Electronic Journal of Probability

Vol. 16 (2011), Paper no. 36, pages 1048–1064.

Journal URL
http://www.math.washington.edu/~ejpecp/

From the Pearcey to the Airy process

M. Adler[*] [†] M. Cafasso [‡] P. van Moerbeke [§¶]

Abstract

Putting dynamics into random matrix models leads to finitely many nonintersecting Brownian motions on the real line for the eigenvalues, as was discovered by Dyson. Applying scaling limits to the random matrix models, combined with Dyson's dynamics, then leads to interesting, infinite-dimensional diffusions for the eigenvalues. This paper studies the relationship between two of the models, namely the Airy and Pearcey processes and more precisely shows how to approximate the multi-time statistics for the Pearcey process by the one of the Airy process with the help of a PDE governing the gap probabilities for the Pearcey process.

Key words: Airy process, Pearcey process, Dyson's Brownian motions.

AMS 2010 Subject Classification: Primary 60K35; Secondary: 60B20.

Submitted to EJP on December 19, 2010, final version accepted May 1, 2011.

[*] *Department of Mathematics, Brandeis University, Waltham, Mass 02454, USA.*adler@brandeis.edu

[†] The support of a National Science Foundation grant # DMS-07- 04271 is gratefully acknowledged

[‡] *Centre de Recherches Mathématiques, Université de Montréal C. P. 6128, succ. centre ville, Montréal, Québec, Canada H3C 3J7 and Department of Mathematics and Statistics, Concordia University 1455 de Maisonneuve W., Montréal, Québec, Canada H3G 1M8.*cafasso@crm.umontreal.ca

[§] *Département de Mathématiques, Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, Mass 02454, USA.* vanmoerbeke@math.ucl.ac.be

[¶] The support of a National Science Foundation grant # DMS-07-04271, a European Science Foundation grant (MISGAM), a Marie Curie Grant (ENIGMA), Nato, FNRS and Francqui Foundation grants is gratefully acknowledged.

1 Introduction

Putting dynamics into random matrix models leads to finitely many nonintersecting Brownian motions on $\mathbb{R}$ for the eigenvalues, as was discovered by Dyson [13]. Applying scaling limits to the random matrix models, combined with Dyson's dynamics, then leads to interesting, infinite-dimensional diffusions for the eigenvalues. This paper studies the relationship between two of the models, namely the Airy and Pearcey processes and more precisely shows how to approximate the Pearcey process by the Airy process with the help of a PDE [2] governing the gap probabilities for the Pearcey process. The Airy process was introduced by Prähofer-Spohn [18] and further developed by K. Johansson [14, 15]. A simple non-linear 3rd order PDE for the transition probabilities for this process was found in [3]; see also [19, 20]. The Pearcey process was introduced in [21, 17] in the context of non-intersecting Brownian motions and plane partitions, also based on prior work on matrix models with external source [16, 10, 22, 23, 11, 12, 5, 7, 8, 9].

Consider n nonintersecting Brownian particles on the real line $\mathbb{R}$,

$$-\infty < x_1(t) < \ldots < x_n(t) < \infty$$

with (local) Brownian transition probability given by

$$p(t;x,y) := \frac{1}{\sqrt{\pi t}} e^{-\frac{(y-x)^2}{t}},$$

all starting from the origin $x = 0$ at time $t = 0$ and such that $\frac{n}{2}$ particles are forced to $\pm\sqrt{\frac{n}{2}}$ at $t = 1$. For very large n, the average mean density of particles has its support, for $t \leq \frac{1}{2}$, on one interval centered about $x = 0$, and for $\frac{1}{2} < t \leq 1$ on two intervals symmetrically located about $x = 0$. The end points of the interval(s) of support describe a heart-shaped region in (x,t)-space, with a cusp at $(x_0, t_0) = (0, 1/2)$. The Pearcey process $\mathscr{P}(\tau)$ is defined (see Figure 1) as the motion of these nonintersecting Brownian motions for large n, about $(x_0, t_0) = (0, 1/2)$ (i.e., near the cusp), with space microscopically rescaled by a factor of $n^{-1/4}$ and time rescaled by a factor $n^{-1/2}$, in tune with the Brownian motion rescaling. A partial differential equation for the Pearcey process was found by Adler-van Moerbeke [4] and in [2] a much better version was obtained, namely a simple third order non-linear PDE for the transition probabilities; it was obtained by a scaling limit on a PDE for non-intersecting Brownian motions with target points. This PDE is related to the Boussinesq equation and its hierarchy; this is part of a general result on integrable kernels, as explained in the paper [1].

Near the boundary of the heart-shaped region of Figure 1, but away from the cusp, the local fluctuations behave as the so-called Airy process, which describe the non-intersecting Brownian motions with space stretched by the customary GUE edge rescaling $n^{1/6}$ and time rescaled by the factor $n^{1/3}$, again in tune with the Brownian motion space-time rescaling.

This paper shows how the Pearcey process statistics tends to the Airy process statistics when one is moving out of the cusp $x = \frac{2}{27}(3(t - t_0))^{3/2}$ very near the boundary, that is at a distance of $(3\tau)^{1/6}$ for τ very large, with τ being the Pearcey time. To be precise, in the two-time case, the times τ_i must be sufficiently near -in a very precise way- for the limit to hold. The main result of the paper can be summarized as follows:

Theorem 1.1. *Given finite parameters $t_1 < t_2$, let both $\tau_1,\ \tau_2 \to \infty$, such that $\tau_2 - \tau_1 \to \infty$ behaves in the following precise way:*

$$\frac{\tau_2 - \tau_1}{2(t_2 - t_1)} = (3\tau_1)^{1/3} + \frac{t_2 - t_1}{(3\tau_1)^{1/3}} + \frac{2t_1 t_2}{3\tau_1} + O(\tau_1^{-5/3}). \tag{1}$$

Let also E_1, E_2 be two arbitrary finite intervals. The parameters t_1 and t_2 provide the Airy times in the following approximation of the Airy process by the Pearcey process:

$$\begin{aligned} &\mathbb{P}\left(\bigcap_{i=1}^{2}\left\{\frac{\mathscr{P}(\tau_i) - \frac{2}{27}(3\tau_i)^{3/2}}{(3\tau_i)^{1/6}} \cap (-E_i) = \emptyset\right\}\right) \\ &\quad = \mathbb{P}\left(\bigcap_{i=1}^{2}\{\mathscr{A}(t_i) \cap (-E_i) = \emptyset\}\right)\left(1 + O(\frac{1}{\tau_1^{4/3}})\right). \end{aligned} \tag{2}$$

The same estimate holds as well for the one-time case. A similar (but different) result was then obtained in [6] for the one–time case in the situation when one is moving out of the cusp following both the two branches of the cusp simultaneously. It is also worth recalling that, in this paper as well as in [6], the statistics of the processes are studied on intervals rather than on general Borel subsets.

The proof of this Theorem proceeds in two steps. At first, Proposition 2.1, stated later, shows that, using the scaling above, the Pearcey kernel tends to the Airy kernel. In a second step, we show, using the PDE for the Pearcey process [2] and Proposition 2.1, the result of Theorem 1.1.

It is a fact that both the Airy and Pearcey processes are determinantal processes, for which the multi-time gap probabilities is given by the matrix Fredholm determinant of the matrix kernel, which will be described here.

