

Homogenization of Fractional Kinetic Equations with Random Initial Data

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Abstract

We present the small-scale limits for the homogenization of a class of spatial-temporal random fields; the field arises from the solution of a certain fractional kinetic equation and also from that of a related two-equation system, subject to given random initial data. The space-fractional derivative of the equation is characterized by the composition of the inverses of the Riesz potential and the Bessel potential. We discuss the small-scale (the micro) limits, opposite to the well-studied large-scale limits, of such spatial-temporal random field. Our scaling schemes involve both the Riesz and the Bessel parameters, and also involve the rescaling in the initial data; our results are completely new-type scaling limits for such random fields.

Key words: Homogenization; Small-scale limits; Riesz-Bessel fractional equation and system; Random initial data; Hermite expansion; Multiple Itô-Wiener integral; Long-range dependence.

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1 Introduction

The purpose of this paper is to present new-type spatial-temporal scaling limits for the homogenization theory associated with a certain fractional kinetic equation and also a related system of reaction-diffusion type formed from it:

$$\begin{cases} \frac{\partial^\beta}{\partial t^\beta} u(t, x) = -\mu_1(I - \Delta)^{\frac{\gamma}{2}}(-\Delta)^{\frac{\alpha}{2}} u(t, x) + b_{11}u + b_{12}v, & u(0, x) = u_0(x), \\ \frac{\partial^\beta}{\partial t^\beta} v(t, x) = -\mu_2(I - \Delta)^{\frac{\gamma}{2}}(-\Delta)^{\frac{\alpha}{2}} v(t, x) + b_{21}u + b_{22}v, & v(0, x) = v_0(x), \end{cases} \quad (1)$$

where $\mu_i > 0, 0 < \beta \leq 1, 0 < \alpha \leq 2, 0 \leq \gamma$, and $t > 0, x \in \mathbb{R}^n$. The parameter β denotes the time-fractional index, and α, γ denote the space-fractional indices, for which we refer as the Riesz parameter and the Bessel parameter respectively (see [24, V.1 and V.3]).

When $(\beta, \alpha, \gamma) = (1, 2, 0)$, the system (1) is reduced to a classical reaction-diffusion system. The time-fractional index $\beta < 1$ means sub-diffusive (super-diffusive in case $\beta > 1$, which we do not study in this paper; see the remarks in Section 4). The spatial-fractional Riesz index α means the jumps of the evolution, and Bessel index γ means the tempering of large jumps; see the now-classic book of Stein [24, V.1 and V.3] for precise mathematical explanations. To our knowledge, fractional kinetic equations of Riesz-Bessel type appear firstly in Anh and Leonenko [1, 2]; subsequent works in this direction by the authors and collaborators can be seen in [3, 4, 5, 13, 15, 16] and the references therein. We mention that fractional differential equations involving more-than-one fractional parameters could be natural mathematical objects to describe long-range dependence and/or intermittency; one can find data exhibiting such characteristics in quite several fields including finance, hydrology, telecommunications, and turbulence.

In this paper, we aim to study the *small-scale limit* of spatial-temporal random field arising from the solution of the fractional Riesz-Bessel equation and its associated system (1) with $\mu_1 = \mu_2 = \mu > 0$, subject to given the random initial data u_0 and v_0 , of which are independent and each one has a certain long-range dependence in its random structure. We mainly focus on the single equation case, and then the derive the system case via a certain decoupling method used in [17, 18]. The opposite large-scale limits have been explored intensively in the above cited literatures; a notable point in such large-scale limits as in [1, 2] is that the Bessel parameter does not play its role; while in our small-scale limits, both the Riesz and the Bessel parameters play their roles, and moreover we need the rescaling on the initial data. Such small-scale spatial-temporal scaling limit is a completely new result in the homogenization theory of random fields, to our best knowledge. Furthermore, our study may show how the theory of Riesz potentials and Bessel potentials, as in the Chapter V of Stein [24], may appear significantly in the homogenization of random fields.

The underlying idea in this paper is motivated by those works in [1, 2, 5, 15, 16] and the references therein. Namely, we use the spectral representations to describe the sample field arising from the initial data, and the relations between Hermite polynomials and homogeneous chaos associated with the initial data, to get representations for the limit field in terms of multiple Itô-Wiener integrals. From limit theorems point-of-view, our results, though it is the small-scale rather than the well-studied large-scale as in those in the above citations, still could be regarded in the realm of non-central limit theorems for convolution type integrals, in which the papers [25, 11] are pioneering; see also the monograph of Major [22] and survey papers in the special volume edited by Doukhan, Oppenheim and Taqqu [9]

The paper is organized as follows. In the preliminary Section 2, we list some result for the subsequent needs; we give the explicit solution of the system (1), which includes automatically the

single equation solution, and our initial data are assumed to be the subordinated Gaussian fields. In the main Section 3, we present main results, the small-scale limit theorems of the solution field of the single equation and the related system, (1), in which we adopt usual time-derivative and the fractional spatial-derivative characterized by the Riesz and the Bessel parameters. The corresponding large-scale limits and time-fractional extensions are described in Section 4, together with two remarks for perspectives. The proofs of main results are give in the final Section 5.

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2 Preliminaries

To begin with, we rewrite the system (1), with $\mu_1 = \mu_2 = \mu > 0$, $\alpha > 0$, $\gamma \geq 0$ and $\beta = 1$, in the matrix form as follows:

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = -\mu(I - \Delta)^{\frac{\gamma}{2}}(-\Delta)^{\frac{\alpha}{2}} \begin{pmatrix} u \\ v \end{pmatrix} + B \begin{pmatrix} u \\ v \end{pmatrix}, \quad (2)$$

subject to some initial conditions

$$\begin{pmatrix} u(0, x) \\ v(0, x) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, \quad x \in \mathbb{R}^n, \quad (3)$$

where $u = u(t, x)$, $v = v(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, Δ is the n -dimensional Laplacian, and B is a 2×2 matrix.

The Green function $G(t, x; \alpha, \gamma)$ associated with the operator $\partial_t + \mu(I - \Delta)^{\frac{\gamma}{2}}(-\Delta)^{\frac{\alpha}{2}}$, for which the spatial part is a hybrid of the Laplace and the Bessel operators, is defined via the spatial Fourier transform as follows; see, [24, Chapter 5] or [2, Section 2].

$$\int_{\mathbb{R}^n} e^{i\langle x, \lambda \rangle} G(t, x; \alpha, \gamma) dx = \exp[-\mu t |\lambda|^\alpha (1 + |\lambda|^2)^{\frac{\gamma}{2}}], \quad \lambda \in \mathbb{R}^n, \quad (4)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathbb{R}^n . We remark that, by (4) it has

$$\int_{\mathbb{R}^n} G(t, x; \alpha, \gamma) dx = 1, \quad \text{for any } t \geq 0. \quad (5)$$

In order to get an explicit representation for the solution of (2), we impose the following assumption on the matrix B .

Condition DM. Suppose the matrix $[b_{ij}]_{1 \leq i, j \leq 2}$ is diagonalizable, i.e., the matrix B can be written as

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = PDP^{-1} \quad \text{with } P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad (6)$$

where P is a real-valued non-degenerate eigenvector matrix associated with the matrix B , and $D = \text{diag}(d_1, d_2)$, $d_1, d_2 \in \mathbb{R}$, where d_j is the eigenvalue associated with the eigenvector $(p_{1,j}, p_{2,j})^T$ (here and henceforth, T denotes the transpose). Without loss of generality, we suppose that $\det(P) = 1$. We always write

$$Q(t; d_1, d_2) := P \begin{pmatrix} e^{d_1 t} & 0 \\ 0 & e^{d_2 t} \end{pmatrix} P^{-1}. \quad (7)$$

Next, we consider the randomness of the initial data. Let $(\Omega, \mathcal{F}, \mathcal{P})$ to be an underlying probability space, such that all random element appeared in this paper are measurable with respect to it. In this paper, we assume that the initial data are a kind of isotropic stationary fields, namely the subordinated Gaussian fields; it is described as follows:

