

Random Gaussian sums on trees

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Abstract

Let T be a tree with induced partial order \preceq . We investigate centered Gaussian processes $X = (X_t)_{t \in T}$ represented as

$$X_t = \sigma(t) \sum_{v \preceq t} \alpha(v) \xi_v$$

for given weight functions α and σ on T and with $(\xi_v)_{v \in T}$ i.i.d. standard normal. In a first part we treat general trees and weights and derive necessary and sufficient conditions for the a.s. boundedness of X in terms of compactness properties of (T, d) . Here d is a special metric defined via α and σ , which, in general, is not comparable with the Dudley metric generated by X . In a second part we investigate the boundedness of X for the binary tree. Assuming some mild regularity assumptions about α , we completely characterize homogeneous weights α and σ with X being a.s. bounded.

Key words: Gaussian processes, processes indexed by trees, bounded processes, summation on trees, metric entropy.

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1 Introduction

The aim of the present paper is to investigate boundedness properties of a special class of centered Gaussian processes $X = (X_t)_{t \in T}$ indexed by some tree T . Those processes are represented as

$$X_t := \sigma(t) \sum_{v \preceq t} \alpha(v) \xi_v, \quad t \in T, \quad (1.1)$$

where α and σ are some (non-negative) weight functions on T , the $(\xi_v)_{v \in T}$ are i.i.d. standard normal and " \preceq " denotes the partial order on T generated by its tree structure. Similar processes (with $\sigma \equiv 1$) were investigated by X. Fernique in his constructions of majorizing measures [4].

More recently, some processes of this class were extensively studied (by methods very different from ours) and applied in relation to various topics, see e.g. the literature on Derrida's random energy model [1] or displacements in random branching walks [11], to mention just a few. Of course, using peculiarities of a specific situation one obtains deeper results about processes (1.1) than those for the general case presented below.

Our interest in this class of processes came from recent investigations about compactness properties of summation operators on trees (cf. [8], [9] and [10]). Recall that each operator from $\ell_2(T)$ into $\ell_\infty(T)$ generates in natural way a centered Gaussian process indexed by T . And in that way processes X as in (1.1) stem from the summation operators treated in the mentioned works (cf. [9] for more details). Fortunately, many tools and methods used for those operators turned out to be useful for the generated processes as well.

The basic question investigated in this paper is as follows: Given a tree T , characterize weights α and σ such that the process X defined by (1.1) is a.s. bounded, i.e., that

$$\mathbb{P} \left(\sup_{t \in T} |X_t| < \infty \right) = 1. \quad (1.2)$$

Let us put this question into the wider context of general Gaussian processes. Take, for a while, an arbitrary index set T and let $X = (X_t)_{t \in T}$ be a centered Gaussian process indexed by T . Then its *covariance function* is given by

$$R_X(t, s) := \mathbb{E} X_t X_s, \quad t, s \in T,$$

and the *Dudley distance* d_X on T is defined by

$$d_X(t, s) := \left(\mathbb{E} |X_t - X_s|^2 \right)^{1/2}, \quad t, s \in T. \quad (1.3)$$

A basic question, going back to A.N. Kolmogorov, about Gaussian processes is as follows: Characterize covariance functions R_X (or, equivalently, metrics d_X) for which X is a.s. bounded. Criteria for many other sample path properties such as continuity, uniform and local Hölder property, behavior of suprema, etc. follow relatively easily once the boundedness problem is settled.

A first crucial step to answer the boundedness question was done by R.M. Dudley in 1966 (see [2]). To formulate his result we have to introduce the concept of *covering numbers*. Thus let ρ be an arbitrary metric on T and, if $\varepsilon > 0$, those numbers are defined by

$$N(T, \rho, \varepsilon) := \inf \left\{ n \geq 1 : T = \bigcup_{j=1}^n B_\varepsilon(t_j) \right\}$$

where $B_\varepsilon(t_j)$ are open ε -balls (w.r.t. the metric ρ) in T . Dudley's theorem asserts that

$$\int_0^\infty \sqrt{\log N(T, d_X, \varepsilon)} d\varepsilon < \infty \quad (1.4)$$

yields the a.s. boundedness of X . Next, in 1969, V.N. Sudakov (see [12]) showed that

$$\sup_{\varepsilon>0} \varepsilon \sqrt{\log N(T, d_X, \varepsilon)} < \infty \quad (1.5)$$

is necessary for the a.s. boundedness of X . There is a small but important gap between conditions (1.4) and (1.5). Consequently, processes X in the critical case, i.e. for which (1.5) holds while (1.4) is violated, are of special interest. Below we will give some examples of bounded as well as unbounded processes corresponding to the critical case (cf. Corollary 6.3).

Giving necessary *and* sufficient conditions for the boundedness required a language different from metric entropy. X. Fernique proved that if there is a probability measure μ (*majorizing measure*) on T for which

$$\sup_{t \in T} \int_0^\infty \sqrt{\log \frac{1}{\mu(B_\varepsilon(t))}} d\varepsilon < \infty, \quad (1.6)$$

where $B_\varepsilon(t)$ denotes the ε -ball centered at t w.r.t. the Dudley distance d_X , then X is a.s. bounded (cf. [3] and [5]).

Finally, in 1987 M. Talagrand ([13]) answered the question about boundedness of Gaussian processes completely. He confirmed an earlier conjecture of Fernique by showing that X is a.s. bounded if and only if a measure satisfying (1.6) exists, thus establishing the *Majorizing Measure Criterion* for boundedness. In subsequent works [15, 16], Talagrand extended his technique to non-Gaussian processes; majorizing measures were replaced by a so called *generic chaining* construction.

The previous results illustrate the crucial role of d_X and its compactness properties for the boundedness of a Gaussian process X . Note that for X defined by (1.1) the Dudley distance equals

$$d_X(t, s)^2 = |\sigma(t) - \sigma(s)|^2 \sum_{v \preceq t \wedge s} \alpha(v)^2 + \sigma(t)^2 \sum_{t \wedge s \prec v \preceq t} \alpha(v)^2 + \sigma(s)^2 \sum_{t \wedge s \prec v \preceq s} \alpha(s)^2 \quad (1.7)$$

where $t \wedge s$ denotes the infimum of $t, s \in T$ in the generated partial order on T .

In our particular setting (1.1), the Majorizing Measure Criterion apparently does not help very much because it looks hopeless to characterize weights α and σ for which a majorizing measure exists. Thus a different approach is needed. In a first part we construct a metric d on T such that Dudley's and Sudakov's theorems hold with respect to $N(T, d, \varepsilon)$, although d is not comparable with d_X . The main advantage of d is that in many cases its covering numbers are easier to handle than those of d_X . The second part is devoted to binary trees. In the case of homogeneous weights, i.e., the weights depend only on the order of an element in T , we get an "almost" complete description of weights α and σ for which X is bounded.

2 Main results

2.1 Some notation and facts about trees

Before stating our main results, let us shortly recall some basic facts about trees needed later on. In the sequel T always denotes a finite or an infinite tree. We suppose that T has a unique root which

we denote by $\mathbf{0}$ and that each element $t \in T$ has a finite number of offsprings. Thereby we do not exclude that some elements do not possess any offspring, i.e., the progeny of some elements may "die out". The tree structure leads in natural way to a *partial order* „ \preceq ” by letting $t \preceq s$, respectively $s \succeq t$, provided there are $t = t_0, t_1, \dots, t_m = s$ in T such that for $1 \leq j \leq m$ the element t_j is an offspring of t_{j-1} . The strict inequalities have the same meaning with the additional assumption $t \neq s$. Two elements $t, s \in T$ are said to be *comparable* provided that either $t \preceq s$ or $s \preceq t$. In the following, $t \wedge s$ denotes the infimum of t and s in the induced partial order on T .

For $t, s \in T$ with $t \preceq s$ the order interval $[t, s]$ is defined by

$$[t, s] := \{v \in T : t \preceq v \preceq s\}$$

and in a similar way we construct $(t, s]$ or (t, s) .

A subset $B \subseteq T$ is said to be a *branch* provided that all elements in B are comparable and, moreover, if $t \preceq v \preceq s$ with $t, s \in B$, then this implies $v \in B$ as well. Of course, finite branches are of the form $[t, s]$ for suitable $t \preceq s$.