The **Airy process** is a determinantal process, for which the multi-time gap probabilities[1] are given by the matrix Fredholm determinant for $t_1 < \ldots < t_\ell$,

$$\mathbb{P}\left(\bigcap_{i=1}^{\ell}(\mathscr{A}(t_i) \cap E_i = \emptyset)\right) = \det\left(I - [\chi_{E_i} K^{\mathscr{A}}_{t_i t_j} \chi_{E_j}]\right)_{1 \le i,j \le \ell} \tag{3}$$

of the matrix kernel in $\vec{x} = (x_1, \ldots, x_\ell)$ and $\vec{y} = (y_1, \ldots, y_\ell)$, denoted as follows:

$$\begin{aligned} &\mathbb{K}^{\mathscr{A}}_{t_1,\ldots,t_\ell}(\vec{x}, \vec{y})\sqrt{d\vec{x}\ d\vec{y}} \\ &:= \begin{pmatrix} K^{\mathscr{A}}_{t_1,t_1}(x_1, y_1)\sqrt{dx_1 dy_1} & \ldots & K^{\mathscr{A}}_{t_1,t_\ell}(x_1, y_\ell)\sqrt{dx_1 dy_\ell} \\ \vdots & & \vdots \\ K^{\mathscr{A}}_{t_\ell,t_1}(x_\ell, y_1)\sqrt{dx_\ell dy_1} & \ldots & K^{\mathscr{A}}_{t_\ell,t_\ell}(x_\ell, y_\ell)\sqrt{dx_\ell dy_\ell} \end{pmatrix}, \end{aligned} \tag{4}$$

where, for arbitrary t_i and t_j, the extended Airy kernel $K^{\mathscr{A}}_{t_i,t_j}(x,y)$ is given by (see Johansson [14, 15])

$$K^{\mathscr{A}}_{t_i,t_j}(x,y) := \tilde{K}^{\mathscr{A}}_{t_i,t_j}(x,y) - \mathbb{1}(t_i < t_j) p^{\mathscr{A}}(t_j - t_i, x, y), \tag{5}$$

[1] χ_E is the indicator function for the set E

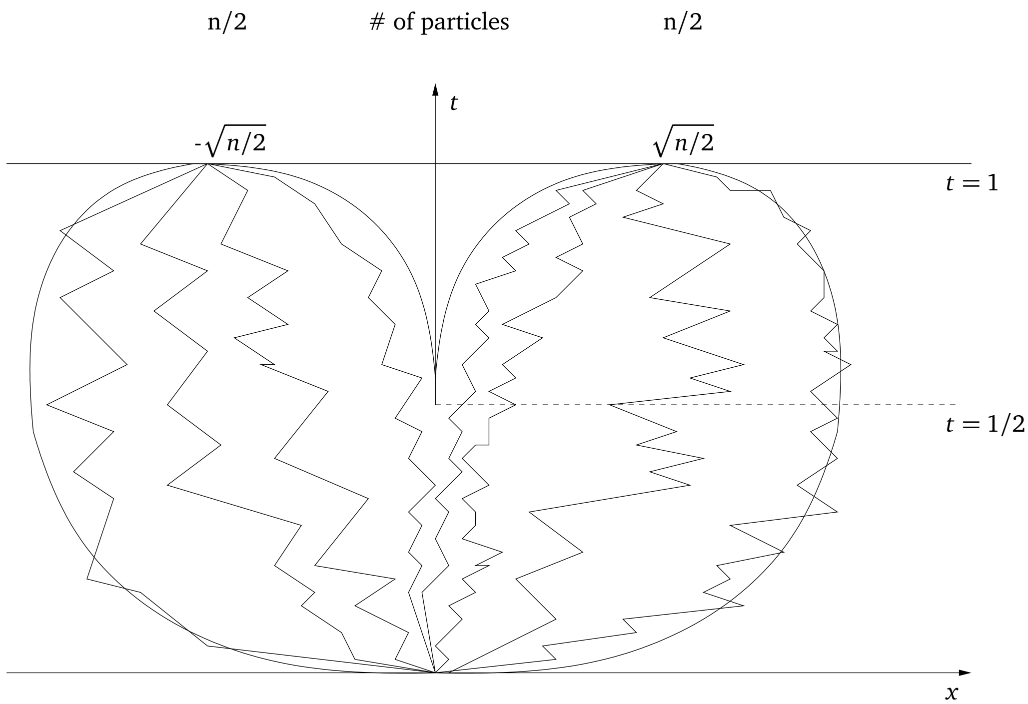


Figure 1: The Pearcey process.

where

$$\tilde{K}^{\mathscr{A}}_{t_i,t_j}(x,y) := \int_0^\infty e^{-\lambda(t_i-t_j)} A(x+\lambda)A(y+\lambda)d\lambda$$
$$= \frac{1}{(2\pi i)^2}\int_{\Gamma_>} dv \int_{\Gamma_<} du\ \frac{e^{-v^3/3+yv}}{e^{-u^3/3+xu}} \frac{1}{(v+t_j)-(u+t_i)}$$

$$p^{\mathscr{A}}(t,x,y) := \frac{1}{\sqrt{4\pi t}} e^{\frac{t^3}{12}-\frac{(x-y)^2}{4t}-\frac{t}{2}(y+x)},\ \text{for } t>0.$$

The contour $u \in \Gamma_<$ consists of two rays emanating from the origin with angles θ_1 and θ_1' with the positive real axis, and the contour $v \in \Gamma_>$ also consists of two rays with angles θ_2 and θ_2' with the negative real axis, as indicated in Figure 2. As is well known, one may choose $\theta_1, \theta_2,\ \theta_1', \theta_2' = \pi/3$. In particular, for $t_i = t_j$, this is the customary Airy kernel

$$K^{\mathscr{A}}_{t_i,t_i} := K^{\mathscr{A}}(x,y) := \int_0^\infty d\lambda A(x+\lambda)A(y+\lambda) = \frac{A(x)A'(y)-A(y)A'(x)}{x-y}. \tag{6}$$

The **Pearcey process** is determinantal as well, for which the multi-time gap probability is given by the Fredholm determinant for $\tau_1 < \ldots < \tau_\ell$,

$$\mathbb{P}\left(\bigcap_{i=1}^{\ell}(\mathscr{P}(\tau_i)\cap E_i = \emptyset)\right) = \det\left(\mathbb{1} - [\chi_{E_i} K^{\mathscr{P}}_{\tau_i\tau_j}\chi_{E_j}]\right)_{1\le i,j\le \ell} \tag{7}$$

for the **Pearcey matrix kernel** in $\vec{\xi}=(\xi_1,\dots,\xi_\ell)$ and $\vec{\eta}=(\eta_1,\dots,\eta_\ell)$, denoted as follows:

$$\mathbb{K}^{\mathscr{P}}_{\tau_1,\dots,\tau_\ell}(\vec{\xi},\vec{\eta})\sqrt{d\vec{\xi}\,d\vec{\eta}}$$

$$:=\begin{pmatrix} K^{\mathscr{P}}_{\tau_1,\tau_1}(\xi_1,\eta_1)\sqrt{d\xi_1 d\eta_1} & \dots & K^{\mathscr{P}}_{\tau_1,\tau_\ell}(\xi_1,\eta_\ell)\sqrt{d\xi_1 d\eta_\ell} \\ \vdots & & \vdots \\ K^{\mathscr{P}}_{\tau_\ell,\tau_1}(\xi_\ell,\eta_1)\sqrt{d\xi_\ell d\eta_1} & \dots & K^{\mathscr{P}}_{\tau_\ell,\tau_\ell}(\xi_\ell,\eta_\ell)\sqrt{d\xi_\ell d\eta_\ell}\end{pmatrix}, \tag{8}$$

