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Representation theorem for generators of BSDEs with monotonic and polynomial-growth generators in the space of processes *

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Abstract

In this paper, on the basis of some recent works of Fan, Jiang and Jia, we establish a representation theorem in the space of processes for generators of BSDEs with monotonic and polynomial-growth generators, which generalizes the corresponding results in Fan (2006, 2007), and Fan and Hu (2008).

Key words: Backward stochastic differential equation; Monotonic generator; Polynomial-growth generator; Representation theorem of generators.

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1 Introduction

By Pardoux and Peng (1990), we know that there exists a unique square-integrable and adapted solution to a backward stochastic differential equation (BSDE for short in the remainder of this paper) of the type

$$y_s = \xi + \int_s^T g(u, y_u, z_u)\mathrm{d}u - \int_s^T z_u \cdot \mathrm{d}B_u \tag{1}$$

provided that g is Lipschitz in both variables y and z and that ξ and $(g(t,0,0))_{t\in[0,T]}$ are square integrable. The g is called the generator of BSDE (1), ξ the terminal data and the triple (ξ, T, g) the parameters of BSDE (1). We denote the unique solution by $(y_s^{\xi,T,g}, z_s^{\xi,T,g})_{s\in[0,T]}$, and often denote $y_t^{\xi,T,g}$ by $\mathscr{E}_{t,T}^g[\xi]$ for every $t \in [0,T]$.

One of the achievements of BSDE theory is the comparison theorem. Recently, many papers have been devoted to studying the converse comparison theorem. For studying the converse comparison theorem, Briand et al. (2000) established the following representation theorem of generators for BSDEs in the space of random variables: For every $(t,y,z) \in [0,T[\times \mathbf{R}^{1+d}$,

$$\lim_{n\to\infty} n\{\mathscr{E}_{t,t+1/n}^g[y + z\cdot(B_{t+1/n} - B_t)] - y\} = g(t,y,z) \tag{2}$$

holds true in (the space of random variables) L^2 when g satisfies two additional assumptions that $\mathbf{E}\left[\sup_{0\le t\le T} |g(t,0,0)|^2\right] < \infty$ and $(g(t,y,z))_{t\in[0,T]}$ is continuous in t for every (y,z). Since then, much effort has been made to weaken and eliminate these two assumptions mentioned above. For instance, after weakening these two assumptions step by step in Jiang (2005a, b, c), under the most elementary conditions that g is Lipschitz in both variables y and z and that ξ and $(g(t,0,0))_{t\in[0,T]}$ are square-integrable, Jiang (2006, 2008) finally proved that (2) holds true in (the space of random variables) L^p $(1 \le p < 2)$ for almost every $t \in [0,T)$. Furthermore, under a continuity condition in t on stochastic differential equations (SDEs in short), Jiang (2005d) generalized this result to the case where the terminal data of BSDEs are solutions of the SDEs.

On the other hand, from the point of view of Fan and Hu (2008), it seems to be more appropriate for this kind of representation theorem to be investigated in the space of processes rather than in the space of random variables, that is to say, without fixing t, we are to investigate whether (2) holds in some kinds of spaces of processes. Accordingly, Fan (2006, 2007) and Fan and Hu (2008) investigated this kind of representation theorem in the space of processes and eliminated the above continuity condition in t on SDEs used in Jiang (2005d).

Furthermore, Mao (1995) established an existence and uniqueness result of solutions for BSDE (1) where g satisfies a non-Lipschitz condition in y, the corresponding representation theorem in L^p $(1 \le p < 2)$ was established in Liu and Jiang (2008). Lepeltier and San Martin (1997) proved the existence and uniqueness of the minimal and maximal solutions for BSDE (1) where $(g(\omega,t,0,0))_{t\in[0,T]}$ is a bounded process and g is continuous with linear growth in (y,z), the corresponding representation theorem in L^p $(1 \le p < 2)$ was obtained in Jia (2008). Very recently, Fan and Jiang (2010a) extended the existence and uniqueness result in Lepeltier and San Martin (1997) by eliminating the condition that $(g(\omega,t,0,0))_{t\in[0,T]}$ is a bounded process, the corresponding representation theorem in the space of processes has also been established in Fan and Jiang (2010b).

It should be noted that all these representation results dealt with the case that the generator g is of linear growth in y. In this paper, we are the first time to consider the case that g is of polynomial growth in y. More precisely, on basis of the existence and uniqueness result of the minimal and maximal solutions for BSDE (1) obtained in Briand et al. (2007), we establish a new representation theorem in the space of processes, where the generator g is continuous in (y,z) and monotonic in y, it has a polynomial growth in y and a linear growth in z, and the terminal data are solutions of SDEs (see Theorem 1 in Section 2). This representation theorem further generalizes the corresponding results in Fan (2006, 2007) and Fan and Hu (2008).

Finally, we would like to mention that the representation theorem has been playing an important role in investigating properties of generators of BSDEs by virtue of solutions of BSDEs. In fact, a lot of problems in BSDE theory and nonlinear mathematical expectation theory are related to the above representation theorem. For example, it was just with the help of the representation theorem that many important results have been obtained in Briand et al. (2000),Chen et al. (2003),Jiang and Chen (2004), Jiang (2004, 2005a, b, c, d, 2006, 2008), Fan (2006, 2007), Fan and Hu (2008) and Fan and Jiang (2010b).

This paper is organized as follows: In section 2, after introducing some notations and assumptions, we put forward our main result–Theorem 1. Section 3 is devoted to the proof of the main result. Finally, some applications are given in Section 4.

2 Notations, assumptions and the main result

Let $(\Omega, \mathscr{F}, P)$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$, and let $(\mathscr{F}_t)_{t\geq 0}$ be the σ-algebra generated by B augmented by the P-null sets of $\mathscr{F}$. Then $(\mathscr{F}_t)_{t\geq 0}$ is right continuous and complete. Let $T > 0$ be a given real number. In this paper, we always work in the space $(\Omega, \mathscr{F}_T, P)$, and only consider processes indexed by $t \in [0, T]$. For every $n \in \mathbf{N}$, let $|z|$ denote the Euclidean norm of $z \in \mathbf{R}^n$. $\mathbf{R}^{m\times d}$ is identified with the space of real matrices with m rows and d columns, and if $z \in \mathbf{R}^{m\times d}$, we have $|z|^2 = \mathrm{trace}(zz^*)$. For every $p \in [1,2]$ and $0 \leq t_1 \leq t_2 \leq T$, we define the following space of processes:

$$\mathscr{H}_p^n(t_1,t_2) = \{\phi \in \mathbf{R}^n \text{ is } (\mathscr{F}_t)-\text{progressively measurable}; \ \ \|\phi(t)\|_p^p = \mathbf{E}\left[\int_{t_1}^{t_2} |\phi(t)|^p \mathrm{d}t\right] < +\infty\}.$$

It is well known that $\mathscr{H}_p^n(t_1,t_2)$ is a Banach space endowed with the norm $\|\cdot\|_p$. For simplicity, $\mathscr{H}_p^n(0,T)$ is also denoted by $\mathscr{H}_p^n$.