Condition SGRID. We assume each component of the initial data (3) $\mathbf{w}_0(x) := (u_0(x), v_0(x)) = (\eta_1(x), \eta_2(x))$, $x \in \mathbb{R}^n$, is a random field of the following form

$$\eta_j(x) := h_j(\zeta_j(x)), \quad x \in \mathbb{R}^n, \quad j \in \{1, 2\}, \quad (8)$$

in which $\zeta_1(x)$ and $\zeta_2(x)$ are independent, mean-square continuous, homogeneous and isotropic Gaussian random fields, each is of mean zero and of variance 1, and for each the spectral measure $F_j(d\lambda)$ has the (spectral) density $f_j(\lambda)$, $\lambda \in \mathbb{R}^n$, $j \in \{1, 2\}$, respectively. Each $f_j(\lambda)$ is decreasing for $|\lambda| > \lambda_0$ for some $\lambda_0 > 0$ and continuous for all $\lambda \neq 0$. Moreover, we assume that $h_j(\cdot)$, $j \in \{1, 2\}$, are real non-random Borel functions satisfy

$$Eh_j^2(\zeta_j(0)) < \infty, \quad j \in \{1, 2\}. \quad (9)$$

Under the Condition SGRID, we have the spectral representation for the sample paths of $\zeta_j(x)$, $j \in \{1, 2\}$, as below:

$$\zeta_j(x) = \int_{\mathbb{R}^n} e^{i\langle x, \lambda \rangle} \sqrt{f_j(\lambda)} W_j(d\lambda), \quad x \in \mathbb{R}^n, \quad j \in \{1, 2\}, \quad (10)$$

where $W_j(\cdot)$, $j = 1, 2$, are two orthogonal standard Gaussian noise measures on \mathbb{R}^n . Moreover, under (9), we can consider the following Hermite expansions of $h_j(u)$ in the Hilbert space $L^2(\mathbb{R}, p(u)du)$ with $p(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$:

$$h_j(u) = C_0^{(j)} + \sum_{\sigma=1}^{\infty} C_{\sigma}^{(j)} \frac{H_{\sigma}(u)}{\sqrt{\sigma!}}, \quad j \in \{1, 2\}, \quad (11)$$

in which the Hermite coefficients

$$C_{\sigma}^{(j)} := \int_{\mathbb{R}} h_j(u) \frac{H_{\sigma}(u)}{\sqrt{\sigma!}} p(u) du, \quad j \in \{1, 2\}, \quad (12)$$

and the Hermite polynomials $\{H_{\sigma}(u), \sigma = 0, 1, 2, \dots\}$

$$H_{\sigma}(u) := (-1)^{\sigma} e^{\frac{u^2}{2}} \frac{d^{\sigma}}{du^{\sigma}} e^{-\frac{u^2}{2}}, \quad \text{for } \sigma \in \{0, 1, 2, \dots\}.$$

The *Hermite rank* of the function $h_j(\cdot)$ is defined by

$$m_j := \inf\{\sigma \geq 1 : C_\sigma^{(j)} \neq 0\}, \quad j \in \{1, 2\}.$$

We also need the following two important properties (see, for example, Major [22], Corollary 5.5 and p. 30) :

$$E[H_{\sigma_1}(\zeta_j(y_1))H_{\sigma_2}(\zeta_{j'}(y_2))] = \delta_j^{j'} \delta_{\sigma_2}^{\sigma_1} \sigma_1! R_{\zeta_j}^{\sigma_1}(y_1 - y_2), \quad y_1, y_2 \in \mathbb{R}^n, \quad (13)$$

and

$$H_\rho(\zeta_j(x)) = \int'_{\mathbb{R}^{n \times \rho}} e^{i\langle x, \lambda_1 + \dots + \lambda_\rho \rangle} \prod_{\sigma=1}^{\rho} \sqrt{f_j(\lambda_\sigma)} W_j(d\lambda_\sigma), \quad (14)$$

in the above, the integral representation (14) is the ρ -fold Itô-Wiener integral; the integration notation \int' means that it excludes the diagonal hyperplanes $z_i = \mp z_j, i, j = 1, \dots, \rho, i \neq j$.

We now can state the following explicit form of the spatial-temporal random fields of the system, subject to the Conditions DM and SGRID. It contains the single-equation solution consequently, which can be seen in [2, Section 2].

Proposition 1. *Let $\mathbf{w}(t, x; \mathbf{w}_0(\cdot)) := (u(t, x; u_0(\cdot)), v(t, x; v_0(\cdot)))$, $t > 0$, $x \in \mathbb{R}^n$ be the mean-square solution of the initial value problem (2) (3), of which satisfies the conditions DM and SGRID, then it is expressed as*

$$\mathbf{w}(t, x; \mathbf{w}_0(\cdot)) = Q(t; d_1, d_2) \left\{ \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} + \sum_{\rho \in \mathbb{N}} \int'_{\mathbb{R}^{n \times \rho}} e^{i\langle x, \lambda_1 + \dots + \lambda_\rho \rangle - \mu t |\lambda_1 + \dots + \lambda_\rho|^\alpha (1 + |\lambda_1 + \dots + \lambda_\rho|^2)^{\frac{\gamma}{2}}} \begin{pmatrix} \frac{C_\rho^{(1)}}{\sqrt{\rho!}} \prod_{\sigma=1}^{\rho} \sqrt{f_1(\lambda_\sigma)} W_1(d\lambda_\sigma) \\ \frac{C_\rho^{(2)}}{\sqrt{\rho!}} \prod_{\sigma=1}^{\rho} \sqrt{f_2(\lambda_\sigma)} W_2(d\lambda_\sigma) \end{pmatrix} \right\},$$

where the coefficients $C_\rho^{(j)}$, $j \in \{1, 2\}$, are defined by (12).

The above display of the solution can be obtained by firstly taking the spatial Fourier transform on both sides of (2). The Condition DM allows us to decouple the system; then we make the inverse Fourier transform to express the solution as the convolution of the Green function and the initial data. Then we make use of the Green function $G(t, y; \alpha, \gamma)$ defined as (4). Finally we plug in the initial data as described in the Condition SGRID.

We also impose the following assumption which is related to the long-range-dependence of the underlying Gaussian fields $\zeta_j(x)$, $j \in \{1, 2\}$; we refer to [9] for the notion and the literatures of long-range dependence. In the following and henceforth, the notation $f(\cdot) \sim g(\cdot)$ means that the ratio $f(\cdot)/g(\cdot)$ tends to 1, as the indicated variable “ \cdot ” tends to ∞ or tends to 0, according to the context.

Condition LD. The Gaussian random fields $\zeta_j(x)$, $j \in \{1, 2\}$, in the Condition SGRID, have their

covariance functions to be regular varying at infinity in the sense that: the covariance function R_{ζ_j} is given by

$$R_{\zeta_j}(x) \sim \frac{L(|x|)}{|x|^{\kappa_j}}, \quad \text{as } |x| \rightarrow \infty, \quad 0 < \kappa_j < \frac{n}{m_j}, \quad j \in \{1, 2\}, \quad (15)$$

where $L : (0, \infty) \rightarrow (0, \infty)$ is a slowly varying function at infinity and is bounded on each finite interval; recall that L is said to be slowly varying at infinity if $\lim_{y \rightarrow \infty} [L(cy)/L(y)] = 1$ uniformly for any $c \in (a, b)$, $0 < a < b < \infty$. Here, the relation “ \sim ” in (15) means that $\lim_{|x| \rightarrow \infty} |x|^{\kappa_j} R_{\zeta_j}(x)/L(|x|) = 1$; the similar notation will be used in this section and also in Section 5.

Under the Condition LD, by a Tauberian theorem (see, for example, the book of Leonenko [14, p. 66]), the spectral density functions of the random fields $\zeta_j(x)$, $j \in \{1, 2\}$, are regular varying near the origin as follows:

$$f_j(\lambda) \sim K(n, \kappa_j) |\lambda|^{\kappa_j - n} L\left(\frac{1}{|\lambda|}\right), \quad \text{as } \lambda \rightarrow 0, \quad j \in \{1, 2\}, \quad (16)$$

where $K(n, \kappa_j)$ denotes the Tauberian constant.