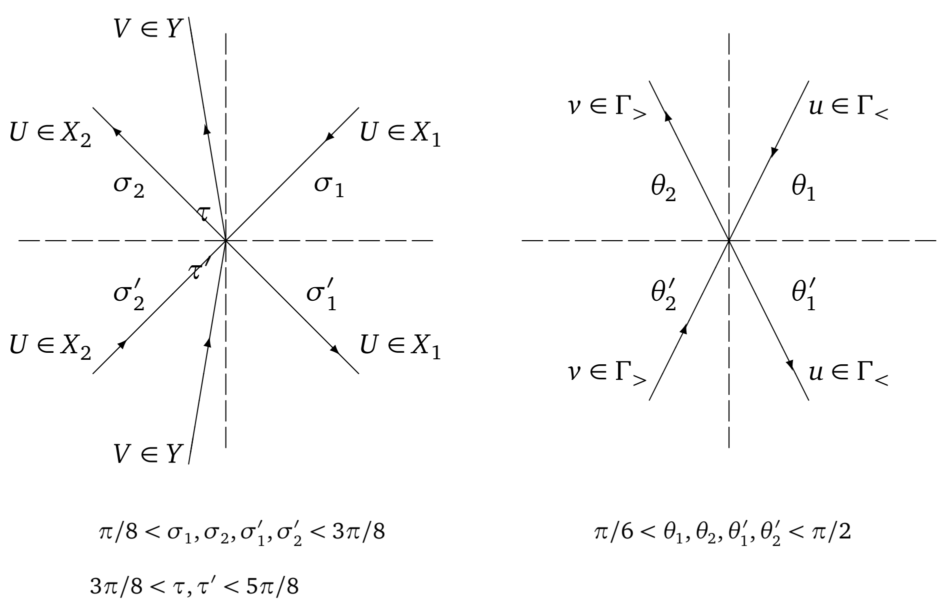
where for arbitrary τ_i and τ_j, (see Tracy-Widom [21])

$$K^{\mathscr{P}}_{\tau_i,\tau_j}(\xi,\eta)=\tilde{K}^{\mathscr{P}}_{\tau_i,\tau_j}(\xi,\eta)-\mathbb{1}(\tau_i<\tau_j)p^{\mathscr{P}}(\tau_j-\tau_i;\xi,\eta), \tag{9}$$

with

$$\tilde{K}^{\mathscr{P}}_{\tau_i,\tau_j}(\xi,\eta):=-\frac{1}{4\pi^2}\int_X dU\int_Y\frac{dV}{V-U}\;\frac{e^{-\frac{V^4}{4}+\frac{\tau_j V^2}{2}-V\eta}}{e^{-\frac{U^4}{4}+\frac{\tau_i U^2}{2}-U\xi}}$$
$$p^{\mathscr{P}}(\tau;\xi,\eta):=\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{(\xi-\eta)^2}{2\tau}},\text{ for }\tau>0,$$

where X and Y are the following contours: $X=X_1\cup X_2$ consists of four rays emanating from the origin with angles σ_1,σ_1' with the positive real axis and σ_2,σ_2' with the negative real axis, as given in Figure 2. The contour Y consists of two rays emanating from the origin with angles τ,τ' with the negative real axis; it is customary to pick $\sigma_1=\sigma_2=\sigma_1'=\sigma_2'=\pi/4$ and $\tau=\tau'=\pi/2$.



Pearcey contour

Airy contour

Figure 2: Integration paths for the kernels.

In [4, 2], it was shown that, given intervals

$$E_i = (\xi_1^{(i)}, \xi_2^{(i)}),$$

the log of the probability

$$Q = Q(\tau_1, \dots, \tau_\ell; E_1, \dots, E_\ell) := \log \mathbb{P}\left(\bigcap_{i=1}^{\ell} \{\mathscr{P}(\tau_i) \cap E_i = \emptyset\}\right)$$

satisfies a non-linear PDE, which we describe here. Given the times and the intervals above, one defines the following operators:

$$\partial_\tau := \sum_i \frac{\partial}{\partial \tau_i}, \qquad \partial_{E_i} := \sum_k \frac{\partial}{\partial \xi_k^{(i)}}, \qquad \partial_E := \sum_i \partial_{E_i}$$

$$\varepsilon_\tau := \sum_i \tau_i \frac{\partial}{\partial \tau_i}, \qquad \varepsilon_E := \sum_i \sum_k \xi_k^{(i)} \frac{\partial}{\partial \xi_k^{(i)}}. \tag{10}$$

Then Q satisfies the Pearcey partial differential equation in its arguments and the boundary of the intervals[2]:

$$2\partial_\tau^3 Q + \frac{1}{4}(2\varepsilon_\tau + \varepsilon_E - 2)\partial_E^2 Q - (\sum_i \tau_i \partial_{E_i})\partial_\tau \partial_E Q + \left\{\partial_\tau \partial_E Q, \partial_E^2 Q\right\}_{\partial_E} = 0. \tag{11}$$

2 From the Pearcey to the Airy kernel

Define the rational functions

$$\begin{aligned} \Phi(x,t;u) &:= -\frac{1}{4(3u)^3} - \frac{t}{(3u)^2} + \frac{x-t^2}{3u} + \frac{4}{3}tx + \frac{u}{6}t^2 x \\ h(x,t;u) &:= \frac{ux}{4}(x+6t^2), \end{aligned} \tag{12}$$

and the diagonal matrix

$$S := \begin{pmatrix} e^{\Phi(x,t_1;z^4) - h(x,t_1;z^4)} & 0 \\ 0 & e^{\Phi(x,t_2;z^4) - h(x,t_2;z^4)} \end{pmatrix}.$$

We now state:

Proposition 2.1. *Given a parameter $z \to 0$, define two times τ_1 and τ_2 blowing up like z^{-6} and depending on two parameters t_1 and t_2,*

$$\tau_i = \frac{1}{3z^6}(1 + 6t_i z^4) + O(z^{10}). \tag{13}$$

[2] Here and below, given two arbitrary functions f, g and a vector field X, we denote $\{f, g\}_X := gX(f) - fX(g)$

Given large ξ_i and η_j, define new space variables x_i and y_j, using τ_i above,

$$\begin{aligned} \xi_i &= \frac{2}{27}(3\tau_i)^{3/2} - (3\tau_i)^{1/6} x_i \\ \eta_j &= \frac{2}{27}(3\tau_j)^{3/2} - (3\tau_j)^{1/6} y_j. \end{aligned} \tag{14}$$

With this space-time rescaling, the following asymptotic expansion holds for the Pearcey kernel in powers of z^4, with polynomial coefficients in $t_1 + t_2$,

$$S\, \mathbb{K}^{\mathscr{P}}_{\tau_1,\tau_2}(\vec{\xi},\vec{\eta})\sqrt{d\vec{\xi}d\vec{\eta}}\, S^{-1} = \Big(1 + (t_1+t_2)O_1(z^4)\Big)\, \mathbb{K}^{\mathscr{A}}_{t_1,t_2}(\vec{x},\vec{y})\sqrt{\vec{d}x\vec{d}y} + O(z^8), \tag{15}$$

where O_1 refers to a differential operator in $\partial/\partial x, \partial/\partial y$, with polynomial coefficients in $x, y, t_1 \pm t_2$ and $(t_1 - t_2)^{-1}$ and varying with the four matrix entries of the matrix kernel.

On general grounds, due to the Fredholm determinant formula, this would give Theorem 1.1, but with a much poorer estimate in (2), namely instead of $O(\tau_1^{-4/3})$, the estimate would be $O(\tau_1^{-2/3})$. The existence of the PDE for the Fredholm determinant of the Pearcey process enables us to bootstrap the $O(\tau_1^{-2/3})$ estimate to $O(\tau_1^{-4/3})$, without tears!