For any $s \in T$ its *order* $|s| \geq 0$ is defined by

$$|s| := \#\{t \in T : t \prec s\}.$$

Let ρ be an arbitrary metric on the tree T . Given $\varepsilon > 0$, a set $\mathcal{O} \subseteq T$ is said to be an ε -*order net* w.r.t. ρ provided that for each $s \in T$ there is a $t \in \mathcal{O}$ with $t \preceq s$ and $\rho(t, s) < \varepsilon$. Let

$$\tilde{N}(T, \rho, \varepsilon) := \inf\{\#\{\mathcal{O}\} : \mathcal{O} \text{ is an } \varepsilon\text{-order net of } T\} \quad (2.1)$$

be the corresponding *order covering numbers*. Clearly, we have

$$N(T, \rho, \varepsilon) \leq \tilde{N}(T, \rho, \varepsilon). \quad (2.2)$$

2.2 Main results

As already mentioned, throughout this paper a special metric d on T , first introduced in [9], plays an important role. Given weight functions α and σ on T with σ non-increasing, it is defined by

$$d(t, s) := \max_{t \prec r \preceq s} \sigma(r) \left(\sum_{t \prec v \preceq r} \alpha(v)^2 \right)^{1/2}, \quad (2.3)$$

whenever $t \preceq s$ and we let $d(t, s) := d(t \wedge s, t) + d(t \wedge s, s)$, if t and s are not comparable. A useful property of d is that a reverse estimate of (2.2) holds. More precisely, as shown in [9, Proposition 3.2]

$$\tilde{N}(T, d, 2\varepsilon) \leq N(T, d, \varepsilon). \quad (2.4)$$

We will prove the following version of the Dudley and the Sudakov theorem using the metric d introduced above.

Theorem 2.1. *Suppose that X is defined by (1.1) with weights α and non-increasing σ . Let d be the metric on T given by (2.3). If*

$$\int_0^\infty \sqrt{\log N(T, d, \varepsilon)} d\varepsilon < \infty, \quad (2.5)$$

then X is a.s. bounded. Conversely, if X is a.s. bounded, then necessarily

$$\sup_{\varepsilon>0} \varepsilon \sqrt{\log N(T, d, \varepsilon)} < \infty.$$

This theorem is not a direct consequence of the above mentioned results due to Dudley and to Sudakov. Indeed, these classical results are based on compactness properties of (T, d_X) and not on those of (T, d) . We shall see below that, in general, the covering numbers w.r.t. d and to d_X may behave differently. More precisely, in Section 4 we construct a tree and weights α and σ such that the associated process X is a.s. bounded, and

$$\lim_{\varepsilon \rightarrow 0} \frac{N(T, d_X, \varepsilon)}{N(T, d, \varepsilon)} = \infty.$$

Moreover, another example shows that in general also $d_X(t, s) \leq c d(t, s)$ with some $c > 0$ cannot be valid, i.e., the metrics d and d_X are not comparable. Thus Theorem 2.1 may be viewed as a special version of the Dudley and Sudakov theorem for processes of type (1.1). Hereby the main advantage of Theorem 2.1 is that in many cases d and its covering numbers, involved in the assertion of the theorem, are easier to handle than those generated by d_X (cf. [9], Sections 6 and 7, for concrete estimates of $N(T, d, \varepsilon)$ and also Proposition 6.1 below).

Although $N(T, d_X, \varepsilon)$ and $N(T, d, \varepsilon)$ may be quite different, at the logarithmic level they must behave similarly. Indeed, combining Theorem 2.1 with the classical results implies

$$\int_0^\infty \sqrt{\log N(T, d_X, \varepsilon)} d\varepsilon < \infty \quad \Rightarrow \quad \sup_{\varepsilon>0} \varepsilon^2 \log N(T, d, \varepsilon) < \infty$$

and

$$\int_0^\infty \sqrt{\log N(T, d, \varepsilon)} d\varepsilon < \infty \quad \Rightarrow \quad \sup_{\varepsilon>0} \varepsilon^2 \log N(T, d_X, \varepsilon) < \infty.$$

Thus it quite natural to ask whether or not the previous implications may even be improved. This is in fact the case. More precisely, in Section 5 we will prove the following:

Proposition 2.2. *We have*

$$\int_0^\infty \sqrt{\log N(T, d_X, u)} du < \infty \quad \Leftrightarrow \quad \int_0^\infty \sqrt{\log N(T, d, u)} du < \infty.$$

and

$$\sup_{\varepsilon>0} \varepsilon^2 \log N(T, d_X, \varepsilon) < \infty \quad \Leftrightarrow \quad \sup_{\varepsilon>0} \varepsilon^2 \log N(T, d, \varepsilon) < \infty.$$

Remark: Of course, in view of the results due to Dudley and Sudakov, Proposition 2.2 immediately implies Theorem 2.1. However, the proofs of both statements are strongly interlaced and proving first Proposition 2.2 turns out to be inconvenient and unnatural. We prefer to begin with proving Theorem 2.1 by rather direct probabilistic methods and then add few necessary deterministic arguments proving Proposition 2.2.

The last part of the paper is devoted to binary trees. First we apply Theorem 2.1 for those trees and next we suppose that the weights are homogeneous, i.e. $\alpha(t)$ and $\sigma(t)$ only depend on the order of t . Our results (cf. Theorems 6.2 and 6.4 below) imply the following:

Theorem 2.3. *Let T be a binary tree and suppose $\alpha(t) = \alpha_{|t|}$ and $\sigma(t) = \sigma_{|t|}$ for two sequences $(\alpha_k)_{k \geq 0}$ and $(\sigma_k)_{k \geq 0}$ of positive numbers with $(\sigma_k)_{k \geq 0}$ non-increasing.*

1. If

$$\sup_{n \geq 1} \sup_{n \leq k \leq 2n} \frac{\alpha_k}{\alpha_n} < \infty, \quad (2.6)$$

then X defined by (1.1) is a.s. bounded if and only if

$$\sup_{n \geq 1} \sigma_n \sum_{k=1}^n \alpha_k < \infty. \quad (2.7)$$

In particular, if $(\alpha_k)_{k \geq 0}$ is non-increasing, then (2.6) is always satisfied, hence in that case X is a.s. bounded if and only if (2.7) is valid.

2. If $(\alpha_k)_{k \geq 0}$ is non-decreasing, then X is a.s. bounded if and only if

$$\sup_{n \geq 1} \sigma_n \sqrt{n} \left(\sum_{k=0}^n \alpha_k^2 \right)^{1/2} < \infty.$$

The organization of the paper is as follows. Section 3 is devoted to the proof of Theorem 2.1. In Section 4 we thoroughly investigate the relation between the two metrics d and d_X . Here the main observation is, as already mentioned, that $N(T, d, \varepsilon)$ and $N(T, d_X, \varepsilon)$ may behave quite differently. Next, in Section 5 we prove Proposition 2.2. In Section 6 we treat processes X indexed by a binary tree. In particular we prove slightly more general results than those stated in Theorem 2.3. Finally, we give some interesting examples of bounded as well as unbounded processes indexed by a binary tree. In particular, these examples show that the boundedness of X may not be described by properties of the product $\alpha\sigma$ only. We investigate the relation between X defined in (1.1) and the one-weight process with weight $\alpha\sigma$ more thoroughly in Section 7.

3 Proof of Theorem 2.1

Let T be an arbitrary tree and let α and σ be weights on T as before. Define $X = (X_t)_{t \in T}$ as in (1.1). Of course, whenever (1.2) holds, then we necessarily have

$$\sup_{t \in T} \left(\mathbb{E} |X_t|^2 \right)^{1/2} = \sup_{t \in T} \sigma(t) \left(\sum_{v \preceq t} \alpha(v)^2 \right)^{1/2} < \infty. \quad (3.1)$$

Thus let us assume that (3.1) is always satisfied.

In order to prove part one of Theorem 2.1 in a first step we replace the process X by a process \hat{X} which is easier to handle.

To this end, if $k \in \mathbb{Z}$, define $I_k \subseteq T$ by

$$I_k := \{t \in T : 2^{-k-1} < \sigma(t) \leq 2^{-k}\}$$

and a new weight $\hat{\sigma}$ by

$$\hat{\sigma} := \sum_{k \in \mathbb{Z}} 2^{-k} \mathbf{1}_{I_k}. \quad (3.2)$$

Let \hat{X} be the process defined by α and $\hat{\sigma}$ via (1.1), i.e., it holds

$$\hat{X}_t = \hat{\sigma}(t) \sum_{v \preceq t} \alpha(v) \xi_v, \quad t \in T, \quad (3.3)$$

and let \hat{d} denote the distance generated via α and $\hat{\sigma}$ as in (2.3). Then the following statement is valid.

Lemma 3.1.

1. If $t \preceq s$, then it holds

$$d(t, s) \leq \hat{d}(t, s) \leq 2d(t, s).$$

Consequently, it follows

$$\tilde{N}(T, d, \varepsilon) \leq \tilde{N}(T, \hat{d}, \varepsilon) \leq \tilde{N}(T, d, \varepsilon/2),$$

where $\tilde{N}(T, d, \varepsilon)$ and $\tilde{N}(T, \hat{d}, \varepsilon)$ are the order covering numbers corresponding to the respective metrics.

