{0\leq s\leq\delta} \|\psi_{t-s}^z - \psi_{u-s}^z\|_\lambda \leq C(\lambda, T) e_\lambda(z) \left\{ (t-u)^{1/2}(u-\delta)^{-3/2} + N^{-1/2}(u-\delta)^{-3/2} \right\}.
$$

(e) Finally, we have $\int_\delta^u \|\psi_{t-s}^z - \psi_{u-s}^z\|_\lambda \, ds \leq C(\lambda, T) e_\lambda(z)(u-\delta)^{1/3}$.

Proof. The proof is a combination of the results of Lemma 3.4.

(a) We have for $n = 1, 2$ and $0 \leq u \leq t$ by Lemma 3.4(c)

$$
\int_u^t \left\|\psi_{t-s}^z\right\|_\lambda^n ds \leq C(\lambda, T) \int_u^t N^{n/2} \wedge (t-s)^{-2n/3} ds \; e_{n\lambda}(z).
$$

For $n = 1$ further bound the integrand by $(t-s)^{-2/3}$, for $n = 2$ and $u = 0$ use the above integrand to obtain the claim.

(b) follows from Lemma 3.4(e).

(c) We further have by Lemma 3.4(c)

$$
\int_\delta^t \|\psi_{t-s}^z - \psi_{t-s}^y\|_\lambda \, ds \leq C(\lambda, T)\left(e_\lambda(z) + e_\lambda(y)\right) \int_\delta^t (t-s)^{-2/3} ds.
$$

(d) follows from Lemma 3.4(f).

(e) Using Lemma 3.4(c) once more, we get

$$\int_\delta^u \|\psi^z_{t-s} - \psi^z_{u-s}\|_\lambda \, ds \le C(\lambda, T) e_\lambda(z) \int_\delta^u (t-s)^{-2/3} + (u-s)^{-2/3} ds,$$

which concludes the proof after some basic calculations. □

We shall need the following technical lemma.

Lemma 3.6. *For $f : N^{-1}\mathbb{Z} \to [0,\infty)$ with $\langle f, 1\rangle < \infty, \lambda \in \mathbb{R}$ we have*

(a) $\langle \nu_s, \psi^z_{t-s}\rangle = \langle A(\xi_s), \bar{\psi}^z_{t-s}\rangle$,

(b) $\left|\langle \nu_t, f\rangle - \langle A(\xi_t), f\rangle\right| \le C(\lambda) \, \|D(f, N^{-1/2})\|_\lambda$.

Proof. *(a)* follows easily from

$$\begin{aligned}
\langle \nu_s, \psi^z_{t-s}\rangle = \langle \xi_s, \psi^z_{t-s}\rangle &= \frac{1}{N}\sum_x \xi_s(x)\psi^x_{t-s}(z) \overset{(40)}{=} \frac{1}{N}\sum_x \xi_s(x)\langle \psi^x_0, \bar{\psi}^z_{t-s}\rangle \\
&= \frac{1}{N}\sum_y \left\{\sum_x \frac{1}{2c(N)N^{1/2}} 1(y \sim x)\xi_s(x)\right\} \bar{\psi}^z_{t-s}(y) \\
&= \frac{1}{N}\sum_y A(\xi_s)(y)\bar{\psi}^z_{t-s}(y) = \langle A(\xi_s), \bar{\psi}^z_{t-s}\rangle.
\end{aligned}$$

Part (b) follows as in the proof of Lemma 5(b) in [13]. Observe in particular that $\langle \nu_t, e_{-\lambda}\rangle \le C(\lambda)$ as will be shown before and in (48) below.

Taken all together this finishes the proof. □

Next use the test function

$$\phi_s \equiv \psi^x_{t-s} \text{ for } s \le t$$

in the semimartingale decomposition (37) and observe that ϕ satisfies (25) and $\partial_s \phi_s = -\Delta(\phi_s)$ by (38). Here the initial condition is chosen so that $\langle \nu_t, \phi_t\rangle = \langle \nu_t, \psi^x_0\rangle = A(\xi_t)(x)$.

The test function chosen in [13] at the beginning of page 526, namely $\phi_s = e^{\theta_c(t-s)}\psi^x_{t-s}$ was chosen so that the drift term $\langle \nu_s, \theta_c \phi_s\rangle ds$ of the semimartingale decomposition (2.9) in [13] would cancel out with the drift term $\langle \nu_s, \partial_s \phi_s\rangle ds$. As we have multiple coefficients, this is not possible. Also, it turned out that the calculations become easier once we consider time differences in Subsection 3.6 to follow.

With the above choice we obtain, for a fixed value of t, an approximate Green's function representation for $A(\xi_t)$, namely

$$\begin{aligned}
A(\xi_t)(x) =& \langle \nu_0, \psi^x_t\rangle + E^{(1)}_t\left(\psi^x_{t-\cdot}\right) + M_t\left(\psi^x_{t-\cdot}\right) + E^{(3)}_t(\psi^x_{t-\cdot}) \\
&+ \sum_{k=0,1}(1-2k) \sum_{m\ge 2, i,j=0,1} q^{(k,m,N)}_{ij} \int_0^t \langle \left(F_j \circ A(\xi_{s-})\right) \\
&\times \left(F_{1-k} \circ A(\xi_{s-})\right)^{m-1} \left(F_i \circ (p^N * \xi_{s-})\right) 1\left(\xi_{s-}(\cdot) = k\right), \psi^x_{t-s}\rangle ds.
\end{aligned} \tag{46}$$

The following lemma is stated analogously to Lemma 4 of [13]. Parts (a) and (c) will follow easily in our setup and so the only significant statement will be part (b).

Lemma 3.7. *Suppose that the initial conditions satisfy $A(\xi_0) \to u_0$ in $\mathscr{C}$ as $N \to \infty$. Then for $T \geq 0, p \geq 2, \lambda > 0$,*

(a) $\mathbb{E}\left[\sup_{t \leq T} \langle \nu_t, e_{-\lambda} \rangle^p\right] \leq C(\lambda, p)$.

(b) We further have

$$\mathbb{E}\left[\left|E_t^{(1)}\left(\psi_{t-\cdot}^z\right)\right|^p \vee \left|E_t^{(3)}\left(\psi_{t-\cdot}^z\right)\right|^p\right] \leq C(\lambda, p, T)\left(1 + C_Q^{p/2}\right) N^{-p/16} e_{\lambda p}(z)$$

for all $t \leq T$ and N big enough, where we set

$$C_Q \equiv \sup_{N \geq N_0} \sum_{m \geq 2, i, j, k = 0,1} q_{ij}^{(k,m,N)}. \tag{47}$$

(c) Finally, $\| \mathbb{E}[A(\xi_t)] \|_{-\lambda p} \leq 1$ for all $t \leq T$.

Proof. First observe that we have $\xi_t \in \{0,1\}^{\mathbb{Z}/N}$ and $0 \leq A(\xi_t) \leq 1$. Therefore, parts (a) and (c) follow immediately. Indeed, for (a) observe that

$$0 \leq \langle \nu_t, e_{-\lambda} \rangle = \langle \xi_t, e^{-\lambda|\cdot|} \rangle \leq \frac{2}{N} \sum_{j=0}^{\infty} e^{-\lambda j/N} = \frac{2}{N} \frac{1}{1 - e^{-\lambda/N}} \overset{N \to \infty}{\to} \frac{2}{\lambda}.$$

Note in particular that we showed that

$$\langle e_{-\lambda}, 1 \rangle \leq \frac{C}{\lambda} \text{ for all } \lambda > 0, \ N = N(\lambda) \text{ big enough,} \tag{48}$$

which will prove useful later.

For (c) we further have

$$\| \mathbb{E}[A(\xi_t)] \|_{-\lambda p} = \sup_x \left| \mathbb{E}[A(\xi_t)(x)] \right| e^{-\lambda p|x|} \leq \sup_x e^{-\lambda p|x|} \leq 1.$$

It only remains to show that (b) holds.

(b) First observe that $C_Q < \infty$ by Hypothesis 2.5.

We shall apply a Burkholder-Davis-Gundy inequality in the form

$$\mathbb{E}\left[\sup_{s \leq t} |X_s|^p\right] \leq C(p) \mathbb{E}\left[\langle X \rangle_t^{p/2} + \sup_{s \leq t} |X_s - X_{s-}|^p\right] \tag{49}$$

for a cadlag martingale X with $X_0 = 0$ (this inequality may be derived from its discrete time version, see Burkholder [3], Theorem 21.1).