Also note that conjugating a kernel does not change its Fredholm determinant. Before proving Proposition 2.1, we shall need the following identities:

Lemma 2.2. *Introducing the polynomial*

$$\Psi(x,s;\omega) := \frac{1}{4}\omega^4 + \frac{3}{2}\omega^2 s^2 - 4s(x\omega - \omega^3), \tag{16}$$

the Airy kernel (6) and the (double) integral part of the extended Airy kernel (5), at $t_1 = s,\ t_2 = -s$, satisfy the following differential equations,

$$\begin{aligned} &\Big(\Psi(x,s,-\partial_x) - \Psi(y,s,\partial_y) + 4s\Big)K^{\mathscr{A}}(x,y) \\ &\qquad\qquad = \frac{1}{4}(x-y)(x+y+6s^2)K^{\mathscr{A}}(x,y), \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\left(\Psi(x,s,-\partial_x) - \Psi(y,-s,\partial_y) - \frac{3}{2}s\frac{\partial}{\partial s}(\partial_x - \partial_y)\right)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y) \\ &\qquad\qquad = \frac{1}{4}(x-y)(x+y+6s^2)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y). \end{aligned} \tag{18}$$

Proof. The operator on the left hand side of (17) reads

$$\begin{aligned} &\Psi((x,s,-\partial_x) - \Psi((y,s,\partial_y) + 4s \\ =\ & 4s(1 + y\partial_y - \partial_y^3 + x\partial_x - \partial_x^3) + \frac{3}{2}s^2(\partial_x^2 - \partial_y^2) + \frac{1}{4}(\partial_x^4 - \partial_y^4). \end{aligned} \tag{19}$$

Using the first representation (6) of the Airy kernel, one checks using the differential equation for the Airy kernel, $A''(x) = xA(x)$ and thus $A'''(x) = xA'(x) + A(x)$ and $A^{(iv)}(x) = 2A'(x) + x^2A(x)$, and using differentiation by parts to establish the last equality,

$$\begin{aligned}
&\Big((y\partial_y-\partial_y^3)+(x\partial_x-\partial_x^3)\Big)K^{\mathscr{A}}(x,y)\\
&= \;-\int_0^\infty dz\left(z\frac{d}{dz}+2\right)A(x+z)A(y+z)=-K^{\mathscr{A}}(x,y).
\end{aligned}$$

In order to take care of the other pieces in (19), one uses the second representation (6) of the kernel $K^{\mathscr{A}}(x,y)$, yielding

$$\begin{aligned}
(\partial_x^2-\partial_y^2)K^{\mathscr{A}}(x,y) &= (x-y)K^{\mathscr{A}}(x,y)\\
(\partial_x^4-\partial_y^4)K^{\mathscr{A}}(x,y) &= (x^2-y^2)K^{\mathscr{A}}(x,y).
\end{aligned}$$

This establishes the first identity (17) of Lemma 2.2. The operator on the left hand side of the second identity (18) reads:

$$\begin{aligned}
&\Psi(x,s,-\partial_x)-\Psi(y,-s,\partial_y)-\frac{3}{2}s\frac{\partial}{\partial s}(\partial_x-\partial_y)\\
&= \frac{1}{4}(\partial_x^4-\partial_y^4)+\frac{3}{2}s^2(\partial_x^2-\partial_y^2)+4s(x\partial_x-\partial_x^3-y\partial_y+\partial_y^3)-\frac{3}{2}s\frac{\partial}{\partial s}(\partial_x-\partial_y),
\end{aligned}$$

of which we will evaluate all the different terms acting on the integral. Notice at first that, since the expression under the differentiation vanishes at 0 and ∞, one has

$$\begin{aligned}
0&=\int_0^\infty dz\frac{\partial}{\partial z}\left(ze^{-2sz}K^{\mathscr{A}}(x+z,y+z)\right)\\
&=\int_0^\infty dz\; e^{-2sz}K^{\mathscr{A}}(x+z,y+z)-\int_0^\infty zdz\; e^{-2sz}A(x+z)A(y+z) \qquad (20)\\
&-2s\int_0^\infty zdz\; e^{-2sz}K^{\mathscr{A}}(x+z,y+z),
\end{aligned}$$

Again, using the second representation (6) of the kernel $K^{\mathscr{A}}(x,y)$, one checks

$$\begin{aligned}
&(\partial_x-\partial_y)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y)=-(x-y)\int_0^\infty dz\; e^{-2sz}K^{\mathscr{A}}(x+z,y+z)\\
&(\partial_x^2-\partial_y^2)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y)=(x-y)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y),
\end{aligned}$$

and also

$$s\frac{\partial}{\partial s}(\partial_x-\partial_y)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y) = 2s(x-y)\int_0^\infty zdz\; e^{-2sz}K^{\mathscr{A}}(x+z,y+z)$$

and, using the differential equation $(x\partial_x-\partial_x^3)A(x)=-A(x)$ and (20),

$$\begin{aligned}
&-2s\Big((x\partial_x-\partial_x^3)-(y\partial_y-\partial_y^3)\Big)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y)\\
&= \;-2s(x-y)\int_0^\infty zdz\; e^{-2sz}K^{\mathscr{A}}(x+z,y+z)
\end{aligned}$$

Using $A^{(iv)}(x) = 2A'(x) + x^2A(x)$ and the Darboux-Christoffel representation (6) of the Airy kernel, one checks

$$
\begin{aligned}
&(\partial_x^4 - \partial_y^4)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y) \\
&= \int_0^\infty e^{-2sz}(A^{(iv)}(x+z)A(y+z) - A(x+z)A^{(iv)}(y+z))dz \\
&= -2(x-y)\int_0^\infty dz\, e^{-2sz}K^{\mathscr{A}}(x+z,y+z) + (x^2-y^2)\tilde{K}^{\mathscr{A}}_{s,-s}(x,y) \\
&\quad +2(x-y)\int_0^\infty zdz\, e^{-2sz}A(x+z)A(y+z).
\end{aligned}
$$

Adding all these different pieces and using the expression for

$$-2s(x-y)\int_0^\infty zdz\, e^{-2sz}K^{\mathscr{A}}(x+z,y+z),$$

given by (20), leads to the statement of Lemma 2.2. □

For the sake of notational convenience in the proof below, set

$$\mathbb{K}^{\mathscr{P}}_{\tau_1,\tau_2}(\vec{\xi},\vec{\eta})\sqrt{d\vec{\xi}\, d\vec{\eta}} = \begin{pmatrix} K^{\mathscr{P}}_{11}(\xi_1,\eta_1)\sqrt{d\xi_1 d\eta_1} & K^{\mathscr{P}}_{12}(\xi_1,\eta_2)\sqrt{d\xi_1 d\eta_2} \\ K^{\mathscr{P}}_{21}(\xi_2,\eta_1)\sqrt{d\xi_2 d\eta_1} & K^{\mathscr{P}}_{22}(\xi_2,\eta_2)\sqrt{d\xi_2 d\eta_2} \end{pmatrix} \tag{21}$$

and similarly for the Airy kernel $\mathbb{K}^{\mathscr{A}}_{t_1 t_2}$; the $\tilde{K}^{\mathscr{P}}_{ij}$ and $\tilde{K}^{\mathscr{A}}_{ij}$ refer, as before, to the (double) integral part.

Proof of Proposition 2.1: Notice that acting with ∂_x and ∂_y on the kernels $\widetilde{K}^{\mathscr{A}}_{\substack{11\\22}}(x,y)$ and $\widetilde{K}^{\mathscr{A}}_{\substack{12\\21}}(x,y)$, amounts to multiplication of the integrand of the kernels with $-u$ and v respectively; i.e., $u \longleftrightarrow -\partial_x$ and $v \longleftrightarrow \partial_y$.