Let $b(\omega,t,x) : \Omega \times [0,T] \times \mathbf{R}^m \to \mathbf{R}^m$, $\sigma(\omega,t,x) : \Omega \times [0,T] \times \mathbf{R}^m \to \mathbf{R}^{m\times d}$ be two functions such that for any $x \in \mathbf{R}^m$, $b(\cdot,x)$ and $\sigma(\cdot,x)$ are $(\mathscr{F}_t)$-progressively measurable. Let b and σ also satisfy the following hypotheses (H1) and (H2):

(H1) There exists a constant $K_1 \geq 0$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K_1|x-y|, \ \ \forall\ x,y \in \mathbf{R}^m.$$

(H2) There exists a constant $K_2 \geq 0$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$,

$$|b(t,x)| + |\sigma(t,x)| \leq K_2(1+|x|), \ \ \forall\ x \in \mathbf{R}^m.$$

Given $(t,x) \in [0,T] \times \mathbf{R}^m$, by classical SDE theory, the following SDE:

$$X_s = x + \int_t^s b(u,X_u)\mathrm{d}u + \int_t^s \sigma(u,X_u)\mathrm{d}B_u,\ s \in [t,T];\ X_s = x,\ s \in [0,t] \tag{3}$$

has a unique s-continuous solution, denoted by $(X_s^{t,x})_{s\in[0,T]}$, with the properties that $(X_s^{t,x})_{s\in[0,T]}$ is $(\mathscr{F}_s)$-adapted and for every $\beta \geq 1$,

$$\mathbf{E}\left[\sup_{0\leq s\leq T}|X_s^{t,x}|^\beta\right] < C_\beta, \text{ and } s \to \mathbf{E}\left[|X_s^{t,x}-x|^\beta\right], \ s\in[0,T] \text{ is continuous,} \tag{4}$$

where the constant C_β depends on $x,\ \beta,\ K_1,\ K_2,\ T$.

In this paper, the generator g of a BSDE is a function $g(\omega,t,y,z):\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^d\to\mathbf{R}$ such that the process $(g(t,y,z))_{t\in[0,T]}$ is $(\mathscr{F}_t)$-progressively measurable for every (y,z) in $\mathbf{R}\times\mathbf{R}^d$. The following Proposition 1 comes from Theorem 4.1 in Briand et al. (2007).

Proposition 1 Let the generator g satisfy the following assumptions:

(A1) $\mathrm{d}P\times\mathrm{d}t - a.e.$, $(y,z)\longmapsto g(t,y,z)$ is continuous.

(A2) g is monotonic with respect to y, i.e., there exists a constant $\mu\geq 0$ such that $\mathrm{d}P\times\mathrm{d}t - a.e.$,

$$(y_1-y_2)(g(t,y_1,z)-g(t,y_2,z))\leq\mu|y_1-y_2|^2, \ \forall\ y_1,y_2,z.$$

(A3') There exists a constant $A\geq 0$, a nonnegative continuous process $(g_t)_{t\in[0,T]}$ which belongs to $\mathscr{H}_\beta^1$ for some $\beta>1$ and a nondecreasing continuous function $\varphi:\mathbf{R}_+\to\mathbf{R}_+$ with $\varphi(0)=0$ such that $\mathrm{d}P\times\mathrm{d}t - a.e.$,

$$|g(t,y,z)|\leq g_t+\varphi(|y|)+A|z|, \ \forall\ y,z.$$

Then, for every $\xi\in L^\beta(\Omega,\mathscr{F}_T,P)$, the BSDE with parameters (ξ,T,g) has a unique minimal solution $(\underline{y}_u,\underline{z}_u)_{u\in[0,T]}$ in $\mathscr{H}_\beta^1\times\mathscr{H}_\beta^d$.

Remark 1 Theorem 4.1 in Briand et al. (2007) pointed out that there also exists a unique maximal solution $(\overline{y}_u,\overline{z}_u)_{u\in[0,T]}$ in $\mathscr{H}_\beta^1\times\mathscr{H}_\beta^d$.

In the remainder of this paper, for notational simplicity, for each $t\in[0,T]$ and $n\in\mathbf{N}$, we denote $(t+1/n)\wedge T$ by t_n, and $(t+1/n_k)\wedge T$ by t_{n_k}. Furthermore, we fix a constant $\alpha\geq 1$ and always assume that the g satisfies (A1), (A2) and the following assumption (A3):

(A3) There exists a constant $C\geq 0$ and a nonnegative continuous process $(f_t)_{t\in[0,T]}$ which belongs to $\mathscr{H}_{2\alpha}^1$ such that $\mathrm{d}P\times\mathrm{d}t - a.e.$,

$$|g(t,y,z)|\leq C\left(f_t+|y|^\alpha+|z|\right), \ \forall\ y,z.$$

Let g satisfy (A1), (A2) and (A3). Given $(x,y,q)\in\mathbf{R}^{m+1+m}$. For every $t\in[0,T]$ and $n\in\mathbf{N}$, in view of (4) and the fact that $2\alpha\geq 2$, it follows from Proposition 1 with $\beta=2$ that the following BSDE:

$$Y_s = y+q\cdot(X_{t_n}^{t,x}-x)+\int_s^{t_n}g(u,Y_u,Z_u)\mathrm{d}u-\int_s^{t_n}Z_u\cdot\mathrm{d}B_u, \ s\in[0,t_n] \tag{5}$$

has a unique minimal solution in the space $\mathscr{H}_2^1(0,t_n)\times\mathscr{H}_2^d(0,t_n)$, denoted by

$$(\underline{Y}_s^{y+q\cdot(X_{t_n}^{t,x}-x),t_n,g},\underline{Z}_s^{y+q\cdot(X_{t_n}^{t,x}-x),t_n,g})_{s\in[0,t_n]}.$$

Moreover, it follows from (4) and Proposition 1 with $\beta=2\alpha$ that this solution also belongs to the space $\mathscr{H}_{2\alpha}^1(0,t_n)\times\mathscr{H}_{2\alpha}^d(0,t_n)$. For notational simplicity, we denote $\underline{Y}_t^{y+q\cdot(X_{t_n}^{t,x}-x),t_n,g}$ by $\underline{\mathscr{E}}_{t,t_n}^g[y+q\cdot$

$(X_{t_n}^{t,x}-x)]$ for every $t\in[0,T]$.

Remark 2 In view of the definition of t_n, we know that for every $n\in\mathbf{N}$, the random variable $X_{t_n}^{t,x}$ with $t\in[0,T]$ and the process $\{\underline{\mathscr{E}}_{t,t_n}^g[y+q\cdot(X_{t_n}^{t,x}-x)]\}_{t\in[0,T]}$ are both well defined. This is exactly why we let $t_n=(t+1/n)\wedge T$.

With respect to the above sequence of processes, we have the following conclusion which is the main result of this paper.

