Electronic Journal of Probability

Vol. 16 (2011), Paper no. 53, pages 1434–1464.

Journal URL
http://www.math.washington.edu/~ejpecp/

Exponential utility maximization in an incomplete market with defaults

Thomas LIM* Marie-Claire QUENEZ†

Abstract

In this paper, we study the exponential utility maximization problem in an incomplete market with a default time inducing a discontinuity in the price of stock. We consider the case of strategies valued in a closed set. Using dynamic programming and BSDEs techniques, we provide a characterization of the value function as the *maximal subsolution of a backward stochastic differential equation* (BSDE) and an optimality criterium. Moreover, in the case of bounded coefficients, the value function is shown to be the *maximal solution of a BSDE*. Moreover, the value function can be written as the *limit of a sequence of processes* which can be characterized as the solutions of Lipschitz BSDEs in the case of bounded coefficients. In the case of convex constraints and under some exponential integrability assumptions on the coefficients, some complementary properties are provided. These results can be generalized to the case of several default times or a Poisson process.

Key words: Optimal investment, exponential utility, default time, incomplete market, dynamic programming, backward stochastic differential equation.

AMS 2010 Subject Classification: Primary 49L20; Secondary: 93E20.

Submitted to EJP on November 9, 2010, final version accepted July 4, 2011.

*Laboratoire de Probabilités et Modèles Aléatoires, CNRS UMR 7599 Université Paris 7, 175 rue Chevaleret, 75013 Paris, France, tlim@math.jussieu.fr. Supported by ALMA Research.

†Laboratoire de Probabilités et Modèles Aléatoires, CNRS UMR 7599 Université Paris 7 and INRIA, 175 rue Chevaleret, 75013 Paris, France, quenez@math.jussieu.fr.

1 Introduction

In this paper, we study the exponential utility maximization problem in an incomplete market with a default time inducing a discontinuity in the price of stock.

Recall that concerning the study of the utility maximization problem from terminal wealth, there exist two possible approaches:

- The first one is the *dual approach* formulated in a *static* way. This dual approach has been largely studied in the literature. Among them, in a Brownian framework, we quote Karatzas *et al.* [19] in a complete market and Karatzas *et al.* [20] in an incomplete market. In the case of general semimartingales, we quote Kramkov and Schachermayer [23], Shachermayer [35] and Delbaen *et al.* [8] for the particular case of the exponential utility function. For the case with a default in a markovian setting we refer to Lukas [26]. Using this approach, these different authors solve the utility maximization problem in the sense of finding the optimal strategy and also give a characterization of this one via the solution of the dual problem.

- The second approach is the *direct* study of the primal problem(s) by using stochastic control tools such as *dynamic programming*. Recall that these techniques had been used in finance but only in a markovian setting for a long time. For example the reference paper of Merton [27] uses the well known Hamilton-Jacobi-Bellman verification theorem to solve the utility maximization problem of consumption/wealth in a complete market. The use in finance of stochastic dynamic techniques (presented in El Karoui's course [12] in a general setting) is more recent. One of the first work in finance using these techniques is that of El Karoui and Quenez [13]. Also, recall that the backward stochastic differential equations (BSDEs) have been introduced by Bismut [5] for the linear case and by Peng [31] and Duffie and Epstein [10] in the non linear case. In the paper of El Karoui *et al.* [14], several applications to finance are provided. One of the achievement of the paper is a *verification* theorem which characterizes the dynamic value function of an optimization problem as the solution of a Lipschitz BSDE. This principle has many applications in finance. One of them can be found in Rouge and El Karoui [33] who study the exponential utility maximization problem in the incomplete Brownian case and characterize the dynamic indifference price as the solution of a quadratic BSDE (introduced by Kobylanski [22]). Concerning the exponential utility maximization problem, there is also the work of Hu *et al.* [18] still in the Brownian case. Using a *verification* theorem (different from the previous one), they characterize the logarithm of the dynamic value function as the solution of a quadratic BSDE.

Due to the presence of jumps, the case of a discontinuous framework is much more involved. Concerning the study of the exponential utility maximization problem in this case, we refer to Morlais [28]. In that paper, the price process of stock is driven by an independent Brownian motion and a Poisson point process. The author considers the particular case of admissible strategies valued in a compact set (not necessarily convex) and assumes that the coefficients of the model are bounded. Using the same approach as in [18], she proves that the logarithm of the associated value function is the unique solution of a quadratic BSDE (for which she shows an existence and a uniqueness result).

In this paper, we consider the more general case of unbounded coefficients and of strategies constrained to be valued in a given closed set. Since this set is not necessarily convex, the dual approach cannot be applied. Using dynamic programming techniques, the value function denoted by J is characterized as the *maximal subsolution* of a BSDE. Moreover, we provide an optimal criterium and another characterization of the value function as the *nonincreasing limit of a sequence of processes* $(J^k)_{k\in\mathbb{N}}$, where for each k, J^k is the value function associated with the subset of admissible strategies bounded by k.

In the case of bounded coefficients, we provide some more precise results. First, in the case of a compact set D, the value function is shown to be the solution of a Lipschitz BSDE. From this, we derive that in the non compact case, the processes J^k, for $k \in \mathbb{N}$, are the solutions of Lipschitz BSDEs. Now, by making a logarithmic change of variables, we are led to a nonincreasing sequence of solutions of quadratic BSDEs. Thanks to the monotone stability convergence property for quadratic BSDEs (see [22] or [28]), we show that the sequence $(J^k)_{k\in\mathbb{N}}$ converges to a limit, which is a solution (and not only a subsolution) of the BSDE associated with the value function J. We then provide that the value function J, equal to this limit, is characterized as the *maximal solution* (and not only the maximal subsolution) of this BSDE.

At last, we study the case of coefficients which only satisfy some *exponential integrability* conditions. If D is a convex and compact set, the value function is shown to be the solution of a BSDE. From this, we derive that in the non compact case, the approximating processes J^k, for $k \in \mathbb{N}$, are the unique solutions of BSDEs.

The outline of the paper is as follows. In Section 2, we present the market model and the maximization problem in the case of only one risky asset. In Section 3, we consider the more simple case studied in [28] that is, where the coefficients are supposed bounded and where the admissible strategies are valued in a *compact* set. Using a verification theorem for BSDEs (different from the one used in [28]), we easily show that the value function can be characterized as the solution of a *Lipschitz BSDE*. In Section 4, we consider the general case where the coefficients are not supposed bounded and where the admissible strategies are valued in a closed set (not necessarily compact). We show that the value function is characterized as the *maximal subsolution* of a BSDE. Second, we provide a characterization of the value function as the nonincreasing limit of a sequence of processes $(J^k)_{k\in\mathbb{N}}$ which are the value functions associated with some subsets of bounded admissible strategies. In Section 5, we consider the case of bounded coefficients. Using the result of Section 3, we derive that the processes J^k, $k \in \mathbb{N}$, are the solutions of Lipschitz BSDEs. We then show that the sequence $(J^k)_{k\in\mathbb{N}}$ converges to a solution (and not only a subsolution) of the BSDE relative to the value function. From this, we derive that the value function is characterized as the *maximal solution* of this BSDE. In Section 6, we consider the case of coefficients satisfying some *exponential integrability* conditions. In the last section, we generalize the previous results to the case of several assets and several default times, and we also extend these results to a Poisson jump model.