2. The process X is a.s. bounded if and only if \hat{X} is a.s. bounded.

Proof. The first assertion follows easily by the definition of d and \hat{d} while the second one is a direct consequence of

$$|X_t| \leq |\hat{X}_t| \leq 2|X_t|, \quad t \in T.$$

□

In view of the preceding lemma, we conclude that it suffices to prove Theorem 2.1 in the case of non-increasing weights σ of the form

$$\sigma := \sum_{k \in \mathbb{Z}} 2^{-k} \mathbf{1}_{I_k}. \quad (3.4)$$

The property that σ is non-increasing reflects in the following properties of the partition $(I_k)_{k \in \mathbb{Z}}$ of T .

1. Whenever $B \subseteq T$ is a branch, then for each $k \in \mathbb{Z}$ either $B \cap I_k = \emptyset$ or it is an order interval in T .

2. If $l < k$, $t \in B \cap I_l$, $s \in B \cap I_k$, then this implies $t \prec s$.
3. $I_k = \emptyset$ whenever $k \leq k_0$ for a certain $k_0 \in \mathbb{Z}$.

Thus from now on we may suppose that the weight σ is as in (3.4) with a partition $(I_k)_{k \in \mathbb{Z}}$ of T possessing properties (1), (2) and (3) stated before.

In a second step of the proof of Theorem 2.1, first part, we define a process $Y := (Y_t)_{t \in T}$ which may be viewed as a localization of X . To this end let us write $t \equiv s$ provided there is a $k \in \mathbb{Z}$ such that $t, s \in I_k$. With this notation we set

$$Y_t := \sigma(t) \sum_{\substack{v \preceq t \\ v \equiv t}} \alpha(v) \xi_v, \quad t \in T. \quad (3.5)$$

It is an easy deal to relate the boundedness of X with that of Y .

Lemma 3.2. *The process Y is a.s. bounded if and only if X is a.s. bounded.*

Proof. Actually, we establish simple linear relations between Y and X , see (3.6) and (3.8) below. For any integers $\ell \leq k$ and any $t \in I_k$ set $B_\ell(t) := [\mathbf{0}, t] \cap I_\ell$. Then we have

$$\begin{aligned} X_t &= 2^{-k} \sum_{\ell \leq k} \sum_{v \in B_\ell(t)} \alpha(v) \xi_v \\ &= \sum_{\ell \leq k} 2^{-(k-\ell)} \cdot 2^{-\ell} \sum_{v \in B_\ell(t)} \alpha(v) \xi_v \\ &= \sum_{\ell} 2^{-(k-\ell)} Y_{\lambda_\ell(t)}, \end{aligned} \quad (3.6)$$

where the last sum is taken over $\ell \leq k$ such that $B_\ell(t) \neq \emptyset$ and $\lambda_\ell(t) := \max\{s : s \in B_\ell(t)\}$. It follows from (3.6) that the boundedness of Y yields that of X .

To prove the converse statement of Lemma 3.2, take an arbitrary $t \in T$ and consider two different cases.

If $t \equiv \mathbf{0}$ (recall that $\mathbf{0}$ denotes the root of T), then by the definition of Y we simply have $Y_t = X_t$.

Otherwise, if $t \not\equiv \mathbf{0}$, let

$$\lambda^-(t) = \max\{s : s \preceq t, s \not\equiv t\}. \quad (3.7)$$

By the definition of Y_t we obtain

$$\begin{aligned} Y_t &= \sigma(t) \sum_{\lambda^-(t) \prec v \preceq t} \alpha(v) \xi_v \\ &= \sigma(t) \left(\sum_{v \preceq t} \alpha(v) \xi_v - \sum_{v \preceq \lambda^-(t)} \alpha(v) \xi_v \right) \\ &= X_t - \frac{\sigma(t)}{\sigma(\lambda^-(t))} X_{\lambda^-(t)}. \end{aligned} \quad (3.8)$$

Since the weight σ is non-increasing, by $\lambda^-(t) \preceq t$ we get $\frac{\sigma(t)}{\sigma(\lambda^-(t))} \leq 1$. It follows from (3.8) that if X is a.s. bounded this is also valid for Y as claimed. This completes the proof. \square

In the next step we calculate the Dudley distance generated by Y and compare $\tilde{N}(T, d_Y, \varepsilon)$ with $\tilde{N}(T, d, \varepsilon)$. Recall that $\tilde{N}(T, d_Y, \varepsilon)$ and $\tilde{N}(T, d, \varepsilon)$ are the corresponding order covering numbers as introduced in (2.1).

Lemma 3.3. *Suppose $\alpha(\mathbf{0}) = 0$, hence $Y_0 = 0$ a.s. Then it follows that*

$$N(T, d_Y, \varepsilon) \leq \tilde{N}(T, d_Y, \varepsilon) \leq \tilde{N}(T, d, \varepsilon) + 1. \quad (3.9)$$

Proof. If $t \preceq s$, then we get

$$d_Y(t, s)^2 = \sigma(s)^2 \sum_{t \prec v \preceq s} \alpha(v)^2 = d(t, s)^2 \quad \text{if } t \equiv s$$

and

$$d_Y(t, s)^2 = \mathbb{E}|Y_t|^2 + \mathbb{E}|Y_s|^2 \quad \text{if } t \not\equiv s.$$

Given $\varepsilon > 0$ let $\mathcal{O} \subseteq T$ be an ε -order net w.r.t. the metric d . Take $s \in T$ arbitrarily. Then there is a $t \in \mathcal{O}$ such that $t \preceq s$ and $d(t, s) < \varepsilon$. If $t \equiv s$, then this implies $d_Y(t, s) = d(t, s) < \varepsilon$ as well. But if $t \not\equiv s$, then we get

$$\begin{aligned} d_Y(\mathbf{0}, s) &= (\mathbb{E}|Y_s - Y_0|^2)^{1/2} = (\mathbb{E}|Y_s|^2)^{1/2} = \sigma(s) \left(\sum_{\substack{v \preceq s \\ v \equiv s}} \alpha(v)^2 \right)^{1/2} \\ &\leq \sigma(s) \left(\sum_{t \prec v \preceq s} \alpha(v)^2 \right)^{1/2} \leq d(t, s) < \varepsilon. \end{aligned}$$

In different words, the set $\mathcal{O} \cup \{\mathbf{0}\}$ is an ε -order net of T w.r.t. d_Y . Of course, this implies the second inequality in (3.9), the first one being trivial. Thus the proof is complete. \square

Proof of Theorem 2.1, first part: Without loosing generality we may assume $\alpha(\mathbf{0}) = 0$. Indeed, write

$$X_t = \sigma(t) \sum_{v \preceq t} \alpha(v) \xi_v = \sigma(t) \sum_{\mathbf{0} \prec v \preceq t} \alpha(v) \xi_v + \sigma(t) \alpha(\mathbf{0}) \xi_{\mathbf{0}}$$

and observe that $\sup_{t \in T} \sigma(t) < \infty$. Moreover, the metric d is independent of $\alpha(\mathbf{0})$. Note that this number never appears in the evaluation of $d(t, s)$ for arbitrary $t, s \in T$.

Thus let us assume now that (2.5) is valid. Then (2.4) implies

$$\int_0^\infty \sqrt{\log \tilde{N}(T, d, \varepsilon)} d\varepsilon < \infty$$

as well. Hence Lemma 3.3 yields

$$\int_0^\infty \sqrt{\log N(T, d_Y, \varepsilon)} d\varepsilon < \infty.$$

Consequently, Dudley's theorem (cf. (1.4) or [7], p.179) applies for Y and d_Y , hence Y possesses a.s. bounded paths. In view of Lemma 3.2, the paths of X are also a.s. bounded and this completes the proof of the first part of Theorem 2.1.