We note that, for each natural number $\rho \geq 2$, the power of the covariance function $(R_{\zeta_j}(x))^\rho$ itself is still the covariance function of some random fields, for which there exists the corresponding spectral density function $(f_j)^{* \rho}(\lambda)$, which is the ρ -th convolution of $f_j(\lambda)$:

$$(f_j)^{* \rho}(\lambda) = \int_{\mathbb{R}^{n \times (\rho-1)}} f_j(\lambda - \lambda_1) f_j(\lambda_1 - \lambda_2) \cdots f_j(\lambda_{\rho-2} - \lambda_{\rho-1}) f_j(\lambda_{\rho-1}) \prod_{l=1}^{\rho-1} d\lambda_l. \quad (17)$$

Note that $L^\rho(|x|)$ is still a slowly varying function for any ρ . Thus, when the ρ satisfies $0 < \rho \kappa_j < n$, we can apply the Tauberian theorem again to get

$$(f_j)^{* \rho}(\lambda) \sim K(n, \rho \kappa_j) |\lambda|^{\rho \kappa_j - n} L^\rho\left(\frac{1}{|\lambda|}\right), \quad \text{as } \lambda \rightarrow 0, \quad 0 < \rho \kappa_j < n, \quad (18)$$

for $j \in \{1, 2\}$. While, if $\rho \kappa_j > n$ then the covariance function $(R_{\zeta_j}(x))^\rho$ belongs to the class $L^1(\mathbb{R}^n)$; thus the corresponding spectral density function is everywhere continuous and satisfies

$$(2\pi)^n (f_j)^{* \rho}(0) = \int_{\mathbb{R}^n} (R_{\zeta_j})^\rho(x) dx \leq \int_{\mathbb{R}^n} |R_{\zeta_j}(x)|^\rho dx \leq \int_{\mathbb{R}^n} |R_{\zeta_j}(x)|^{\rho^*} dx < \infty, \quad (19)$$

where $\rho^* := \inf\{\rho \in \mathbb{N} \mid \rho \kappa_j > n\}$; we note that $|R_{\zeta_j}(\cdot)| \leq 1$.

The displays (17), (18) and (19) will be used in the proofs in Section 5.

3 Main results: the small-scale limits

In this section, we present the main results of this paper, which concerns with the small-scale (or say the micro) limits of the homogenization of the spatial-temporal random field described in Section 2. It features that both the Riesz parameter α and the Bessel parameter γ play their roles in the scaling procedure, and we also need to rescale the initial data.

Firstly, we state our result for the small-scale limit of a single fractional kinetic equation. The equation is then of the form:

$$\frac{\partial s}{\partial t}(t, x) = -\mu(I - \Delta)^{\frac{\gamma}{2}}(-\Delta)^{\frac{\alpha}{2}}s(t, x), \quad s(0, x) = h(\zeta(x)). \quad (20)$$

To our knowledge, the homogenization presented below is completely new spatial-temporal scalings. In the below, the notation imposed on ζ wants to mean that the variable of ζ is under the indicated dilation factor.

Theorem 1. *Let $s := s(t, x; s_0(\cdot))$, $t > 0$, $x \in \mathbb{R}^n$, be the mean-square solution of (20), which satisfies the Conditions SGRID and LD, with $\kappa \in (0, \frac{n}{m})$, where m denotes the Hermite rank of the non-random function $h(\cdot)$ on \mathbb{R} , which has the Hermite coefficients $C_i(h)$, $i = 0, 1, \dots$ (i.e., $h_1(x) = h(x)$, $\zeta_1(x) = \zeta(x)$, $f_1(\lambda) = f(\lambda)$ and $\kappa_1 = \kappa$, etc. in Section 2). Then, for any fixed parameter $\chi > 0$, (1) the covariance function of the rescaled random field*

$$s^\varepsilon(t, x) := [\varepsilon^{m\kappa\chi} L^m(\varepsilon^{-\chi})]^{-\frac{1}{2}} \left\{ s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; h(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot))) - C_0(h) \right\}, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (21)$$

satisfies

$$\lim_{\varepsilon \rightarrow 0} \text{Cov}(s^\varepsilon(t, x) s^\varepsilon(t', x')) = (C_m(h))^2 K(n, m\kappa) \int_{\mathbb{R}^n} e^{i\langle x-x', \tau \rangle} \frac{e^{-\mu(t+t')|\tau|^{\alpha+\gamma}}}{|\tau|^{n-m\kappa}} d\tau. \quad (22)$$

(2) When $\varepsilon \rightarrow 0$, the rescaled random field $s^\varepsilon(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, converges to the limiting spatial-temporal random field $s_m(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, in the finite dimensional distribution sense, and $s_m(t, x)$ is represented by the multiple-Wiener integral

$$s_m(t, x) = \frac{C_m(h)}{\sqrt{m!}} K(n, \kappa)^{\frac{m}{2}} \int'_{\mathbb{R}^{n \times m}} \frac{e^{i\langle x, z_1 + \dots + z_m \rangle - \mu t |z_1 + \dots + z_m|^{\alpha+\gamma}}}{(|z_1| \cdots |z_m|)^{\frac{n-\kappa}{2}}} \prod_{l=1}^m W(dz_l), \quad (23)$$

where $\int' \cdots$ denotes a m -fold Wiener integral with respect to the complex Gaussian white noise $W(\cdot)$ on \mathbb{R}^n .

Remark. The requirement for rescaling the initial data could be figured intuitively as that, the small-scale is meant to “freeze down” both the time t and the space x , to ensure the overall renormalization we have to “heat up” the initial data, so that it can be achieved a non-degenerate (though singular) limiting field. The term “small-scale limits”, opposite to large-scale (other naming such as micro v.s. macro, local v.s. asymptotic), is adapted from the recent studies on the multi-scaling behavior of fractional non-Gaussian random fields, notably in [6, 7]. It has also been used in the study of the fractal nature of the intersection local time measure [23].

Next, we state the small-scale limit for the system as follows. It shows that there is a nontrivial combination limit, namely the case (3) below, caused by the imposed relation on various parameters.

Theorem 2. *Let $\mathbf{w}(t, x; \mathbf{w}_0(\cdot)) := (u(t, x; u_0(\cdot)), v(t, x; v_0(\cdot)))$, $t > 0$, $x \in \mathbb{R}^n$, be the solution-vector of the initial value problem (2) and (3), satisfying the Conditions DM, SGRID, and LD. In the following, χ is a positive parameter; $Q(t; d_1, d_2)$ is a matrix defined in (7), and the two Gaussian noise fields W_j , $j \in \{1, 2\}$ are totally independent. Let m_1 , m_2 , κ_1 and κ_2 denote respectively the Hermite ranks*

and the long-range-dependence indices mentioned in the Conditions SGRID and LD for u_0 and v_0 .

(1) If $m_2\kappa_2 > m_1\kappa_1$, then the finite-dimensional distributions of the rescaled random field

$$[\varepsilon^{m_1\kappa_1\lambda} L^{m_1}(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ \mathbf{w}(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; \mathbf{w}_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$\begin{pmatrix} Y_1^*(t, x) \\ Y_2^*(t, x) \end{pmatrix} = \begin{pmatrix} \tilde{X}_{m_1}^{(1)}(t, x) \\ 0 \end{pmatrix}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

where

$$\tilde{X}_{m_1}^{(1)}(t, x) := \frac{C_{m_1}^{(1)}}{\sqrt{m_1!}} K(n, \kappa_1)^{\frac{m_1}{2}} \int'_{\mathbb{R}^{n \times m_1}} \frac{e^{i\langle x, z_1 + \dots + z_{m_1} \rangle - \mu t |z_1 + \dots + z_{m_1}|^{\alpha+\gamma}}}{(|z_1| \cdots |z_{m_1}|)^{\frac{n-\kappa_1}{2}}} \prod_{l=1}^{m_1} W_1(dz_l), \quad (24)$$

with $W_1(\cdot)$ is a complex Gaussian white noise on \mathbb{R}^n (i.e., (23) with m , κ and W replaced by m_1 , κ_1 and W_1 , respectively).

(2) If $m_1\kappa_1 > m_2\kappa_2$, then the finite-dimensional distributions of the rescaled random field

$$[\varepsilon^{m_2\kappa_2\lambda} L^{m_2}(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ \mathbf{w}(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; \mathbf{w}_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$\begin{pmatrix} Y_1^{**}(t, x) \\ Y_2^{**}(t, x) \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{X}_{m_2}^{(2)}(t, x) \end{pmatrix}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

where

$$\tilde{X}_{m_2}^{(2)}(t, x) := \frac{C_{m_2}^{(2)}}{\sqrt{m_2!}} K(n, \kappa_2)^{\frac{m_2}{2}} \int'_{\mathbb{R}^{n \times m_2}} \frac{e^{i\langle x, z_1 + \dots + z_{m_2} \rangle - \mu t |z_1 + \dots + z_{m_2}|^{\alpha+\gamma}}}{(|z_1| \cdots |z_{m_2}|)^{\frac{n-\kappa_2}{2}}} \prod_{l=1}^{m_2} W_2(dz_l), \quad (25)$$

and $W_2(\cdot)$ is a complex Gaussian white noise on \mathbb{R}^n (i.e., (23) with m , κ and W replaced by m_2 , κ_2 and W_2 , respectively).