2 The market model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that all processes are defined on a finite time horizon $[0, T]$, with $T < \infty$ and we also suppose the space to be equipped with two stochastic processes: a unidimensional standard Brownian motion W and a jump process N defined by $N_t = \mathbb{1}_{\tau\leq t}$ for any $t \in [0, T]$, where τ is a random variable which stands for a default time (see Section 7.1 for several default times). We assume that this default can appear at any time that is $\mathbb{P}(\tau > t) > 0$ for any $t \in [0, T]$. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these

processes. The filtration is supposed to be right-continuous and W is a $\mathbb{G}$-Brownian motion.
We denote by M the compensated martingale of the process N and by Λ its compensator which is assumed to be absolutely continuous w.r.t. Lebesgue's measure, so that there exists a process λ such that $\Lambda_t = \int_0^t \lambda_s ds$. Hence, the $\mathbb{G}$-martingale M satisfies

$$M_t \quad = \quad N_t - \int_0^t \lambda_s ds \,. \tag{2.1}$$

We introduce the following sets which are used throughout the sequel:

- $\mathscr{S}^{+,\infty}$ is the set of positive $\mathbb{G}$-adapted $\mathbb{P}$-essentially bounded rcll processes.
- $\mathscr{S}^2$ is the set of $\mathbb{G}$-adapted rcll processes φ such that $\mathbb{E}[\sup_t |\varphi_t|^2] < +\infty$.
- $L^{1,+}$ is the set of positive $\mathbb{G}$-adapted rcll processes φ such that $\mathbb{E}[\varphi_t] < \infty$ for any $t \in [0, T]$.
- $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{G}$-predictable processes Z such that

$$\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty \quad \text{(resp. } \int_0^T |Z_t|^2 dt < \infty \ a.s.\,)\,.$$

- $L^2(M)$ (resp. $L^2_{loc}(M)$, $L^1_{loc}(M)$) is the set of $\mathbb{G}$-predictable processes U such that

$$\mathbb{E}\Big[\int_0^T \lambda_t |U_t|^2 dt\Big] < \infty \quad \text{(resp. } \int_0^T \lambda_t |U_t|^2 dt < \infty\,, \int_0^T \lambda_t |U_t| dt < \infty \ a.s.\,)\,.$$

We recall the useful martingale representation theorem (see for example Jeanblanc *et al.* [16]) which is paramount throughout the sequel:

Lemma 2.1. *Any $(\mathbb{P}, \mathbb{G})$-local martingale m has the representation*

$$m_t \quad = \quad m_0 + \int_0^t a_s dW_s + \int_0^t b_s dM_s\,, \quad \forall\, t \in [0, T] \quad a.s.\,, \tag{2.2}$$

where $a \in L^2_{loc}(W)$ and $b \in L^1_{loc}(M)$. If m is a square integrable martingale, each term on the right-hand side of the representation (2.2) is square integrable.

We now consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at any date, and one risky asset with price process S which admits a discontinuity at time τ (we give the results for n assets and p default times in Section 7.1). Throughout the sequel, we consider that the price process S evolves according to the equation

$$dS_t \quad = \quad S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dN_t)\,, \tag{2.3}$$

with the classical assumptions:

Assumption 2.1.

(i) μ, σ and β are $\mathbb{G}$-predictable processes such that $\sigma_t > 0$ and

$$\int_0^T \Big(|\mu_t| + |\sigma_t|^2 + \lambda_t|\beta_t|^2\Big)dt \ < \ \infty \quad a.s.$$

(ii) β satisfies $\beta_\tau > -1$.

The condition (*ii*) ensures that the process S is positive. This condition is equivalent to $\beta_t > -1$ for any $0 \leq t \leq T$ a.s. (see Jeanblanc *et al.* [17]).

We also suppose that $\mathbb{E}[\exp(-\int_0^T \alpha_s dW_s - \frac{1}{2}\int_0^T \alpha_t^2 dt)] = 1$ where $\alpha_t = (\mu_t + \lambda_t\beta_t)/\sigma_t$, which ensures the existence of a martingale probability measure and hence the absence of arbitrage.

Throughout the sequel, a process π is called a trading strategy if it is a $\mathbb{G}$-predictable process and if $\int_0^T \frac{\pi_t}{S_{t^-}} dS_t$ is well defined e.g. $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t|\pi_t\beta_t|^2 dt < \infty$ a.s. In this case, π_t describes the amount of money invested in the risky asset at time t. Under the assumption that the trading strategy is self-financing, the wealth process $X^{x,\pi}$ associated with the trading strategy π and an initial capital x satisfies

$$\begin{cases} dX_t^{x,\pi} \ = \ \pi_t\big(\mu_t dt + \sigma_t dW_t + \beta_t dN_t\big)\,, \\ X_0^{x,\pi} \ = \ x\,. \end{cases} \tag{2.4}$$

For a given initial time t and an initial capital x, the wealth process associated with a trading strategy π is denoted by $X^{t,x,\pi}$.

We assume that the investor in this financial market faces some liability, which is modeled by a $\mathcal{G}_T$-measurable random variable ξ (for example, ξ may be a contingent claim written on a default event, which itself affects the price of the underlying asset). We suppose that $\xi \in L^2(\mathcal{G}_T)$ and is non-negative (all the results still hold under the assumption that ξ is only bounded from below).

Our aim is to study the classical optimization problem

$$V(x,\xi) \ = \ \sup_{\pi\in\mathcal{D}} \mathbb{E}\big[U(X_T^{x,\pi} + \xi)\big]\,, \tag{2.5}$$

where $\mathcal{D}$ is a set of admissible strategies (independent of x) which will be specified throughout the sequel and U is the utility function

$$U(x) \ = \ -\exp(-\gamma x)\,, \quad x \in \mathbb{R}\,,$$

where $\gamma > 0$ is a given constant, which can be seen as a coefficient of absolute risk aversion. The optimization problem (2.5) can be written as

$$V(x,\xi) \ = \ e^{-\gamma x} V(0,\xi)\,.$$

Hence, it is sufficient to study the case $x = 0$. To simplify notation, we will denote X^π (resp. $X^{t,\pi}$) instead of $X^{0,\pi}$ (resp. $X^{t,0,\pi}$). Also, we have

$$V(0,\xi) \ = \ -\inf_{\pi\in\mathcal{D}} \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi + \xi)\big)\big]\,. \tag{2.6}$$

3 Strategies valued in a compact set in the case of bounded coefficients

In this section, we consider the particular case of bounded coefficients and strategies valued in a compact set, which has been already studied by Morlais ([28]). In her paper, by using quite sophisticated techniques of quadratic BSDEs, Morlais shows that the value function (or more precisely its logarithm) is the unique solution of a BSDE. Throughout the sequel, we propose another method which gives a very short proof of this result. This result will be used in Section 5 to provide a characterization of the value function in the case of general constraints.

As in [28], the coefficients of the model are supposed to be bounded and the strategies are constrained to take their values in a given non empty compact set C of $\mathbb{R}$. The set of admissible strategies denoted by $\mathscr{C}$ is thus defined as the set of predictable processes π taking their values in C.

This case cannot be solved by using the dual approach because the set of admissible strategies is not necessarily convex. In this context, we address the problem of characterizing dynamically the value function associated with the exponential utility maximization problem. We give a dynamic extension of the initial problem (2.6) (with $\mathscr{D} = \mathscr{C}$). For any initial time $t \in [0,T]$, we define the value function $J(t,\xi)$ (also denoted by $J(t)$) by the following random variable

$$J(t,\xi) \;\; = \;\; \underset{\pi\in\mathscr{C}_t}{\operatorname{ess\,inf}} \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi}+\xi)\big)\big|\mathscr{G}_t\big], \tag{3.1}$$

where $\mathscr{C}_t$ is the set of all restrictions to $[t,T]$ of the strategies of $\mathscr{C}$. We have $V(0,\xi) = -J(0,\xi)$.

Throughout the sequel, we want to characterize this dynamic value function $J(.)$ as the solution of a BSDE. Since the coefficients are supposed to be bounded and the strategies are constrained to take their values in a compact set, it is possible to solve very simply this problem by using a *verification* principle. For that, for any $\pi \in \mathscr{C}$, we introduce the process J^π satisfying

$$J_t^\pi \;\; = \;\; \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi}+\xi)\big)\big|\mathscr{G}_t\big], \quad \forall\, t \in [0,T].$$

By classical techniques of linear BSDEs (see El Karoui *et al.* [14] in the Brownian case), this process can be easily shown to be the solution of a linear Lipschitz BSDE. More precisely, there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$, such that (J^π, Z^π, U^π) is the unique solution in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the linear BSDE with bounded coefficients

$$-\,dJ_t^\pi \;\; = \;\; f^\pi(t, J_t^\pi, Z_t^\pi, U_t^\pi)dt - Z_t^\pi dW_t - U_t^\pi dM_t\,; \qquad J_T^\pi \; = \; \exp(-\gamma\xi), \tag{3.2}$$

where $f^\pi(s,y,z,u) = \frac{\gamma^2}{2}|\pi_s\sigma_s|^2 y - \gamma\pi_s(\mu_s y + \sigma_s z) - \lambda_s(1-e^{-\gamma\pi_s\beta_s})(y+u)$.