Proof of Theorem 2.1, second part: Take $\varepsilon > 0$. As proved in [9, Proposition 5.2] there are at least $m := N(T, d, 2\varepsilon) - 1$ disjoint order intervals $(t_i, s_i]$ in T with $d(t_i, s_i) \geq \varepsilon$. By the definition of d we find $t_i \prec r_i \preceq s_i$ such that

$$\sigma(r_i) \left(\sum_{t_i \prec v \preceq r_i} \alpha(v)^2 \right)^{1/2} \geq \varepsilon, \quad 1 \leq i \leq m.$$

Next, set

$$\eta_i := X_{r_i} - \frac{\sigma(r_i)}{\sigma(t_i)} X_{t_i}, \quad 1 \leq i \leq m. \quad (3.10)$$

Then it follows that

$$\eta_i = \sigma(r_i) \left[\sum_{v \preceq r_i} \alpha(v) \xi_v - \sum_{v \preceq t_i} \alpha(v) \xi_v \right] = \sigma(r_i) \left[\sum_{t_i \prec v \preceq r_i} \alpha(v) \xi_v \right]$$

and, consequently, the η_i are independent centered Gaussian with

$$(\mathbb{E} |\eta_i|^2)^{1/2} = \sigma(r_i) \left(\sum_{t_i \prec v \preceq r_i} \alpha(v)^2 \right)^{1/2} \geq \varepsilon. \quad (3.11)$$

Since σ is assumed to be non-increasing, we get

$$\sup_{1 \leq i \leq m} |\eta_i| \leq 2 \sup_{t \in T} |X_t|. \quad (3.12)$$

Suppose now that X is a.s. bounded. By Fernique's theorem (cf. [5] or [7], p.142), this implies

$$C := \mathbb{E} \sup_{t \in T} |X_t| < \infty,$$

hence (3.12) leads to

$$\mathbb{E} \sup_{1 \leq i \leq m} |\eta_i| \leq 2C,$$

and by the choice of m the assertion follows by

$$c \varepsilon \sqrt{\log m} \leq \mathbb{E} \sup_{1 \leq i \leq m} |\eta_i|$$

where we used (3.11) and the classical Fernique–Sudakov bound recalled below in (6.4). \square

Remark: One can give a geometric interpretation of the relations between the processes that we constructed in the proof of Theorem 2.1. Recall that an arbitrary Gaussian process $X = (X_t)_{t \in T}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ can be regarded as a subset of the Hilbert space $L_2(\Omega, \mathcal{A}, \mathbb{P})$, i.e., we identify X with $\{X_t : t \in T\}$. We denote by $\text{aco}(X)$ the absolutely convex hull of X in $L_2(\Omega, \mathcal{A}, \mathbb{P})$. It is easy to prove the processes X , \hat{X} and Y defined in (1.1), (3.3), and (3.5), respectively, are in this sense connected by the simple geometric relation

$$\text{aco}(X) \subseteq \text{aco}(\hat{X}) \subseteq 2 \text{aco}(Y) \subseteq 4 \text{aco}(\hat{X}) \subseteq 8 \text{aco}(X).$$

This again shows that the boundedness of one of these processes implies the boundedness of the two others.

4 Compactness properties of (T, d) versus those of (T, d_X)

The aim of this section is to compare the metric d on T defined in (2.3) with the Dudley distance d_X introduced in (1.7). Note that by (1.7) for $t \preceq s$ we have

$$d_X(t, s)^2 = |\sigma(t) - \sigma(s)|^2 \sum_{v \preceq t} \alpha(v)^2 + \sigma(s)^2 \sum_{t \prec v \preceq s} \alpha(v)^2. \quad (4.1)$$

while in that case

$$d(t, s) = \max_{t \prec r \preceq s} \sigma(r) \left(\sum_{t \prec v \preceq r} \alpha(v)^2 \right)^{1/2}. \quad (4.2)$$

Comparing (4.1) with (4.2), it is not clear at all how these two distances are related in general.

In a first result we show that the covering numbers w.r.t. d and to d_X may be of quite different order.

Proposition 4.1. *There are non-increasing weights α and σ on a tree T such that the generated process X is a.s. bounded and, moreover,*

$$\lim_{\varepsilon \rightarrow 0} \frac{N(T, d_X, \varepsilon)}{N(T, d, \varepsilon)} = \infty.$$

Proof. Take $T = \mathbb{N}_0 = \{0, 1, \dots\}$ and let $\alpha(0) = \sigma(0) = 1$. If $k \geq 1$ set

$$\alpha(k) = k^{-\nu} \quad \text{and} \quad \sigma(k) = k^{-\theta}$$

for some $\theta, \nu > 0$, i.e.,

$$X_k = k^{-\theta} \left[\sum_{j=1}^k j^{-\nu} \xi_j + \xi_0 \right], \quad k \geq 1. \quad (4.3)$$

The law of iterated logarithm tells us that the process X is a.s. bounded if and only if $\theta + \nu > 1/2$. Thus let us assume that this is satisfied.

Take now any $1 \leq k < l$. Then by (4.1) it follows

$$d_X(k, l) \geq k^{-\theta} - l^{-\theta} \geq k^{-\theta} - (k+1)^{-\theta} \geq c_\theta k^{-\theta-1}.$$

Hence, if $1 \leq k < l \leq n$ for some $n \geq 2$, this implies

$$d_X(k, l) \geq c_\theta n^{-\theta-1}$$

which yields

$$N(T, d_X, \varepsilon) \geq c \varepsilon^{-1/(\theta+1)} \quad (4.4)$$

for some $c > 0$ only depending on θ .

On the other hand, we have $\alpha(k)\sigma(k) = k^{-(\theta+\nu)}$. As shown in [9, Proposition 6.3] (apply this proposition with $q = 2$, $H = 0$ and $\gamma = 2(\theta + \nu)$) a bound $\alpha(k)\sigma(k) \leq k^{-(\theta+\nu)}$ implies

$$N(T, d, \varepsilon) \leq c \varepsilon^{-1/(\theta+\nu)}. \quad (4.5)$$

Of course, if $\nu > 1$, then (4.4) and (4.5) lead to

$$\lim_{\varepsilon \rightarrow 0} \frac{N(T, d_X, \varepsilon)}{N(T, d, \varepsilon)} = \infty, \quad (4.6)$$

completing the proof. □

Let us state an interesting consequence of the preceding proposition. To this end, recall a result due to M. Talagrand (cf. [14] and [6]). Suppose $X = (X_t)_{t \in T}$ is a centered Gaussian process on an arbitrary index set T and let d_X , as in (1.3), be the Dudley metric on T generated by X . If $N(T, d_X, \varepsilon) \leq \psi(\varepsilon)$ for a non-increasing function ψ satisfying

$$c_1 \psi(\varepsilon) \leq \psi(\varepsilon/2) \leq c_2 \psi(\varepsilon) \tag{4.7}$$

for certain $1 < c_1 < c_2$, then this implies

$$-\log \mathbb{P} \left(\sup_{t \in T} |X_t| < \varepsilon \right) \leq c \psi(\varepsilon) \tag{4.8}$$

for some $c > 0$.

We claim now that in the case of processes X defined by (4.3) even holds

$$-\log \mathbb{P} \left(\sup_{k \geq 1} \left| k^{-\theta} \left[\sum_{j=1}^k j^{-\nu} \xi_j + \xi_0 \right] \right| < \varepsilon \right) \approx \varepsilon^{-1/(\theta+\nu)}. \tag{4.9}$$

Here and later on " \approx " has to be understood as follows: Given functions f and g on $(0, \varepsilon_0)$, we write $f(\varepsilon) \approx g(\varepsilon)$ provided that

$$0 < \liminf_{\varepsilon \rightarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} \leq \limsup_{\varepsilon \rightarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} < \infty.$$

To verify (4.9), apply Proposition 7.1 in [9] with $\varphi(x) = x^{-\gamma}$ where $\gamma = 2(\theta + \nu)$. Then it follows that (4.5) is sharp, i.e., we obtain

$$N(T, d, \varepsilon) \approx \varepsilon^{-1/(\theta+\nu)}.$$

Consequently, (4.9) follows by Proposition 9.1 in [9].

Comparing (4.9) with (4.4) shows that for $\nu > 1$ estimate (4.8) cannot lead to sharp estimates while, as seen above, the use of $N(T, d, \varepsilon)$ does so. In some sense this observation proves that the metric d fits better to those processes X than d_X does.

One may ask now whether or not there are examples of trees and weights such that the quotient in (4.6) tends to zero, i.e., whether there are examples with

$$\lim_{\varepsilon \rightarrow 0} \frac{N(T, d, \varepsilon)}{N(T, d_X, \varepsilon)} = \infty. \tag{4.10}$$

Although we do not know the answer to this question let us shortly indicate why such examples are hard to construct provided they exist. Indeed, if $N(T, d, \varepsilon) \approx \varepsilon^{-a} |\log \varepsilon|^b$ for some $a > 0$ and $b \geq 0$, then by Proposition 9.1 in [9] this implies

$$-\log \mathbb{P} \left(\sup_{t \in T} |X_t| < \varepsilon \right) \approx \varepsilon^{-a} |\log \varepsilon|^b.$$

Consequently, whenever $N(T, d_X, \varepsilon) \approx \psi(\varepsilon)$ with ψ satisfying (4.7), then by (4.8) we get

$$N(T, d, \varepsilon) \leq c \psi(\varepsilon) \leq c' N(T, d_X, \varepsilon),$$

hence in that situation examples satisfying (4.10) cannot exist.