Theorem 1 (Representation Theorem I) Let (A1), (A2) and (A3) hold true for the generator g; let (H1) and (H2) hold true for b and σ. Then for every $(x,y,q)\in\mathbf{R}^{m+1+m}$ and every $p\in[1,2)$, the following equality

$$\lim_{n\to\infty} n\{\underline{\mathscr{E}}_{t,t_n}^g[y+q\cdot(X_{t_n}^{t,x}-x)]-y\}=g(t,y,\sigma^*(t,x)q)+q\cdot b(t,x) \tag{6}$$

holds true in the space of processes $\mathscr{H}_p^1$. And, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\mathrm{d}P\times \mathrm{d}t-a.e.$,

$$\lim_{k\to\infty} n_k\{\underline{\mathscr{E}}_{t,t_{n_k}}^g[y+q\cdot(X_{t_{n_k}}^{t,x}-x)]-y\}=g(t,y,\sigma^*(t,x)q)+q\cdot b(t,x). \tag{7}$$

Moreover, if the process $(f_t)_{t\in[0,T]}$ defined in (A3) also satisfies

$$\mathbf{E}\left[\sup_{0\le t\le T}|f_t|^2\right]<+\infty, \tag{8}$$

then (6) holds true in the space of processes $\mathscr{H}_2^1$ and (7) also holds true.

By letting $m=d$, $b\equiv 0$, $\sigma\equiv 1$ and $q=z$ in Theorem 1, the following Theorem 2 follows immediately.

Theorem 2 (Representation Theorem II) Let (A1), (A2) and (A3) hold true for the generator g. Then for every $(y,z)\in\mathbf{R}^{1+d}$ and every $p\in[1,2)$, the equality

$$\lim_{n\to\infty} n\{\underline{\mathscr{E}}_{t,t_n}^g[y+z\cdot(B_{t_n}-B_t)]-y\}=g(t,y,z) \tag{9}$$

holds true in the space of processes $\mathscr{H}_p^1$. And, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\mathrm{d}P\times \mathrm{d}t-a.e.$,

$$\lim_{k\to\infty} n_k\{\underline{\mathscr{E}}_{t,t_{n_k}}^g[y+z\cdot(B_{t_n}-B_t)]-y\}=g(t,y,z). \tag{10}$$

Moreover, if (8) is also satisfied, then (9) holds true in the space of processes $\mathscr{H}_2^1$ and (10) also holds true.

Remark 3 Obviously, the Lipschitz assumptions of g in Pardoux and Peng (1990) can imply the assumptions (A1), (A2) and (A3) with $\alpha=1$. Hence, Theorems 1-2 generalize the corresponding results in Fan (2006, 2007) and Fan and Hu (2008).

3 Proof of the main result

This section aims at giving a proof of our main result–Theorem 1. let us first introduce some Lemmas which will play important roles in the proof of Theorem 1. The following Lemma 1 is a direct corollary of Proposition 3.2 in Briand et al. (2003).

Lemma 1 Let the generator $\bar{g}$ satisfy the following assumption:

(A) There exists two constants $\bar{\mu}, \bar{C} \geq 0$ and a nonnegative continuous process $(\bar{f}_t)_{t\in[0,T]}$ which belongs to $\mathscr{H}^1_{2\alpha}$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$,

$$y \cdot \bar{g}(t,y,z) \leq \bar{\mu}|y|^2 + |y|(\bar{f}_t + \bar{C}|z|), \ \ \forall \ y, z.$$

Then, there exists a constant $K > 0$ depending only on $(\bar{C}, \bar{\mu}, \alpha, T)$ such that for every $0 \leq t_1 \leq t_2 \leq T$, $\xi \in L^{2\alpha}(\Omega, \mathscr{F}_{t_2}, P)$ and $1 < \beta \leq 2\alpha$,

$$\mathbf{E}\left[\sup_{t_1\leq s\leq t_2} |y_s|^\beta + \left(\int_{t_1}^{t_2} |z_s|^2 \mathrm{d}s\right)^{\beta/2}\right] \leq K\ \mathbf{E}\left[|\xi|^\beta + \left(\int_{t_1}^{t_2} \bar{f}_s \mathrm{d}s\right)^\beta\right],$$

where $(y_s, z_s)_{s\in[t_1,t_2]} \in \mathscr{H}^1_{2\alpha}(t_1,t_2) \times \mathscr{H}^d_{2\alpha}(t_1,t_2)$ solves the following BSDE:

$$y_s = \xi + \int_s^{t_2} \bar{g}(u, y_u, z_u)\mathrm{d}u - \int_s^{t_2} z_u \cdot \mathrm{d}B_u,\ s \in [t_1, t_2].$$

The following Lemma 2 comes from the Lemma 5 together with its proof in Fan and Jiang (2010b).

Lemma 2 Assume that $\{\phi(t)\}_{t\in[0,T]} \in \mathscr{H}^1_2$. We have

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^{T-1/n} \left(n\int_t^{t+1/n} |\phi(s)|^2 \mathrm{d}s\right) \mathrm{d}t\right] = \mathbf{E}\left[\int_0^T |\phi(t)|^2 \mathrm{d}t\right], \tag{11}$$

$$\lim_{n\to\infty} \mathbf{E}\left[\int_{T-1/n}^{T} \left(n\int_t^{T} |\phi(s)|^2 \mathrm{d}s\right) \mathrm{d}t\right] = 0. \tag{12}$$

And, for every $p \in [1,2)$,

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^{T-1/n} \left|n\int_t^{t+1/n} \left(\phi(s) - \phi(t)\right) \mathrm{d}s\right|^p \mathrm{d}t\right] = 0. \tag{13}$$

Moreover, if $\mathbf{E}\left[\sup_{0\leq t\leq T} |\phi(t)|^2\right] < +\infty$, then

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^{T-1/n} \left|n\int_t^{t+1/n} \left(\phi(s) - \phi(t)\right) \mathrm{d}s\right|^2 \mathrm{d}t\right] = 0. \tag{14}$$

Combining (11) with (12), in view of $t_n = (t + 1/n) \wedge T$, one can obtain the following Corollary.

Corollary 1 Assume that $\{\phi(t)\}_{t\in[0,T]} \in \mathscr{H}^1_2$. We have

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^{T} \left(n\int_t^{t_n} |\phi(s)|^2 \mathrm{d}s\right) \mathrm{d}t\right] = \mathbf{E}\left[\int_0^T |\phi(t)|^2 \mathrm{d}t\right].$$

Inspired by Lepeltier and San Martin (1997), we can establish the following Lemma 3.

Lemma 3 Assume that $f(\cdot):\mathbf{R}^k \to \mathbf{R}$ with $k\in\mathbf{N}$ is a continuous function with polynomial growth, i.e., there exists constants $\bar{K}_1,\bar{K}_2 \geq 0$ and $\beta \geq 1$ such that

$$|f(x)| \leq \bar{K}_1(\bar{K}_2 + |x|^\beta), \ \ \forall\ x. \tag{15}$$

Let f_n be the function defined as follows:

$$f_n(x) := \inf_{u\in\mathbf{R}^k}\{f(u) + n2^{\beta-1}\bar{K}_1|u-x|^\beta\}. \tag{16}$$

Then the sequence of functions f_n is well defined for every $n \geq 1$, and it satisfies:

(i) Polynomial growth: $|f_n(x)| \leq 2^{\beta-1}\bar{K}_1(\bar{K}_2 + |x|^\beta), \ \ \forall\ x$;

(ii) Monotonicity in n: $f_n(x)$ increases in n, $\forall\ x$;

(iii) Convergence: If $x_n \longrightarrow x$, then $f_n(x_n) \longrightarrow f(x)$.