(3) If $m_1 = m_2 := m$, $\kappa_1 = \kappa_2 := \kappa$, then the finite-dimensional distributions of the rescaled random field

$$[\varepsilon^{m\kappa\lambda} L^m(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ \mathbf{w}(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; \mathbf{w}_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$\begin{pmatrix} Y_1^{***}(t, x) \\ Y_2^{***}(t, x) \end{pmatrix} = \begin{pmatrix} \tilde{X}_m^{(1)}(t, x) \\ \tilde{X}_m^{(2)}(t, x) \end{pmatrix}, \quad t > 0, \quad x \in \mathbb{R}^n, \quad (26)$$

where $\tilde{X}_m^{(1)}$ and $\tilde{X}_m^{(2)}$, are defined in (24) and (25) with $m_1 = m_2 = m$ and $\kappa_1 = \kappa_2 = \kappa$.

To understand the stochastic structure of the limiting fields, we state, for instance, the following covariance result of $(Y_1^{***}(t, x) \ Y_2^{***}(t, x))$.

Proposition 2. For each fixed $t > 0$, the limiting vector field $\left(Y_1^{***}(t, x) \ Y_2^{***}(t, x) \right)$ in the case (3) of Theorem 2 is spatial-homogeneous and its covariance matrix has the following spectral representation

$$E \begin{pmatrix} Y_1^{***}(t, x) \\ Y_2^{***}(t, x) \end{pmatrix} \begin{pmatrix} Y_1^{***}(t', x') & Y_2^{***}(t', x') \end{pmatrix} = \int_{\mathbb{R}^n} e^{i\langle x-x', \lambda \rangle} S(\lambda; t, t', \alpha, \gamma) d\lambda,$$

where $S(\lambda; t, t', \alpha, \gamma) = K(n, m\kappa) \frac{e^{-\mu(t+t')|\lambda|^{\alpha+\gamma}}}{|\lambda|^{n-m\kappa}} \text{diag}((C_m^{(1)})^2, (C_m^{(2)})^2)$.

Remark. In view of the singularity of the (diagonal) spectral matrix near the origin, we may conclude that, for limiting vector field in the case (3), the long-range-dependence only exists within each individual component. In Proposition 3 below, we will find that there is also long-range-dependence between the different components of the large-scale limiting fields.

4 Extensions

4.1 Large-scalings

In this subsection, we state the large-scale (or say the macro) limits of the system, in which only the Riesz parameter α plays its role in the scaling scheme. The result is compared to the single-equation case in And and Leonenko [2, Theorems 2.2 and 2.3]; it shows various limit fields may happen because of the different relations of the various parameters. The proof can be proceeded by the decoupling method.

Proposition 3. Let $\mathbf{w}(t, x; \mathbf{w}_0(\cdot)) := (u(t, x; u_0(\cdot)), v(t, x; v_0(\cdot)))$, $t > 0$, $x \in \mathbb{R}^n$, be the solution-vector of the initial value problem (2) and (3), satisfying the Conditions MD, SGRID, and LD. In the following, $Q(t; d_1, d_2)$ is the matrix defined in (7), p_{ij} is the entry in (6), and the two Gaussian noise fields W_j , $j \in \{1, 2\}$, are totally independent. Again let m_1 , m_2 , κ_1 and κ_2 denote the parameters in the Conditions SGRID and LD for u_0 and v_0 .

(1) If $m_2\kappa_2 > m_1\kappa_1$ and $d_1 > d_2$, then the finite-dimensional distributions of the rescaled random field

$$[\varepsilon^{\frac{m_1\kappa_1}{\alpha}} L^{m_1}(\varepsilon^{-\frac{1}{\alpha}})]^{-\frac{1}{2}} e^{-d_1 \frac{t}{\varepsilon}} \left\{ \mathbf{w}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}; \mathbf{w}_0(\cdot)\right) - Q\left(\frac{t}{\varepsilon}; d_1, d_2\right) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$\mathbf{T}_{m_1}^{(1)}(t, x) := \begin{pmatrix} p_{11}p_{22}X_{m_1}^{(1)}(t, x) \\ p_{21}p_{22}X_{m_1}^{(1)}(t, x) \end{pmatrix}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

where

$$X_{m_1}^{(1)}(t, x) := \frac{C_{m_1}^{(1)}}{\sqrt{m_1!}} K(n, \kappa_1)^{\frac{m_1}{2}} \int_{\mathbb{R}^{n \times m_1}} \frac{e^{i\langle x, z_1 + \dots + z_{m_1} \rangle - \mu t |z_1 + \dots + z_{m_1}|^\alpha}}{(|z_1| \cdots |z_{m_1}|)^{\frac{n-\kappa_1}{2}}} \prod_{l=1}^{m_1} W_1(dz_l), \quad (27)$$

with $W_1(\cdot)$ is a complex Gaussian white noise on \mathbb{R}^n .

(2) If $m_1\kappa_1 > m_2\kappa_2$ and $d_1 > d_2$, then the finite-dimensional distributions of the rescaled random field

$$[\varepsilon^{\frac{m_2\kappa_2}{\alpha}} L^{m_2}(\varepsilon^{-\frac{1}{\alpha}})]^{-\frac{1}{2}} e^{-d_1 \frac{t}{\varepsilon}} \left\{ \mathbf{w}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\frac{1}{\alpha}}}; \mathbf{w}_0(\cdot)\right) - Q\left(\frac{t}{\varepsilon}; d_1, d_2\right) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$\mathbf{T}_{m_2}^{(2)}(t, x) := \begin{pmatrix} -p_{11}p_{12}X_{m_2}^{(2)}(t, x) \\ -p_{21}p_{12}X_{m_2}^{(2)}(t, x) \end{pmatrix}, \quad t > 0, x \in \mathbb{R}^n,$$

where

$$X_{m_2}^{(2)}(t, x) := \frac{C_{m_2}^{(2)}}{\sqrt{m_2!}} K(n, \kappa_2)^{\frac{m_2}{2}} \int_{\mathbb{R}^{n \times m_2}} \frac{e^{i\langle x, z_1 + \dots + z_{m_2} \rangle - \mu t |z_1 + \dots + z_{m_2}|^\alpha}}{(|z_1| \cdots |z_{m_2}|)^{\frac{n-\kappa_2}{2}}} \prod_{l=1}^{m_2} W_2(dz_l), \quad (28)$$

and $W_2(\cdot)$ is a complex Gaussian white noise on \mathbb{R}^n .

(3) If $m_1 = m_2 := m$, $\kappa_1 = \kappa_2 := \kappa$, and $d_1 > d_2$, then the finite-dimensional distributions of the rescaled random field

$$[\varepsilon^{\frac{m\kappa}{\alpha}} L^m(\varepsilon^{-\frac{1}{\alpha}})]^{-\frac{1}{2}} e^{-d_1 \frac{t}{\varepsilon}} \left\{ \mathbf{w}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon^{\frac{1}{\alpha}}}; \mathbf{w}_0(\cdot)\right) - Q\left(\frac{t}{\varepsilon}; d_1, d_2\right) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$\mathbf{T}_m^{(3)}(t, x) := \mathbf{T}_m^{(1)}(t, x) + \mathbf{T}_m^{(2)}(t, x), \quad (29)$$

where $\mathbf{T}_m^{(1)}(t, x)$ and $\mathbf{T}_m^{(2)}(t, x)$, are defined in the case (1) and the case (2) with $m_1 = m_2 = m$ and $\kappa_1 = \kappa_2 = \kappa$.

4.2 The subdiffusive case

We extend the above results to the subdiffusive case, meant for which the time-fractional derivative is $\frac{\partial^\beta}{\partial t^\beta}$, $\beta \in (0, 1)$, (see Section 1) in the system (2), that is,

$$\frac{\partial^\beta}{\partial t^\beta} \begin{pmatrix} u \\ v \end{pmatrix} = -\mu(I - \Delta)^{\frac{\gamma}{2}} (-\Delta)^{\frac{\alpha}{2}} \begin{pmatrix} u \\ v \end{pmatrix} + B \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mu, \alpha, \gamma > 0, \quad (30)$$

The time-fractional derivative $\frac{\partial^\beta}{\partial t^\beta}$, for any $\beta > 0$, can be seen, for example, the book of Djrbashian [8]). For $0 < \beta < 1$, it is

$$\frac{d^\beta f}{dt^\beta}(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\beta} d\tau, \quad (31)$$

where $f(t)$ is causal (i.e., f vanishes for $t < 0$).