Using the fact that $J(t) = \operatorname{ess\,inf}_{\pi\in\mathscr{C}_t} J_t^\pi$ for any $0 \le t \le T$, we show that $J(.)$ corresponds to the solution of a BSDE, whose generator is the essential infimum over π of the generators of $(J^\pi)_{\pi\in\mathscr{C}}$. More precisely,

Proposition 3.1. *The following properties hold:*

– *Let (Y,Z,U) be the solution in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the following BSDE*

$$\begin{cases} -dY_t = \underset{\pi\in\mathscr{C}}{\operatorname{ess\,inf}} \Big\{ \dfrac{\gamma^2}{2}|\pi_t\sigma_t|^2 Y_t - \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(Y_t+U_t)\Big\} dt \\ \qquad\qquad -\, Z_t dW_t - U_t dM_t\,, \\ \quad Y_T = \exp(-\gamma\xi)\,. \end{cases} \tag{3.3}$$

Then, $J(t) = Y_t$ for any $0 \leq t \leq T$ a.s.

– *There exists an optimal strategy for $J(0) = \inf_{\pi \in \mathscr{C}} \mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))]$. Moreover, $\hat{\pi}$ is optimal if and only if $\hat{\pi}_t$ attains the essential infimum in (3.3) $dt \otimes d\mathbb{P} - a.e.$*

Proof. Let us introduce the generator f which satisfies $ds \otimes d\mathbb{P} - a.e.$

$$f(s,y,z,u) \quad = \quad \operatorname*{ess\,inf}_{\pi \in \mathscr{C}} f^\pi(s,y,z,u).$$

Since the generator f is written as an infimum of linear generators f^π w.r.t. (y,z,u) with uniformly bounded coefficients, f is Lipschitz. That is true since the supremum of affine functions, whose coefficients are bounded by a constant $c > 0$, is Lipschitz and the Lipschitz constant can be taken to be equal to c. Hence, by Tang and Li's results [36], BSDE (3.3) with Lipschitz generator f

$$-dY_t \quad = \quad f(t,Y_t,Z_t,U_t)dt - Z_t dW_t - U_t dM_t\,; \qquad Y_T \;=\; \exp(-\gamma\xi)$$

admits a unique solution $(Y,Z,U) \in \mathscr{S}^2 \times L^2(W) \times L^2(M)$.
Since we have

$$f^\pi(t,y,z,u) - f^\pi(t,y,z,u') \quad = \quad \lambda_t(u-u')\gamma_t\,, \tag{3.4}$$

with $\gamma_t = e^{-\gamma\pi_t\beta_t} - 1$, and since there exist some constants $-1 < C_1 \leq 0$ and $0 \leq C_2$ such that $C_1 \leq \gamma_t \leq C_2$, the comparison theorem in case of jumps (see for example Theorem 2.5 in Royer [34]) can be applied. It implies that $Y_t \leq J_t^\pi$, $\forall\, t \in [0,T]$ a.s. As this inequality is satisfied for any $\pi \in \mathscr{C}$, it follows that $Y_t \leq \operatorname{ess\,inf}_{\pi \in \mathscr{C}} J_t^\pi$ a.s.
Also, by applying a measurable selection theorem (see for e.g. [9] or Benes [1]), one shows that there exists $\hat{\pi}$, $\hat{\pi} \in \mathscr{C}$, such that $dt \otimes d\mathbb{P}$-a.s.

$$\begin{aligned}\operatorname*{ess\,inf}_{\pi \in \mathscr{C}} \Big\{ \frac{\gamma^2}{2}|\pi_t\sigma_t|^2 Y_t - \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t\big(1 - e^{-\gamma\pi_t\beta_t}\big)(Y_t + U_t)\Big\} \\ = \frac{\gamma^2}{2}|\hat{\pi}_t\sigma_t|^2 Y_t - \gamma\hat{\pi}_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t\big(1 - e^{-\gamma\hat{\pi}_t\beta_t}\big)(Y_t + U_t).\end{aligned}$$

Thus, (Y,Z,U) is a solution of BSDE (3.2) associated with $\hat{\pi}$. By uniqueness of the solution of BSDE (3.2), we have $Y_t = J_t^{\hat{\pi}}$, $0 \leq t \leq T$ a.s. Hence, $Y_t = \operatorname{ess\,inf}_{\pi \in \mathscr{C}_t} J_t^\pi = J_t^{\hat{\pi}}$, $0 \leq t \leq T$ a.s., and $\hat{\pi}$ is an optimal strategy. □

Remark 3.1. Let us make the following change of variables: $y_t = \frac{1}{\gamma}\log(Y_t)$, $z_t = \frac{1}{\gamma}\frac{Z_t}{Y_t}$, $u_t = \frac{1}{\gamma}\log\left(1 + \frac{U_t}{Y_{t^-}}\right)$. One can verify that the process (y,z,u) is the solution of the following quadratic BSDE

$$-\,dy_t \quad = \quad g(t,z_t,u_t)dt - z_t dW_t - u_t dM_t\,; \qquad y_T = -\xi\,, \tag{3.5}$$

where

$$g(s,z,u) = \operatorname*{ess\,inf}_{\pi \in \mathscr{C}} \Big(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z + \frac{\mu_s + \lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u - \pi_s\beta_s|_\gamma\Big) - (\mu_s + \lambda_s\beta_s)z - \frac{|\mu_s + \lambda_s\beta_s|^2}{2\gamma}\,,$$

with $|u-\pi\beta_t|_\gamma = \lambda_t \frac{\exp(\gamma(u-\pi\beta_t))-1-\gamma(u-\pi\beta_t)}{\gamma}$. Hence, our result yields the existence and the uniqueness of the quadratic BSDE (3.5) and also gives that the logarithm of the value function is the solution of this BSDE. This corresponds exactly to Morlais's result [28].

Recall that the proof given in [28] consists in showing first an existence and uniqueness result for BSDE (3.5) by using a sophisticated approximation method in the vein of Kobylanski [22] but adapted to the case of jumps. Then, by using a verification theorem quite similar to Hu *et al.*' theorem [18], the logarithm of the value function is proved to be the solution of the quadratic BSDE (3.5).

The proof given here is thus much shorter. It is based on a verification principle via BSDEs in the vein of [14].

The previous result will be used in Section 5, devoted to the case of bounded coefficients and general constraints, in order to prove that the value function is the maximal solution of a BSDE (see Theorem 5.4).

4 The general case

Throughout the sequel, we consider the utility maximization problem in the case of unbounded coefficients and general constraints on the admissible strategies. More precisely, the admissible strategies are required to take their values in a set which is not necessarily compact. Recall that since the utility function is the exponential utility function, the set of admissible strategies is not standard in the literature. The next subsection studies the choice of a suitable set of admissible strategies which will allow to dynamize the problem and to characterize the associated dynamic value function.

4.1 The set of admissible strategies

Recall that in the case of the power or logarithmic utility functions defined (or restricted) on $\mathbb{R}_+$, the strategies are required to make the associated wealth positive. Since we consider the exponential utility function which is finitely valued for all $x \in \mathbb{R}$, the wealth process is no longer required to be positive. However, from a financial point of view, it is natural to consider strategies such that the associated wealth process is uniformly bounded from below (see for example Schachermayer [35]) or even such that any increment of the wealth is bounded from below.

More precisely, let D be a closed subset of $\mathbb{R}$ which contains 0. We introduce the set of admissible trading strategies $\mathscr{D}$ which consists of all $\mathbb{G}$-predictable processes π which take their values in D and satisfy $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t|\pi_t\beta_t|^2 dt < \infty$ a.s. and such that for any fixed π and any $s \in [0,T]$, there exists a real constant $K_{s,\pi}$ such that

$$X_t^\pi - X_s^\pi \quad \geq \quad -K_{s,\pi}\,, \qquad s \leq t \leq T\,, \quad a.s. \tag{4.1}$$

Our aim is to give a characterization of the value function $V(0,\xi)$ associated with $\mathscr{D}$ defined by

$$V(0,\xi) \quad = \quad -\inf_{\pi\in\mathscr{D}} \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi+\xi)\big)\big]\,. \tag{4.2}$$

Our approach consists in giving a dynamic extension of this optimization problem and in providing a characterization of the dynamic value function. The set $\mathscr{D}$ (in particular condition (4.1)) has been chosen so that it is closed by *binding*, that is: if π^1, π^2 are two strategies of $\mathscr{D}$ and if $s \in [0, T]$, then the strategy π^3 defined by

$$\pi^3_t \quad = \quad \begin{cases} \pi^1_t & \text{if } t \leq s\,, \\ \pi^2_t & \text{if } t > s\,, \end{cases}$$

belongs to $\mathscr{D}$. Let us now give a *dynamic extension* of the initial problem: for any initial time $t \in [0, T]$, the value function $J(t, \xi)$ is defined by

$$J(t,\xi) \quad = \quad \underset{\pi\in\mathscr{D}_t}{\operatorname{ess\,inf}} \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi}+\xi)\big)\big|\mathscr{G}_t\big]\,, \tag{4.3}$$

where the set $\mathscr{D}_t$ is the set of the restrictions to $[t, T]$ of the strategies of $\mathscr{D}$. We have $J(0, \xi) = -V(0, \xi)$.