To get an upper bound on the second term of the r.h.s. of (49) for the martingales we consider, observe that the largest possible jumps of the martingales $E_t^{(1)}(\psi_{t-\cdot}^z)$ respectively $E_t^{(3)}(\psi_{t-\cdot}^z)$ are bounded a.s. by $CN^{-1/2}$. Indeed,

$$E_t^{(1)}(\psi_{t-\cdot}^z) = \frac{1}{N} \sum_x \sum_{y \sim x} \int_0^t \xi_{s-}(y) \left(\psi_{t-s}^z(x) - \psi_{t-s}^z(y)\right) \left(dP_s(x; y) - d\langle P(x; y) \rangle_s\right)$$

and thus, using Lemma 3.4(a), the maximal jump size is bounded by

$$\frac{1}{N} 2 \sup_{t \leq T} \| \psi_t^z \|_0 \leq \frac{C}{N^{1/2}} \tag{50}$$

(the maximal number of jumps at a fixed time is 1). The bound on the maximal jump size of $E_t^{(3)}(\psi_{t-\cdot}^z)$ follows analogously.

Now choose $t \leq T$. We shall start with $E_t^{(3)}(\psi_{t-\cdot}^z)$. By (49), (50) and (36) we have

$$\begin{aligned}
\mathbb{E}\left[\left| E_t^{(3)}(\psi_{t-\cdot}^z) \right|^p \right] &\leq C(p) \left(\sum_{m \geq 2, i,j,k=0,1} q_{ij}^{(k,m,N)} \frac{1}{N} \int_0^t \left\| \psi_{t-s}^z \right\|_\lambda^2 \langle e_{-2\lambda}, 1 \rangle ds \right)^{p/2} + C(p) N^{-p/2} \\
&\overset{(48)}{\leq} C(\lambda, p) C_Q^{p/2} N^{-p/2} \left\{ \left(\int_0^t \left\| \psi_{t-s}^z \right\|_\lambda^2 ds \right)^{p/2} + 1 \right\}.
\end{aligned}$$

By Corollary 3.5(a) this is bounded from above by

$$C(\lambda, p, T) C_Q^{p/2} N^{-p/2} \left\{ N^{p/8} e_{\lambda p}(z) + 1 \right\} = C(\lambda, p, T) C_Q^{p/2} N^{-3p/8} e_{\lambda p}(z).$$

It remains to investigate $E_t^{(1)}(\psi_{t-\cdot}^z)$. Here (49), (50), (30) and (31) yield

$$\begin{aligned}
\mathbb{E}\left[\left| E_t^{(1)}(\psi_{t-\cdot}^z) \right|^p \right] &\leq C(p) \left(\int_0^t \left[\| \psi_{t-s}^z \|_0 \langle \psi_{t-s}^z, 1 \rangle \right] \wedge \left[\left\| D\left(\psi_{t-s}^z, \frac{1}{\sqrt{N}} \right) \right\|_\lambda^2 \langle 1, e_{-2\lambda} \rangle \right] ds \right)^{p/2} \\
&\quad + C(p) N^{-p/2}.
\end{aligned}$$

This in turn is bounded from above by

$$C(p) \left(\int_0^t \left[C(T)(t-s)^{-2/3} \right] \wedge \left[\left\| D\left(\psi_{t-s}^z, \frac{1}{\sqrt{N}} \right) \right\|_\lambda^2 C(\lambda) \right] ds \right)^{p/2} + C(p) N^{-p/2},$$

where we used Lemma 3.4(a), (c) and (48). To apply Lemma 3.4(g) to the second part of the integrand, we need to ensure that $N^{-3/4} \leq t-s$. As $N^{-3/4} \leq N^{-3/8}$ we get as a further upper bound

$$\begin{aligned}
&C(p) \left(\int_0^{N^{-3/8}} C(T) s^{-2/3} ds + \int_{N^{-3/8} \wedge t}^t \left(C(\lambda, T) e_\lambda(z) N^{-1/4} s^{-1} \right)^2 C(\lambda) ds \right)^{p/2} + C(p) N^{-p/2} \\
&\leq C(\lambda, p, T) e_{\lambda p}(z) \left\{ \left(\left(N^{-3/8} \right)^{1/3} + N^{-1/2} \left(N^{-3/8} \right)^{-1} \right)^{p/2} + N^{-p/2} \right\} \\
&\leq C(\lambda, p, T) N^{-p/16} e_{\lambda p}(z).
\end{aligned}$$

This finishes the proof. □

3.6 Tightness

In what follows, we shall derive estimates on p^{th}-moment differences of

$$\hat{A}(\xi_t)(z) \equiv A(\xi_t)(z) - \langle \nu_0, \psi_t^z \rangle.$$

Recall the assumption $A(\xi_0) \to u_0$ in $\mathscr{C}$ from Theorem 2.10. Also note that Lemma 3.9(b) to come will yield that $\psi_t^z(x)$ converges to $p\left(\frac{t}{3}, z-x\right)$. The estimates of Lemma 3.8 and the convergence of ψ_t^z taken together will be sufficient to show C-tightness of the approximate densities $A(\xi_t)(z)$ at the end of this section.

Lemma 3.8. *For $0 \le s \le t \le T, y, z \in N^{-1}\mathbb{Z}, |t-s| \le 1, |y-z| \le 1, \lambda > 0$ and $p \ge 2$ we have*

$$\mathbb{E}\left[\left|\hat{A}(\xi_t)(z) - \hat{A}(\xi_s)(y)\right|^p\right] \le C(\lambda, p, T)\left(1 + C_Q^p\right) e_{\lambda p}(z) \left(|t-s|^{p/24} + |z-y|^{p/24} + N^{-p/24}\right).$$

Proof. Fix $s, t, T, y, z, \lambda, p$ as in the statement. We decompose the increment $\hat{A}(\xi_t)(z) - \hat{A}(\xi_s)(y)$ into a space increment $\hat{A}(\xi_t)(z) - \hat{A}(\xi_t)(y)$ and a time increment $\hat{A}(\xi_t)(y) - \hat{A}(\xi_s)(y)$.

We consider first the space differences. From the Green's function representation (46), the estimates obtained in Lemma 3.7(b) for the error terms $E^{(1)}$ and $E^{(3)}$ and the linearity of $M_t(\phi)$ and $E_t^{(1)}(\phi), E_t^{(3)}(\phi)$ in ϕ, we get

$$\begin{aligned} &\mathbb{E}\left[\left|\hat{A}(\xi_t)(z) - \hat{A}(\xi_t)(y)\right|^p\right] \\ &\le C(\lambda, p, T)\left(1 + C_Q^{p/2}\right) N^{-p/16} e_{\lambda p}(z) + \mathbb{E}\left[\left|M_t\left(\psi_{t-\cdot}^z - \psi_{t-\cdot}^y\right)\right|^p\right] \\ &\quad + \mathbb{E}\Big[\Big|\sum_{k=0,1}(1-2k) \sum_{m\ge 2, i,j=0,1} q_{ij}^{(k,m,N)} \int_0^t \langle \left(F_j \circ A(\xi_{s-})\right) \\ &\qquad \times \left(F_{1-k} \circ A(\xi_{s-})\right)^{m-1} \left(F_i \circ (p^N * \xi_{s-})\right) 1\left(\xi_{s-}(\cdot) = k\right), \left(\psi_{t-s}^z - \psi_{t-s}^y\right) \rangle ds\Big|^p\Big]. \end{aligned} \tag{51}$$

Recall Definition (35) and observe that $0 \le \left(p^N * \xi_{s-}\right)(x) \le 1$ follows from $\xi_{s-} \in \{0,1\}^{\mathbb{Z}/N}$. Use this and $0 \le A(\xi_{s-})(x) \le 1$ together with the definition of F_k from Notation 2.9 to get

$$\begin{aligned} &\mathbb{E}\left[\left|\hat{A}(\xi_t)(z) - \hat{A}(\xi_t)(y)\right|^p\right] \\ &\le C(\lambda, p, T)\left(1 + C_Q^{p/2}\right) N^{-p/16} e_{\lambda p}(z) + \mathbb{E}\left[\left|M_t\left(\psi_{t-\cdot}^z - \psi_{t-\cdot}^y\right)\right|^p\right] \\ &\quad + \mathbb{E}\Big[\Big(\sum_{m\ge 2, i,j,k=0,1} q_{ij}^{(k,m,N)} \int_0^t \langle \left(F_{1-k} \circ A(\xi_{s-})\right) 1\left(\xi_{s-}(\cdot) = k\right), \left|\psi_{t-s}^z - \psi_{t-s}^y\right| \rangle ds\Big)^p\Big] \\ &\le C(\lambda, p, T)\left(1 + C_Q^{p/2}\right) N^{-p/16} e_{\lambda p}(z) + \mathbb{E}\left[\left|M_t\left(\psi_{t-\cdot}^z - \psi_{t-\cdot}^y\right)\right|^p\right] \\ &\quad + C_Q^p \mathbb{E}\Big[\Big(\int_0^t \langle A(\xi_{s-}) + \xi_{s-}, \left|\psi_{t-s}^z - \psi_{t-s}^y\right| \rangle ds\Big)^p\Big]. \end{aligned} \tag{52}$$

Note that this is the main step to see why the fixed kernel interaction does not impact our results on tightness.