In spite of this observation we will show now that $d_X(t, s)$ may become arbitrarily small while $d(t, s) \geq C > 0$. Hence an estimate $d(t, s) \leq c d_X(t, s)$ cannot be valid in general. Recall that in view of Proposition 4.1 a relation $d_X(t, s) \leq c d(t, s)$ is impossible as well.

Proposition 4.2. *There are weights α and σ on $T = \mathbb{N}_0$ such that the corresponding process X is a.s. bounded and such that $\lim_{k \rightarrow \infty} d_X(\mathbf{0}, k) = 0$ while $d(\mathbf{0}, k) = C > 0$ for all $k \geq 1$.*

Proof. For $k \in \mathbb{N}_0$ choose $\sigma(k) = 2^{-k}$ while $\alpha(0) = 0$ and $\alpha(k) = k^{-1}$ for $k \geq 1$. Of course, the generated process X is a.s. bounded. Moreover, if $k \geq 1$, then it follows that

$$d_X(\mathbf{0}, k) = 2^{-k} \left(\sum_{v=1}^k v^{-2} \right)^{1/2}.$$

In particular, $d_X(\mathbf{0}, k) \rightarrow 0$ quite rapidly as $k \rightarrow \infty$. On the other hand,

$$d(\mathbf{0}, k) = 2^{-1} \alpha(1) = 2^{-1}$$

and this completes the proof with $C = 2^{-1}$. □

5 Proof of Proposition 2.2

Before proving Proposition 2.2, let us come back to a geometric interpretation of Gaussian processes briefly mentioned at the end of Section 3. We identify a process $X = (X_t)_{t \in T}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with the subset $\{X_t : t \in T\}$ of the Hilbert space $L_2(\Omega, \mathcal{A}, \mathbb{P})$. The induced distance equals

$$\|X_t - X_s\|_2 = (\mathbb{E}|X_t - X_s|^2)^{1/2} = d_X(t, s),$$

hence also

$$N(T, d_X, \varepsilon) = N(X, \|\cdot\|_2, \varepsilon).$$

Proof of Proposition 2.2: We first give the lower bounds for $N(T, d_X, \varepsilon)$. By (3.10) and (3.11) it follows that

$$N(X - [0, 1] \cdot X, \|\cdot\|_2, \frac{\varepsilon}{\sqrt{2}}) \geq N(T, d, 2\varepsilon) - 1. \quad (5.1)$$

Our next task is to replace $X - [0, 1] \cdot X$ by X in (5.1) by using the following trivial fact.

Lemma 5.1. *Let X be a subset of a normed space and $M_X := \sup_{x \in X} \|x\|$. Then*

$$N([0, 1] \cdot X, \|\cdot\|, 2\varepsilon) \leq N(X, \|\cdot\|, \varepsilon) \frac{M_X}{\varepsilon} \quad \text{and} \quad (5.2)$$

$$N(X - [0, 1] \cdot X, \|\cdot\|, 3\varepsilon) \leq N(X, \|\cdot\|, \varepsilon)^2 \frac{M_X}{\varepsilon}. \quad (5.3)$$

of the lemma. Let B be an ε -net for X and set

$$C = \left\{ \frac{j\varepsilon}{M_X} y : y \in B, j \in \mathbb{N}, 1 \leq j \leq \frac{M_X}{\varepsilon} \right\}.$$

Clearly,

$$\#\{C\} \leq \#\{B\} \frac{M_X}{\varepsilon}.$$

Take any $z = \theta x \in [0, 1] \cdot X$ with $x \in X$, $\theta \in [0, 1]$. Find $y \in B$ and a positive integer $j \leq \frac{M_X}{\varepsilon}$ such that

$$\|x - y\| < \varepsilon, \quad \left| j - \frac{\theta M_X}{\varepsilon} \right| \leq 1.$$

Then $z' := \frac{j\varepsilon}{M_X} y \in C$ and observe that

$$\begin{aligned} \|z - z'\| &= \left\| \theta x - \frac{j\varepsilon}{M_X} y \right\| \\ &\leq \theta \|x - y\| + \left| \theta - \frac{j\varepsilon}{M_X} \right| \|y\| \\ &< \varepsilon + \frac{\varepsilon}{M_X} M_X = 2\varepsilon. \end{aligned}$$

Hence, C is a 2ε -net for $[0, 1] \cdot X$ and the first claim of the lemma is proved. The second one follows immediately. \square

We may proceed now with the proof of Proposition 2.2. Combining (5.3) with (5.1) leads to

$$N(T, d_X, \varepsilon)^2 \geq \frac{\varepsilon}{M_X} \left[N(T, d, 6\sqrt{2}\varepsilon) - 1 \right].$$

Hence, we conclude

$$\int_0^\infty \sqrt{\log N(T, d_X, u)} du < \infty \quad \Rightarrow \quad \int_0^\infty \sqrt{\log N(T, d, u)} du < \infty$$

and

$$\sup_{\varepsilon > 0} \varepsilon^2 \log N(T, d_X, \varepsilon) < \infty \quad \Rightarrow \quad \sup_{\varepsilon > 0} \varepsilon^2 \log N(T, d, \varepsilon) < \infty.$$

Conversely, we will move now towards an upper bound for $N(T, d_X, \varepsilon)$. By Lemma 3.3 and Proposition 3.2 of [9] we have

$$N(T, d_Y, \varepsilon) \leq \tilde{N}(T, \hat{d}, \varepsilon) + 1 \leq \tilde{N}(T, d, \varepsilon/2) + 1 \leq N(T, d, \varepsilon/4) + 1. \quad (5.4)$$

Next, if \hat{X} is defined as in (3.3), then (3.6) yields

$$\hat{X} \subseteq \sum_{m=0}^{\infty} 2^{-m} Y,$$

and from this inclusion we trivially obtain that

$$\begin{aligned}
N(T, d_{\hat{X}}, 2\varepsilon) &\leq N\left(T, d_{\hat{X}}, \left(\sum_{m=0}^{\infty} (m+1)^{-2}\right) \varepsilon\right) \\
&\leq \prod_{m=0}^{\infty} N(T, d_{2^{-m}Y}, (m+1)^{-2}\varepsilon) \\
&= \prod_{m=0}^{\infty} N(T, d_Y, 2^m(m+1)^{-2}\varepsilon). \\
&\leq \prod_{m=0}^{\infty} N_*(T, d, 2^{m-2}(m+1)^{-2}\varepsilon),
\end{aligned}$$

where we used (5.4) on the last step and

$$N_*(T, d, r) := \begin{cases} N(T, d, r) + 1 & : N(T, d_Y, 4r) > 1, \\ 1 & : N(T, d_Y, 4r) = 1. \end{cases}$$

It follows that

$$\log N(T, d_{\hat{X}}, 2\varepsilon) \leq \sum_{\{m \geq 0: 2^m(m+1)^{-2}\varepsilon \leq M_Y\}} \log(N(T, d, 2^{m-2}(m+1)^{-2}\varepsilon) + 1), \quad (5.5)$$

where $M_Y := \sup_{t \in T} \|Y_t\|_2$. For the Dudley integral this implies

$$\begin{aligned}
\int_0^\infty \sqrt{\log N(T, d_{\hat{X}}, 2\varepsilon)} d\varepsilon &\leq \sum_{m=0}^{\infty} \int_0^{2^{m(m+1)^{-2}M_Y}} \sqrt{\log(N(T, d, 2^{m-2}(m+1)^{-2}\varepsilon) + 1)} d\varepsilon \\
&\leq \sum_{m=0}^{\infty} \frac{(m+1)^2}{2^{m-2}} \int_0^\infty \sqrt{\log(N(T, d, u) + 1)} du \\
&= C \int_0^\infty \sqrt{\log(N(T, d, u) + 1)} du.
\end{aligned}$$

Hence,

$$\int_0^\infty \sqrt{\log N(T, d, u)} du < \infty \quad \Rightarrow \quad \int_0^\infty \sqrt{\log N(T, d_{\hat{X}}, u)} du < \infty.$$

Moreover, (5.5) yields

$$\sup_{\varepsilon > 0} \varepsilon^2 \log N(T, d, \varepsilon) < \infty \quad \Rightarrow \quad \sup_{\varepsilon > 0} \varepsilon^2 \log N(T, d_{\hat{X}}, \varepsilon) < \infty.$$

The final passage goes from \hat{X} to X . Since $X \subseteq [0, 1] \cdot \hat{X}$, by applying (5.2) to \hat{X} we obtain

$$\log N(T, d_X, 2\varepsilon) \leq \log N([0, 1] \cdot \hat{X}, \|\cdot\|_2, 2\varepsilon) \leq \log N(T, d_{\hat{X}}, \varepsilon) + \log\left(\frac{M_{\hat{X}}}{\varepsilon}\right).$$

Hence

$$\int_0^\infty \sqrt{\log N(T, d_{\hat{X}}, u)} du < \infty \quad \Rightarrow \quad \int_0^\infty \sqrt{\log N(T, d_X, u)} du < \infty,$$

as well as

$$\sup_{\varepsilon > 0} \varepsilon^2 \log N(T, d_{\hat{X}}, \varepsilon) < \infty \quad \Rightarrow \quad \sup_{\varepsilon > 0} \varepsilon^2 \log N(T, d_X, \varepsilon) < \infty.$$

By combining the preceding estimates we finish the proof. \square

6 Applications to binary trees

6.1 General weights

Let us first demonstrate how the general estimates of $N(T, d, \varepsilon)$ in [9] combined with Theorem 2.1 lead to concrete assertions about the boundedness of processes X defined by (1.1) in the case of a binary tree.