For the sake of brevity, we shall denote $J(t)$ instead of $J(t, \xi)$. The random variable $J(t)$ is defined uniquely only up to $\mathbb{P}$-almost sure equivalent. The process $J(.)$ will be called the *dynamic value function*. This process is adapted but not necessarily rcll and not even progressive.

However, there exist some other possible sets which are closed by binding as for example the set $\mathscr{D}^{'}$ defined as the set of $\mathbb{G}$-predictable processes π, with $\int_0^T |\pi_t \sigma_t|^2 dt + \int_0^T \lambda_t |\pi_t \beta_t|^2 dt < \infty$ a.s., which are valued in D and such that for any $t \in [0, T]$ and for any $p > 1$, the following integrability condition holds

$$\mathbb{E}\Big[\sup_{s\in[t,T]} \exp\Big(-\gamma p X_s^{t,\pi}\Big)\Big] \quad < \quad \infty\,. \tag{4.4}$$

We have $\mathscr{D} \subset \mathscr{D}^{'}$.

The property of closedness by binding of the set $\mathscr{D}^{'}$ can be verified by using the assumption of p-integrability (4.4) and Cauchy-Schwarz inequality (see Appendix C for details).

From a mathematical point of view, the set $\mathscr{D}^{'}$ is a relevant admissibility set which ensures the closedness property by binding, the dynamic programming principle in the multiplicative form of Proposition 4.1 below (see Remark 4.1) and also the characterization of the dynamic value function via a BSDE (see Remark 4.5), for which a uniformly integrability condition is required. Some additional comments on this point are given in Appendix A.

As for $\mathscr{D}$, a dynamic extension of the value function associated with $\mathscr{D}'$ can be given. Using a localization argument, one verifies (see Appendix B) that

Lemma 4.1. *If β is bounded, then the dynamic value function $J(.)$ associated with $\mathscr{D}$ coincides a.s. with the one associated with $\mathscr{D}'$.*

Hence, concerning the dynamic study of the value function, one can choose $\mathscr{D}$ or $\mathscr{D}'$ as set of admissible strategies. The choice of $\mathscr{D}$ is justified since it appears as the natural set of admissible strategies from a financial point of view. However, all the results in this paper still hold with $\mathscr{D}'$ instead of $\mathscr{D}$.

After this dynamic extension of the value function, the aim is to characterize the dynamic value function via a BSDE. It is no longer possible to use a verification theorem like the one in Section

3 because the associated BSDE is no longer Lipschitz and there is no existence result for it. One could think to use a verification theorem as the on of [18], but it is no longer possible since there is no existence and uniqueness results for the associated BSDE because of the presence of jumps. Therefore, as it seems not possible to derive a *sufficient condition* so that a given process corresponds to the dynamic value function, we will now provide some *necessary* conditions satisfied by $J(.)$ and more precisely a dynamic programming principle. Then, using this property, we will derive a first characterization of the value function via a BSDE.

4.2 First properties of the dynamic value function

In this section, we provide a first characterization of the dynamic value function and also an optimality criterium.

Proposition 4.1. *The process $J(.)$ is the largest $\mathbb{G}$-adapted process such that $e^{-\gamma X^\pi}J(.)$ is a submartingale for any admissible strategy $\pi \in \mathcal{D}$ with $J(T) = \exp(-\gamma\xi)$. More precisely, if $\hat{J}$ is a $\mathbb{G}$-adapted process such that $\exp(-\gamma X^\pi)\hat{J}$ is a submartingale for any $\pi \in \mathcal{D}$ with $\hat{J}_T = \exp(-\gamma\xi)$, then we have $J(t) \geq \hat{J}_t$ a.s., for any $t \in [0, T]$.*

Also, for each $\hat{\pi} \in \mathcal{D}$, the strategy $\hat{\pi} \in \mathcal{D}$ is optimal for $J(0)$ if and only if the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ is a martingale.

Proof. We introduce the family of random variables $(J_t^\pi)_{\pi\in\mathcal{D}_t}$ such that

$$J_t^\pi = \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathcal{G}_t\big]\,.$$

The proof is divided in 4 steps.
We introduce the family of random variables $(J_t^\pi)_{\pi\in\mathcal{D}_t}$ such that

$$J_t^\pi \;=\; \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathcal{G}_t\big]\,.$$

Step 1: The set $\{J_t^\pi,\ \pi \in \mathcal{D}_t\}$ is *stable by pairwise minimization* for any $t \in [0, T]$: for every $\pi^1,\ \pi^2 \in \mathcal{D}_t$ there exists $\pi \in \mathcal{D}_t$ such that $J_t^\pi = J_t^{\pi^1} \wedge J_t^{\pi^2}$. Indeed, if we fix $t \in [0, T]$ and introduce the set $E = \{J_t^{\pi^1} \leq J_t^{\pi^2}\}$ which belongs to $\mathcal{G}_t$, we can define π for any $s \in [t, T]$ by $\pi_s = \pi_s^1 \mathbb{1}_E + \pi_s^2 \mathbb{1}_{E^c}$. By construction of π, we get $J_t^\pi = J_t^{\pi^1} \wedge J_t^{\pi^2}$ a.s. Moreover, $\pi \in \mathcal{D}_t$ because $X_s^{t,\pi} = X_s^{t,\pi^1}\mathbb{1}_E + X_s^{t,\pi^2}\mathbb{1}_{E^c}$ and the sum of two random variables bounded from below is bounded from below.
Using the classical results on the essential infimum (see Neveu [29]), there exists a sequence $(\pi^n)_{n\in\mathbb{N}} \in \mathcal{D}_t$ such that

$$J(t) \;=\; \lim_{n\to\infty} \downarrow J_t^{\pi^n} \ a.s.$$

Step 2: For each $\pi \in \mathcal{D}$, the process $\exp(-\gamma X^\pi)J(.)$ is a submartingale. Indeed, from Step 1, there exists a sequence $(\pi^n)_{n\in\mathbb{N}} \in \mathcal{D}_t$ such that $J(t) = \lim_{n\to\infty} \downarrow J_t^{\pi^n}$ a.s.

Without loss of generality, we can suppose that $\pi^0 = 0$. Thus, for each $n \in \mathbb{N}$, we have $J_t^{\pi^n} \leq J_t^{\pi^0} \leq 1$

a.s. Moreover, the integrability property $\mathbb{E}[\exp(-\gamma X_t^{s,\pi})] < \infty$ holds because $\pi \in \mathcal{D}$. Together with the Lebesgue theorem, it gives

$$\mathbb{E}\Big[\lim_{n\to\infty}\exp(-\gamma X_t^{s,\pi})J_t^{\pi^n}\Big|\mathcal{G}_s\Big] \;=\; \lim_{n\to\infty}\mathbb{E}\Big[\exp(-\gamma X_t^{s,\pi})J_t^{\pi^n}\Big|\mathcal{G}_s\Big]. \tag{4.5}$$

Recall that $X_t^{s,\pi} = \int_s^t \frac{\pi_u}{S_{u^-}} dS_u$. Now, we have a.s.

$$\exp\Big(-\gamma\int_s^t \frac{\pi_u}{S_{u^-}}dS_u\Big)J_t^{\pi^n} \;=\; \mathbb{E}\Big[\exp\Big(-\gamma\Big(\int_s^T \frac{\tilde{\pi}_u^n}{S_{u^-}}dS_u + \xi\Big)\Big)\Big|\mathcal{G}_t\Big], \tag{4.6}$$

where the strategy $\tilde{\pi}^n$ is defined by

$$\tilde{\pi}_u^n \;=\; \begin{cases} \pi_u & \text{if } 0 \le u \le t\,, \\ \pi_u^n & \text{if } t < u \le T\,. \end{cases}$$

From the closedness property by binding, $\tilde{\pi}^n \in \mathcal{D}$ for each $n \in \mathbb{N}$. By (4.5) and (4.6), we have a.s.