Remark 4 Note that (16) does not contain the constant $\bar{K}_2$ in (15). This fact will be made full use of in the proof of the following Proposition 2, which explains why we use (15) rather than the usual expression (i.e., $|f(x)| \leq K(1+|x|^\beta), \ \forall\ x$) although they are equivalent.

Remark 5 The case of $\beta = 1$ in Lemma 3 has been proved in Lepeltier and San Martin (1997). In addition, in view of the continuity of the f, we can use $\mathbf{Q}^k$ instead of $\mathbf{R}^k$ in (16).

Proof of Lemma 3 The case of $\bar{K}_1 = 0$ being trivial, we assume that $\bar{K}_1 > 0$. Note that $(a+b)^\beta \leq 2^{\beta-1}(a^\beta + b^\beta)$ holds true for every $a,b \geq 0$. It follows from (15) that for every $n \geq 1$ and $x \in \mathbf{R}^k$, we have

$$\begin{aligned} f_n(x) &\geq \inf_{u\in\mathbf{R}^k}\{-\bar{K}_1(\bar{K}_2 + |(u-x)+x|^\beta) + 2^{\beta-1}\bar{K}_1|u-x|^\beta\} \\ &\geq -\bar{K}_1(\bar{K}_2 + 2^{\beta-1}|x|^\beta) \geq -2^{\beta-1}\bar{K}_1(\bar{K}_2+|x|^\beta) \end{aligned}$$

and

$$f_n(x) \leq f(x) \leq 2^{\beta-1}\bar{K}_1(\bar{K}_2 + |x|^\beta).$$

Thus, for every $n \geq 1$, f_n is well defined and (i) holds true. It is clear from (16) that (ii) holds true. Hence, it suffices to show (iii). Indeed, assume that $x_n \longrightarrow x$. In view of (16), (15) and the inequality that $(a+b)^\beta \leq 2^{\beta-1}(a^\beta+b^\beta)$, we can take a sequence $\{u_n\}$ such that

$$\begin{aligned} f_n(x_n) &\geq f(u_n) + n2^{\beta-1}\bar{K}_1|u_n - x_n|^\beta - 1/n \\ &\geq -\bar{K}_1(\bar{K}_2 + |(u_n - x_n) + x_n|^\beta) + n2^{\beta-1}\bar{K}_1|u_n-x_n|^\beta - 1/n \\ &\geq -\bar{K}_1\bar{K}_2 - 2^{\beta-1}\bar{K}_1|x_n|^\beta + (n-1)2^{\beta-1}\bar{K}_1|u_n-x_n|^\beta - 1/n, \end{aligned} \tag{17}$$

which means, in view of (i), that

$$(n-1)2^{\beta-1}\bar{K}_1|u_n - x_n|^\beta \leq 2^\beta\bar{K}_1(\bar{K}_2 + |x_n|^\beta) + 1/n$$

and then $\limsup_{n\to\infty} n2^{\beta-1}\bar{K}_1|u_n - x_n|^\beta < +\infty$. Therefore, $\lim\limits_{n\to\infty} u_n = x$. It then follows from (17) and the continuity of f that

$$\liminf_{n\to\infty} f_n(x_n) \geq \liminf_{n\to\infty} f(u_n) = f(x).$$

On the other hand, from (16) and the continuity of f we know that

$$\limsup_{n\to\infty} f_n(x_n) \leq \limsup_{n\to\infty} f(x_n) = f(x).$$

Hence, (iii) follows and the proof of Lemma 3 is complete. □

With Lemma 3 in hand, we can establish the following proposition which will play a key role in the proof of Theorem 1.

Proposition 2 Let the generator g satisfy (A1) and (A3), let σ satisfy (H1) and (H2), and let $(x,y,q)\in\mathbf{R}^{m+1+m}$. Then there exists a non-negative process sequence $\{(\psi^n(t))_{t\in[0,T]}\}_{n=1}^{\infty}$ in $\mathscr{H}^1_{2\alpha}$ depending on (x,y,q) such that $\lim_{n\to\infty}\|\psi^n(t)\|_{2\alpha}=0$ and $\mathrm{d}P\times\mathrm{d}t-a.e.$, for every $n\in\mathbf{N}$ and $(\bar{y},\bar{z},\bar{x})\in\mathbf{R}^{1+d}$,

$$|g(t,\bar{y},\bar{z}+\sigma^*(t,\bar{x})q)-g(t,y,\sigma^*(t,x))q|\le n2^{\alpha}\tilde{C}(|\bar{y}-y|^{\alpha}+|\bar{z}|+|\bar{x}-x|)+\psi^n(t),$$

where the constant $\tilde{C}=C(1+|q|K_2)$.

Proof. Let $(x,y,q)\in\mathbf{R}^{m+1+m}$ and define $\tilde{g}(t,\tilde{y},\tilde{z},\tilde{x}):=g\left(t,\tilde{y},\tilde{z}+\sigma^*(t,\tilde{x})q\right)$. It is easy to see from (A1) and (H1) that $\tilde{g}$ is continuous with respect to the variables $(\tilde{y},\tilde{z},\tilde{x})$. Moreover, it follows from (A3) and (H2) that

$$\begin{array}{rcl}|\tilde{g}(t,\tilde{y},\tilde{z},\tilde{x})| & \le & C(f_t+|\tilde{y}|^{\alpha}+|\tilde{z}|+|q|K_2(1+|\tilde{x}|))\\ & \le & \tilde{C}(1+f_t+|\tilde{y}|^{\alpha}+|\tilde{z}|+|\tilde{x}|),\end{array}$$

where $\tilde{C}=C(1+|q|K_2)$ is a constant. Thus, similar to Lemma 3, we can prove that the following processes $\psi_1^n(t)$ and $\psi_2^n(t)$ are well defined for every $n\in\mathbf{N}$:

$$\psi_1^n(t)=\sup_{(u,v,w)\in\mathbf{R}^{1+d+m}}\{\tilde{g}(t,u,v,w)-n2^{\alpha-1}\tilde{C}(|u-y|^{\alpha}+|v|+|w-x|)\},$$

$$\psi_2^n(t)=\inf_{(u,v,w)\in\mathbf{R}^{1+d+m}}\{\tilde{g}(t,u,v,w)+n2^{\alpha-1}\tilde{C}(|u-y|^{\alpha}+|v|+|w-x|)\}.$$