In what follows, we shall employ a similar strategy to the proof of Lemma 6 in [13] to obtain estimates on the above. We nevertheless give full calculations as we proceeded in a different logical order to highlight the ideas for obtaining bounds. Minor changes in the exponents of our bounds ensued, both due to the different logical order and the different setup.

Let us first derive a bound on $\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}-\psi^y_{t-\cdot}\right)\right|^p\right]$. Using the Burkholder-Davis-Gundy inequality (49) from above and observing that the jumps of the martingales $M_t(\psi^x_{t-\cdot})$ are bounded a.s. by $CN^{-1/2}$ we have for any $0 \le \delta \le t$

$$\begin{aligned}\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}-\psi^y_{t-\cdot}\right)\right|^p\right] &\overset{(33)}{\le} C(\lambda,p)\mathbb{E}\left[\left(\left(\int_0^\delta \|\psi^z_{t-s}-\psi^y_{t-s}\|^2_\lambda \langle 1, e_{-2\lambda}\rangle ds\right.\right.\right. \\ &\qquad \left.\left.+\int_\delta^t \|\psi^z_{t-s}-\psi^y_{t-s}\|_0 \langle \xi_{s-}, \left|\psi^z_{t-s}-\psi^y_{t-s}\right|\rangle ds\right)^{p/2}\right] + C(p)N^{-p/2} \\ &\overset{(48)}{\le} C(\lambda,p)\mathbb{E}\left[\left(T\sup_{0\le s\le\delta}\|\psi^z_{t-s}-\psi^y_{t-s}\|^2_\lambda \frac{1}{\lambda}\right.\right. \\ &\qquad \left.\left.+\int_\delta^t \|\psi^z_{t-s}-\psi^y_{t-s}\|_0 \langle \xi_{s-}, \left|\psi^z_{t-s}-\psi^y_{t-s}\right|\rangle ds\right)^{p/2}\right] + C(p)N^{-p/2}.\end{aligned} \tag{53}$$

Now observe that by Lemma 3.6(a) and Lemma 3.4(a),

$$\langle \xi_{s-}, \left|\psi^z_{t-s}-\psi^y_{t-s}\right|\rangle \le \langle A(\xi_{s-}), \bar\psi^z_{t-s}+\bar\psi^y_{t-s}\rangle \le \langle 1, \bar\psi^z_{t-s}+\bar\psi^y_{t-s}\rangle = 2. \tag{54}$$

We can therefore apply the estimates from Corollary 3.5(b) to the first term in (53) and Corollary 3.5(c) to the second term, assuming $\delta \le \left(t-N^{-3/4}\right)\vee 0$ and using $|y-z|\le 1$ to obtain

$$\begin{aligned}&\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}-\psi^y_{t-\cdot}\right)\right|^p\right] \\ &\le C(\lambda,p,T)e_{\lambda p}(z)\left\{|z-y|^{p/2}(t-\delta)^{-p}+N^{-p/2}(t-\delta)^{-3p/2}+(t-\delta)^{p/6}\right\}+C(p)N^{-p/2}.\end{aligned}$$

Now set

$$\delta = t-\left(\left(|z-y|^{1/4}\vee N^{-1/4}\right)\wedge t\right)$$

and observe that $\delta \le \left(t-N^{-3/4}\right)\vee 0$ follows. We obtain $t-\delta = \left(|z-y|^{1/4}\vee N^{-1/4}\right)\wedge t$ and

$$\begin{aligned}|z-y|^{1/4}\le N^{-1/4} \Rightarrow \quad & |z-y|^{p/2}(t-\delta)^{-p}+N^{-p/2}(t-\delta)^{-3p/2}+(t-\delta)^{p/6} \\ &\le |z-y|^{p/4}+N^{-p/8}+N^{-p/24}, \\ |z-y|^{1/4} > N^{-1/4} \Rightarrow \quad & |z-y|^{p/2}(t-\delta)^{-p}+N^{-p/2}(t-\delta)^{-3p/2}+(t-\delta)^{p/6} \\ &\le |z-y|^{p/4}+N^{-p/8}+|z-y|^{p/24}.\end{aligned} \tag{55}$$

Plugging this back in the above estimate we finally have

$$\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}-\psi^y_{t-\cdot}\right)\right|^p\right] \le C(\lambda,p,T)e_{\lambda p}(z)\left\{|z-y|^{p/24}+N^{-p/24}\right\}.$$

Next we shall get a bound on the last term of (52). Recall that $\langle \xi_t, \phi \rangle = \langle \nu_t, \phi \rangle$. We get

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^t \langle A(\xi_{s-}) + \xi_{s-}, \big|\psi^z_{t-s} - \psi^y_{t-s}\big|\rangle ds\Big)^p\Big] \\
&\leq C(p)\left\{\mathbb{E}\left[\left(\int_0^\delta \langle A(\xi_{s-}) + \nu_{s-}, e_{-\lambda}\rangle ds \sup_{0\leq s\leq \delta} \|\psi^z_{t-s} - \psi^y_{t-s}\|_\lambda\right)^p\right]\right. \\
&\qquad \left. + \mathbb{E}\left[\left(\int_\delta^t \langle A(\xi_{s-}) + \nu_{s-}, e_{-\lambda}\rangle \, \|\psi^z_{t-s} - \psi^y_{t-s}\|_\lambda \, ds\right)^p\right]\right\}.
\end{aligned}
$$

Now use that

$$
\langle A(\xi_{s-}) + \nu_{s-}, e_{-\lambda}\rangle = \langle A(\xi_{s-}) + \xi_{s-}, e_{-\lambda}\rangle \leq \langle 2, e_{-\lambda}\rangle \overset{(48)}{\leq} C(\lambda) \tag{56}
$$

to obtain that the above is bounded by

$$
\begin{aligned}
&C(p)\left\{\left(T C(\lambda) \sup_{0\leq s\leq \delta} \|\psi^z_{t-s} - \psi^y_{t-s}\|_\lambda\right)^p + \left(\int_\delta^t C(\lambda) \, \|\psi^z_{t-s} - \psi^y_{t-s}\|_\lambda \, ds\right)^p\right\} \\
&\leq C(\lambda, p, T) e_{\lambda p}(z) \left\{|z-y|^{p/2}(t-\delta)^{-p} + N^{-p/2}(t-\delta)^{-3p/2} + (t-\delta)^{p/3}\right\},
\end{aligned}
$$

where we used Corollary 3.5(b),(c) and $|y-z| \leq 1$. Here we assumed $\delta \leq \left(t - N^{-3/4}\right) \vee 0$ when we applied Corollary 3.5(b). Now choose $\delta = t - \left(\left(|z-y|^{1/4} \vee N^{-1/4}\right) \wedge t\right) \leq \left(t - N^{-3/4}\right) \vee 0$ as before. Reasoning as in (55), we get

$$
C(\lambda, p, T) e_{\lambda p}(z) \left(N^{-p/8} + |z-y|^{p/12}\right)
$$

as an upper bound.

Now we can take all the above bounds together and plug them back into (52) to obtain (recall that $|z-y| \leq 1$)

$$
\mathbb{E}\left[\left|\hat{A}(\xi_t)(z) - \hat{A}(\xi_t)(y)\right|^p\right] \leq C(\lambda, p, T)\left(1 + C_Q^{p/2} + C_Q^p\right) e_{\lambda p}(z) \left(|z-y|^{p/24} + N^{-p/24}\right).
$$

Next we derive a similar bound on the time differences. We start by subtracting the two Green's function representations again, this time for the time differences, using (46) and Lemma 3.7(b) for the error terms.