Proposition 6.1. *Let T be a binary tree and suppose that*

$$\alpha(t)\sigma(t) \leq c |t|^{-\gamma}, \quad t \in T,$$

for some $\gamma > 1$. Then X defined by (1.1) is a.s. bounded. Conversely, if

$$\alpha(t) \geq c |t|^{-\gamma}$$

for some $\gamma < 1$ and $\sigma(t) \equiv 1$, then the generated process X is a.s. unbounded.

Proof. As shown in [9], an estimate $\alpha(t)\sigma(t) \leq c |t|^{-\gamma}$ implies $\log N(T, d, \varepsilon) \leq c \varepsilon^{-2/(2\gamma-1)}$ for each $\gamma > 1/2$. Hence, if $\gamma > 1$, then (2.5) holds, hence Theorem 2.1 applies and completes the proof of the first part.

The second part follows by $\log N(T, d, \varepsilon) \geq c \varepsilon^{-2/(2\gamma-1)}$ whenever $\alpha(t) \geq c |t|^{-\gamma}$ for some $\gamma > 1/2$ and $\sigma(t) \equiv 1$ (cf. [9, Proposition 7.7]). Thus, by Theorem 2.1 the process X cannot be bounded if $\gamma < 1$. \square

Remark: The second part of Proposition 6.1 does no longer hold for non-constant weights σ . In different words, an estimate $\alpha(t)\sigma(t) \geq c |t|^{-\gamma}$ with $1/2 < \gamma < 1$ does not always imply that X is unbounded (cf. the remark after Corollary 6.5 below).

6.2 Homogeneous weights

Before investigating Gaussian processes with homogeneous weights let us shortly recall some basic facts about suprema of Gaussian sequences.

Let (X_1, \dots, X_n) be a centered Gaussian random vector. Introduce the following notations:

$$\sigma_1^2 := \min_j \mathbb{E}X_j^2, \quad \sigma_2^2 := \max_j \mathbb{E}X_j^2, \quad S := \max_j X_j,$$

and let m_S be a median of S . Then the following is well known.

- It is true that

$$m_S \leq \mathbb{E}S. \tag{6.1}$$

See [7], p.143.

- The following concentration principle is valid:

$$\mathbb{P}(S > m_S + r) \leq \hat{\Phi}(r/\sigma_2) \leq \exp(-r^2/2\sigma_2^2), \quad \forall r > 0,$$

where

$$\hat{\Phi}(r) = \frac{1}{\sqrt{2\pi}} \int_r^\infty e^{-\frac{u^2}{2}} du$$

is the standard Gaussian tail. See [7], p.142. By combining this with (6.1) we also have

$$\mathbb{P}(S > \mathbb{E}S + r) \leq \exp(-r^2/2\sigma_2^2), \quad \forall r > 0. \quad (6.2)$$

- It is true that

$$\mathbb{E}S \leq \sqrt{2 \log n} \sigma_2. \quad (6.3)$$

See [7], p.180.

- If X_1, \dots, X_n are independent, then

$$\mathbb{E}S \geq c \sqrt{\log n} \sigma_1. \quad (6.4)$$

with $c = 0.64$. See [7], p.193–194.

Remark that the same properties hold true for

$$S' := \max_{j \leq n} |X_j| = \max_{j \leq n} \max\{X_j, -X_j\}.$$

Let T be a binary tree and suppose that the weights depend only on the level numbers, i.e. $\alpha(t) = \alpha_{|t|}$ and $\sigma(t) = \sigma_{|t|}$ for some sequences $(\alpha_k)_{k \geq 0}$ and $(\sigma_k)_{k \geq 0}$ of positive numbers with $(\sigma_k)_{k \geq 0}$ non-increasing. The following two theorems give, with a certain overlap, necessary and sufficient conditions for the boundedness of $(X_t)_{t \in T}$ in that case.

Theorem 6.2. a) If $X = (X_t)_{t \in T}$ is a.s. bounded, then

$$G := \sup_{n \geq 1} \sigma_n \sum_{k=1}^n \alpha_k < \infty. \quad (6.5)$$

b) Moreover, if $(\alpha_k)_{k \geq 0}$ satisfies the regularity assumption

$$Q := \sup_{n \geq 1} \sup_{n \leq k \leq 2n} \frac{\alpha_k}{\alpha_n} < \infty, \quad (6.6)$$

then X is a.s. bounded if and only if (6.5) holds.

Proof. a) Let us construct a random sequence $(t_n)_{n \geq 0}$ in T and a sequence of random variables $(\zeta_n)_{n \geq 1}$ by the following inductive procedure. Let $t_0 = \mathbf{0}$. Next, assuming that t_n is constructed, let t' and t'' be the two offsprings of t_n . We let

$$\zeta_{n+1} := \max\{\xi_{t'}, \xi_{t''}\}, \quad t_{n+1} := \operatorname{argmax}\{\xi_{t'}, \xi_{t''}\}.$$

It is obvious that (ζ_n) are i.i.d. random variables with strictly positive expectation. Our construction yields

$$X_{t_n} = \sigma_n \left(\alpha_0 \xi_0 + \sum_{j=1}^n \alpha_j \zeta_j \right), \quad n \geq 1.$$

It follows that

$$\mathbb{E} \sup_{t \in T} X_t \geq \sup_{n \geq 1} \mathbb{E} X_{t_n} = C \sup_{n \geq 1} \sigma_n \sum_{j=1}^n \alpha_j,$$

where $C := \mathbb{E} \zeta_j > 0$. Since the assumption " $(X_t)_{t \in T}$ is a.s. bounded" implies $\mathbb{E} \sup_{t \in T} X_t < \infty$, we obtain (6.5).

b) Let us assume that $G < \infty$, $Q < \infty$ and prove that $(X_t)_{t \in T}$ is a.s. bounded. For any $m \geq 0$ set $B_m = [2^m, 2^{m+1})$ and $J_m := \{t \in T : |t| \in B_m\}$. For any $M \geq 1$ and $t \in J_M$ write

$$\sum_{\substack{v \leq t \\ |v| \geq 2}} \alpha(v) \xi_v = \sum_{m=1}^M \sum_{\substack{v \leq t \\ v \in J_m}} \alpha(v) \xi_v \leq \sum_{m=1}^M U_m, \quad (6.7)$$

where

$$U_m := \sup_{u \in J_m} \left| \sum_{\substack{v \leq u \\ v \in J_m}} \alpha(v) \xi_v \right|.$$

Let

$$\tilde{X}_t = \sigma_t \sum_{\substack{v \leq t \\ |v| \geq 2}} \alpha(v) \xi_v, \quad t \in T,$$

be the process that differs from X by the two terms of order 0 and 1. By using that $(\sigma_k)_{k \geq 0}$ is non-increasing, we infer from (6.7) for any $M \geq 1$ and $t \in J_M$

$$\begin{aligned} \tilde{X}_t &\leq \sigma_{2^M} \sum_{m=1}^M U_m = \sigma_{2^M} \sum_{m=1}^M (\mathbb{E} U_m + (U_m - \mathbb{E} U_m)) \\ &\leq \sigma_{2^M} \sum_{m=1}^M (\mathbb{E} U_m + (U_m - \mathbb{E} U_m)_+) \\ &\leq \sigma_{2^M} \sum_{m=1}^M \mathbb{E} U_m + \sum_{m=1}^{\infty} \sigma_{2^m} (U_m - \mathbb{E} U_m)_+. \end{aligned}$$

Hence,

$$\sup_{t \in T} \tilde{X}_t \leq \sup_{M \geq 1} \sigma_{2^M} \sum_{m=1}^M \mathbb{E} U_m + \sum_{m=1}^{\infty} \sigma_{2^m} (U_m - \mathbb{E} U_m)_+. \quad (6.8)$$