We can also prove that

$$|\psi_1^n(t)|\le 2^{\alpha-1}\tilde{C}(1+f_t+|y|^{\alpha}+|x|)\in\mathscr{H}^1_{2\alpha},$$
$$|\psi_2^n(t)|\le 2^{\alpha-1}\tilde{C}(1+f_t+|y|^{\alpha}+|x|)\in\mathscr{H}^1_{2\alpha},$$

and $\mathrm{d}P\times\mathrm{d}t-a.e.$,

$$\lim_{n\to\infty}\psi_1^n(t)=\lim_{n\to\infty}\psi_2^n(t)=\tilde{g}(t,y,0,x).$$

Furthermore, it follows from Lebesgue's dominated convergence theorem that the above limit also holds true in the process space $\mathscr{H}^1_{2\alpha}$.
On the other hand, it is clear that for every $n\in\mathbf{N}$ and $(\bar{y},\bar{z})\in\mathbf{R}^{1+d}$,

$$\tilde{g}(t,\bar{y},\bar{z},\bar{x})-\tilde{g}(t,y,0,x)\le n2^{\alpha-1}\tilde{C}(|\bar{y}-y|^{\alpha}+|\bar{z}|+|\bar{x}-x|)+\psi_1^n(t)-\tilde{g}(t,y,0,x),$$

$$\tilde{g}(t,\bar{y},\bar{z},\bar{x})-\tilde{g}(t,y,0,x)\ge -n2^{\alpha-1}\tilde{C}(|\bar{y}-y|^{\alpha}+|\bar{z}|+|\bar{x}-x|)+\psi_2^n(t)-\tilde{g}(t,y,0,x).$$

Thus, by letting

$$\psi^n(t)=|\psi_1^n(t)-\tilde{g}(t,y,0,x)|+|\psi_2^n(t)-\tilde{g}(t,y,0,x)|,$$

we have

$$|\tilde{g}(t,\bar{y},\bar{z},\bar{x})-\tilde{g}(t,y,0,x)|\le n2^{\alpha-1}\tilde{C}(|\bar{y}-y|^{\alpha}+|\bar{z}|+|\bar{x}-x|)+\psi^n(t),$$

which is the desired result. The proof of Proposition 2 is complete. □

Now we are in a position to prove our main result–Theorem 1.

The Proof of Theorem 1. Given $(x,y,q)\in\mathbf{R}^{m+1+m}$ and $p\in[1,2)$. For notational simplicity, we denote the unique solution of SDE (3) by $(X_s^t)_{s\in[0,T]}$ for every $t\in[0,T]$, and denote the minimal solution of BSDE (5) in $\mathscr{H}^1_{2\alpha}(0,t_n)\times\mathscr{H}^d_{2\alpha}(0,t_n)$ by $(\underline{Y}_s^{t,n},\underline{Z}_s^{t,n})_{s\in[0,t_n]}$ for every $n\in\mathbf{N}$. For every $s\in[t,t_n]$, set

$$\widetilde{Y}_s^{t,n}:=\underline{Y}_s^{t,n}-(y+q\cdot(X_s^t-x)),\quad \widetilde{Z}_s^{t,n}:=\underline{Z}_s^{t,n}-\sigma^*(s,X_s^t)q,$$

then applying Itô's formula to $\widetilde{Y}_u^{t,n}$ yields that

$$\begin{aligned}\widetilde{Y}_s^{t,n} \;=\; & \int_s^{t_n} g\left(u,\widetilde{Y}_u^{t,n}+y+q\cdot(X_u^t-x),\widetilde{Z}_u^{t,n}+\sigma^*(u,X_u^t)q\right)\mathrm{d}u \\ & +\int_s^{t_n} q\cdot b(u,X_u^t)\mathrm{d}u-\int_s^{t_n}\widetilde{Z}_u^{t,n}\cdot\mathrm{d}B_u,\quad s\in[t,t_n].\end{aligned}\tag{18}$$

Let

$$\begin{aligned}M_t^n &:= n\mathbf{E}\left[\left.\int_t^{t_n} g\left(u,\widetilde{Y}_u^{t,n}+y+q\cdot(X_u^t-x),\widetilde{Z}_u^{t,n}+\sigma^*(u,X_u^t)q\right)\mathrm{d}u\right|\mathscr{F}_t\right],\\ N_t^n &:= n\mathbf{E}\left[\left.\int_t^{t_n} g\left(u,y,\sigma^*(u,x)q\right)\mathrm{d}u\right|\mathscr{F}_t\right].\end{aligned}$$

By letting $s=t$ in (18) and then taking the conditional expectation with respect to $\mathscr{F}_t$, it follows that $n\{\underline{\mathscr{E}}^g_{t,t_n}[y+q\cdot(X_{t_n}^t-x)]-y\}=n(\underline{Y}_t^{t,n}-y)=n\widetilde{Y}_t^{t,n}=M_t^n+n\mathbf{E}\left[\left.\int_t^{t_n}q\cdot b(u,X_u^t)\mathrm{d}u\right|\mathscr{F}_t\right]$ and then $\mathrm{d}P\times\mathrm{d}t-a.e.$ in $\Omega\times[0,T]$,

$$\begin{aligned}& n\{\underline{\mathscr{E}}^g_{t,t_n}[y+q\cdot(X_{t_n}^t-x)]-y\}-\left[g(t,y,\sigma^*(t,x)q)+q\cdot b(t,x)\right]\\ = \;& M_t^n-N_t^n+N_t^n-g(t,y,\sigma^*(t,x)q)\\ & +n\mathbf{E}\left[\left.\int_t^{t_n}q\cdot b(u,X_u^t)\mathrm{d}u\right|\mathscr{F}_t\right]-q\cdot b(t,x).\end{aligned}\tag{19}$$

Thus, in view of the relation between the moment convergence and almost sure convergence, for completing the proof of Theorem 1 it suffices to prove that the right hand side of equality (19) tends to 0 in the space of process $\mathscr{H}^1_p$ as $n\to\infty$, and that if (8) also holds true, then the right hand side of equality (19) tends to 0 in $\mathscr{H}^1_2$ as $n\to\infty$.