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\hat{A}(\xi_t)(z)-\hat{A}(\xi_u)(z)\Big|^p\Big] \qquad (57)\\
&\le C(\lambda,p,T)\left(1+C_Q^{p/2}\right)N^{-p/16}e_{\lambda p}(z)+\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}\right)-M_u\left(\psi^z_{u-\cdot}\right)\right|^p\right]\\
&\quad+\mathbb{E}\Big[\Big(\sum_{m\ge 2,i,j,k=0,1} q_{ij}^{(k,m,N)}\Big\{\int_u^t\langle\left(F_j\circ A(\xi_{s-})\right)\left(F_{1-k}\circ A(\xi_{s-})\right)^{m-1}\left(F_i\circ(p^N*\xi_{s-})\right)\\
&\qquad\times 1\left(\xi_{s-}(\cdot)=k\right),\psi^z_{t-s}\rangle ds\\
&\qquad+\int_0^u\langle\left(F_j\circ A(\xi_{s-})\right)\left(F_{1-k}\circ A(\xi_{s-})\right)^{m-1}\left(F_i\circ(p^N*\xi_{s-})\right)\\
&\qquad\times 1\left(\xi_{s-}(\cdot)=k\right),\left|\psi^z_{t-s}-\psi^z_{u-s}\right|\rangle ds\Big\}\Big)^p\Big]\\
&\le C(\lambda,p,T)\left(1+C_Q^{p/2}\right)N^{-p/16}e_{\lambda p}(z)+\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}\right)-M_u\left(\psi^z_{u-\cdot}\right)\right|^p\right]\\
&\quad+\mathbb{E}\Big[\Big(\sum_{m\ge 2,i,j,k=0,1} q_{ij}^{(k,m,N)}\Big\{\int_u^t\langle\left(F_{1-k}\circ A(\xi_{s-})\right)1\left(\xi_{s-}(\cdot)=k\right),\psi^z_{t-s}\rangle ds\\
&\qquad+\int_0^u\langle\left(F_{1-k}\circ A(\xi_{s-})\right)1\left(\xi_{s-}(\cdot)=k\right),\left|\psi^z_{t-s}-\psi^z_{u-s}\right|\rangle ds\Big\}\Big)^p\Big]\\
&\le C(\lambda,p,T)\left(1+C_Q^{p/2}\right)N^{-p/16}e_{\lambda p}(z)+\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}\right)-M_u\left(\psi^z_{u-\cdot}\right)\right|^p\right]\\
&\quad+C_Q^p\mathbb{E}\Big[\Big(\int_u^t\langle A(\xi_{s-})+\xi_{s-},\psi^z_{t-s}\rangle ds+\int_0^u\langle A(\xi_{s-})+\xi_{s-},\left|\psi^z_{t-s}-\psi^z_{u-s}\right|\rangle ds\Big)^p\Big].
\end{aligned}
$$

For the martingale term we now further get via the Burkholder-Davis-Gundy inequality (49)

$$
\begin{aligned}
&\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}\right)-M_u\left(\psi^z_{t-\cdot}\right)\right|^p\right]\\
&\le C(p)\left\{\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}\right)-M_u\left(\psi^z_{t-\cdot}\right)\right|^p\right]+\mathbb{E}\left[\left|M_u\left(\psi^z_{t-\cdot}\right)-M_u\left(\psi^z_{u-\cdot}\right)\right|^p\right]\right\}\\
&\le C(p)\mathbb{E}\left[\left|\langle M_\cdot\left(\psi^z_{t-\cdot}\right)\rangle_t-\langle M_\cdot\left(\psi^z_{t-\cdot}\right)\rangle_u\right|^{p/2}\right]+C(p)\mathbb{E}\left[\left|\langle M_\cdot\left(\psi^z_{t-\cdot}-\psi^z_{u-\cdot}\right)\rangle_u\right|^{p/2}\right]+C(p)N^{-p/2}\\
&\le C(\lambda,p)\mathbb{E}\left[\left(\int_u^t\|\psi^z_{t-s}\|_0\,\langle\xi_{s-},\psi^z_{t-s}\rangle ds\right)^{p/2}\right]+C(\lambda,p)\left(\int_0^{\delta\wedge u}\|\psi^z_{t-s}-\psi^z_{u-s}\|_\lambda^2\,\langle 1,e_{-2\lambda}\rangle ds\right)^{p/2}\\
&\quad+C(\lambda,p)\mathbb{E}\left[\left(\int_{\delta\wedge u}^u\|\psi^z_{t-s}-\psi^z_{u-s}\|_0\,\langle\xi_{s-},\left|\psi^z_{t-s}-\psi^z_{u-s}\right|\rangle ds\right)^{p/2}\right]+C(p)N^{-p/2},
\end{aligned}
$$

where we used equation (33) to bound the first and second term. Using (48) and reasoning as in (54) the above can further be bounded by

$$
\begin{aligned}
&C(\lambda,p)\left(\int_u^t\|\psi^z_{t-s}\|_0\,ds\right)^{p/2}+C(\lambda,p,T)\sup_{0\le s\le\delta\wedge u}\|\psi^z_{t-s}-\psi^z_{u-s}\|_\lambda^p\\
&\quad+C(\lambda,p)\left(\int_{\delta\wedge u}^u\|\psi^z_{t-s}-\psi^z_{u-s}\|_0\,ds\right)^{p/2}+C(p)N^{-p/2}.
\end{aligned}
$$

Under the assumption $N^{-3/4} \wedge u \leq u - (\delta \wedge u)$ we obtain from Corollary 3.5(a), (d), (e) that

$$\begin{aligned}
&\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}\right) - M_u\left(\psi^z_{u-\cdot}\right)\right|^p\right] \qquad (58)\\
&\leq C(\lambda,p,T)e_{\lambda p}(z)\left\{(t-u)^{p/6} + \left(|t-u|^{p/2} + N^{-p/2}\right)(u-(\delta\wedge u))^{-3p/2}\right.\\
&\qquad \left. + (u-(\delta\wedge u))^{p/6} + N^{-p/2}\right\}.
\end{aligned}$$

Finally observe that with

$$\delta = u - \left(\left(|t-u|^{1/4} \vee N^{-1/4}\right) \wedge u\right)$$

we get $N^{-3/4} \wedge u \leq u - \delta$ and by proceeding as in (55) we obtain

$$\mathbb{E}\left[\left|M_t\left(\psi^z_{t-\cdot}\right) - M_u\left(\psi^z_{u-\cdot}\right)\right|^p\right] \leq C(\lambda,p,T)e_{\lambda p}(z)\left\{(t-u)^{p/6} + |t-u|^{p/24} + N^{-p/24} + N^{-p/2}\right\}.$$

Finally, we can bound the last expectation of the last line of (57) by using

$$\langle A(\xi_{t-s}) + \xi_{s-}, \psi^z_{t-s}\rangle \ \leq\ \langle 1+1, \psi^z_{t-s}\rangle = 2.$$

Here the last equality followed from Lemma 3.4(a). We thus obtain as an upper bound on the last expectation of the last line of (57),

$$C(p)\left\{|t-u|^p + \mathbb{E}\left[\left(\int_0^u \langle A(\xi_{s-}) + \nu_{s-}, \left|\psi^z_{t-s} - \psi^z_{u-s}\right|\rangle ds\right)^p\right]\right\}.$$

We further have for the second term

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\int_0^u \langle A(\xi_{s-}) + \nu_{s-}, \left|\psi^z_{t-s} - \psi^z_{u-s}\right|\rangle ds\Big)^p\Big]\\
&\leq C(p)\left\{\mathbb{E}\left[\left(\int_0^{\delta\wedge u} \langle A(\xi_{s-}) + \nu_{s-}, e_{-\lambda}\rangle ds \sup_{0\leq s\leq\delta} \|\psi^z_{t-s} - \psi^z_{u-s}\|_\lambda\right)^p\right]\right.\\
&\qquad \left. + \mathbb{E}\left[\left(\int_{\delta\wedge u}^u \langle A(\xi_{s-}) + \nu_{s-}, e_{-\lambda}\rangle\, \|\psi^z_{t-s} - \psi^z_{u-s}\|_\lambda\, ds\right)^p\right]\right\}\\
&\overset{(56)}{\leq} C(\lambda,p,T)\left\{\left(\sup_{0\leq s\leq\delta\wedge u} \|\psi^z_{t-s} - \psi^z_{u-s}\|_\lambda\right)^p + \left(\int_{\delta\wedge u}^u \|\psi^z_{t-s} - \psi^z_{u-s}\|_\lambda\, ds\right)^p\right\}\\
&\leq C(\lambda,p,T)e_{\lambda p}(z)\left\{(t-u)^{p/2}(u-(\delta\wedge u))^{-3p/2} + N^{-p/2}(u-(\delta\wedge u))^{-3p/2} + (u-(\delta\wedge u))^{p/3}\right\},
\end{aligned}$$

where we assumed $N^{-3/4} \wedge u \leq u - (\delta \wedge u)$ when we applied Corollary 3.5(d) together with Corollary 3.5(e) in the last line. Now reason as from (58) on to obtain

$$C(\lambda,p,T)e_{\lambda p}(z)\left\{|t-u|^{p/24} + N^{-p/24}\right\}$$

as an upper bound.