Given the (small) parameter $z \in \mathbb{R}$, consider the following (i,j)-dependent change of integration variables $(U,V) \mapsto (u,v)$ in the four Pearcey kernels $K^{\mathscr{P}}_{ij}$ in (21), namely

$$U = \frac{1}{3z^3}(1+3uz^4)(1+3t_iz^4), \quad V = \frac{1}{3z^3}(1+3vz^4)(1+3t_jz^4), \tag{22}$$

together with the changes of variables $(\xi,\eta,\tau_i,\tau_j) \mapsto (x,y,t_i,t_j)$, in accordance with (13) and (14),

$$
\begin{aligned}
\tau_i &= \frac{1}{3z^6}(1+6t_iz^4) + O(z^{10}), \qquad \tau_j = \frac{1}{3z^6}(1+6t_jz^4) + O(z^{10}) \\
\xi &= \frac{2}{27}(3\tau_i)^{3/2} - (3\tau_i)^{1/6}x, \qquad \eta = \frac{2}{27}(3\tau_j)^{3/2} - (3\tau_j)^{1/6}y.
\end{aligned} \tag{23}
$$

To be precise, for $K^{\mathscr{P}}_{kk}$, one sets in the transformations above $i=j=k$, for $K^{\mathscr{P}}_{12}$ and $K^{\mathscr{P}}_{21}$, one sets $i=1,\ j=2$ and $i=2,\ j=1$ respectively. Then, remembering the expressions Φ and Ψ, defined in

(12) and (16), the following estimate holds for small z:

$$
\begin{aligned}
\frac{e^{-\Phi(y,t_j;z^4)}}{e^{-\Phi(x,t_i;z^4)}}\frac{e^{-\frac{V^4}{4}+\frac{\tau_j V^2}{2}-V\eta}}{e^{-\frac{U^4}{4}+\frac{\tau_i U^2}{2}-U\xi}} &= \frac{e^{-z^4\Psi(y,t_j;v)}}{e^{-z^4\Psi(x,t_i;u)}}\frac{e^{-\frac{v^3}{3}+yv}}{e^{-\frac{u^3}{3}+xu}}(1+t_iO(z^8)+t_jO(z^8)) \\
&= \frac{e^{-z^4\Psi(y,t_j;\partial_y)}}{e^{-z^4\Psi(x,t_i;-\partial_x)}}\frac{e^{-\frac{v^3}{3}+yv}}{e^{-\frac{u^3}{3}+xu}}(1+t_iO(z^8)+t_jO(z^8));
\end{aligned} \tag{24}
$$

replacing in the latter expression u and v by differentiations ∂_x and ∂_y has the advantage that upon doubly integrating in u and v, the fraction of exponentials in front can be taken out of the integration. Moreover, setting $t_1 = t + s$ and $t_2 = t - s$, one also checks[3] :

$$
\begin{aligned}
&\frac{e^{-\Phi(y,t_j;z^4)}}{e^{-\Phi(x,t_i;z^4)}}p^{\mathscr{P}}(\tau_j-\tau_i;\xi,\eta)\sqrt{d\xi d\eta} \\
&= p^{\mathscr{A}}(-2s,x,y)\sqrt{dxdy}\left(1+\frac{z^4}{4}\left[(x-y)(x+y+6s^2)+\frac{t}{s}r(x,y,s)\right]+O(z^8)\right).
\end{aligned} \tag{25}
$$

The multiplication by the quotient of the exponentials on the left hand side of the expressions above will amount to a conjugation of the kernel, which will not change the Fredholm determinant. Setting $t_1 = t + s$ and $t_2 = t - s$, with $t = 0$, one finds for the non-exponential part in the kernel

$$
\begin{aligned}
&\sqrt{d\xi d\eta}\frac{dUdV}{V-U} \\
&= \frac{\sqrt{dxdy}\; dudv(1+\frac{7}{2}(t_i+t_j)z^4)}{v+t_j-u-t_i+3z^4(vt_j-ut_i)}+O(z^8) \\
&= \begin{cases} \frac{\sqrt{dxdy}\; dudv}{v-u}\left(1\pm 4sz^4+O(z^8)\right) & \text{for } \begin{cases} i=1, & j=1 \\ i=2, & j=2 \end{cases} \\[2ex] \frac{\sqrt{dxdy}\; dudv}{v-u\mp 2s}\left(1\pm \frac{3z^4 s(u+v)}{v-u\mp 2s}+O(z^8)\right) & \text{for } \begin{cases} i=1, & j=2 \\ i=2, & j=1. \end{cases} \end{cases}
\end{aligned} \tag{26}
$$

Along the same vein as the remark in the beginning of the proof of this Theorem, one notices that multiplication of the integrand of the kernel $\tilde{K}^{\mathscr{A}}_{\substack{12\\21}}(x,y)$ with the fraction, appearing in the last formula of (26), amounts to an appropriate differentiation, to wit:

$$
\pm\frac{3s(u+v)}{v-u\mp 2s} \longleftrightarrow \frac{3}{2}s\frac{\partial}{\partial s}(\partial_y-\partial_x). \tag{27}
$$

[3] The precise expression for $r(x,y,s) := (x-y)^2 - 8s^2(x+y-2s^2)$ will be irrelevant in the sequel.

So we have, using (24), (25), (26), and the differential equation for $K^{\mathscr{A}}$ of Lemma 2.2,

$$\begin{aligned}
&\frac{e^{-\Phi(y,t_{\frac{1}{2}};z^4)}}{e^{-\Phi(x,t_{\frac{1}{2}};z^4)}} K^{\mathscr{P}}_{\substack{11\\22}}(\xi,\eta)\sqrt{d\xi d\eta}\Big|_{t=0} - K^{\mathscr{A}}(x,y)\sqrt{dxdy} \\
&= z^4\left(\pm 4s + \Psi((x,\pm s,-\partial_x) - \Psi((y,\pm s,\partial_y)\right) K^{\mathscr{A}}(x,y)\sqrt{dxdy} + O(z^8) \\
&= \frac{z^4}{4}(x-y)(x+y+6s^2)K^{\mathscr{A}}(x,y)\sqrt{dxdy} + O(z^8);
\end{aligned} \tag{28}$$

the upper(lower)-indices correspond to the upper(lower)-signs. Using (27), and the differential equation for $\tilde{K}^{\mathscr{A}}_{\substack{12\\21}}$ of Lemma 2.2,

$$\begin{aligned}
&\frac{e^{-\Phi(y,t_{\frac{2}{1}};z^4)}}{e^{-\Phi(x,t_{\frac{1}{2}};z^4)}} \tilde{K}^{\mathscr{P}}_{\substack{12\\21}}(\xi,\eta)\sqrt{d\xi d\eta} - \tilde{K}^{\mathscr{A}}_{\substack{12\\21}}(x,y)\sqrt{dxdy}\Big|_{t=0} \\
&= z^4\left(\frac{3}{2}s\frac{\partial}{\partial s}(\partial_y-\partial_x) + \Psi(x,\pm s,-\partial_x) - \Psi(y,\mp s,\partial_y)\right)\tilde{K}^{\mathscr{A}}_{\substack{12\\21}}(x,y)\sqrt{dxdy}\Big|_{t=0} \\
&\qquad + O(z^8) \\
&= \frac{z^4}{4}(x-y)(x+y+6s^2)\tilde{K}^{\mathscr{A}}_{\substack{12\\21}}(x,y)\sqrt{dxdy}\Big|_{t=0} + O(z^8).
\end{aligned} \tag{29}$$