The solution of (30) can be obtained via the fractional (both in time and in space) procedure (see, for example, [20, 21]); under the Condition DM, we express the solution as the convolution of the fractional (both in time and in space) Green function and the initial data, as follows.

$$\mathbf{w}(t, x; \mathbf{w}_0(\cdot)) = \int_{\mathbb{R}^n} P \begin{pmatrix} G_\beta(t, x - y; d_1) & 0 \\ 0 & G_\beta(t, x - y; d_2) \end{pmatrix} P^{-1} \begin{pmatrix} u_0(y) \\ v_0(y) \end{pmatrix} dy \quad (32)$$

with the fractional Green function $G_\beta(t, x; d_j)$ is defined via the transformation

$$E_\beta(-\mu|\lambda|^\alpha(1 + |\lambda|^2)^{\frac{\gamma}{2}}t^\beta + d_j t^\beta) = \int_{\mathbb{R}^n} e^{i\langle x, \lambda \rangle} G_\beta(t, x; d_j) dx, \quad j \in \{1, 2\},$$

where $E_\beta(\cdot)$ is the Mittag-Leffler function defined by (see, for example, [2] or [8, Chapter 1])

$$E_\beta(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(\beta p + 1)}, \quad z \in \mathbb{C}. \quad (33)$$

For the properties of the Mittag-Leffler function, we refer the classic book by Erdélyi *et al.* [10] (pp. 206-212, in particular p. 206 (7) and p. 210 (21)).

The following results are time-fractional versions of those in the above; *however*, the sub-diffusivity brings some new feature. For the large-scalings of homogenization of the system, we need to take an additional scaling on the matrix B in the system (30) in order to compromise the effect of this sub-diffusivity upon the interaction between u and v ; while the sub-diffusivity has no influence on the small-scalings and thus it is the same as that in Section 3. To make the situation clear, in the following we denote the vector solution by $\mathbf{w}(t, x; \mathbf{w}_0(\cdot), B)$. We only state some partial assertions in the below.

Proposition 4. *Let $\{\mathbf{w}(t, x; \mathbf{w}_0(\cdot), B)$, $t > 0$, $x \in \mathbb{R}^n\}$ be the solution-vector of the initial value problem (30) and (3), satisfying Conditions DM, SGRID, and LD. Denote again the long-range-dependence indices κ_j and the Hermite ranks m_j , $j \in \{1, 2\}$. Assume that $m_1\kappa_1 < m_2\kappa_2$.*

Large-scalings: under $m_1\kappa_1, m_2\kappa_2 < \min\{2\alpha, n\}$ the finite-dimensional distributions of the rescaled random field

$$T_\varepsilon^{(1)}(t, x) := (\varepsilon^{\frac{m_1\kappa_1}{\alpha}} L^{m_1}(\varepsilon^{-\frac{\beta}{\alpha}}))^{-\frac{1}{2}} \left\{ \mathbf{w}(\varepsilon^{-1}t, \varepsilon^{-\frac{\beta}{\alpha}}x; \mathbf{w}_0(\cdot), \varepsilon^\beta B) - \mathbf{C}(\varepsilon^{-1}t; \varepsilon^\beta B) \right\},$$

$t > 0$, $x \in \mathbb{R}^n$, converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$T^{(1)}(t, x) = \begin{pmatrix} p_{11}p_{22}T^{(1)}(t, x; d_1) - p_{12}p_{21}T^{(1)}(t, x; d_2) \\ p_{21}p_{22}T^{(1)}(t, x; d_1) - p_{21}p_{22}T^{(1)}(t, x; d_2) \end{pmatrix}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

where

$$\mathbf{C}(\varepsilon^{-1}t; \varepsilon^\beta B) = \mathbf{C}(t; B) = P \begin{pmatrix} E_\beta(d_1 t^\beta) & 0 \\ 0 & E_\beta(d_2 t^\beta) \end{pmatrix} P^{-1} \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix},$$

and for $j \in \{1, 2\}$ $T^{(1)}(t, x; d_j)$ is expressed by the following multiple-Wiener integral:

$$\frac{C_{m_1}^{(1)}}{\sqrt{m_1!}} K(n, \kappa_1)^{\frac{m_1}{2}} \int_{\mathbb{R}^{n \times m_1}} e^{i\langle x, \lambda_1 + \dots + \lambda_{m_1} \rangle} \frac{E_\beta(-\mu|\lambda_1 + \dots + \lambda_{m_1}|^\alpha t^\beta + d_j t^\beta)}{(|\lambda_1| \dots |\lambda_{m_1}|)^{\frac{n-\kappa_1}{2}}} \prod_{l=1}^{m_1} W_1(d\lambda_l). \quad (34)$$

Small-scalings: under $m_1\kappa_1, m_2\kappa_2 < \min\{2(\alpha + \gamma), n\}$ the finite-dimensional distributions of the rescaled random field

$$\mathbf{M}_\varepsilon^{(1)} := [\varepsilon^{m_1\kappa_1} L^{m_1}(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ \mathbf{w}(\varepsilon t, \varepsilon^{\frac{\beta}{\alpha+\gamma}} x; \mathbf{w}_0(\varepsilon^{-\frac{\beta}{\alpha+\gamma}-\lambda} \cdot), B) - \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \rightarrow 0$, to the finite-dimensional distributions of the random field

$$\mathbf{M}^{(1)}(t, x) = \begin{pmatrix} M^{(1)}(t, x) \\ 0 \end{pmatrix}, \quad t > 0, \quad x \in \mathbb{R}^n,$$

where

$$M^{(1)} = \frac{C_{m_1}^{(1)}}{\sqrt{m_1!}} K(n, \kappa_1)^{\frac{m_1}{2}} \int'_{\mathbb{R}^{n \times m_1}} e^{i \langle x, \lambda_1 + \dots + \lambda_{m_1} \rangle} \frac{E_\beta(-\mu |\lambda_1 + \dots + \lambda_{m_1}|^{\alpha+\gamma} t^\beta)}{(|\lambda_1| \dots |\lambda_{m_1}|)^{\frac{n-\kappa_1}{2}}} \prod_{l=1}^{m_1} W_1(d\lambda_l).$$

4.3 Two remarks for future study

1 Superdiffusive case. The time-fractional index $\beta < 1$ indicates the sub-diffusivity, and it changes to be the super-diffusivity if we consider $\beta > 1$ (see Section 1). In [18], the time-fractional reaction-wave type system with random initial data are studied, in which the first-order initial time-derivatives of u and v play the crucial role. To consider spatial-temporal fractional kinetic systems which is super-diffusive in time and Riesz-Bessel in space will be a task of tremendous calculations. We also mention that, for the classical, *i.e.* non-fractional, heat-type system with random initial condition, the solution vector-field and the scaling limit are expressed in terms of heat kernels; this more explicit and simpler case is treated in [17].

2 Relativistic diffusion equation. That role of the Bessel parameter could be somewhat intriguing. It plays no role in the large-scaling [1, 2], in which only the Riesz parameter appears; the Bessel parameter does appear in our small-scaling Theorem 1, and it plays jointly with the Riesz parameter. Perhaps, an appropriate viewpoint on this intriguing situation is to consider the "physically correct" Bessel operator, that is, to consider the prominent relativistic diffusion equation and its fractional version. This is addressed in [19].

5 Proofs of main results

We concentrate on the proofs of the main results in Section 3, that is, Theorem 1 and Theorem 2. The proofs of those propositions mentioned in Section 4 are skipped and left to the readers.

In the following proofs, \Rightarrow denotes the convergence of random variables (or random families) in distributional sense, and $\stackrel{d}{=}$ denotes the equality of random variables (or random families) in distributional sense. Moreover, we also denote $f(t, x; \varepsilon) \asymp g(t, x; \varepsilon)$ if there exists a constant $c := c(t, x) > 0$ such that $cg(t, x; \varepsilon) < f(t, x; \varepsilon) < c^{-1}g(t, x; \varepsilon)$ when $\varepsilon \rightarrow 0$.

The following lemma will play an important role in the proof of our main results; its proof is a easy consequence of the Cramer-Wold argument for two-dimensional random vectors (see, for example, [14, p. 6.]).

Lemma 1. Let $X_\varepsilon := \{[X_\varepsilon^{(1)} X_\varepsilon^{(2)}]^T(t, x), x \in \mathbb{R}^n, t > 0\}$ be a \mathbb{R}^2 -valued random field which is generated by $X_\varepsilon(t, x) = Q_\varepsilon(t)[U_\varepsilon V_\varepsilon]^T(t, x)$, where $U_\varepsilon(t, x)$ and $V_\varepsilon(t, x)$ are independent random fields on $\mathbb{R}^n \times \mathbb{R}^+$ and $Q_\varepsilon(t)$ is a non-random 2×2 matrix. If there exist two random fields U_0 and V_0 such that $U_\varepsilon(t, x) \Rightarrow U_0(t, x)$ and $V_\varepsilon(t, x) \Rightarrow V_0(t, x)$, respectively, and $Q_\varepsilon(t)$ converges to $Q(t)$ in the usual sense when $\varepsilon \rightarrow 0$, then the finite dimensional distributions of $X_\varepsilon(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, converge to the finite dimensional distributions of $X := \{Q(t)[U_0 V_0]^T(t, x), t > 0, x \in \mathbb{R}^n\}$.