Taking all bounds together we have for the time differences from (57)

$$\mathbb{E}\left[\left|\hat{A}(\xi_t)(z) - \hat{A}(\xi_u)(z)\right|^p\right] \leq C(\lambda,p,T)\left(1 + C_Q^{p/2} + C_Q^p\right) e_{\lambda p}(z)\left\{|t-u|^{p/24} + N^{-p/24}\right\}.$$

The bounds on the space difference and the time difference taken together complete the proof. □

We now show that these moment estimates imply C-tightness of the approximate densities. We shall start including dependence on N again to clarify the tightness argument. First define

$$\tilde{A}(\xi_t^N)(z) = \hat{A}(\xi_t^N)(z) \text{ on the grid } z \in N^{-1}\mathbb{Z}, t \in N^{-2}\mathbb{N}_0.$$

Linearly interpolate first in z and then in t to obtain a continuous $\mathscr{C}$-valued process. Note in particular that we can use Lemma 3.8 to show that for $0 \le s \le t \le T, |t-s| \le 1$ and $y, z \in \mathbb{R}, |y-z| \le 1$,

$$\mathbb{E}\left[\left|\tilde{A}(\xi_t^N)(z) - \tilde{A}(\xi_s^N)(y)\right|^p\right] \le C(\lambda, p, T)\left(1 + C_Q^p\right) e_{\lambda p}(z)\left(|t-s|^{p/48} + |z-y|^{p/24}\right)$$

for $\lambda > 0, p \ge 2$ arbitrarily fixed.

The next lemma shows that $\tilde{A}(\xi_t^N)$ and $\hat{A}(\xi_t^N)$ remain close. The advantage of using $\tilde{A}(\xi_t^N)$ is that it is continuous.

Using Kolmogorov's continuity theorem (see for instance Corollary 1.2 in Walsh [19]) on compacts $R_1^{(i_1,i_2)} \equiv \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} : (t,x) \in (i_1,i_2) + [0,1]^2\}$ for $i_1 \in \mathbb{N}_0, i_2 \in \mathbb{Z}$ we obtain tightness of $\tilde{A}(\xi_t^N)(x)$ in the space of continuous functions on $\left\{(t,x) : (t,x) \in R_1^{(i_1,i_2)}\right\}$. Indeed, we can use the Arzelà-Ascoli theorem. With arbitrarily high probability, part (ii) of Corollary 1.2 of [19] provides a uniform (in N) modulus of continuity for all $N \ge N_0$. Pointwise boundedness follows from the boundedness of $A(\xi_t^N)(x)$ together with Lemma 3.9(b) below. Now use a diagonalization argument to obtain tightness of $(\tilde{A}(\xi_t^N)(x) : t \in \mathbb{R}^+, x \in \mathbb{R})_{N\in\mathbb{N}}$ in the space of continuous functions from $\mathbb{R}^+ \times \mathbb{R}$ to $\mathbb{R}$ equipped with the topology of uniform convergence on compact sets. Next observe that if we consider instead the space of continuous functions from $\mathbb{R}^+$ to the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, both equipped with the topology of uniform convergence on compact sets, tightness of $(\tilde{A}(\xi_t^N)(x) : t \in \mathbb{R}^+, x \in \mathbb{R})_{N\in\mathbb{N}}$ in the former space is equivalent to tightness of $(\tilde{A}(\xi_t^N)(\cdot) : t \in \mathbb{R}^+)_{N\in\mathbb{N}}$ in the latter.

Finally, tightness of $(A(\xi_t^N) : t \in \mathbb{R}^+)_{N\in\mathbb{N}}$ as cadlag $\mathscr{C}_1$-valued processes (recall that $0 \le A(\xi_t^N)(x) \le 1$ by construction) and also the continuity of all weak limit points follow from the next lemma.

Lemma 3.9. *For any $\lambda > 0, T < \infty$ we have*

(a) $\mathbb{P}\left(\sup_{t\le T} \|\tilde{A}(\xi_t^N) - \hat{A}(\xi_t^N)\|_{-\lambda} \ge 7N^{-1/4}\right) \to 0$ *as* $N \to \infty$.

(b) $\sup_{t\le T} \|\langle \nu_0^N, \psi_t^{\cdot}\rangle - P_{t/3}u_0\|_{-\lambda} \to 0$ *as* $N \to \infty$.

Proof. The proof is very similar to the proof of Lemma 7 in [13]. We shall only give some additional steps for part (a) to complement the proof of the given reference.

(a) For $0 \le s \le t$ we have

$$\|\langle \nu_0^N, \psi_t^{\cdot}\rangle - \langle \nu_0^N, \psi_s^{\cdot}\rangle\|_{-\lambda} = \sup_z \left|\langle A(\xi_0^N), \bar{\psi}_t^z - \bar{\psi}_s^z\rangle\right| e^{-\lambda|z|} \le 2N|t-s|.$$

Here we used Lemma 3.6(a), $0 \le A(\xi_0^N) \le 1$ and Lemma 3.4(d). Hence, this only changes by

$O(N^{-1})$ between the (time-)grid points in $N^{-2}\mathbb{N}_0$. We obtain that

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\le T}\|\tilde{A}(\xi_t^N)-\hat{A}(\xi_t^N)\|_{-\lambda}\ge 7N^{-1/4}\right)\\
&\le \mathbb{P}\Big(\exists t\in[0,T]\cap N^{-2}\mathbb{N}_0, s\in[0,T], |s-t|\le N^{-2} \text{ such that}\\
&\qquad \|A(\xi_t^N)-A(\xi_s^N)\|_{-\lambda}+\Big\|\langle \nu_0^N,\psi_t^{\cdot}-\psi_s^{\cdot}\rangle\Big\|_{-\lambda}\ge 7N^{-1/4}\Big)\\
&\le \mathbb{P}\Big(\exists t\in[0,T]\cap N^{-2}\mathbb{N}_0, s\in[0,T], |s-t|\le N^{-2} \text{ such that } \|A(\xi_t^N)-A(\xi_s^N)\|_{-\lambda}\ge 6N^{-1/4}\Big)
\end{aligned}$$

for N big enough.

Next note that the value of $A(\xi_t^N)(x)$ changes only at jump times of $P_t(x;y)$ or $Q_t^{m,i,j,k}(x;y_1,\dots,y_m;z), i,j,k=0,1, m\ge 2$ for some $y\sim x$ respectively for some $y_1,\dots,y_m\sim x$ and arbitrary $z\in N^{-1}\mathbb{Z}$ and that each jump of $A(\xi_t^N)$ is by definition of $A(\xi_t^N)$ bounded by $N^{-1/2}$. Then, writing $\mathscr{P}(a)$ for a Poisson variable with mean a, we get as a further bound on the above

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}}\mathbb{P}\Big(\exists z\in N^{-1}\mathbb{Z}\cap(l,l+1], \exists t\in[0,T]\cap N^{-2}\mathbb{N}_0, \exists s\in[t,t+N^{-2}] \text{ with}\\
&\qquad\Big\{\Big|A(\xi_t^N)(z)-A(\xi_s^N)(z)\Big|\wedge\Big|A(\xi_{t+N^{-2}}^N(z)-A(\xi_s^N)(z)\Big|\Big\}\ge N^{-1/4}e^{\lambda(|l|-1)}\Big)\\
&\le \sum_{l\in\mathbb{Z}}N(N^2T)\mathbb{P}\left(CN^{-1/2}\left(\sum_{y\sim 0}P_{N^{-2}}(0;y)\right.\right.\\
&\qquad\left.\left.+\sum_{i,j,k=0,1,m\ge 2}\ \sum_{y_1,\dots,y_m\sim 0}\ \sum_u Q_{N^{-2}}^{m,i,j,k}(0;y_1,\dots,y_m;u)\right)\ge N^{-1/4}e^{\lambda(|l|-1)}\right)\\
&\le \sum_{l\in\mathbb{Z}}C(T)N^3\mathbb{P}\Big(CN^{-1/2}\mathscr{P}\Big(N^{-2}\Big(\Big(N-\theta^{(N)}\Big)+C_Q\Big)\Big)\ge N^{-1/4}e^{\lambda(|l|-1)}\Big)\\
&\le \sum_{l\in\mathbb{Z}}C(T)N^3\mathbb{P}\Big(\Big(\mathscr{P}\Big(N^{-2}\Big(N+C_Q\Big)\Big)\Big)^p\ge CN^{p/4}e^{\lambda p(|l|-1)}\Big)
\end{aligned}$$

for some $p>0$. Now apply Chebyshev's inequality. Choose $p>0$ such that $3-p/4<0$. Then the resulting sum is finite and goes to zero for $N\to\infty$.