We will use now standard Gaussian techniques in order to evaluate the quantities on the r.h.s. Note that on the binary tree

$$\#\{J_m\} \leq \#\{t : |t| < 2^{m+1}\} \leq 2^{2^{m+1}}, \quad \text{as } m \geq 0.$$

Moreover, we have

$$h_m^2 := \sup_{u \in J_m} \sum_{\substack{v \leq u \\ v \in J_m}} \alpha(v)^2 \leq \sum_{k \in B_m} \alpha_k^2, \quad \text{as } m \geq 0.$$

Assuming (6.6) to hold, we obtain

$$h_m^2 \leq \sum_{k \in B_m} \alpha_k^2 \leq Q^2 2^m \alpha_{2^m}^2, \quad \text{as } m \geq 0.$$

Using (6.6) again we arrive at

$$h_m \leq Q 2^{m/2} \alpha_{2^m} \leq Q^2 2^{1-m/2} \sum_{k \in B_{m-1}} \alpha_k, \quad \text{as } m \geq 1. \quad (6.9)$$

Now by (6.3) it follows that

$$\mathbb{E}U_m \leq \sqrt{2 \log(\#\{J_m\})} h_m \leq 4Q^2 \sum_{k \in B_{m-1}} \alpha_k, \quad \text{as } m \geq 1.$$

Hence, for any $M \geq 1$ we get

$$\sigma_{2^M} \sum_{m=1}^M \mathbb{E}U_m \leq \sigma_{2^M} 4Q^2 \sum_{m=1}^M \sum_{k \in B_{m-1}} \alpha_k = \sigma_{2^M} 4Q^2 \sum_{k=1}^{2^M-1} \alpha_k \leq 4Q^2 G.$$

On the other hand, by the Gaussian concentration principle (6.2),

$$\mathbb{E}(U_m - \mathbb{E}U_m)_+ = \int_0^\infty \mathbb{P}(U_m - \mathbb{E}U_m > r) dr \leq \int_0^\infty \exp(-r^2/2h_m^2) dr \leq 2h_m.$$

From (6.9) it follows that

$$\begin{aligned} \sigma_{2^m} \mathbb{E}(U_m - \mathbb{E}U_m)_+ &\leq 2\sigma_{2^m} h_m \\ &\leq 2\sigma_{2^m} Q^2 2^{1-m/2} \sum_{k \in B_{m-1}} \alpha_k \\ &\leq 2^{2-m/2} Q^2 \sigma_{2^m} \sum_{k=1}^{2^m} \alpha_k \\ &\leq 2^{2-m/2} Q^2 G, \quad \text{as } m \geq 1. \end{aligned}$$

By plugging this into (6.8), we arrive at

$$\mathbb{E} \sup_{t \in T} \tilde{X}_t \leq \sup_{M \geq 1} \sigma_{2^M} \sum_{m=1}^M \mathbb{E}U_m + \sum_{m=1}^\infty \mathbb{E} \sigma_{2^m} (U_m - \mathbb{E}U_m)_+ \leq 4Q^2 G + Q^2 G \sum_{m=1}^\infty 2^{2-m/2} < \infty$$

and $(\tilde{X}_t)_{t \in T}$ is a.s. bounded. Since we have a bound

$$|X_t - \tilde{X}_t| = \sigma_t \left| \sum_{\substack{v \leq t \\ |v| \leq 1}} \alpha(v) \xi_v \right| \leq \sigma_0 (\alpha_0 + \alpha_1) \max_{|v| \leq 1} |\xi_v|$$

uniformly over $t \in T$, we conclude that $(X_t)_{t \in T}$ is also a.s. bounded. \square

Let us give an example where Theorem 6.2 applies efficiently. Take the binary tree T and suppose that either $\alpha(t) = (|t| + 1)^{-1}$ and $\sigma(t) \equiv 1$ or that $\alpha(t) \equiv 1$ and $\sigma(t) = (|t| + 1)^{-1}$. Note that these weights lead to critical cases, namely, we have $\log N(T, d, \varepsilon) \approx \varepsilon^{-2}$ for both pairs of weights.

Corollary 6.3. *The process*

$$X'_t := (|t| + 1)^{-1} \sum_{v \preceq t} \xi_v, \quad t \in T,$$

is a.s. bounded, while

$$X''_t := \sum_{v \preceq t} (|v| + 1)^{-1} \xi_v, \quad t \in T,$$

is a.s. unbounded.

Proof. In the first case (6.5) and (6.6) are satisfied while in the second one (6.5) fails. Thus both assertions follow by Theorem 6.2. \square

Remark: The preceding corollary is of special interest because $\alpha(t)\sigma(t) = (|t| + 1)^{-1}$ in both cases. Consequently, the boundedness of the process X cannot be described by the behavior of $\alpha\sigma$. This is in contrast to the main results about metric entropy in [9] which only depend on this product behavior.

Theorem 6.2 does not apply in the case of rapidly increasing sequences $(\alpha_k)_{k \geq 0}$ because (6.6) fails for them. The next theorem fills this gap.

Theorem 6.4. *a) If $X = (X_t)_{t \in T}$ is a.s. bounded, then*

$$G_1 := \sup_n \sup_{m \leq n} \sigma_n \sqrt{m} \left(\sum_{k=m}^n \alpha_k^2 \right)^{1/2} < \infty. \quad (6.10)$$

b) If

$$G_2 := \sup_n \sigma_n \sqrt{n} \left(\sum_{k=0}^n \alpha_k^2 \right)^{1/2} < \infty, \quad (6.11)$$

then $(X_t)_{t \in T}$ is a.s. bounded.

c) Moreover, if $(\alpha_k)_{k \geq 0}$ is non-decreasing, then the conditions (6.10) and (6.11) are equivalent, thus X is a.s. bounded if and only if either of them holds.

Proof. a) Let us fix a pair of integers $m \leq n$. Take any mapping $L : \{t : |t| = m\} \rightarrow \{t : |t| = n\}$ such that $t \preceq L(t)$ for all t . Consider

$$Y_t := \sigma_n \sum_{t \preceq s \preceq L(t)} \alpha_{|s|} \xi_s, \quad |t| = m.$$

Notice that the variables $(Y_t)_{|t|=m}$ are independent and that

$$\mathbb{E}Y_t^2 = \sigma_n^2 \sum_{m \leq k \leq n} \alpha_k^2.$$

By (6.4) it follows

$$\mathbb{E} \max_{|t|=m} Y_t \geq c \sqrt{\log(2^m)} \sigma_n \left(\sum_{m \leq k \leq n} \alpha_k^2 \right)^{1/2} = \tilde{c} \sqrt{m} \sigma_n \left(\sum_{m \leq k \leq n} \alpha_k^2 \right)^{1/2}.$$

On the other hand

$$Y_t = X_{L(t)} - \frac{\sigma_n}{\sigma_m} X_t,$$

hence

$$\max_{|t|=m} Y_t \leq 2 \sup_{t \in T} |X_t|.$$

We arrive at

$$2 \mathbb{E} \sup_{t \in T} |X_t| \geq \tilde{c} \sqrt{m} \sigma_n \left(\sum_{m \leq k \leq n} \alpha_k^2 \right)^{1/2},$$

and achieve the proof of a) by taking the supremum over m and n .

b) Let $S_n := \max_{|t|=n} X_t$. By (6.3) we have

$$\mathbb{E} S_n \leq \sqrt{2 \log(2^n)} \sigma_n \left(\sum_{k=0}^n \alpha_k^2 \right)^{1/2} \leq 2G_2. \quad (6.12)$$

We also have

$$\mathbb{E} X_t^2 = \sigma_n^2 \sum_{k=0}^n \alpha_k^2 \leq \frac{G_2^2}{n}, \quad |t| = n. \quad (6.13)$$

Since $\sup_{t \in T} X_t = \sup_n S_n$, for any $r > 0$ it follows that

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in T} X_t > 2G_2 + r \right) &\leq \sum_{n=0}^{\infty} \mathbb{P} (S_n \geq 2G_2 + r) \\ &\leq \sum_{n=0}^{\infty} \mathbb{P} (S_n \geq \mathbb{E} S_n + r) \quad (\text{by (6.12)}) \\ &\leq \mathbb{P} (S_0 \geq \mathbb{E} S_0 + r) + \sum_{n=1}^{\infty} \exp \left(-\frac{r^2 n}{2G_2^2} \right) \quad (\text{by (6.13) and (6.2)}) \\ &= \mathbb{P} (X_0 \geq r) + \frac{\exp \left(-\frac{r^2}{2G_2^2} \right)}{1 - \exp \left(-\frac{r^2}{2G_2^2} \right)} \rightarrow 0, \quad \text{as } r \rightarrow \infty. \end{aligned}$$