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\gamma\int_s^t \frac{\pi_u}{S_{u^-}}dS_u\Big)J(t)\Big|\mathcal{G}_s\Big] \;&=\; \lim_{n\to\infty}\mathbb{E}\Big[\exp\Big(-\gamma\Big(\int_s^T \frac{\tilde{\pi}_u^n}{S_{u^-}}dS_u + \xi\Big)\Big)\Big|\mathcal{G}_s\Big] \\ &=\; \lim_{n\to\infty} J_s^{\tilde{\pi}^n} \quad\ge\quad J(s)\,,\end{aligned}$$

from the definition of $J(s)$. Hence, the process $\exp(-\gamma X^\pi)J(.)$ is a submartingale for any $\pi \in \mathcal{D}$.

Step 3: The process $J(.)$ is the largest $\mathbb{G}$-adapted process satisfying the property of Step 2 and such that $J(T) = \exp(-\gamma\xi)$. Indeed, suppose a process $\hat{J}$ such that for any $t \in [0,T]$ and $\pi \in \mathcal{D}$, we have

$$\mathbb{E}\big[\exp\big(-\gamma X_T^\pi\big)\hat{J}_T\big|\mathcal{G}_t\big] \;\;\ge\;\; \exp\big(-\gamma X_t^\pi\big)\hat{J}_t \;\; a.s.$$

Then, we have

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{D}_t}\mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi}+\xi)\big)\big|\mathcal{G}_t\big] \;\;\ge\;\; \hat{J}_t \;\; a.s.,$$

which implies that $J(t) \ge \hat{J}_t$ a.s.

Step 4: Suppose that $\hat{\pi}$ is optimal for $J(0)$. Hence,

$$J(0) \;=\; \inf_{\pi\in\mathcal{D}}\mathbb{E}\big[\exp\big(-\gamma(X_T^\pi+\xi)\big)\big] \;=\; \mathbb{E}\big[\exp\big(-\gamma(X_T^{\hat{\pi}}+\xi)\big)\big].$$

Since the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ is a submartingale and since $J(0) = \mathbb{E}[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))]$, the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ is a martingale.

Suppose now that the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ is a martingale. Then, $\mathbb{E}[\exp(-\gamma X_T^{\hat{\pi}})J(T)] = J(0)$. Also, since for any $\pi \in \mathcal{D}$, the process $\exp(-\gamma X^\pi)J(.)$ is a submartingale and $J(T) = \exp(-\gamma\xi)$, it is clear that $J(0) \le \inf_{\pi\in\mathcal{D}} \mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))]$. Consequently,

$$J(0) \;=\; \inf_{\pi\in\mathcal{D}}\mathbb{E}\big[\exp\big(-\gamma(X_T^\pi+\xi)\big)\big] \;=\; \mathbb{E}\big[\exp\big(-\gamma(X_T^{\hat{\pi}}+\xi)\big)\big].$$

In other words, $\hat{\pi}$ is an optimal strategy.

□

Remark 4.1. The integrability property $\mathbb{E}[\exp(-\gamma X_t^{s,\pi})] < \infty$ is required in the proof of this property. Indeed, if it is not satisfied, equality (4.5) does not hold since the Lebesgue theorem (and the monotone convergence theorem) cannot be applied. We stress on that the importance of the integrability condition is due to the fact that we study an *essential infimum* of positive random variables. In the case of an *essential supremum* of positive random variables, the dynamic programming principle holds without any integrability condition (see for example the case of the power utility function in Lim and Quenez [25]). Consequently, the set of $\mathbb{G}$-predictable processes π such that for any $p > 1$, for any $s \in [0, T]$ and for any $t \in [s, T]$, $\mathbb{E}[\exp(-\gamma p X_t^{s,\pi})] < \infty$, appears as a set of strategies which allows to have the closedness property by binding and the above dynamic programming principle. The set $\mathscr{D}'$ is nearly the same but with an integrability condition which is uniform w.r.t. $t \in [s, T]$ (see (4.4)). This uniform integrability in time will be useful to ensure a characterization of the dynamic value function via a BSDE (see Remark 4.5).

With this property, it is possible to show that there exists a rcll version of the dynamic value function $J(.)$. More precisely, we have:

Proposition 4.2. *There exists a $\mathbb{G}$-adapted rcll process J such that for any $t \in [0, T]$,*

$$J_t \quad = \quad J(t) \quad a.s.$$

Moreover, the two processes are indistinguishable.

A direct proof is given in Appendix D.

Remark 4.2. Proposition 4.1 can be written under the form: J is the largest $\mathbb{G}$-adapted rcll process such that the process $\exp(-\gamma X^\pi)J$ is a submartingale for any $\pi \in \mathscr{D}$ with $J_T = \exp(-\gamma\xi)$.

Moreover, the process J is bounded.

Lemma 4.2. *The process J verifies*

$$0 \le J_t \le 1\,, \quad \forall\, t \in [0, T] \quad a.s.$$

Proof. Fix $t \in [0, T]$. The first inequality is easy to prove, because it is obvious that

$$0 \quad \le \quad \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathscr{G}_t\big] \quad a.s.,$$

for any $\pi \in \mathscr{D}_t$, which implies $0 \le J_t$.
Since the strategy defined by $\pi_s = 0$ for any $s \in [t, T]$ is admissible, we can see that $J_t \le \mathbb{E}[\exp(-\gamma\xi)|\mathscr{G}_t]$ a.s. Moreover, as the contingent claim ξ is supposed to be non negative, we have $J_t \le 1$ a.s. □

Remark 4.3. If ξ is only bounded from below by a real constant $-K$, then J is still upper bounded but by $\exp(\gamma K)$ instead of 1.

4.3 Characterization of the dynamic value function via a BSDE

Using the previous characterization of the dynamic value function (see Proposition 4.1), we prove that this one is characterized by a BSDE. Since we work in terms of *necessary conditions* satisfied by the dynamic value function, the study is more technical than in the cases where a verification theorem can be applied.

Since J is a bounded rcll submartingale, it admits a unique Doob-Meyer decomposition (see Dellacherie and Meyer [9], Chapter 7)

$$dJ_t \quad = \quad dm_t + dA_t\,,$$

where m is a square integrable martingale and A is an increasing $\mathbb{G}$-predictable process with $A_0 = 0$. From the martingale representation theorem (see Lemma 2.1), the previous Doob-Meyer decomposition can be written under the form

$$dJ_t \quad = \quad Z_t dW_t + U_t dM_t + dA_t\,, \tag{4.7}$$

with $Z \in L^2(W)$ and $U \in L^2(M)$.

Using the dynamic programming principle (Proposition 4.1), we precise the process A of (4.7). This allows to show that the dynamic value function J is a subsolution of a BSDE. For that let us introduce the set $\mathscr{A}^2$ consisting in all the nondecreasing adapted rcll processes K with $K_0 = 0$ and $\mathbb{E}|K_T|^2 < \infty$.

Theorem 4.1.

− There exists a process $K \in \mathscr{A}^2$ such that the process $(J, Z, U, K) \in \mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathscr{A}^2$ is a solution of the following BSDE

$$\begin{cases} -dJ_t = \operatorname*{ess\,inf}_{\pi\in\mathscr{D}}\Big\{\dfrac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t - \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) - \lambda_t(1 - e^{-\gamma\pi_t\beta_t})(J_t + U_t)\Big\}dt \\ \qquad\quad - dK_t - Z_t dW_t - U_t dM_t\,, \\ \quad J_T = \exp(-\gamma\xi)\,. \end{cases} \tag{4.8}$$

− Also, (J, Z, U, K) is the maximal solution *in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathscr{A}^2$ of BSDE (4.8) that is, for any solution $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ of the BSDE in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathscr{A}^2$, we have $\bar{J}_t \le J_t$, $\forall\, t \in [0, T]$ a.s.*

− Moreover, an admissible strategy $\hat{\pi}$ is optimal for $J(0) = \inf_{\pi\in\mathscr{D}} \mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))]$ if and only if $K = 0$ and $\hat{\pi}_t$ attains the essential infimum in (4.8) $dt \otimes d\mathbb{P} - a.s.$

Remark 4.4. Due to the presence of the nondecreasing process K, the process J is said to be a *subsolution* (and even the maximal one) of the BSDE associated with the terminal condition $\exp(-\gamma\xi)$ and the generator given by the above essinf.