(b) The proof of part (b) follows as the proof of Lemma 7(b) of [13]. □

3.7 Characterizing limit points

To conclude the proof of Theorem 2.10 we can proceed as in Section 4 in [13], except for the proof of weak uniqueness of (21). We shall give a short overview in what follows. The interested reader is referred to [13] for complete explanations.

In short, Lemma 3.6(b) implies for all $\phi\in\mathscr{C}_c$ that

$$\sup_t\Big|\langle \nu_t^N,\phi\rangle-\langle A(\xi_t^N),\phi\rangle\Big|\le C(\lambda)\,\|D(\phi,N^{-1/2})\|_\lambda\overset{N\to\infty}{\to}0. \tag{59}$$

The C-tightness of $(A(\xi_t^N):t\ge 0)$ in $\mathscr{C}_1$ follows from the results of Subsection 3.6.

This in turn implies the C-tightness of $(\nu_t^N : t \geq 0)$ as cadlag Radon measure valued processes with the vague topology. Indeed, let $\varphi_k, k \in \mathbb{N}$ be a sequence of smooth functions from $\mathbb{R}$ to $[0,1]$ such that $\varphi_k(x)$ is 1 for $|x| \leq k$ and 0 for $|x| \geq k+1$. Then a diagonalization argument shows that C-tightness of $(\nu_t^N : t \geq 0)$ as cadlag Radon measure valued processes with the vague topology holds if and only if C-tightness of $(\varphi_k d\nu_t^N : t \geq 0)$ as cadlag $\mathscr{M}_F([-(k+1), k+1])$-valued processes with the weak topology holds. Here, $\mathscr{M}_F([-(k+1), k+1])$ denotes the space of finite measures on $[-(k+1), k+1]$. Now use Theorem II.4.1 in Perkins [15] (also see the Remark following it and the proof of sufficiency on pages 157-159) to obtain C-tightness of $(\varphi_k d\nu_t^N : t \geq 0)$ in $D(\mathscr{M}_F([-(k+1), k+1]))$. The compact containment condition (i) in [15] is obvious. The second condition (ii) in [15] follows from (59) and the C-tightness of $(A(\xi_t^N) : t \geq 0)$ in $\mathscr{C}_1$ together with Lemma 3.7(a).

Observe in particular, that (59) implies the existence of a subsequence $\left(A(\xi_t^{N_k}), \nu_t^{N_k}\right)$ that converges to (u_t, ν_t). Hence, we can define variables with the same distributions on a different probability space such that with probability one, for all $T < \infty, \lambda > 0, \phi \in \mathscr{C}_c$,

$$\sup_{t \leq T} \left\| A(\xi_t^{N_k}) - u_t \right\|_{-\lambda} \to 0 \text{ as } k \to \infty,$$
$$\sup_{t \leq T} \left| \langle \phi, \nu_t^{N_k} \rangle - \langle \phi, \nu_t \rangle \right| \to 0 \text{ as } k \to \infty,$$

where we used $0 \leq A(\xi_t^{N_k}) \leq 1$ and thus $0 \leq u_t(x) \leq 1$ a.s. for the first limit. We obtain in particular

$$\nu_t(dx) = u_t(x)dx \text{ for all } t \geq 0.$$

It remains to investigate u_t in the special case with no short-range competition, i.e. where $q_{0j}^{(k,m,N)} = q_{1j}^{(k,m,N)}$, $j = 0, 1$. Take $\phi_t \equiv \phi \in \mathscr{C}_c^3$ in (37). We get

$$\begin{aligned} M_t^{(N)}(\phi) =& \langle \nu_t^N, \phi \rangle - \langle \nu_0^N, \phi \rangle - \int_0^t \langle \nu_{s-}^N, \Delta(\phi) \rangle ds - E_t^{(1)}(\phi) \\ & - \sum_{k=0,1} (1-2k) \sum_{m \geq 2, j=0,1} q_{0j}^{(k,m,N)} \int_0^t \langle \left(F_j \circ A\left(\xi_{s-}^N\right)\right) \\ & \times \left(F_{1-k} \circ A\left(\xi_{s-}^N\right)\right)^{m-1} 1\left(\xi_{s-}^N(\cdot) = k\right), \phi \rangle ds - E_t^{(3)}(\phi). \end{aligned} \tag{60}$$

From (30) and (36) and the Burkholder-Davis-Gundy inequality (49) we obtain that the error terms converge to zero for all $0 \leq t \leq T$ almost surely. Taylor's theorem further shows that (replace N_k by N for notational ease)

$$\Delta(\phi)(x_N) = \frac{N - \theta^{(N)}}{c(N)N} \frac{\sqrt{N}}{2} \sum_{y \sim x_N} (\phi(y) - \phi(x_N)) \to \frac{\Delta\phi}{6}(x)$$

as $x_N \to x$ and $N \to \infty$ on the support of ϕ. Using this in (60) we can show that $M_t^{(N)}(\phi)$ converges to a continuous martingale $M_t(\phi)$ satisfying

$$\begin{aligned} M_t(\phi) =& \int \phi(x) u_t(x) dx - \int \phi(x) u_0(x) dx - \int_0^t \int \frac{\Delta\phi(x)}{6} u_s(x) dx ds \\ & - \sum_{k=0,1} (1-2k) \sum_{m \geq 2, j=0,1} q_{0j}^{(k,m)} \int_0^t \int F_j(u_s(x)) \left(F_{1-k}(u_s(x))\right)^{m-1} F_k(u_s(x)) \phi(x) dx ds. \end{aligned} \tag{61}$$

To exchange the limit in $N \to \infty$ with the infinite sum we used [16], Proposition 11.18 together with Hypothesis 2.7. Recall in particular, that $0 \le F_l(u_s(x)) \le 1$ for $l = 0,1$. To show that $M_t(\phi)$ is indeed a martingale we used in particular (33) to see that $\langle M^{(N)}(\phi)\rangle_t \le C(\lambda)t \parallel \phi \parallel_\lambda^2 \langle 1, e_{-2\lambda}\rangle$ is uniformly bounded. Therefore, $(M_t^{(N)}(\phi) : N \ge N_0)$ and all its moments are uniformly integrable, using the Burkholder-Davis-Gundy inequality of the form (49) once more.

We can further calculate its quadratic variation by making use of (32) for $N \to \infty$ together with the uniform integrability of $((M_t^{(N)}(\phi))^2 : N \ge N_0)$.

Use our results for $\phi \in \mathscr{C}_c^3$, note that $\mathscr{C}_c^3$ is dense in $\mathscr{C}_c^2$ with respect to the norm $\parallel f \parallel \equiv \parallel f \parallel_\infty + \parallel f' \parallel_\infty + \parallel \Delta f \parallel_\infty$, and use (61) to see that u_t solves the martingale problem associated with the SPDE (21). It is now straightforward to show that, with respect to some white noise, u_t is actually a solution to (21) (see Rogers and Williams [17], V.20 for the similar argument in the case of SDEs).