First, it should be noted that the following statement has been proved in Fan and Hu (2008)(see (3.11) in Fan and Hu (2008)):

$$\lim_{n\to\infty}\mathbf{E}\left[\int_0^T\left|n\mathbf{E}\left[\left.\int_t^{t_n}q\cdot b(u,X_u^t)\mathrm{d}u\right|\mathscr{F}_t\right]-q\cdot b(t,x)\right|^2\mathrm{d}t\right]=0.\tag{20}$$

Second, it follows from (3.16) and (3.19) in Fan and Hu (2008) that

$$\begin{array}{rl} & \mathbf{E}\left[\int_0^T \left|N_t^n - g(t,y,\sigma^*(t,x)q)\right|^p dt\right] \\ \leq & \mathbf{E}\left[\int_0^{T-1/n} \left|n\int_t^{t+1/n} \left[g(u,y,\sigma^*(u,x)q) - g(t,y,\sigma^*(t,x)q)\right] \mathrm{d}u\right|^p \mathrm{d}t\right] \\ & + \mathbf{E}\left[\int_{T-1/n}^T \left|n\mathbf{E}\left[\int_t^T g(u,y,\sigma^*(u,x)q)\mathrm{d}u\middle|\mathscr{F}_t\right] - g(t,y,\sigma^*(t,x)q)\right|^p \mathrm{d}t\right] \end{array} \tag{21}$$

and

$$\begin{array}{rl} & \mathbf{E}\left[\int_{T-1/n}^T \left|n\mathbf{E}\left[\int_t^T g(u,y,\sigma^*(u,x)q)\mathrm{d}u\middle|\mathscr{F}_t\right] - g(t,y,\sigma^*(t,x)q)\right|^2 \mathrm{d}t\right] \\ \leq & 2\mathbf{E}\left[\int_{T-1/n}^T \left(n\int_t^T \left|g(u,y,\sigma^*(u,x)q)\right|^2 \mathrm{d}u\right) \mathrm{d}t\right] + 2\mathbf{E}\left[\int_{T-1/n}^T \left|g(t,y,\sigma^*(t,x)q)\right|^2 \mathrm{d}t\right]. \end{array} \tag{22}$$

Since g satisfies (A3) and σ satisfies (H2), it is not difficult to verify that the process $(g\left(t,y,\sigma^*(t,x)q\right))_{t\in[0,T]}$ belongs to $\mathscr{H}^1_{2\alpha}$ and then $\mathscr{H}^1_2$. Then, it follows from the absolute continuity of integral that the second term of the right hand side of (22) tends to zero as $n\to\infty$. Applying (12) with $\phi(t) = g\left(t,y,\sigma^*(t,x)q\right)$ yields that the first term of the right hand side of (22) also tends to zero as $n\to\infty$. Thus, we have

$$\lim_{n\to\infty} \mathbf{E}\left[\int_{T-1/n}^T \left|n\mathbf{E}\left[\int_t^T g(u,y,\sigma^*(u,x)q)\mathrm{d}u\middle|\mathscr{F}_t\right] - g(t,y,\sigma^*(t,x)q)\right|^2 \mathrm{d}t\right] = 0, \tag{23}$$

and then the second term of the right hand side of (21) tends to zero as $n\to\infty$. Furthermore, applying (13) with $\phi(t) = g\left(t,y,\sigma^*(t,x)q\right)$ yields that the first term of the right hand side of (21) also tends to zero as $n\to\infty$. Consequently, we can conclude that

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^T \left|N_t^n - g(t,y,\sigma^*(t,x)q)\right|^p \mathrm{d}t\right] = 0. \tag{24}$$

Third, let us prove that

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^T |M_t^n - N_t^n|^2 \mathrm{d}t\right] = 0. \tag{25}$$

It follows from Proposition 2 that there exists a non-negative process sequence $\{(\psi^k(t))_{t\in[0,T]}\}_{k=1}^\infty$ in $\mathscr{H}^1_{2\alpha}$ depending on (x,y,q) such that $\lim_{k\to\infty}\|\psi^k(t)\|_{2\alpha} = 0$ and for every $k\in\mathbf{N}$, $\mathrm{d}P\times\mathrm{d}t - a.e.$,

$$\begin{array}{rcl} P_u^{t,n} & := & \left|g\left(u,\widetilde{Y}_u^{t,n} + y + q\cdot(X_u^t - x), \widetilde{Z}_u^{t,n} + \sigma^*(u,X_u^t)q\right) - g\left(u,y,\sigma^*(u,x)q\right)\right| \\ & \leq & k\tilde{C}2^\alpha\left[|\widetilde{Y}_u^{t,n} + q\cdot(X_u^t - x)|^\alpha + |\widetilde{Z}_u^{t,n}| + |X_u^t - x|\right] + \psi^k(u) \\ & \leq & kC_1(|\widetilde{Y}_u^{t,n}|^\alpha + |X_u^t - x|^\alpha + |\widetilde{Z}_u^{t,n}| + |X_u^t - x|) + \psi^k(u), \end{array} \tag{26}$$

where the constant $\tilde{C} = C(1+|q|K_2)$ and the constant C_1 depends only on $(\alpha, q, \tilde{C})$. Note that we also have $\lim\limits_{k\to\infty} \|\psi^k(t)\|_2 = 0$. By Fubini's Theorem, Jensen's inequality and Hölder's inequality, we can deduce that

$$\mathbf{E}\left[\int_0^T |M_t^n - N_t^n|^2 \mathrm{d}t\right] = \int_0^T \left\{\mathbf{E}[|M_t^n - N_t^n|^2]\right\} \mathrm{d}t \le \int_0^T \left[\mathbf{E}\left(n\int_t^{t_n} |P_u^{t,n}|^2 \mathrm{d}u\right)\right] \mathrm{d}t,$$

then, it follows from (26) that there exists a constant $C_2 > 0$ depending only on C_1 such that for every $k \in \mathbf{N}$,

$$\begin{aligned}
& \mathbf{E}\left[\int_0^T |M_t^n - N_t^n|^2 \mathrm{d}t\right] \\
\le\ & k^2 C_2 \int_0^T \left\{\mathbf{E}\left[n\int_t^{t_n} \left(|\widetilde{Y}_u^{t,n}|^{2\alpha} + |\widetilde{Z}_u^{t,n}|^2\right) \mathrm{d}u\right]\right\} \mathrm{d}t \\
& +2\int_0^T \left[\mathbf{E}\left(n\int_t^{t_n} |\psi^k(u)|^2 \mathrm{d}u\right)\right] \mathrm{d}t \\
& +k^2 C_2 \int_0^T \left[\mathbf{E}\left(n\int_t^{t_n} |X_u^t - x|^2 \mathrm{d}u\right)\right] \mathrm{d}t + k^2 C_2 \int_0^T \left[\mathbf{E}\left(n\int_t^{t_n} |X_u^t - x|^{2\alpha} \mathrm{d}u\right)\right] \mathrm{d}t.
\end{aligned} \tag{27}$$

Furthermore, it follows from (4) and (H2) that $(\widetilde{Y}_s^{t,n},\ \widetilde{Z}_s^{t,n})_{s\in[t,t_n]} \in \mathscr{H}_{2\alpha}^1(t,t_n) \times \mathscr{H}_{2\alpha}^d(t,t_n)$, and from (18) that it solves the following BSDE:

$$\widetilde{Y}_s^{t,n} = \int_s^{t_n} \tilde{g}(u,\ \widetilde{Y}_u^{t,n},\ \widetilde{Z}_u^{t,n}) \mathrm{d}u - \int_s^{t_n} \widetilde{Z}_u^{t,n} \cdot \mathrm{d}B_u, \quad s \in [t, t_n], \tag{28}$$

where for every $(\omega, u, \tilde{y}, \tilde{z}) \in \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d$,

$$\tilde{g}(u,\tilde{y},\tilde{z}) := g(u, \tilde{y} + y + q\cdot(X_u^t - x), \tilde{z} + \sigma^*(u, X_u^t)q) + q\cdot b(u, X_u^t).$$