Proof. We prove the first point of this theorem. Applying first Itô's formula (see for example [32]) to the semimartingale $\exp(-X^\pi)J$, we obtain

$$d(e^{-\gamma X_t^\pi} J_t) = dA_t^\pi + dm_t^\pi\,, \tag{4.9}$$

with $A_0^\pi = 0$ and

$$\begin{cases} dA_t^\pi = e^{-\gamma X_t^\pi}\Big[dA_t + \Big\{\dfrac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t - \lambda_t\big(1 - e^{-\gamma\pi_t\beta_t}\big)(U_t + J_t) - \gamma\pi_t(\sigma_t Z_t + \mu_t J_t)\Big\}dt\Big]\,, \\ dm_t^\pi = e^{-\gamma X_{t^-}^\pi}\big[(Z_t - \gamma\pi_t\sigma_t J_t)dW_t + \big(U_t + \big(e^{-\gamma\pi_t\beta_t} - 1\big)(U_t + J_{t^-})\big)dM_t\big]\,. \end{cases}$$

Using then the DP (see Proposition 4.1), we argue that $\exp(-X^\pi)J$ is a submartingale for any $\pi \in \mathscr{D}$ which yields

$$dA_t \quad \geq \quad \operatorname*{ess\,sup}_{\pi\in\mathscr{D}} \Big\{ \lambda_t \big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t)+\gamma\pi_t(\sigma_t Z_t+\mu_t J_t)-\frac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t \Big\} dt\,. \qquad (4.10)$$

We then define the process K by $K_0 = 0$ and

$$dK_t \quad = \quad dA_t - \operatorname*{ess\,sup}_{\pi\in\mathscr{D}} \Big\{ \lambda_t \big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t)+\gamma\pi_t(\sigma_t Z_t+\mu_t J_t)-\frac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t \Big\} dt\,.$$

It is clear that the process K is nondecreasing from (4.10). Since the strategy defined by $\pi_t = 0$ for any $t \in [0, T]$ is admissible, we have

$$\operatorname*{ess\,sup}_{\pi\in\mathscr{D}} \Big\{ \lambda_t \big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t)+\gamma\pi_t(\sigma_t Z_t+\mu_t J_t)-\frac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t \Big\} \quad \geq \quad 0\,.$$

Hence, $0 \leq K_t \leq A_t$ a.s. As $\mathbb{E}|A_T|^2 < \infty$, we have $K \in \mathscr{A}^2$. Thus, the Doob-Meyer decomposition (4.7) of J can be written as follows

$$\begin{aligned} dJ_t = {} & \operatorname*{ess\,sup}_{\pi\in\mathscr{D}} \Big\{ \lambda_t \big(1-e^{-\gamma\pi_t\beta_t}\big)(U_t+J_t)+\gamma\pi_t(\sigma_t Z_t+\mu_t J_t)-\frac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t \Big\} dt \\ & + dK_t + Z_t dW_t + U_t dM_t\,, \end{aligned}$$

with $Z \in L^2(W)$, $U \in L^2(M)$ and $K \in \mathscr{A}^2$.

We now prove the second point. Let $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ be a solution of (4.8) in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathscr{A}^2$. Let us prove that the process $\exp(-\gamma X^\pi)\bar{J}$ is a submartingale for any $\pi \in \mathscr{D}$.
From the product rule, we get

$$d\big(e^{-\gamma X_t^\pi}\bar{J}_t\big) = d\bar{M}_t^\pi + d\bar{A}_t^\pi + e^{-\gamma X_t^\pi} d\bar{K}_t\,, \qquad (4.11)$$

with $\bar{A}_0^\pi = 0$ and

$$\begin{cases} d\bar{A}_t = -\operatorname*{ess\,inf}_{\pi\in\mathscr{D}} \Big\{ \dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t) \Big\} dt\,, \\ d\bar{A}_t^\pi = e^{-\gamma X_t^\pi} \Big\{ \Big[\dfrac{\gamma^2}{2}|\pi_t\sigma_t|^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t) \Big] dt + d\bar{A}_t \Big\}\,, \\ d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi} \big[(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t)dW_t + \big(\bar{U}_t + \big(e^{-\gamma\pi_t\beta_t}-1\big)(\bar{U}_t+\bar{J}_{t^-})\big)dM_t \big]\,. \end{cases}$$

Since $\bar{J}$ is bounded and since the strategy π belongs to $\mathscr{D}$ (but it still holds for $\pi \in \mathscr{D}'$), we have

$$\mathbb{E}[\sup_{t\in[0,T]} \exp(-\gamma X_t^\pi)\bar{J}_t] < +\infty \quad \text{and} \quad \mathbb{E}[\int_0^T \exp(-\gamma X_t^\pi) d\bar{K}_t] < +\infty\,.$$

By a classical localization argument and by using (4.11), we derive that

$$\mathbb{E}[\bar{A}_T^\pi] \leq \mathbb{E}[\sup_{t\in[0,T]} \exp(-\gamma X_t^\pi)\bar{J}_t] < +\infty.$$

Hence, $\mathbb{E}[\sup_{t\in[0,T]}|\bar{M}_t^\pi|] < +\infty$, which yields that the local martingale $\bar{M}^\pi$ is a martingale. It follows that the process $\exp(-\gamma X^\pi)\bar{J}$ is a submartingale.
Recall now that J is the largest process such that $\exp(-\gamma X^\pi)J$ is a submartingale for any $\pi \in \mathcal{D}$ with $J_T = \exp(-\gamma\xi)$ (see Proposition 4.1). Therefore, we get

$$\bar{J}_t \quad \leq \quad J_t\,, \quad \forall\, t \in [0,T] \quad a.s.$$

It remains to show the third assertion. Suppose that $\hat{\pi}$ is an admissible strategy which is optimal for $J(0)$. Recall that equality (4.9) holds with π replaced by $\hat{\pi}$. Thanks to the optimality criterion (see the second assertion in Proposition 4.1), we argue that $\exp(-X^{\hat{\pi}})J$ is a martingale, which yields that $A^{\hat{\pi}} = 0$ that is, $K = 0$ and $\hat{\pi}_t$ attains the essential infimum in (4.8) $dt \otimes d\mathbb{P} - a.s.$ Also, one can easily verify that the converse holds (by using the fact that $\mathbb{E}[\sup_{t\in[0,T]}\exp(-\gamma X_t^{\hat{\pi}})] < +\infty$). The proof is thus complete.

□

Remark 4.5. Note that the integrability property $\mathbb{E}[\sup_{t\in[0,T]}\exp(-\gamma X_t^\pi)] < +\infty$, for each admissible strategy π, is used in the proof of the second and the third assertions.

4.4 Approximation of the dynamic value function

In this part, the dynamic value function is characterized as the *limit of a nonincreasing sequence* of processes $(J^k)_{k\in\mathbb{N}}$ as k tends to $+\infty$, where for each k, J^k corresponds to the dynamic value function associated with the subset of admissible strategies bounded by k.
For each k, we denote by $\mathcal{D}_t^k$ the subset of strategies of $\mathcal{D}_t$ uniformly bounded by k, and we consider the associated dynamic value function $J^k(.)$ defined for any $t \in [0,T]$ by

$$J^k(t) \quad = \quad \operatorname*{ess\,inf}_{\pi\in\mathcal{D}_t^k} \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi}+\xi)\big)\big|\mathcal{G}_t\big]\,. \tag{4.12}$$

By similar argument as for J, there exists a rcll version of $J^k(.)$ denoted by J^k. As previously, the following dynamic programming principle holds:

Lemma 4.3. *The process* $\exp(-\gamma X^\pi)J^k$ *is a submartingale for any* $\pi \in \mathcal{D}^k$.