Also, using (25),

$$\begin{aligned}
&\frac{e^{-\Phi(y,t_2;z^4)}}{e^{-\Phi(x,t_1;z^4)}} p^{\mathscr{P}}(\tau_2-\tau_1;\xi,\eta)\sqrt{d\xi d\eta} - p^{\mathscr{A}}(-2s,x,y)\sqrt{dxdy}\Big|_{t=0} \\
&= \frac{z^4}{4}(x-y)(x+y+6s^2)p^{\mathscr{A}}(-2s,x,y)\sqrt{dxdy} + O(z^8).
\end{aligned} \tag{30}$$

Since $h(y,t_i,z^4) - h(x,t_j;z^4) = -\frac{z^4}{4}(x-y)(x+y+6s^2)$ for arbitrary $t_{i,j} = t \pm s$ at $t=0$, and thus

$$\frac{e^{-h(y,t_i;z^4)}}{e^{-h(x,t_j;z^4)}} = 1 + \frac{z^4}{4}(x-y)(x+y+6s^2) + O(z^8) \tag{31}$$

Then the following approximations follow upon combining the three estimates above (28), (29), (30) and using (31):

$$\frac{e^{-\Phi(y,t_{\frac{1}{2}};z^4)+h(y,t_{\frac{1}{2}};z^4)}}{e^{-\Phi(x,t_{\frac{1}{2}};z^4)+h(x,t_{\frac{1}{2}};z^4)}} K^{\mathscr{P}}_{\substack{11\\22}}(\xi,\eta)\sqrt{d\xi d\eta}\Big|_{t=0} - K^{\mathscr{A}}(x,y)\sqrt{dxdy} = O(z^8)$$

$$\frac{e^{-\Phi(y,t_{\frac{2}{1}};z^4)+h(y,t_{\frac{2}{1}};z^4)}}{e^{-\Phi(x,t_{\frac{1}{2}};z^4)+h(x,t_{\frac{1}{2}};z^4)}} \tilde{K}^{\mathscr{P}}_{\substack{12\\21}}(\xi,\eta)\sqrt{d\xi d\eta}\Big|_{t=0} - \tilde{K}^{\mathscr{A}}_{\substack{12\\21}}(x,y)\sqrt{dxdy} = O(z^8)$$

$$\frac{e^{-\Phi(y,t_2;z^4)+h(y,t_2;z^4)}}{e^{-\Phi(x,t_1;z^4)+h(x,t_1;z^4)}} p^{\mathscr{P}}(\tau_2-\tau_1;\xi,\eta)\sqrt{d\xi d\eta}\Big|_{t=0} - p^{\mathscr{A}}(-2s,x,y)\sqrt{dxdy} = O(z^8).$$

The reader is reminded that this estimate sofar is done at $t=0$. It then follows for $t\neq 0$ from the estimates (22), (23), (24), (25), (26), (27) that the right hand side of (15) is an asymptotic series in z^4 with polynomial coefficients in $t_1+t_2=2t$, proving the claim about O_1.

So far an important point was omitted, namely to analyze how the Pearcey contour turns in the limit into an Airy contour; a detailed description of the possible contours was given in Figure 2. The Pearcey-rays X_1, with angles $\sigma_1,\sigma_1'\in(\pi/8,3\pi/8)$ can be deformed into two acceptable rays $\theta_1,\theta_1'\in(\pi/6,\pi/2)$ for the $\Gamma_<$-Airy contour, since the two intervals have a non-empty intersection. Also, since $(3\pi/8,5\pi/8)\cap(\pi/6,\pi/2)\neq\emptyset$, the Pearcey Y-contour can be deformed into an acceptable $v\in\Gamma_>$-Airy contour. Again, since the admissible interval $(\pi/8,3\pi/8)$ for σ_2 and σ_2' in the X_2-Pearcey contour contains $\pi/6$, one ends up in the limit integrating the function $e^{u^3/3}$ along a contour of the form $\Gamma_>$, which we may choose to have an angle of $\pi/6-\delta$ with the negative real axis, for small arbitrary $\delta>0$. In the sector $\Gamma_>$, with $0<\theta_2=\theta_2'\leq\pi/6-\delta$, the function $e^{u^3/3}$ decays exponentially fast. Therefore, the contribution, due to the u-integration of $e^{u^3/3}\times$ (lower order terms) over $\Gamma_>$, having an angle of $\pi/6-\delta$ with the negative real axis, vanishes by applying Cauchy's Theorem in that sector. This ends the proof of Proposition 2.1. $\square$

3 From the Pearcey to the Airy statistics

This section concerns itself with proving Theorem 1.1.

Proof of Theorem 1.1: For any intervals E_1 and E_2, the function

$$Q(\tau_1,\tau_2;E_1,E_2)=\log\mathbb{P}\left(\bigcap_{i=1}^{2}(\mathscr{P}(\tau_i)\cap E_i=\emptyset)\right)\tag{32}$$

satisfies the Pearcey PDE (11). Reparametrizing, without loss of generality,

$$\tau_1=\tau+\sigma,\quad \tau_2=\tau-\sigma,\quad E_1=(\xi+\eta+\mu,\xi+\eta-\mu),\quad E_2=(\xi-\eta+\nu,\xi-\eta-\nu),\tag{33}$$

leads to a manageable PDE for the function

$$F(\tau,\sigma;\xi,\eta,\mu,\nu):=Q(\tau_1,\tau_2;E_1,E_2),\tag{34}$$

namely the PDE:

$$\begin{aligned}2\frac{\partial^3F}{\partial\tau^3}+\frac{1}{4}\left(2(\sigma\frac{\partial}{\partial\sigma}-\tau\frac{\partial}{\partial\tau})+\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}+\mu\frac{\partial}{\partial\mu}+\nu\frac{\partial}{\partial\nu}-2\right)\frac{\partial^2F}{\partial\xi^2}\\ -\sigma\frac{\partial^3F}{\partial\tau\partial\xi\partial\eta}+\left\{\frac{\partial^2F}{\partial\tau\partial\xi},\frac{\partial^2F}{\partial\xi^2}\right\}_{\partial_\xi}=0.\end{aligned}\tag{35}$$

To go from the Pearcey PDE (11) to the PDE (35), one notices that the operators (10), appearing in the Pearcey PDE (11), have simple expressions in terms of the variables $\tau,\sigma,\xi,\eta,\mu,\nu$,

$$\partial_\tau Q=\frac{\partial F}{\partial\tau},\qquad \partial_E Q=\frac{\partial F}{\partial\xi},\qquad \varepsilon_E Q=(\xi\frac{\partial}{\partial\xi}+\eta\frac{\partial}{\partial\eta}+\mu\frac{\partial}{\partial\mu}+\nu\frac{\partial}{\partial\nu})F,$$

$$\varepsilon_\tau Q := (\sigma\frac{\partial}{\partial\sigma}+\tau\frac{\partial}{\partial\tau})F, \quad \sum_i \tau_i \partial_{E_i} Q = (\tau\frac{\partial}{\partial\xi}+\sigma\frac{\partial}{\partial\eta})F.$$