3.8 Uniqueness in law

To show uniqueness of all limit points of Subsection 3.7 in the case with no short-range competition and with $\langle u_0, 1\rangle < \infty$, we need to show uniqueness in law of $[0,1]$-valued solutions to (21). Indeed, as $0 \le A(\xi_t^N)(x) \le 1$ by definition, any limit point has to satisfy $u_t(x) \in [0,1]$. Rewrite (21) as

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\Delta u}{6} + u(1-u)\sum_{k=0,1}(1-2k)\sum_{m\ge 2, j=0,1} q_{0j}^{(k,m)} F_j(u)\left(F_{1-k}(u)\right)^{m-2} + \sqrt{2u(1-u)}\dot{W} \qquad (62)\\ &\equiv \frac{\Delta u}{6} + u(1-u)Q(u) + \sqrt{2u(1-u)}\dot{W}.\end{aligned}$$

Observe that $|Q(u_s(x))| \le C_Q$ with C_Q as in (47) because $0 \le u_s(x) \le 1$.

To check uniqueness in law of $[0,1]$-valued solutions we apply a version of Dawson's Girsanov theorem in what follows, cf. Theorem IV.1.6 in [15], p. 252. The idea is to use change of measure techniques to deduce uniqueness in law of solutions u to (62) from the uniqueness in law of solutions v to a better understood SPDE.

Let $\mathbb{P}^u$ denote the law of a solution to the SPDE (62) and $\mathbb{P}^v$ denote the unique law of the $[0,1]$-valued solution to the SPDE

$$\frac{\partial v}{\partial t} = \frac{\Delta v}{6} + \sqrt{2v(1-v)}\dot{W} \qquad (63)$$

with $v_0 = u_0$. Reasons for existence and uniqueness of a $[0,1]$-valued solution to the latter can be found in Shiga [18], Example 5.2, p. 428. Note in particular that the solution v_t takes values in $\mathscr{C}_1$.

To prove weak uniqueness, we shall follow the reasoning of the proof of Theorem IV.1.6(a),(b) in [15] in a univariate setup. We obtain as a result the following lemma.

Lemma 3.10. *If $\langle u_0, 1\rangle < \infty$ the weak $[0,1]$-valued solution to (62) is unique in law. If we let*

$$\begin{aligned}R_t \equiv \exp\Bigg\{&\int_0^t \int \frac{Q(v_s(x))}{2}\sqrt{2v_s(x)(1-v_s(x))}dW(x,s)\\ &-\frac{1}{2}\int_0^t\int \frac{(1-v_s(x))\left(Q(v_s(x))\right)^2}{2} v_s(x)dxds\Bigg\},\end{aligned}$$

then

$$\left.\frac{d\mathbb{P}^u}{d\mathbb{P}^v}\right|_{\mathscr{F}_t} = R_t \text{ for all } t > 0, \tag{64}$$

where $\mathscr{F}_t$ is the canonical filtration of the process $v(t,x)$.

Proof. We proceed analogously to the proof of Theorem IV.1.6(a),(b) in [15]. Observe in particular that we take

$$T_n = \inf\left\{ t \geq 0 : \int_0^t \int \frac{(1-u_s(x))\left(Q(u_s(x))\right)^2}{2} u_s(x)dx + 1\ ds \geq n \right\}.$$

Lemma C.1 shows that under $\mathbb{P}^u$

$$\int_0^t \int \frac{(1-u_s(x))\left(Q(u_s(x))\right)^2}{2} u_s(x)dxds \leq \frac{\left(C_Q\right)^2}{2} \int_0^t \langle u_s, 1\rangle ds < \infty$$

for all $t > 0$ $\mathbb{P}^u$−a.s. and so $T_n \uparrow \infty$ $\mathbb{P}^u$-a.s. As in Theorem IV.1.6(a) of [15] this gives uniqueness of the law $\mathbb{P}^u$ of a solution to (62). As in Theorem IV.1.6(b) of [15] the fact that $T_n \uparrow \infty$ $\mathbb{P}^v$-a.s. (from Lemma C.1) shows that (64) defines a probability $\mathbb{P}^u$ which satisfies (62). □

A Proof of Remark 2.1

In what follows we shall prove the claim of Remark 2.1. See for instance Theorem B3, p.3 in Liggett [11] and note the uniform boundedness assumption on the rates from p.1 of [11]. Following the notation in [11], let $c(x,\xi^N)$ denote the rate at which the coordinate $\xi^N(x)$ flips from 0 to 1 or from 1 to 0 when the system is in state ξ^N. Then using $(G_i^{(N)} : N \in \mathbb{N})$, $(H_i^{(N)} : N \in \mathbb{N}) \in \vec{\mathscr{P}}_0, i = 0,1,$ (15), (16) and $0 \leq f_i^{(N)}, g_i^{(N)} \leq 1, i = 0,1$ yield

$$\begin{aligned}
\sup_{x\in\mathbb{Z}/N} \sup_{\xi^N\in\{0,1\}^{\mathbb{Z}/N}} c\left(x,\xi^N\right) &\leq N + \left(\sum_{m=0}^{\infty} \left|\alpha_0^{(m+1,N)}\right| + \left|\beta_0^{(m+1,N)}\right|\right) \vee \left(\sum_{m=0}^{\infty} \left|\alpha_1^{(m+1,N)}\right| + \left|\beta_1^{(m+1,N)}\right|\right) \\
&\equiv N + C_0(N) < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{x\in\mathbb{Z}/N} \sum_{u\in\mathbb{Z}/N} \sup_{\xi^N\in\{0,1\}^{\mathbb{Z}/N}} \left|c\left(x,\xi^N\right) - c\left(x,\xi_u^N\right)\right| \\
&\leq \sup_{x\in\mathbb{Z}/N} \sum_{u\sim x} \sup_{\xi^N\in\{0,1\}^{\mathbb{Z}/N}} \left|c\left(x,\xi^N\right) - c\left(x,\xi_u^N\right)\right| \\
&\quad + \sup_{x\in\mathbb{Z}/N} \sum_{u\in\mathbb{Z}/N} \sup_{\xi^N\in\{0,1\}^{\mathbb{Z}/N}} \sum_{i=0,1} \left|g_i^{(N)}\left(x,\xi^N\right) - g_i^{(N)}\left(x,\xi_u^N\right)\right| C_0(N) \\
&\leq 2c(N)N^{1/2}2(N + C_0(N)) + \sup_{x\in\mathbb{Z}/N} \sum_{u\in\mathbb{Z}/N} 2p(N(x-u))C_0(N) \\
&< \infty,
\end{aligned}$$

where

$$\xi_u^N(v) = \begin{cases} \xi^N(v), & v \neq u, \\ 1-\xi^N(v), & v = u. \end{cases}$$

Following [11], the two conditions are sufficient to ensure the claim holds.

B Auxiliary results to prove Lemma 3.4

The following lemma and corollary are necessary to prove Lemma 3.4.

Lemma B.1. *There exists $N_0 < \infty$ such that for all $N \geq N_0, k \geq 1$*

(a) $\left|\rho^k(t) - \exp\left(-c_2(N)\frac{kt^2}{6N}\right)\right| \leq C\frac{1}{k}\exp\left(-c_2(N)\frac{kt^2}{12N}\right)$ *for* $t \leq \sqrt{\frac{N}{3}}$,

(b) $|\rho(t)| \leq \exp\left(-C\frac{t^2}{12N}\right)$ *for* $t \leq \left(\frac{6N}{c_2(N)}\right)^{1/2}$,

(c) There exists $\delta > 0$ *such that* $|\rho(t)| \leq 1-\delta$ *for* $t \in \left[\left(\frac{6N}{c_2(N)}\right)^{1/2}, \pi N\right]$.

Proof. The proof mainly follows along the lines of the proof of Lemma 8 in [13]. Some small changes ensued due to the different setup. Recall the definition of $\rho(t)$ from equation (42).

For (b), we could not find the reference mentioned in [13] but the following reasoning in [13] based on applying Taylor's theorem at $t = 0$ works well without it.

For (a), first observe that $\rho^k(t) = \mathbb{E}\left[e^{itS_k}\right]$ and use Bhattacharya and Rao [2], (8.11), (8.13) and [2], Theorem 8.5. as suggested in [13]. We used that $\mathbb{E}[Y_1] = \mathbb{E}[Y_1^3] = 0$.