Proof of Theorem 1

(1) Firstly, for simplification, we set $N_1(\varepsilon) = \varepsilon^{\chi m \kappa} L^m(\varepsilon^{-\chi})$. By the Hermite expansion, we can rewrite $s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; h(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot))) - C_0(h)$ as

$$s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; h(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot))) - C_0(h) = \sum_{\rho=m}^{\infty} s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; \frac{C_\rho(h)}{\sqrt{\rho!}} H_\rho(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot))), \quad (35)$$

where the summation is in $L^2(\Omega)$ sense. Hence, in accordance to the definition (21) about the random field $s^\varepsilon(t, x)$, it can be rewritten as

$$s^\varepsilon(t, x) = \sum_{\rho=m}^{\infty} I_\rho^\varepsilon(t, x), \quad (36)$$

with $I_\rho^\varepsilon(t, x) := (N_1(\varepsilon))^{-\frac{1}{2}} s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; \frac{C_\rho(h)}{\sqrt{\rho!}} H_\rho(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot)))$.

By the solution form given in Section 2, with $d_1 = d_2 = 0$, we have

$$\begin{aligned} & \left(\frac{C_\rho(h)}{\sqrt{\rho!}}\right)^{-1} (N_1(\varepsilon))^{\frac{1}{2}} I_\rho^\varepsilon(t, x) \\ &= \int_{\mathbb{R}^n} G(\varepsilon t, y; \alpha, \gamma) H_\rho(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi}(\varepsilon^{\frac{1}{\alpha+\gamma}} x - y))) dy \\ &= \int_{\mathbb{R}^n} G(\varepsilon t, y; \alpha, \gamma) \int_{\mathbb{R}^{n \times \rho}} e^{i \langle \varepsilon^{-\frac{1}{\alpha+\gamma}-\chi}(\varepsilon^{\frac{1}{\alpha+\gamma}} x - y), \lambda_1 + \dots + \lambda_\rho \rangle} \prod_{\sigma=1}^{\rho} \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) dy \\ &= \int_{\mathbb{R}^{n \times \rho}} e^{i \langle \varepsilon^{-\chi} x, \lambda_1 + \dots + \lambda_\rho \rangle} \left\{ \int_{\mathbb{R}^n} G(\varepsilon t, y; \alpha, \gamma) e^{-i \langle \varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} y, \lambda_1 + \dots + \lambda_\rho \rangle} dy \right\} \prod_{\sigma=1}^{\rho} \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma). \end{aligned} \quad (37)$$

For the bracket above, by substituting $t \rightarrow \varepsilon t$ and $\lambda \rightarrow \varepsilon^{-\frac{1}{\alpha+\gamma}-\chi}(\lambda_1 + \dots + \lambda_\rho)$ into (4), we have

$$\int_{\mathbb{R}^n} G(\varepsilon t, y; \alpha, \gamma) e^{-i \langle \varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} y, \lambda_1 + \dots + \lambda_\rho \rangle} dy = e^{-\mu \varepsilon t \varepsilon^{-\frac{\alpha}{\alpha+\gamma}-\alpha \chi} |\lambda_1 + \dots + \lambda_\rho|^\alpha (1 + \varepsilon^{-\frac{2}{\alpha+\gamma}-2\chi} |\lambda_1 + \dots + \lambda_\rho|^2)^{\frac{\gamma}{2}}},$$

so (37) is equal to

$$\begin{aligned}
& \int_{\mathbb{R}^{n \times \rho}} e^{i \langle \varepsilon^{-\chi} x, \lambda_1 + \dots + \lambda_\rho \rangle} e^{-\mu \varepsilon t \varepsilon^{-\frac{\alpha}{\alpha+\gamma} - \alpha \chi} |\lambda_1 + \dots + \lambda_\rho|^\alpha (1 + \varepsilon^{-\frac{2}{\alpha+\gamma} - 2\chi} |\lambda_1 + \dots + \lambda_\rho|^2)^{\frac{\gamma}{2}}} \prod_{\sigma=1}^{\rho} \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) \\
& \stackrel{d}{=} \varepsilon^{\frac{\chi \rho n}{2}} \int_{\mathbb{R}^{n \times \rho}} e^{i \langle x, \lambda'_1 + \dots + \lambda'_\rho \rangle} e^{-\mu \varepsilon t \varepsilon^{-\frac{\alpha}{\alpha+\gamma}} |\lambda'_1 + \dots + \lambda'_\rho|^\alpha (1 + \varepsilon^{-\frac{2}{\alpha+\gamma}} |\lambda'_1 + \dots + \lambda'_\rho|^2)^{\frac{\gamma}{2}}} \prod_{\sigma=1}^{\rho} \sqrt{f(\varepsilon^\chi \lambda'_\sigma)} W(d\lambda'_\sigma) \\
& := \left(\frac{C_\rho(h)}{\sqrt{\rho!}} \right)^{-1} (N_1(\varepsilon))^{\frac{1}{2}} \tilde{I}_\rho^\varepsilon(t, x), \tag{38}
\end{aligned}$$

where we have used the self-similar property for Gaussian random measure on \mathbb{R}^n in the last equality. Therefore, by the orthogonal property for the Gaussian white noise, we can get

$$\begin{aligned}
& (C_\rho(h))^{-2} N_1(\varepsilon) \text{Cov}(I_\rho^\varepsilon(t, x) I_\rho^\varepsilon(t', x')) = (C_\rho(h))^{-2} N_1(\varepsilon) \text{Cov}(\tilde{I}_\rho^\varepsilon(t, x) \tilde{I}_\rho^\varepsilon(t', x')) \\
& = \varepsilon^{\chi \rho n} \int_{\mathbb{R}^{n \times \rho}} e^{i \langle x - x', \lambda'_1 + \dots + \lambda'_\rho \rangle} e^{-\mu \varepsilon (t+t') \varepsilon^{-\frac{\alpha}{\alpha+\gamma}} |\lambda_1 + \dots + \lambda_\rho|^\alpha (1 + \varepsilon^{-\frac{2}{\alpha+\gamma}} |\lambda_1 + \dots + \lambda_\rho|^2)^{\frac{\gamma}{2}}} \prod_{\sigma=1}^{\rho} f(\varepsilon^\chi \lambda_\sigma) d\lambda_\sigma \\
& = \varepsilon^{\chi \rho n} \int_{\mathbb{R}^n} e^{i \langle x - x', \tau_1 \rangle} e^{-\mu \varepsilon (t+t') \varepsilon^{-\frac{\alpha}{\alpha+\gamma}} |\tau_1|^\alpha (1 + \varepsilon^{-\frac{2}{\alpha+\gamma}} |\tau_1|^2)^{\frac{\gamma}{2}}} \frac{f^{*\rho}(\varepsilon^\chi \tau_1)}{\varepsilon^{\chi(\rho-1)n}} d\tau_1 \\
& = \varepsilon^{n\chi} \int_{\mathbb{R}^n} e^{i \langle x - x', \tau \rangle} e^{-\mu(t+t') |\tau|^\alpha (\varepsilon^{\frac{2}{\alpha+\gamma}} + |\tau|^2)^{\frac{\gamma}{2}}} f^{*\rho}(\varepsilon^\chi \tau) d\tau, \tag{39}
\end{aligned}$$

where $f^{*\rho}(\cdot)$ is defined in (17).