The change of variables, considered in (13) and (14), combined with a linear change of variables, in parallel with (33),

$$\begin{aligned} \tau_i &= \frac{1}{3z^6}(1+6t_i z^4) \quad \text{with } \begin{cases} t_1 = t+s \\ t_2 = t-s \end{cases} \\ E_i &= \frac{2}{27}(3\tau_i)^{3/2} - (3\tau_i)^{1/6}\tilde{E}_i \quad \text{with } \begin{cases} \tilde{E}_1 = (x+y+u, x+y-u) \\ \tilde{E}_2 = (x-y+v, x-y-v) \end{cases} \end{aligned} \tag{36}$$

yields a z-dependent invertible map,

$$T: \ (t,s,x,y,u,v) \mapsto (\tau,\sigma,\xi,\eta,\mu,\nu), \tag{37}$$

and thus the function F in (34) leads to a new z-dependent function G:

$$F(\tau,\sigma;\xi,\eta,\mu,\nu) = F(T(t,s,x,y,u,v)) =: G(t,s,x,y,u,v),$$

of which we compute, in principle, the series in z. From Proposition 2.1, it follows that the z^4-term in the asymptotic expansion in (15) vanishes when $t=0$; so, omitting $\sqrt{dxdy}$, one has

$$S\mathbb{K}^{\mathscr{P}}_{\tau_1,\tau_2}S^{-1} = \mathbb{K}^{\mathscr{A}}_{t_1,t_2} + z^4\mathbb{K}_1 + \sum_2^\infty z^{4i}\mathbb{K}_i, \ \text{ with } \mathbb{K}_1 = t\mathbb{H},$$

for some kernel $\mathbb{H}$, with the $\mathbb{K}_i$ polynomial in t. Then defining

$$L_i := (I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{K}_i, \tag{38}$$

one finds:

$$\begin{aligned} &G(t,s,x,y,u,v) \\ &= F(\tau,\sigma;\xi,\eta,\mu,\nu) \\ &= \log \mathbb{P}\Big(\bigcap_{i=1}^2 \{\mathscr{P}(\tau_i)\cap E_i = \emptyset\}\Big) \\ &= \log\det(I-\mathbb{K}^{\mathscr{P}}_{\tau_1,\tau_2})_{E_1\times E_2} \\ &= \log\det(I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})_{\tilde{E}_1\times\tilde{E}_2} - z^4\operatorname{Tr}L_1 - z^8\operatorname{Tr}(L_2+\frac{1}{2}L_1^2) - \dots \\ &=: G_0(s;\tilde{E}_1,\tilde{E}_2) - G_1(t,s;\tilde{E}_1,\tilde{E}_2)z^4 - G_2(t,s;\tilde{E}_1,\tilde{E}_2)z^8 + O(z^{12}), \end{aligned} \tag{39}$$

where

$$\begin{aligned} G_0(s;\tilde{E}_1,\tilde{E}_2) &= \log\det(I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})_{\tilde{E}_1\times\tilde{E}_2}, \\ G_1(t,s;\tilde{E}_1,\tilde{E}_2) &= t\operatorname{Tr}((I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{H})_{\tilde{E}_1\times\tilde{E}_2}. \end{aligned} \tag{40}$$

Note $G_0(s;\tilde{E}_1,\tilde{E}_2)$ is t-independent, since the Airy process is stationary. Then setting

$$F(\tau,\sigma;\xi,\eta,\mu,\nu) = G(T^{-1}(\tau,\sigma;\xi,\eta,\mu,\nu)) = G(t,s,x,y,u,v)$$

into the PDE (35) yields, using differentiation by parts, a new PDE for $G(t,s,x,y,u,v)$. Upon setting the expansion (39) of G for small z,

$$G(t,s,x,y,u,v) = G_0(s;\tilde{E}_1,\tilde{E}_2) - G_1(t,s;\tilde{E}_1,\tilde{E}_2)z^4 + O(z^8),$$

into this new PDE leads to

$$z^2\left(2s\frac{\partial^3 G_0}{\partial s\partial x^2} - \frac{\partial^3 G_1}{\partial t\partial x^2}\right) + O(z^6) = 0, \tag{41}$$

with G_1 being polynomial in t. From the term $\frac{\partial^3 F}{\partial\tau^3}$ in the PDE (35), it would seem like the leading term would be of order z^{-6}; in fact, by explicit but non-enlightening calculations, there are two consecutive cancellations, so that the first non-trivial term has order z^2, thus leading to a simple PDE connecting G_1 with G_0. But since G_0 is t-independent, from equation (41) we deduce

$$\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\left(G_1 - 2ts\frac{\partial}{\partial s}G_0\right) = 0 \tag{42}$$

From (40), one finds [4]

$$\begin{aligned} G_1 - 2ts\frac{\partial}{\partial s}G_0 &= t\left(\operatorname{Tr}((I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{H})_{\tilde{E}_1\times\tilde{E}_2} - 2s\frac{\partial}{\partial s}G_0\right) \\ &= \sum_1^{\ell} t^i a_i(s;x,y,u,v), \end{aligned}$$

which substituted back in (42), leads to the PDE's for the a_i, namely

$$\left(\frac{\partial}{\partial x}\right)^2 a_i(s;x,y,u,v) = 0, \tag{43}$$

implying

$$a_i(s;x,y,u,v) = x b_i(s;y,u,v) + c_i(s;y,u,v). \tag{44}$$

Letting the intervals $\tilde{E}_1$ and $\tilde{E}_2$ go to ∞, while keeping their relative position $2y$ fixed and widths $-2u$ and $-2v$ fixed as well, is achieved by letting $x\to\infty$, as follows from (33). Remember $t_1,t_2,\tilde{E}_1,\tilde{E}_2$ from (36). But in the limit $x\to\infty$, the expressions G_0, namely

$$G_0(s;\tilde{E}_1,\tilde{E}_2) = \log\det(I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})_{\tilde{E}_1\times\tilde{E}_2}$$

tends to 0 exponentially fast using the exponential decay of the four components of the matrix Airy kernel (5); note that the term $p^{\mathscr{A}}(t,x,y)$ tends to 0 as well, when x and y tend to ∞, due to the presence of $e^{-t(x+y)/2}$. Next, we sketch the proof that

$$G_1(t,s;\tilde{E}_1,\tilde{E}_2) = t\operatorname{Tr}((I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{H})_{\tilde{E}_1\times\tilde{E}_2}$$

tends to 0 exponentially fast, when $x\to\infty$. Indeed, using the identity ($\mathbf{R} := \mathbb{K}^{\mathscr{A}}_{t_1,t_2}(I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}$ is the resolvent of the Airy kernel $\mathbb{K}^{\mathscr{A}}_{t_1,t_2}$)

$$(I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{H} = \mathbb{H} + \mathbb{K}^{\mathscr{A}}_{t_1,t_2}(I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{H} = \mathbb{H} + \mathbb{K}^{\mathscr{A}}_{t_1,t_2}(I+\mathbf{R})\mathbb{H},$$

[4]The series is finite as $\mathbb{K}^{\mathscr{A}}_{t_1,t_2}$ depends on $t_2-t_1=-2s$, G_0 is t–independent and $\mathbb{H}$, by Prop. 2.1 is a polynomial in t.

one computes

$$\begin{aligned} G_1 &= \mathrm{Tr}((I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{K}_1)_{\tilde{E}_1\times\tilde{E}_2} \\ &= t\,\mathrm{Tr}((I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1}\mathbb{H})_{\tilde{E}_1\times\tilde{E}_2} \\ &= t\,\mathrm{Tr}\,\mathbb{H}_{\tilde{E}_1\times\tilde{E}_2} + t\,\mathrm{Tr}(\mathbb{K}^{\mathscr{A}}_{t_1,t_2}\tilde{\mathbb{H}})_{\tilde{E}_1\times\tilde{E}_2}, \text{ with } \tilde{\mathbb{H}} := (I+\mathbf{R})\mathbb{H}, \quad \mathbf{R} := \mathbb{K}^{\mathscr{A}}_{t_1,t_2}(I-\mathbb{K}^{\mathscr{A}}_{t_1,t_2})^{-1} \end{aligned}$$

where

$$\begin{aligned} \mathrm{Tr}\,\mathbb{H}_{\tilde{E}_1\times\tilde{E}_2} &= \sum_{i=1}^{2}\int_{\tilde{E}_i} H_{ii}(u,u)du \\ \mathrm{Tr}(\mathbb{K}^{\mathscr{A}}_{t_1,t_2}\tilde{\mathbb{H}})_{\tilde{E}_1\times\tilde{E}_2} &= \sum_{i=1}^{2}\iint_{\tilde{E}_i\times\tilde{E}_i} K^{\mathscr{A}}_{t_it_i}(u,v)\tilde{H}_{ii}(v,u)dudv \\ &\quad + \iint_{\tilde{E}_1\times\tilde{E}_2} K^{\mathscr{A}}_{t_1t_2}(u,v)\tilde{H}_{21}(v,u)dudv \\ &\quad + \iint_{\tilde{E}_2\times\tilde{E}_1} K^{\mathscr{A}}_{t_2t_1}(u,v)\tilde{H}_{12}(v,u)dudv. \end{aligned}$$