It is not difficult to verify that $\tilde{g}$ satisfies assumption (A). In fact, for every $(\tilde{y},\tilde{z})$, we write

$$\begin{aligned}
\tilde{y}\cdot\tilde{g}(u,\tilde{y},\tilde{z}) \ =\ & \tilde{y}\cdot\Big(g(u,\tilde{y}+y+q\cdot(X_u^t - x), \tilde{z}+\sigma^*(u,X_u^t)q) \\
& \quad -g(u, y+q\cdot(X_u^t - x), \tilde{z}+\sigma^*(u,X_u^t)q)\Big) \\
& +\tilde{y}\cdot\Big(g(u, y+q\cdot(X_u^t - x), \tilde{z}+\sigma^*(u,X_u^t)q) + q\cdot b(u,X_u^t)\Big),
\end{aligned}$$

then, it follows from (A2), (A3) and (H2) that $\mathrm{d}P\times\mathrm{d}t - a.e.$, $\forall\ \tilde{y},\tilde{z}$,

$$\begin{aligned}
\tilde{y}\cdot\tilde{g}(u,\tilde{y},\tilde{z}) \ \le\ & \mu|\tilde{y}|^2 + C|\tilde{y}|\Big(f_u + |y+q\cdot(X_u^t - x)|^\alpha + |\tilde{z}+\sigma^*(u,X_u^t)q|\Big) + |\tilde{y}||q\cdot b(u,X_u^t)| \\
\le\ & \mu|\tilde{y}|^2 + |\tilde{y}|(\tilde{f}_u + C|\tilde{z}|)
\end{aligned}$$

with $\tilde{f}_u = Cf_u + C2^\alpha[|y|^\alpha + |2q|^\alpha(|X_u^t|^\alpha + |x|^\alpha)] + (C+1)K_2|q|(1+|X_u^t|)$. Since $(f_t)_{t\in[0,T]}$ belongs to $\mathscr{H}_{2\alpha}^1$ and $\mathbf{E}\left[\sup_{0\le u\le T}|X_u^t|^\beta\right] < C_\beta$ for every $\beta \ge 1$ (see (4)), the process $(\tilde{f}_t)_{t\in[0,T]}$ belongs to the space $\mathscr{H}_{2\alpha}^1$. Consequently, $\tilde{g}$ satisfies the assumption (A) with $\bar{\mu} = \mu$, $\bar{C} = C$ and $\bar{f}_t = \tilde{f}_t$. Thus, applying Lemma 1 with $t_1 = t$, $t_2 = t_n$, $\beta = 2\alpha$ and $\beta = 2$ for BSDE (28) yields that there exists a constant $\bar{K}>0$ depending only on (μ, C, α, T) such that for every $t\in[0,T]$,

$$\begin{aligned}
& n\mathbf{E}\left[\int_t^{t_n}\left(|\widetilde{Y}_u^{t,n}|^{2\alpha}+|\widetilde{Z}_u^{t,n}|^2\right)\mathrm{d}u\right] \\
\leq\ & n\bar{K}\left\{\mathbf{E}\left[\left(\int_t^{t_n}\tilde{f}_u\mathrm{d}u\right)^{2\alpha}\right]+\mathbf{E}\left[\left(\int_t^{t_n}\tilde{f}_u\mathrm{d}u\right)^{2}\right]\right\} \\
\leq\ & n\bar{K}\left\{\mathbf{E}\left[\int_t^{t_n}|\tilde{f}_u|^{2\alpha}\mathrm{d}u\right]\cdot(t_n-t)^{2\alpha-1}+\mathbf{E}\left[\int_t^{t_n}|\tilde{f}_u|^{2}\mathrm{d}u\right]\cdot(t_n-t)\right\} \\
\leq\ & \bar{K}\left\{\mathbf{E}\left[\int_t^{t_n}|\tilde{f}_u|^{2\alpha}\mathrm{d}u\right]+\mathbf{E}\left[\int_t^{t_n}|\tilde{f}_u|^{2}\mathrm{d}u\right]\right\},
\end{aligned}\tag{29}$$

where we have used Hölder's inequality and the fact that $0\leq t_n-t\leq 1/n$ and $2\alpha-1\geq 1$. Combining (27) with (29) implies that for every $n\in\mathbf{N}$ and $k\in\mathbf{N}$,

$$\begin{aligned}
& \mathbf{E}\left[\int_0^T|M_t^n-N_t^n|^2\mathrm{d}t\right] \\
\leq\ & k^2C_2\bar{K}\int_0^T\left\{\mathbf{E}\left[\int_t^{t_n}|\tilde{f}_u|^{2\alpha}\mathrm{d}u\right]+\mathbf{E}\left[\int_t^{t_n}|\tilde{f}_u|^{2}\mathrm{d}u\right]\right\}\mathrm{d}t \\
& +2\mathbf{E}\left[\int_0^T\left(n\int_t^{t_n}|\psi^k(u)|^2\mathrm{d}u\right)\mathrm{d}t\right] \\
& +k^2C_2\int_0^T\left[\mathbf{E}\left(n\int_t^{t_n}|X_u^t-x|^2\mathrm{d}u\right)\right]\mathrm{d}t+k^2C_2\int_0^T\left[\mathbf{E}\left(n\int_t^{t_n}|X_u^t-x|^{2\alpha}\mathrm{d}u\right)\right]\mathrm{d}t.
\end{aligned}\tag{30}$$

Note that the process $(\tilde{f}_t)$ belongs to $\mathscr{H}^1_{2\alpha}$ and $\mathscr{H}^1_2$. It follows from the absolute continuity of integral and Lebesgue's dominated convergence theorem that the first term of the right hand side in (30) tends to zero as $n\to\infty$. Note that the process $(\psi^k(t))$ belongs to $\mathscr{H}^1_{2\alpha}$ and then $\mathscr{H}^1_2$. Applying Corollary 1 with $\phi(t)=\psi^k(t)$ yields that the second term of the right hand side in (30) tends to $2\|\psi^k(t)\|_2^2$ as $n\to\infty$. And, Fan and Hu (2008) has proved that the third term of the right hand side in (30) tends to zero as $n\to\infty$ (see (3.5) in Fan and Hu (2008)), similar to their proof we can show that the last term of the right hand side in (30) tends to zero as $n\to\infty$. In fact, noticing that $\mathbf{E}\left[|X_t^t-x|^{2\alpha}\right]=0$, by (4) and Fubini's Theorem we can get that for every $t\in[0,T)$,

$$\begin{aligned}
\lim_{n\to\infty}\mathbf{E}\left[n\int_t^{t_n}|X_u^t-x|^{2\alpha}\mathrm{d}u\right] &= \lim_{n\to\infty}n\int_t^{t_n}\mathbf{E}\left\{\left[|X_u^t-x|^{2\alpha}\right]\right\}\mathrm{d}u \\
&= \lim_{n\to\infty}\mathbf{E}\left[|X_{t_n}^t-x|^{2\alpha}\right]=0
\end{aligned}$$

and

$$\forall\ n\in\mathbf{N},\ \ \mathbf{E}\left[n\int_t^{t_n}|X_u^t-x|^{2\alpha}\mathrm{d}u\right]\leq 2^{2\alpha}(C_{2\alpha}+|x|^{2\alpha}),$$

then the desired result follows from Lebesgue's dominated convergence theorem. Consequently, for every $k\in\mathbf{N}$, we have

$$\limsup_{n\to\infty}\mathbf{E}\left[\int_0^T|M_t^n-N_t^n|^2\mathrm{d}t\right]\leq 2\mathbf{E}\left[\int_0^T|\psi^k(t)|^2\mathrm{d}t\right]=2\|\psi^k(t)\|_2^2,$$

from which (25) follows immediately by letting $k \to \infty$. Thus, combining (20), (24) with (25) yields that the right hand side of equality (19) tends to 0 in the space of process $\mathcal{H}_p^1$ as $n \to \infty$.