(i) For $\rho \in \mathbb{N}$ with $m\kappa \leq \rho\kappa < n$ and any $\delta > 0$, by (39) and (18),

$$\text{Cov}(I_\rho^\varepsilon(t, x) I_\rho^\varepsilon(t', x')) = (C_\rho(h))^2 (A_1(\varepsilon) + A_2(\varepsilon)), \tag{40}$$

with

$$\begin{aligned}
|A_1(\varepsilon)| &= N_1(\varepsilon)^{-1} \varepsilon^{n\chi} \left| \int_{|\varepsilon^\chi \tau| > \delta} e^{i \langle x - x', \tau \rangle} e^{-\mu(t+t') |\tau|^\alpha (\varepsilon^{\frac{2}{\alpha+\gamma}} + |\tau|^2)^{\frac{\gamma}{2}}} f^{*\rho}(\varepsilon^\chi \tau) d\tau \right| \\
&\leq N_1(\varepsilon)^{-1} \varepsilon^{n\chi} \sup\{f^{*\rho}(\tilde{\lambda}) \mid |\tilde{\lambda}| > \delta\} \int_{|\varepsilon^\chi \tau| > \delta} e^{-\mu(t+t') |\tau|^{\alpha+\gamma}} d\tau \\
&= (\varepsilon^{\chi m\kappa} L^m(\varepsilon^{-\chi}))^{-1} \varepsilon^{n\chi} \sup\{f^{*\rho}(\tilde{\lambda}) \mid |\tilde{\lambda}| > \delta\} \int_{|\tau| > \delta \varepsilon^{-\chi}} e^{-\mu(t+t') |\tau|^{\alpha+\gamma}} d\tau \\
&\rightarrow 0, \text{ as } \varepsilon \rightarrow 0 \text{ } (\because m\kappa \leq \rho\kappa < n),
\end{aligned}$$

and, by choosing δ small enough and (18),

$$\begin{aligned}
A_2(\varepsilon) &= N_1(\varepsilon)^{-1} \varepsilon^{n\chi} \int_{|\varepsilon^\chi \tau| \leq \delta} e^{i \langle x - x', \tau \rangle} e^{-\mu(t+t') |\tau|^\alpha (\varepsilon^{\frac{2}{\alpha+\gamma}} + |\tau|^2)^{\frac{\gamma}{2}}} f^{*\rho}(\varepsilon^\chi \tau) d\tau \\
&= N_1(\varepsilon)^{-1} \varepsilon^{n\chi} \int_{|\varepsilon^\chi \tau| \leq \delta} e^{i \langle x - x', \tau \rangle} e^{-\mu(t+t') |\tau|^\alpha (\varepsilon^{\frac{2}{\alpha+\gamma}} + |\tau|^2)^{\frac{\gamma}{2}}} (1 + o(1)) K(n, \rho\kappa) \frac{L^\rho(|\varepsilon^\chi \tau|^{-1})}{|\varepsilon^\chi \tau|^{n-\rho\kappa}} d\tau \\
&\sim (N_1(\varepsilon))^{-1} \varepsilon^{\chi \rho\kappa} L^\rho(\varepsilon^{-\chi}) K(n, \rho\kappa) \int_{\mathbb{R}^n} \frac{e^{i \langle x - x', \tau \rangle} e^{-\mu(t+t') |\tau|^{\alpha+\gamma}}}{|\tau|^{n-\rho\kappa}} d\tau, \text{ as } \varepsilon \rightarrow 0,
\end{aligned}$$

where the asymptotic equivalence is guaranteed by the uniform convergence theorem for the slowly varying function (see, for example, Leonenko [14, Section 1.4]). We remark that the f , defined as a spectral density function, is bounded outside of zero (while the singularity at zero is from the long-range-dependence assumption, as employed in subsection 3.2); therefore its ρ -th convolution remains to have, at most, the singularity only at zero.

The conclusion of (i): Apart from the term $\text{Cov}(I_m^\varepsilon(t, x)I_m^\varepsilon(t', x'))$,

$$\lim_{\varepsilon \rightarrow 0} \sum_{\rho: m < \rho < n/\kappa} \text{Cov}(I_\rho^\varepsilon(t, x)I_\rho^\varepsilon(t', x')) = 0 \quad (41)$$

where we have used the fact that $\{l \in \mathbb{N} | m\kappa \leq l\kappa < n\}$ is a finite set and on this set $\lim_{\varepsilon \rightarrow 0} N_1(\varepsilon)^{-1} \varepsilon^{\lambda \rho \kappa} L^\rho(\varepsilon^{-\lambda}) = 0$ except for $\rho = m$.

(ii) For $\rho \in \mathbb{N}$ with $\rho\kappa > n$, by (39), (19) and $\sum_{\rho=m}^{\infty} (C_\rho(h))^2 \leq \|h\|_{L^2(p(u)du)}^2 < \infty$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \sum_{\rho: \rho\kappa > n} \text{Cov}(I_\rho^\varepsilon(t, x)I_\rho^\varepsilon(t', x')) \quad (42) \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon^{n\lambda} N_1(\varepsilon)^{-1} \sum_{\rho: \rho\kappa > n} (C_\rho(h))^2 \int_{\mathbb{R}^n} e^{i\langle x-x', \tau \rangle - \mu(t+t')|\tau|^{\alpha+\gamma}} f^{*\rho}(0) d\tau \\ &\leq \lim_{\varepsilon \rightarrow 0} \varepsilon^{n\lambda} N_1(\varepsilon)^{-1} h \|_{L^2(p(u)du)}^2 f^{*\tilde{\rho}}(0) \int_{\mathbb{R}^n} e^{-\mu(t+t')|\tau|^{\alpha+\gamma}} d\tau = 0, \end{aligned}$$

since by (19) $f^{*\rho}(0)$, $\rho > n/\kappa$, are bounded by $f^{*\tilde{\rho}}(0)$ with $\tilde{\rho} = \inf\{l \in \mathbb{N} | l\kappa > n\}$.

Finally, from the expansion (36) for the random field $s^\varepsilon(t, x)$ and combining the observations (41) and (42) we know that only the component $I_m^\varepsilon(t, x)$ in (36) do contribute to the covariance function of the random field $s^\varepsilon(t, x)$, that is,

$$\lim_{\varepsilon \rightarrow 0} \text{Cov}(s^\varepsilon(t, x)s^\varepsilon(t', x')) = (C_m(h))^2 K(n, m\kappa) \int_{\mathbb{R}^n} e^{i\langle x-x', \tau \rangle} \frac{e^{-\mu(t+t')|\tau|^{\alpha+\gamma}}}{|\tau|^{n-m\kappa}} d\tau. \quad \square$$

(2) From the above discussion, we may apply Chebyshev's inequality to obtain that:

$$\sum_{\rho=m+1}^{\infty} I_\rho^\varepsilon(t, x) \xrightarrow{P} 0.$$

Therefore, in view of Slutsky's argument (for example, see again [14, p. 6.]) , we suffice to focus our attention on the term $I_m^\varepsilon(t, x)$. In the following we will prove that $I_m^\varepsilon(t, x)$ converges in the distributional sense to $s_m(t, x)$, which is defined in (23), for each fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. By the definition of $N_1(\varepsilon)$ and replacing the letter ρ by m in (38), we can rewrite (38) as follows

$$I_m^\varepsilon(t, x) \stackrel{d}{=} \tilde{I}_m^\varepsilon(t, x) = \frac{C_m(h)}{\sqrt{m!}} \int_{\mathbb{R}^{n \times m}} e^{i\langle x, \lambda'_1 + \dots + \lambda'_m \rangle} M_\varepsilon(\lambda) \prod_{\sigma=1}^m W(d\lambda'_\sigma), \quad (43)$$

with

$$M_\varepsilon(\lambda) = \varepsilon^{\frac{\lambda m(n-\kappa)}{2}} L^{-\frac{m}{2}}(\varepsilon^{-\lambda}) e^{-\mu t (\varepsilon^{\frac{2}{\alpha+\gamma}} + |\lambda_1 + \dots + \lambda_m|^2)^{\frac{\gamma}{2}} |\lambda_1 + \dots + \lambda_m|^\alpha} \prod_{\sigma=1}^m \sqrt{f(\varepsilon^\lambda \lambda_\sigma)},$$

which, when $\varepsilon \rightarrow 0$, satisfies

$$\lim_{\varepsilon \rightarrow 0} M_\varepsilon(\lambda) \stackrel{(16)}{=} (K(n, \kappa))^{\frac{m}{2}} \frac{e^{-\mu t |\lambda_1 + \dots + \lambda_m|^{\alpha+\gamma}}}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n-\kappa}{2}}}. \quad (44)$$

Now, applying the isometric property of the multiple Wiener integrals to the difference of (43) and (23), we have

$$E|\tilde{I}_m^\varepsilon(t, x) - s_m(t, x)|^2 = \lim_{\varepsilon \rightarrow 0} (C_m(h))^2 \int_{\mathbb{R}^{n \times m}} |M_\varepsilon(\lambda) - (K(n, \kappa))^{\frac{m}{2}} \frac{e^{-\mu t |\lambda_1 + \dots + \lambda_m|^{\alpha+\gamma}}}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n-\kappa}{2}}}|^2 \prod_{\sigma=1}^m d\lambda_\sigma$$