Each of these integrals tend to 0 because each of them contains the Airy kernel and also because $\mathbb{H}$ is obtained by acting with differential operators on the Airy kernel, as explained in section 2. This ends the proof that $G_1(t,s;\tilde{E}_1,\tilde{E}_2)\to 0$, when the $\tilde{E}_i$ tend to ∞. This fact together with the form (44) of the a_i imply $b_i(s;y,u,v)=c_i(s;y,u,v)=0$ for $i\geq 1$, and thus $a_i=0$ for $i\geq 1$, implying

$$G_1 = 2ts\frac{\partial}{\partial s}G_0.$$

Summarizing, this implies that for $E_i=\frac{2}{27}(3\tau_i)^{3/2}-(3\tau_i)^{1/6}\tilde{E}_i$, substituting $G_1=2ts(\partial G_0/\partial s)$ into (39),

$$\begin{aligned} &\log\mathbb{P}\left(\bigcap_{i=1}^{2}\left\{\frac{\mathscr{P}(\tau_i)-\frac{2}{27}(3\tau_i)^{3/2}}{(3\tau_i)^{1/6}}\cap\left(-\tilde{E}_i\right)=\emptyset\right\}\right) \\ &= \quad G_0(s;\tilde{E}_1,\tilde{E}_2)-2z^4ts\frac{\partial G_0}{\partial s}+O(z^8) \\ &= \quad G_0(s-2tsz^4;\tilde{E}_1,\tilde{E}_2)+O(z^8) \\ &= \quad \log\mathbb{P}\left(\bigcap_{i=1}^{2}\left\{\mathscr{A}(t_i(1-t_iz^4))\cap(-\tilde{E}_i)=\emptyset\right\}\right)+O(z^8), \end{aligned}$$

in view of the change of variables given in (36), one has that $s-2tsz^4=\frac{1}{2}(t_1(1-t_1z^4)-t_2(1-t_2z^4))$. Then, substituting $t_i=u_i(1+u_iz^4)$ into $t_i(1-t_iz^4)$ and $\tau_i=(1+6t_iz^4)/(3z^6)+O(z^{10})$, as in (13), yields

$$t_i(1-t_iz^4)=u_i-2u_i^3z^8-u_i^4z^{12} \text{ and } \tau_i=\frac{1}{3z^6}+\frac{2u_i}{z^2}+2u_i^2z^2+O(z^{10}) \text{ for } i=1,2.$$

Then eliminating z between the two expressions above for τ_1 and τ_2, by first expressing z as a series in τ_1, yields (1) and (2), with t_i replaced by u_i. This ends the proof of Theorem 1.1. □

References

[1] M. Adler, M. Cafasso, and P. van Moerbeke. Fredholm determinants of general $(1, p)$–kernels and reductions of non–linear integrable PDE's. *arXiv:1104.4268*, 2011.

[2] Mark Adler, Nicolas Orantin, and Pierre van Moerbeke. Universality of the Pearcey process. *Physica D*, 239:(924-941), 2010. MR2639611

[3] Mark Adler and Pierre van Moerbeke. PDEs for the joint distributions of the Dyson, Airy and sine processes. *Ann. Probab.*, 33(4):1326–1361, 2005. MR2150191

[4] Mark Adler and Pierre van Moerbeke. PDEs for the Gaussian ensemble with external source and the Pearcey distribution. *Comm. Pure Appl. Math.*, 60(9):1261–1292, 2007. MR2337504

[5] Alexander I. Aptekarev, Pavel M. Bleher, and Arno B. J. Kuijlaars. Large n limit of Gaussian random matrices with external source. II. *Comm. Math. Phys.*, 259(2):367–389, 2005. MR2172687

[6] M. Bertola and M. Cafasso. The Riemann-Hilbert approach to the transition between the gap probabilities from the Pearcey to the Airy process. *International Mathematics Research Notices*, doi: 10.1093/imrn/rnr066, 2011.

[7] P. M. Bleher and A. B. J. Kuijlaars. Random matrices with external source and multiple orthogonal polynomials. *Int. Math. Res. Not.*, (3):109–129, 2004. MR2038771

[8] Pavel Bleher and Arno B. J. Kuijlaars. Large n limit of Gaussian random matrices with external source. I. *Comm. Math. Phys.*, 252(1-3):43–76, 2004. MR2103904

[9] Pavel M. Bleher and Arno B. J. Kuijlaars. Large n limit of Gaussian random matrices with external source. III. Double scaling limit. *Comm. Math. Phys.*, 270(2):481–517, 2007. MR2276453

[10] Mark J. Bowick and Édouard Brézin. Universal scaling of the tail of the density of eigenvalues in random matrix models. *Phys. Lett. B*, 268(1):21–28, 1991. MR1134369

[11] E. Brézin and S. Hikami. Correlations of nearby levels induced by a random potential. *Nuclear Phys. B*, 479(3):697–706, 1996. MR1418841

[12] E. Brézin and S. Hikami. Level spacing of random matrices in an external source. *Phys. Rev. E (3)*, 58(6, part A):7176–7185, 1998. MR1662382

[13] Freeman J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Mathematical Phys.*, 3:1191–1198, 1962. MR0148397

[14] Kurt Johansson. Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.*, 242(1-2):277–329, 2003. MR2018275

[15] Kurt Johansson. The arctic circle boundary and the Airy process. *Ann. Probab.*, 33(1):1–30, 2005. MR2118857

[16] Gregory Moore. Matrix models of 2D gravity and isomonodromic deformation. In *Random surfaces and quantum gravity (Cargèse, 1990)*, volume 262 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 157–190. Plenum, New York, 1991. MR1214388

[17] Andrei Okounkov and Nicolai Reshetikhin. Random skew plane partitions and the Pearcey process. *Comm. Math. Phys.*, 269(3):571–609, 2007. MR2276355

[18] Michael Prähofer and Herbert Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.*, 108(5-6):1071–1106, 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. MR1933446

[19] Craig A. Tracy and Harold Widom. A system of differential equations for the Airy process. *Electron. Comm. Probab.*, 8:93–98 (electronic), 2003. MR1987098

[20] Craig A. Tracy and Harold Widom. Differential equations for Dyson processes. *Comm. Math. Phys.*, 252(1-3):7–41, 2004. MR2103903

[21] Craig A. Tracy and Harold Widom. The Pearcey process. *Comm. Math. Phys.*, 263(2):381–400, 2006. MR2207649

[22] P. Zinn-Justin. Random Hermitian matrices in an external field. *Nuclear Phys. B*, 497(3):725–732, 1997. MR1463645

[23] P. Zinn-Justin. Universality of correlation functions of Hermitian random matrices in an external field. *Comm. Math. Phys.*, 194(3):631–650, 1998. MR1631489