It follows that $(X_t)_{t \in T}$ is a.s. bounded. Thus assertion b) is proved.

c) The inequality $G_1 \leq G_2$ is obvious for any $(\alpha_k)_{k \geq 0}$. We only need to show that a bound in the opposite direction holds, too. Let

$$m_n := \begin{cases} \frac{n}{2} & : n \text{ even} \\ \frac{n+1}{2} & : n \text{ odd.} \end{cases}$$

Assuming that $(\alpha_k)_{k \geq 0}$ is non-decreasing, we have

$$\sum_{m_n \leq k \leq n} \alpha_k^2 \geq \sum_{0 \leq k < m_n} \alpha_k^2,$$

hence

$$2 \sum_{m_n \leq k \leq n} \alpha_k^2 \geq \sum_{0 \leq k \leq n} \alpha_k^2.$$

It follows that

$$G_1 \geq \sup_n \sigma_n \sqrt{m_n} \left(\sum_{m_n \leq k \leq n} \alpha_k^2 \right)^{1/2} \geq \frac{1}{2} \sup_n \sigma_n \sqrt{n} \sum_{0 \leq k \leq n} \alpha_k^2 = \frac{G_2}{2}.$$

□

Corollary 6.5. Let $\alpha_k = k^b 2^k$ for some $b \in \mathbb{R}$. Then $(X_t)_{t \in T}$ is a.s. bounded if and only if

$$\sup_n \sigma_n n^{1/2+b} 2^n < \infty.$$

Remark: Note that criterion (6.5) from Theorem 6.2 fails to work in that case. Moreover, letting $b = -\gamma$ with $1/2 < \gamma < 1$ and $\sigma_n = 2^{-n}$, by Corollary 6.5 the corresponding process is bounded although $\alpha(t)\sigma(t) \geq |t|^{-\gamma}$ for $t \in T$. This shows that the second part of Proposition 6.1 is no longer valid for non-constant weights σ .

Another example where Theorem 6.2 does not apply is as follows.

Corollary 6.6. Let $\alpha_k^2 = \exp((\log k)^\beta)$ with $\beta > 1$. Then $(X_t)_{t \in T}$ is a.s. bounded if and only if

$$\sup_n \sigma_n \frac{n}{(\log n)^{\frac{\beta-1}{2}}} \exp((\log n)^\beta / 2) < \infty.$$

Proof. Easy calculation shows that

$$\begin{aligned} \sum_{k=0}^n \alpha_k^2 &\sim \int_1^n \exp((\log u)^\beta) du = \int_0^{(\log n)^\beta} \exp(z + z^{1/\beta}) \frac{dz}{\beta z^{1-1/\beta}} \\ &\sim \frac{n}{\beta (\log n)^{\beta-1}} \exp((\log n)^\beta). \end{aligned}$$

An application of Theorem 6.4 yields the result. □

Our message is that Theorems 6.2 and 6.4 should *jointly* cover any reasonable case. Let us illustrate this by the following example. Recall that by the first part of Proposition 6.1, if T is the binary tree and $\alpha(t)\sigma(t) \leq c|t|^{-\gamma}$ for some $\gamma > 1$, then the generated process X is a.s. bounded. For homogeneous (level-dependent) weights this means that $\alpha_k \sigma_k \leq c k^{-\gamma}$ for some $\gamma > 1$ yields the a.s. boundedness of X . Let us see how this fact is related to Theorems 6.2 and 6.4.

Essentially, we have the following

- If $(\alpha_k)_{k \geq 0}$ is decreasing, then

$$\sigma_n \sum_{k \leq n} \alpha_k \leq \sigma_n \sum_{k \leq n} \frac{c k^{-\gamma}}{\sigma_k} \leq \sum_{k \leq n} c k^{-\gamma} \leq c \sum_{k=1}^{\infty} k^{-\gamma},$$

hence (6.5) and (6.6) hold and Theorem 6.2 yields the boundedness.

- If $(\alpha_k)_{k \geq 0}$ is increasing, then

$$\sigma_n \sqrt{n} \left(\sum_{k \leq n} \alpha_k^2 \right)^{1/2} \leq \sigma_n \sqrt{n} (n \alpha_n^2)^{1/2} = \sigma_n \alpha_n n \leq c n^{1-\gamma},$$

thus (6.11) holds even for $\gamma \geq 1$, and Theorem 6.4 yields the boundedness.

Finally let us relate the results in Theorems 6.2 and 6.4 to those about compactness properties of (T, d) with d defined in (2.3). Here we have the following partial result.

Proposition 6.7. *The expression G_1 in (6.10) is finite if and only if there is a constant $c > 0$ such that*

$$d(t, s) \leq c |t|^{-1/2} \tag{6.14}$$

for all $t, s \in T$ with $t \prec s$.

Proof. First note that in the case of homogeneous weights we get

$$d(t, s) = \max_{|t| < l \leq |s|} \sigma_l \left(\sum_{k=|t|+1}^l \alpha_k^2 \right)^{1/2}.$$

Next we remark that $G_1 < \infty$ if and only if there is a constant $c > 0$ such that

$$\sigma_n \left(\sum_{k=m+1}^n \alpha_k^2 \right)^{1/2} \leq c m^{-1/2} \tag{6.15}$$

for all $0 \leq m < n < \infty$.

Suppose now that (6.14) holds and take integers $m < n$. Next, choose two elements $t, s \in T$ with $t \prec s$ such that $m = |t|$ and $n = |s|$. Note that (6.14) implies

$$\sigma_n \left(\sum_{k=m+1}^n \alpha_k^2 \right)^{1/2} \leq d(t, s) \leq c |t|^{-1/2} = c m^{-1/2}$$

which proves (6.15).

Conversely, assume (6.15) and take any two elements $t \prec s$ in T . Furthermore, let $v \in (t, s]$ be a node where

$$d(t, s) = \sigma_{|v|} \left(\sum_{k=|t|+1}^{|v|} \alpha_k^2 \right)^{1/2}.$$

Applying (6.15) with $m := |t|$ and $n := |v|$ leads to

$$d(t, s) \leq c m^{-1} = c |t|^{-1/2}$$

as claimed. This completes the proof. □

Remark: Clearly (6.14) implies $\log N(T, d, \varepsilon) \leq c \varepsilon^{-2}$, as we already know by combining Theorems 2.1 and 6.4. But it says a little bit more. Namely, an ε -net giving this order may be chosen as $\{t \in T : |t| \leq c \varepsilon^{-1/2}\}$ for a certain $c > 0$. Of course, this heavily depends on the fact that we deal with homogeneous weights.

7 Two-weight processes vs one-weight ones

Proposition 6.1 suggests that the boundedness of a process with weights $\alpha(t)$ and $\sigma(t)$ might be determined by the product $\sigma(t)\alpha(t)$. In other words, it is natural to ask what is the relation between the boundedness of this process and the process generated by the weights $\tilde{\sigma}(t) := 1$ and $\tilde{\alpha}(t) := \sigma(t)\alpha(t)$. It turns out that (only) a one-sided implication is valid.

To investigate this question, write now $X^{\alpha, \sigma}$ for the process defined in (1.1).

Proposition 7.1. *If $X^{\alpha\sigma, 1}$ is a.s. bounded, then this is also true for $X^{\alpha, \sigma}$.*

Proof. We only give a sketch of the proof.

1. Recall a general fact from the theory of Gaussian processes: If X and Y are two independent centered Gaussian processes, then $X + Y$ bounded yields X bounded. This is an immediate consequence of Anderson's inequality (cf. [7, p.135]).
2. By applying this fact we obtain: If two weights are related by $\alpha_1 \leq c \alpha_2$ and $X^{\alpha_2, 1}$ is bounded, then $X^{\alpha_1, 1}$ is bounded as well.
3. Suppose now that $X^{\alpha\sigma, 1}$ is bounded, then $X^{\alpha\hat{\sigma}, 1}$ is bounded, where the binary weight $\hat{\sigma}$ is defined in (3.2). Set $X' := X^{\alpha\hat{\sigma}, 1}$.
4. Let now Y be the process constructed in the article associated to $X^{\alpha, \sigma}$. Then we get $Y_t = X'_t - X'_{\lambda^-(t)}$ with $\lambda^-(t)$ defined by (3.7). From this we see that if X' is bounded, then Y is bounded as well.
5. Recall that we know that the boundedness of Y is equivalent to that of $X^{\alpha, \sigma}$.

□

The examples in Corollary 6.3 show that the statement of Proposition 7.1 cannot be reversed, i.e., in general the boundedness of $X^{\alpha, \sigma}$ does not yield that of $X^{\alpha\sigma, 1}$.

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