$\rightarrow 0$, as $\varepsilon \rightarrow 0$, by (44), the assumption $f(\lambda)$ on Condition SGRID, and

$$\int_{\mathbb{R}^{n \times m}} \frac{e^{-2\mu t |\lambda_1 + \dots + \lambda_m|^{\alpha+\gamma}}}{(|\lambda_1| \cdots |\lambda_m|)^{n-\kappa}} \prod_{\sigma=1}^m d\lambda_\sigma = \pi^{\frac{(m-1)n}{2}} \left(\frac{\Gamma(\frac{\kappa}{2})}{\Gamma(\frac{n-\kappa}{2})} \right)^m \frac{\Gamma(\frac{n-m\kappa}{2})}{\frac{m\kappa}{2}} \int_{\mathbb{R}^n} \frac{e^{-2\mu t |\lambda|^{\alpha+\gamma}}}{|\lambda|^{n-m\kappa}} d\lambda < \infty,$$

for $m\kappa < n$, by the Riesz composition formula (see, for example, [4, Appendix A]). Finally, the assertion (2) follows from Slutsky's and the Cramer-Wold arguments. \square

Proof of Theorem 2(1)

From the expression of the solution, we have

$$\begin{pmatrix} u(t, x; u_0(\cdot)) \\ v(t, x; v_0(\cdot)) \end{pmatrix} - Q(t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} = Q(t; d_1, d_2) \left[\begin{pmatrix} U(t, x) \\ V(t, x) \end{pmatrix} - \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right], \quad (45)$$

where $Q(t; d_1, d_2)$ is the matrix defined by (7), and $U(t, x)$ and $V(t, x)$ are convolutions of the Green function with the respective initial data.

By (45),

$$\begin{aligned} & [\varepsilon^{m_1 \kappa_1 \chi} L^{m_1}(\varepsilon^{-\chi})]^{-\frac{1}{2}} \left\{ \begin{pmatrix} u(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; u_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot)) \\ v(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; v_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot)) \end{pmatrix} - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\} \\ &= Q(\varepsilon t; d_1, d_2) [\varepsilon^{m_1 \kappa_1 \chi} L^{m_1}(\varepsilon^{-\chi})]^{-\frac{1}{2}} \begin{pmatrix} U(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; u_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot)) - C_0^{(1)} \\ V(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; v_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\chi} \cdot)) - C_0^{(2)} \end{pmatrix} \end{aligned} \quad (46)$$

$$:= Q(\varepsilon t; d_1, d_2) [\varepsilon^{m_1 \kappa_1 \chi} L^{m_1}(\varepsilon^{-\chi})]^{-\frac{1}{2}} \begin{pmatrix} U_\varepsilon(t, x) \\ V_\varepsilon(t, x) \end{pmatrix}. \quad (47)$$

We make use of the assumption that $m_2 \kappa_2 > m_1 \kappa_1$.

Firstly, by Theorem 1(2), we have

$$U_\varepsilon(t, x) \Rightarrow \tilde{X}_{m_1}^{(1)}(t, x), \quad (48)$$

where $\tilde{X}_{m_1}^{(1)}$ is defined in (24).

Secondly, by Theorem 1(1), we can obtain

$$V_\varepsilon(t, x) \xrightarrow{P} 0 \quad (49)$$

since we can apply Chebyshev's inequality to observe that for any $c > 0$, as $\varepsilon \rightarrow 0$,

$$P(|V_\varepsilon(t, x)| > c) \leq c^{-2} \text{Var}(V_\varepsilon(t, x)) \asymp c^{-2} [\varepsilon^{-m_1 \kappa_1 \lambda} L^{-m_1}(\varepsilon^{-\lambda})] \cdot [\varepsilon^{-m_2 \kappa_2 \lambda} L^{-m_2}(\varepsilon^{-\lambda})] \rightarrow 0.$$

Meanwhile, since

$$\lim_{\varepsilon \rightarrow 0} Q_\varepsilon(t) := \lim_{\varepsilon \rightarrow 0} Q(\varepsilon t; d_1, d_2) = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} = I_{2 \times 2}. \quad (50)$$

Therefore, we may apply Lemma 1 to those $U_\varepsilon(t, x)$, $V_\varepsilon(t, x)$ and $Q_\varepsilon(t)$ on the above to obtain that

$$\begin{aligned} & [\varepsilon^{m_1 \kappa_1 \lambda} L^{m_1}(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ \begin{pmatrix} u(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; u_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) \\ v(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; v_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) \end{pmatrix} - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\} \\ & \Rightarrow I_{2 \times 2} \begin{pmatrix} \tilde{X}_{m_1}^{(1)}(t, x) \\ 0 \end{pmatrix}, \quad t > 0, x \in \mathbb{R}^n. \quad \square \end{aligned}$$

Proof of Theorem 2(2)

The proof is proceeded as the case (1), with the roles of $m_1, \kappa_1, m_2, \kappa_2$ are interchanged, and thus we skip it. \square

Proof of Theorem 2(3)

The assumption that $m_1 = m_2 = m$, $\kappa_1 = \kappa_2 = \kappa$ will enforce the limiting field to be the combination of those of (1) and (2), as we show below. By Theorem 1(2), we have

$$\begin{aligned} U_\varepsilon(t, x) &:= [\varepsilon^{m \kappa \lambda} L^m(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ U(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; u_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) - C_0^{(1)} \right\} \Rightarrow \tilde{X}_m^{(1)}(t, x), \\ V_\varepsilon(t, x) &:= [\varepsilon^{m \kappa \lambda} L^m(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ V(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; v_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) - C_0^{(2)} \right\} \Rightarrow \tilde{X}_m^{(2)}(t, x), \end{aligned}$$

where $\tilde{X}_m^{(j)}$, $j \in \{1, 2\}$, are defined in (24) and (25) with $m_1 = m_2 = m$.

Because in this case the equality $\lim_{\varepsilon \rightarrow 0} Q_\varepsilon(t) = I$ is unchanged; in the same way, we obtained

$$\begin{aligned} & [\varepsilon^{m \kappa \lambda} L^m(\varepsilon^{-\lambda})]^{-\frac{1}{2}} \left\{ \begin{pmatrix} u(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; u_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) \\ v(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; v_0(\varepsilon^{-\frac{1}{\alpha+\gamma}-\lambda} \cdot)) \end{pmatrix} - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\} \\ & \Rightarrow I_{2 \times 2} \begin{pmatrix} \tilde{X}_m^{(1)}(t, x) \\ \tilde{X}_m^{(2)}(t, x) \end{pmatrix} = \begin{pmatrix} \tilde{X}_m^{(1)}(t, x) \\ \tilde{X}_m^{(2)}(t, x) \end{pmatrix}, \quad t > 0, x \in \mathbb{R}^n. \quad \square \end{aligned}$$

Proof of Proposition 2

$$\begin{aligned} & \mathbb{E} \begin{pmatrix} Y_1^{***}(t, x) \\ Y_2^{***}(t, x) \end{pmatrix} \begin{pmatrix} Y_1^{***}(t', x') & Y_2^{***}(t', x') \end{pmatrix} \\ & = \mathbb{E} \begin{pmatrix} \tilde{X}_m^{(1)}(t, x) \\ \tilde{X}_m^{(2)}(t, x) \end{pmatrix} \begin{pmatrix} \tilde{X}_m^{(1)}(t', x') & \tilde{X}_m^{(2)}(t', x') \end{pmatrix} \\ & = \begin{pmatrix} \mathbb{E} \tilde{X}_m^{(1)}(t, x) \tilde{X}_m^{(1)}(t', x') & 0 \\ 0 & \mathbb{E} \tilde{X}_m^{(2)}(t, x) \tilde{X}_m^{(2)}(t', x') \end{pmatrix} \quad (51) \end{aligned}$$

Because the representation for the limiting fields $\tilde{X}_m^{(1)}(t, x)$ and $\tilde{X}_m^{(2)}(t, x)$ is the same as the limiting field $s_m(t, x)$, defined in (23), we can apply the result (22) to get

$$\mathbb{E}\tilde{X}_m^{(j)}(t, x)\tilde{X}_m^{(j)}(t', x') = (C_m^{(j)})^2 K(n, \kappa m) \int_{\mathbb{R}^n} e^{i\langle x-x', \tau \rangle} \frac{e^{-\mu(t+t')|\tau|^{\alpha+\gamma}}}{|\tau|^{n-m\kappa}} d\tau.$$

Therefore, the covariance matrix (51) is equal to

$$\int_{\mathbb{R}^n} e^{i\langle x-x', \tau \rangle} K(n, \kappa m) \frac{e^{-\mu(t+t')|\tau|^{\alpha+\gamma}}}{|\tau|^{n-m\kappa}} \begin{pmatrix} (C_m^{(1)})^2 & 0 \\ 0 & (C_m^{(2)})^2 \end{pmatrix} d\tau. \quad \square$$

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