Finally, we assume that (8) also holds true. It is easy to see from (A3), (H2) and (8) that $\mathbf{E}\left[\sup_{0\le t\le T} |g\left(t,y,\sigma^*(t,x)q\right)|^2\right] < +\infty$. Then, applying (14) with $\phi(t) = g\left(t,y,\sigma^*(t,x)q\right)$ yields that

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^{T-1/n} \left| n\int_t^{t+1/n} \left(g\left(u,y,\sigma^*(u,x)q\right) - g\left(t,y,\sigma^*(t,x)q\right)\right) \mathrm{d}u\right|^2 \mathrm{d}t\right] = 0.$$

Note that (21) also holds true in the case of $p = 2$. We can derive from the above equality and (23) that

$$\lim_{n\to\infty} \mathbf{E}\left[\int_0^T \left|N_t^n - g(t,y,\sigma^*(t,x)q)\right|^2 dt\right] = 0. \tag{31}$$

Thus, combining (20), (31) with (25) yields that the right hand side of equality (19) tends to 0 in $\mathcal{H}_2^1$ as $n \to \infty$. The proof of Theorem 1 is then completed. □

4 Some Applications

In this section, we will give some applications relating to Theorem 1 and Theorem 2. The following Theorem 3 gives a converse comparison theorem for generators of BSDEs with monotonic and polynomial-growth generators.

Theorem 3 (Converse Comparison Theorem) Let the generators g_i $(i = 1,2)$ satisfy (A1), (A2) and (A3). If for every $t \in [0,T]$ and $\xi \in L^2(\Omega,\mathcal{F}_t,P)$, the minimal solutions $(\underline{y}_u^{\xi,t,g_i},\underline{z}_u^{\xi,t,g_i})_{u\in[0,t]} \in \mathcal{H}_2^1(0,t)\times\mathcal{H}_2^d(0,t)$ $(i = 1,2)$ of BSDEs with parameters (ξ,t,g_i) satisfy that for every $s \in [0,t]$,

$$\underline{y}_s^{\xi,t,g_1} \ge \underline{y}_s^{\xi,t,g_2}, \ \ \mathrm{d}P - a.s., \tag{32}$$

then for every $(y,z)\in\mathbf{R}^{1+d}$, we have

$$g_1(t,y,z) \ge g_2(t,y,z), \ \mathrm{d}P\times\mathrm{d}t - a.e.. \tag{33}$$

Proof. For every given $(y,z) \in \mathbf{R}^{1+d}$, it follows from the condition (32) that for every $n \in \mathbf{N}$ and $t\in[0,T]$, the minimal solutions $(\underline{y}_u^{\xi,t_n,g_i},\underline{z}_u^{\xi,t_n,g_i})_{u\in[0,t_n]} \in \mathcal{H}_2^1(0,t_n)\times\mathcal{H}_2^d(0,t_n)$ $(i = 1,2)$ of BSDEs with parameters $(\xi := y+z\cdot(B_{t_n}-B_t),t_n,g_i)$ satisfy

$$\underline{\mathcal{E}}_{t,t_n}^{g_1}[y+z\cdot(B_{t_n}-B_t)] \ge \underline{\mathcal{E}}_{t,t_n}^{g_2}[y+z\cdot(B_{t_n}-B_t)], \ \mathrm{d}P - a.s..$$

Then, $\mathrm{d}P\times\mathrm{d}t - a.e.$,

$$\underline{\mathcal{E}}_{t,t_n}^{g_1}[y+z\cdot(B_{t_n}-B_t)] - y \ge \underline{\mathcal{E}}_{t,t_n}^{g_2}[y+z\cdot(B_{t_n}-B_t)] - y. \tag{34}$$

It follows from Theorem 2 that there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $\mathrm{d}P\times\mathrm{d}t - a.e.$,

$$\lim_{k\to\infty} n_k\{\underline{\mathcal{E}}_{t,t_{n_k}}^{g_1}[y+z\cdot(B_{t_{n_k}}-B_t)] - y\} = g_1(t,y,z). \tag{35}$$

$$\lim_{k\to\infty} n_k\{\underline{\mathscr{E}}^{g_2}_{t,t_{n_k}}[y+z\cdot(B_{t_{n_k}}-B_t)]-y\}=g_2(t,y,z). \tag{36}$$

Thus, coming back to (34), by (35) and (36) we can easily get (33). □

Like the representation theorem for generators of BSDEs with Lipschitz generators, Theorem 2 can be used to investigate properties of generators of BSDEs with monotonic and polynomial-growth generators by virtue of their solutions. The following Theorem 4 and Theorem 5 are two specific examples which are both direct corollaries of Theorem 2. Some further results can be obtained like Section 2.3.2 in Jia (2008).

Theorem 4 (Self-financing Condition) Let the generator g satisfy (A1), (A2) and (A3). If the minimal solution $(\underline{y}^{0,T,g}_t, \underline{z}^{0,T,g}_t)_{t\in[0,T]} \in \mathscr{H}^1_2 \times \mathscr{H}^d_2$ of the BSDE with parameters $(0,T,g)$ satisfies that for every $t\in[0,T]$,

$$\underline{y}^{0,T,g}_t = 0, \ \ \mathrm{d}P - a.s.,$$

then $\mathrm{d}P\times\mathrm{d}t - a.e.$, $g(t,0,0)=0$.

Theorem 5 (Zero-interest Condition) Let the generator g satisfy (A1), (A2) and (A3). For every constant c, if the minimal solution $(\underline{y}^{c,T,g}_t, \underline{z}^{c,T,g}_t)_{t\in[0,T]} \in \mathscr{H}^1_2 \times \mathscr{H}^d_2$ of the BSDE with parameters (c,T,g) satisfies that for every $t\in[0,T]$,

$$\underline{y}^{c,T,g}_t = c, \ \ \mathrm{d}P - a.s.,$$

then $\mathrm{d}P\times\mathrm{d}t - a.e.$, for every y, $g(t,y,0)=0$.

Remark 6 It is easy to see that if the minimal solutions of BSDEs are replaced by the maximal solutions (see Remark 1), Theorems 1-5 also hold true.

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