It remains to prove (c). We have to change the proof of [13], Lemma 8(c) slightly, as we used $x \not\sim x$. We get

$$|\rho(t)| = \left|\frac{1}{2c(N)N^{1/2}} \sum_{0<j\leq c(N)\sqrt{N}} 2\mathrm{Re}\left[e^{it\frac{j}{N}}\right]\right| = \left|\frac{1}{c(N)N^{1/2}}\mathrm{Re}\left[\frac{e^{it\frac{1}{N}} - e^{it\frac{c(N)\sqrt{N}+1}{N}}}{1-e^{it\frac{1}{N}}}\right]\right|$$
$$\leq \frac{1}{c(N)N^{1/2}}\left|\frac{2}{2\sin\left(\frac{t}{2N}\right)}\right|.$$

For $\frac{1+\epsilon}{c(N)N^{1/2}} \leq \frac{t}{2N} \leq \frac{\pi}{2}$ with $\epsilon > 0$ fixed we get as an upper bound

$$\frac{1}{c(N)N^{1/2}}\left|\frac{1}{\sin\left(\frac{1+\epsilon}{c(N)N^{1/2}}\right)}\right| \leq \frac{1}{1+\epsilon} < 1,$$

given N big enough. Finally use that $2 < \sqrt{6}$ to obtain the claim. □

Corollary B.2. *For $N \geq N_0$, $y \in N^{-1}\mathbb{Z}$ we have*

$$\left| N\mathbb{P}(S_k = y) - p\left(c_2(N)\frac{k}{3N}, y\right)\right| \leq C_1 \left\{N \exp\{-kC_2\} + N^{1/2}k^{-3/2}\right\},$$

where $C_1, C_2 > 0$ are some positive constants.

Proof. This result corresponds to Corollary 9 in [13]. The proof works similarly. Instead of the reference given at the beginning of the proof of Corollary 9 in [13], we used Durrett [7], p. 95, Ex. 3.2(ii) and [7], Thm. (3.3).

Note in particular that the result of Lemma B.1(c) can be extended to $t \in \left[\sqrt{\frac{N}{3}}, \pi N\right]$ if we choose $\delta > 0$ small enough. Indeed, using Lemma B.1(b) we obtain

$$|\rho(t)| \leq e^{-C\frac{t^2}{12N}} \leq e^{-C\frac{N/3}{12N}} \leq (1-\delta)$$

as claimed. □

C Auxiliary results to prove Lemma 3.10

The following lemma is used to prove Lemma 3.10. Let u and $Q(u)$ be as in (62) and v as in (63). Also recall the definitions of $\mathbb{P}^u, \mathbb{P}^v$ and C_Q in between (62) and (63).

Lemma C.1. *Given $u_0 = v_0$ satisfying $\langle u_0, 1\rangle < \infty$, we have $\mathbb{P}^u$-a.s. $\int_0^t \langle u_s, 1\rangle ds < \infty$ for all $t \geq 0$ and $\mathbb{P}^v$-a.s. $\int_0^t \langle v_s, 1\rangle ds < \infty$ for all $t \geq 0$.*

Proof. We shall prove the claim for $\mathbb{P}^u$. The other claim then follows by considering the special case $Q \equiv 0$. As a first step we shall use a generalization of the weak form of (62) to functions in two variables. In the proof of Theorem 2.1 on p. 430 of [18] it is shown that for every $\psi \in \mathscr{D}^2_{rap}(T)$ and $0 < t < T$ we have

$$\begin{aligned}\langle u_t, \psi_t\rangle =& \langle u_0, \psi_0\rangle + \int_0^t \langle u_s, \left(\frac{\partial}{\partial s} + \frac{\Delta}{6}\right)\psi_s\rangle ds + \int_0^t \langle u_s(1-u_s)Q(u_s), \psi_s\rangle ds \\ &+ \int_0^t \int \sqrt{2u_s(x)(1-u_s(x))}\psi_s(x) dW(x,s).\end{aligned} \tag{65}$$

Here we have for $T > 0$,

$$\begin{aligned}\mathscr{C}(\mathbb{R}) &= \{f : \mathbb{R} \to \mathbb{R} \text{ continuous}\}, \\ \mathscr{C}_{rap} &= \left\{f \in \mathscr{C}(\mathbb{R}) \text{ such that } \sup_x e^{\lambda|x|}|f(x)| < \infty \text{ for all } \lambda > 0\right\}, \\ \mathscr{C}^2_{rap} &= \left\{\psi \in \mathscr{C}^2(\mathbb{R}) \text{ such that } \psi, \psi', \psi'' \in \mathscr{C}_{rap}\right\}, \\ \mathscr{D}^2_{rap}(T) &= \Big\{\psi \in \mathscr{C}^{1,2}([0,T) \times \mathbb{R}) \text{ such that } \psi(t,\cdot) \text{ is } \mathscr{C}^2_{rap}\text{-valued continuous and} \\ &\qquad \frac{\partial \psi}{\partial t}(t,\cdot) \text{ is } \mathscr{C}_{rap}\text{-valued continuous in } 0 \leq t < T\Big\}.\end{aligned}$$

Also observe that the condition (2.2) of [18] is satisfied as we have $0 \le u_s(x) \le 1$ and therefore $|Q(u_s(x))| \le C_Q$.

Now recall that the Brownian transition density is $p(s,x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}$. Let $\left(P_s\phi\right)(x) = \int p\left(\frac{s}{3}, y-x\right)\phi(y)dy$ with $\phi \in \mathscr{C}_c^\infty, \phi \ge 0$ and let

$$\psi(s,x) = \psi_s(x) = e^{C_Q(T-s)}\left(P_{T-s}\phi\right)(x) \text{ and thus } \psi \in \mathscr{D}^2_{rap}(T).$$

Note that $\frac{\partial}{\partial s}\left(P_{T-s}\phi\right)(x) = -\frac{\Delta}{6}\left(P_{T-s}\phi\right)(x)$, where we used that $\frac{\partial}{\partial s}p(s,x) = \frac{1}{2}\Delta p(s,x)$. We obtain for the drift term in (65) that

$$\begin{aligned}
&\langle u_s, \left(\frac{\partial}{\partial s} + \frac{\Delta}{6}\right)\psi_s\rangle + \langle u_s(1-u_s)Q(u_s), \psi_s\rangle \\
&= \langle u_s, -C_Q\psi_s - \frac{\Delta}{6}\psi_s + \frac{\Delta}{6}\psi_s\rangle + \langle u_s(1-u_s)Q(u_s), \psi_s\rangle \\
&\le 0
\end{aligned}$$

using that $\psi(s,x) \ge 0$ for $\phi \ge 0$. Additionally, the local martingale in (65) is a true martingale as

$$\begin{aligned}
&\left\langle \int_0^\cdot \int \sqrt{2u_s(x)(1-u_s(x))}\psi_s(x)dW(x,s)\right\rangle_t \\
&= \int_0^t \langle 2u_s(1-u_s), \psi_s^2\rangle ds \le 2e^{2C_QT}\int_0^t \langle 1, \left(P_{T-s}\phi\right)^2\rangle ds \\
&\le 2e^{2C_QT}\,\|\phi\|_0 \int_0^t \langle 1, P_{T-s}\phi\rangle ds = 2e^{2C_QT}\,\|\phi\|_0\,\langle 1,\phi\rangle t \\
&< \infty.
\end{aligned}$$

Hence we obtain from (65) for all $0 < t < T$ after taking expectations

$$\mathbb{E}\left[\langle u_t, \psi_t\rangle\right] \le \langle u_0, \psi_0\rangle,$$

i.e.

$$e^{C_Q(T-t)}\mathbb{E}\left[\langle u_t, (P_{T-t}\phi)\rangle\right] \le e^{C_QT}\langle u_0, (P_T\phi)\rangle.$$

Now choose an increasing sequence of non-negative functions $\phi^n \in \mathscr{C}_c^\infty$ such that $\phi^n \uparrow 1$ for $n \to \infty$. Using the monotone convergence theorem, we obtain from the above

$$e^{C_Q(T-t)}\mathbb{E}\left[\langle u_t, 1\rangle\right] = \lim_{n\to\infty} e^{C_Q(T-t)}\mathbb{E}\left[\langle u_t, (P_{T-t}\phi^n)\rangle\right] \le \lim_{n\to\infty} e^{C_QT}\langle u_0, (P_T\phi^n)\rangle = e^{C_QT}\langle u_0, 1\rangle.$$

Hence by the Fubini-Tonelli theorem,

$$\mathbb{E}\left[\int_0^t \langle u_s, 1\rangle ds\right] \le \langle u_0, 1\rangle \int_0^t e^{C_Qs}ds < \infty$$

for all $t \ge 0$, which proves the claim. □

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