We show that the sequence $(J^k)_{k\in\mathbb{N}}$ converges to J. More precisely, we have:

Theorem 4.2. *(Approximation of the dynamic value function) For any* $t \in [0,T]$*, we have*

$$J_t \quad = \quad \lim_{k\to\infty} \downarrow J_t^k \quad a.s.$$

Proof. Fix $t \in [0,T]$. From the definition of sets $\mathcal{D}_t$ and $\mathcal{D}_t^k$, we have $\mathcal{D}_t^k \subset \mathcal{D}_t$ for each $k \in \mathbb{N}$ and hence

$$J_t \quad \leq \quad J_t^k \quad a.s.$$

Moreover, since $\mathcal{D}_t^k \subset \mathcal{D}_t^{k+1}$ for each $k \in \mathbb{N}$, the sequence of positive random variables $(J_t^k)_{k\in\mathbb{N}}$ is nonincreasing. Let us define the random variable

$$\bar{J}(t) \quad = \quad \lim_{k\to\infty} \downarrow J_t^k \quad a.s.$$

From the previous inequality, we get that $J_t \leq \bar{J}(t)$ a.s., and this holds for any $t \in [0,T]$.
It remains to prove that $J_t \geq \bar{J}(t)$ a.s. for any $t \in [0,T]$.
Step 1: Let us prove that the process $\exp(-\gamma X^\pi)\bar{J}(.)$ is a submartingale for any bounded strategy $\pi \in \mathcal{D}$. Let π be a bounded admissible strategy and fix $0 \leq s < t \leq T$. Then, there exists $n \in \mathbb{N}$ such that π is uniformly bounded by n. For each $k \geq n$, since $\pi \in \mathcal{D}^k$, $\exp(-\gamma X^\pi)J^k$ is a submartingale from Lemma 4.3

$$\mathbb{E}\big[\exp(-\gamma X_t^\pi)J_t^k \big| \mathcal{G}_s\big] \quad \geq \quad \exp(-\gamma X_s^\pi)J_s^k \ \geq \ \exp(-\gamma X_s^\pi)\bar{J}(s) \quad a.s.$$

The dominated convergence theorem (which can be applied since $\pi \in \mathcal{D}$ and $0 \leq J_t^k \leq 1$ for each $k \in \mathbb{N}$) gives

$$\mathbb{E}\big[\exp(-\gamma X_t^\pi)\bar{J}(t) \big| \mathcal{G}_s\big] \quad = \quad \lim_{k\to\infty} \mathbb{E}\big[\exp(-\gamma X_t^\pi)J_t^k \big| \mathcal{G}_s\big] \ \geq \ \exp(-\gamma X_s^\pi)\bar{J}(s) \quad a.s.,$$

which gives step 1.

Step 2: The process $\bar{J}(.)$ is a submartingale not necessarily rcll. However, by a theorem of Dellacherie-Meyer [9] (see VI.18), we know that the nonincreasing limit of a sequence of rcll submartingales is indistinguishable from a rcll adapted process. Hence, there exists a rcll version of $\bar{J}(.)$ which will be denoted by $\bar{J}$. Moreover, $\bar{J}$ is still a submartingale.

Step 3: Let us show that $\bar{J}_t \leq J_t$, $\forall\, t \in [0,T]$ a.s.
The process $\bar{J}$ is clearly bounded by 1 because for each $k \in \mathbb{N}$, J^k is bounded by 1. Hence, by the previous steps, it is a bounded rcll submartingale. It thus admits the following Doob-Meyer decomposition

$$d\bar{J}_t \quad = \quad \bar{Z}_t dW_t + \bar{U}_t dM_t + d\bar{A}_t\,,$$

where $\bar{Z} \in L^2(W)$, $\bar{U} \in L^2(M)$ and $\bar{A}$ is a nondecreasing $\mathbb{G}$-predictable process with $\bar{A}_0 = 0$.
As before, we use that the process $\exp(-\gamma X^\pi)\bar{J}$ is a submartingale for any bounded strategy $\pi \in \mathcal{D}$ to give some necessary conditions satisfied by the process $\bar{A}$.
Let $\pi \in \mathcal{D}$ be a uniformly bounded strategy. The product rule gives

$$d(e^{-\gamma X_t^\pi}\bar{J}_t) \quad = \quad d\bar{M}_t^\pi + d\bar{A}_t^\pi\,,$$

with $\bar{A}_0^\pi = 0$ and

$$\begin{cases} d\bar{A}_t^\pi = e^{-\gamma X_t^\pi}\Big\{d\bar{A}_t + \Big[\dfrac{\gamma^2}{2}|\pi_t\sigma_t|^2\bar{J}_t + \lambda_t\big(e^{-\gamma\pi_t\beta_t} - 1\big)(\bar{U}_t + \bar{J}_t) - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t)\Big]dt\Big\}\,, \\ d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi}\big[(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t)dW_t + \big(\bar{U}_t + \big(e^{-\gamma\pi_t\beta_t} - 1\big)(\bar{U}_t + \bar{J}_{t^-})\big)dM_t\big]\,. \end{cases}$$

Let $\bar{\mathcal{D}}$ be the set consisting in all the uniformly bounded admissible strategies. Since the process $e^{-\gamma X^\pi}\bar{J}$ is a submartingale for any $\pi \in \bar{\mathcal{D}}$, we have $d\bar{A}_t^\pi \geq 0$ a.s. for any $\pi \in \bar{\mathcal{D}}$. Hence, there exists a process $\bar{K} \in \mathcal{A}^2$ such that

$$d\bar{A}_t \quad = \quad -\operatorname*{ess\,inf}_{\pi\in\bar{\mathcal{D}}}\Big\{\frac{\gamma^2}{2}|\pi_t\sigma_t|^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1 - e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t + \bar{U}_t)\Big\}dt + d\bar{K}_t\,.$$

The following equality holds $dt \otimes d\mathbb{P} - a.e.$ (see Appendix E for details)

$$\operatorname*{ess\,inf}_{\pi\in\bar{\mathscr{D}}}\Big\{\frac{\gamma^2}{2}|\pi_t\sigma_t|^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big\}$$
$$= \operatorname*{ess\,inf}_{\pi\in\mathscr{D}}\Big\{\frac{\gamma^2}{2}|\pi_t\sigma_t|^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big\}. \quad (4.13)$$

Hence, $(\bar{J},\bar{Z},\bar{U},\bar{K})$ is a solution of BSDE (4.8), and Theorem 4.1 implies that

$$\bar{J}_t \quad \leq \quad J_t\,, \quad \forall\, t\in[0,T] \quad a.s.$$

□

Remark 4.6. By the above proof, it follows that J coincides with the value function associated with the set consisting in all the bounded admissible strategies.

5 Case of bounded coefficients

In this section, we assume that λ, μ, σ and β are uniformly bounded. We prove that for each $k\in\mathbb{N}$, J^k is characterized as the solution of a Lipschitz BSDE. Then, by using this property and the two previous characterizations of the dynamic value function (Theorem 4.1 and Theorem 4.2), we derive that this one is the maximal solution of the associated BSDE (4.8) and not only the maximal subsolution.

5.1 Approximation of the dynamic value function by Lipschitz BSDEs

In the case of bounded coefficients, thanks to Proposition 3.1 and Theorem 4.2, we obtain the following property:

Theorem 5.3.

- *For any $t\in[0,T]$, we have*

$$J_t \quad = \quad \lim_{k\to\infty}\downarrow J_t^k \quad a.s.$$

- *For each $k\in\mathbb{N}$, the process J^k is the solution of the* Lipschitz *BSDE (3.3) with $\mathscr{C}$ equal to the set of all strategies taking their values in the compact set $[-k,k]\cap D$.*

Proof. The first point corresponds to Theorem 4.2. It thus remains to prove the second point, that is, for each $k\in\mathbb{N}$, J^k is characterized as the solution of a *Lipschitz BSDE*.

First, recall that the dynamic value function associated with $\mathscr{D}^{'}$ is equal to J, the one which is associated with $\mathscr{D}$. Similarly, the dynamic value function associated with $\mathscr{D}_k^{'}$, the subset of strategies of $\mathscr{D}^{'}$ bounded by k, is equal to J^k.
Hence, we can consider that $\mathscr{D}^{'}$ is the set of admissibility. Using martingale inequalities, one can easily show that all the bounded strategies belong to $\mathscr{D}^{'}$ (but not necessarily in $\mathscr{D}$) because the coefficients are bounded. It follows that $\mathscr{D}_k^{'}$ is equal to the set of all strategies taking their values in the compact set $[-k,k]\cap D$. Thanks to Proposition 3.1 applied to the compact set $[-k,k]\cap D$, it follows that the process J^k is the solution of the *Lipschitz* BSDE (3.3) with $\mathscr{C}$ replaced by $\mathscr{D}_k^{'}$. □

5.2 Characterization of the dynamic value function as the maximal solution of a BSDE

In this subsection, we add the following assumption:

Assumption 5.1. *ξ is bounded.*

In this case, the dynamic value function J can be proved to be a solution of a BSDE (and not only a subsolution). More precisely,

Theorem 5.4. *(Characterization of the dynamic value function)*
J is the maximal solution *in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the following BSDE:*

$$\begin{cases} -dJ_t = \operatorname*{ess\,inf}_{\pi\in\mathscr{D}} \Big\{ \frac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t - \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(J_t+U_t) \Big\} dt \\ \qquad\qquad - Z_t dW_t - U_t dM_t\,, \\ J_T = \exp(-\gamma\xi)\,. \end{cases} \tag{5.14}$$

Also, an admissible strategy $\hat{\pi}$ is optimal for J_0 if and only if $\hat{\pi}_t$ attains the essential infimum in (4.8) $dt \otimes d\mathbb{P} - a.e.$

Remark 5.7. Recall that $\mathscr{D}$ can be replaced by $\mathscr{D}'$ in the above essential infimum.

Proof. For each $k \in \mathbb{N}$, let us denote by (J^k, Z^k, U^k) the solution of the associated *Lipschitz* BSDE (3.3) with $\mathscr{C}$ replaced by $\mathscr{D}'_k$. We make the following change of variables

$$\begin{cases} y_t^k = \frac{1}{\gamma}\log(J_t^k)\,, \\ z_t^k = \frac{1}{\gamma}\frac{Z_t^k}{J_t^k}\,, \\ u_t^k = \frac{1}{\gamma}\log\Big(1+\frac{U_t^k}{J_{t^-}^k}\Big)\,. \end{cases}$$

It is clear that the process (y^k, z^k, u^k) is a solution of the following quadratic BSDE

$$-dy_t^k \;=\; g^k(t, z_t^k, u_t^k)dt - z_t^k dW_t - u_t^k dM_t\,; \qquad y_T^k \;=\; -\xi\,,$$

where

$$g^k(s,z,u) = \operatorname*{ess\,inf}_{\pi\in\mathscr{D}'_k} \Big(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z + \frac{\mu_s+\lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u-\pi_s\beta_s|_\gamma \Big) - (\mu_s+\lambda_s\beta_s)z - \frac{|\mu_s+\lambda_s\beta_s|^2}{2\gamma}$$

with $|u-\pi\beta_s|_\gamma = \lambda_s \frac{\exp(\gamma(u-\pi\beta_s))-1-\gamma(u-\pi\beta_s)}{\gamma}$.

Since the sequence $(y^k)_{k\in\mathbb{N}}$ is nonincreasing, we can use a monotone stability convergence property for quadratic BSDEs: the sequence $(y^k, z^k, u^k)_{k\in\mathbb{N}}$ converges to (y, z, u) in the following sense

$$\mathbb{E}\Big(\sup_{t\in[0,T]}|y_t^k - y_t|\Big) + |z^k - z|_{L^2(W)} + |u^k - u|_{L^2(M)} \quad\to\quad 0\,,$$

where (y,z,u) is solution of

$$-dy_t \;=\; g(t,z_t,u_t)dt - z_t dW_t - u_t dM_t\,; \qquad y_T \;=\; -\xi\,,$$

with

$$g(s,z,u) = \operatorname*{ess\,inf}_{\pi\in\tilde{\mathscr{D}}}\Big(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z + \frac{\mu_s+\lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u - \pi_s\beta_s|_\gamma\Big) - (\mu_s+\lambda_s\beta_s)z - \frac{|\mu_s+\lambda_s\beta_s|^2}{2\gamma}\,,$$

where $\tilde{\mathscr{D}} = \cup_k \mathscr{D}'_k$. The proof of this result is based on the same arguments as those used in the proof of Kobylanski's monotone stability convergence property for quadratic BSDEs [22] adapted to the case of jumps (as done in [28] in the case $D = \mathbb{R}$).
By localization arguments (as in Appendix E), one can show that in the above essinf, the set $\tilde{\mathscr{D}}$ can be replaced by $\mathscr{D}$ (or even by $\mathscr{D}'$).

Let us now define the following processes

$$\begin{cases} J_t^* \;=\; e^{\gamma y_t}\,, \\ Z_t^* \;=\; \gamma J_t^* z_t\,, \\ U_t^* \;=\; (e^{\gamma u_t} - 1)J_{t^-}^*\,. \end{cases}$$

Then, (J^*, Z^*, U^*) is a solution of BSDE (5.14).
Also, using the above convergence property and our characterization of J as the *nonincreasing limit* of $(J^k)_{k\in\mathbb{N}}$ (see Theorem 4.2), we have

$$J_t \;=\; \lim_{k\to\infty} J_t^k \;=\; \lim_{k\to\infty} e^{\gamma y_t^k} \;=\; e^{\gamma y_t} \;=\; J_t^* \quad a.s.$$

Moreover, the uniqueness of the Doob-Meyer decomposition (4.7) of J implies that $Z_t^* = Z_t$ and $U_t^* = U_t\ dt\otimes d\mathbb{P} - a.s.$ Hence, (J,Z,U) is a solution of BSDE (5.14). In other words, J is not only a subsolution but a solution of this BSDE. Since by Theorem 4.1, J is the maximal subsolution of BSDE (5.14), it follows that J is the maximal solution of BSDE (5.14). □

6 Case of coefficients satisfying some exponential integrability conditions

In this section, we consider the case of coefficients not necessarily bounded but satisfying some integrability conditions. We first show that, in the particular case of strategies valued in a convex-compact set, the value function is the unique solution of a BSDE. From this, we derive that in the case of a convex set (but not necessarily compact), for each k, the process J^k, is characterized as the unique solution of a BSDE.

6.1 Case of strategies valued in a convex-compact set

Suppose that the set of admissible strategies is given by $\mathscr{C}$ (see Section 3), where C is a convex-compact set with $0 \in C$. Here, it simply corresponds to a closed interval of $\mathbb{R}$ since we are in the one dimensional case. However, the following results clearly still hold in the multidimensional case (see Section 7.1). Let $J(.)$ be the dynamic value function associated with $\mathscr{C}_t$ defined as in Section 3. Using some classical results of convex analysis (see for example Ekeland and Temam [11]), we derive the following existence property:

Proposition 6.3. *There exists a unique optimal strategy $\hat{\pi} \in \mathscr{C}$ for the optimization problem (2.5), that is*

$$J(0) \;=\; \inf_{\pi\in\mathscr{C}} \mathbb{E}\big[\exp\big(-\gamma(X_T^{\pi}+\xi)\big)\big] \;=\; \mathbb{E}\big[\exp\big(-\gamma(X_T^{\hat{\pi}}+\xi)\big)\big]\,.$$

Proof. $\mathscr{C}$ is strongly closed and convex in $L^2([0,T]\times\Omega)$. Hence, $\mathscr{C}$ is closed for the weak topology. Moreover, since $\mathscr{C}$ is bounded, it is a compact for the weak topology.
We define the function $\phi(\pi)=\mathbb{E}[\exp(-\gamma(X_T^{\pi}+\xi))]$ on $L^2([0,T]\times\Omega)$. This function is convex and continuous for the strong topology in $L^2([0,T]\times\Omega)$. By classical results of convex analysis, it is l.s.c. for the weak topology. Now, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathscr{C}$ such that $\phi(\pi^n)\to\min_{\pi\in\mathscr{C}}\phi(\pi)$ as $n\to\infty$.
Since $\mathscr{C}$ is weakly compact, there exists an extracted sequence still denoted by $(\pi^n)_{n\in\mathbb{N}}$ which converges for the weak topology to $\hat{\pi}$ for some $\hat{\pi}\in\mathscr{C}$. Now, since ϕ is l.s.c. for the weak topology, it implies that

$$\phi(\hat{\pi}) \;\le\; \liminf \phi(\pi^n) \;=\; \min_{\pi\in\mathscr{C}}\phi(\pi)\,.$$

Therefore, $\phi(\hat{\pi})=\inf_{\pi\in\mathscr{C}}\phi(\pi)$. The uniqueness of the optimal strategy derives from the convexity property of the set C and the strict convexity property of the function $x\mapsto\exp(-\gamma x)$. □

We now want to characterize the value function $J(.)$ as the unique solution of a BSDE. For that, we cannot apply the same techniques as in the case of bounded coefficients. Indeed, since the coefficients are not necessarily bounded, the generators of the associated BSDEs are no longer Lipschitz. Hence, the existence and uniqueness properties do not a priori hold. Therefore, in order to show the desired characterization of $J(.)$, we will use the dynamic programming principle and also the existence of an optimal strategy.
In order to have a dynamic programming principle similar to Proposition 4.1, we suppose that the coefficients satisfy the following integrability condition:

Assumption 6.2. *β is uniformly bounded and*

$$\mathbb{E}\Big[\exp\Big(a\int_0^T|\mu_t|dt\Big)\Big]+\mathbb{E}\Big[\exp\Big(b\int_0^T|\sigma_t|^2dt\Big)\Big] \;<\; \infty\,,$$

with $a=2\gamma||\mathscr{C}||_\infty$ and $b=8\gamma^2||\mathscr{C}||_\infty^2$.

We remark that such an assumption is satisfied in some stochastic volatility models.

By classical computations, one derives that for any $t\in[0,T]$ and any $\pi\in\mathscr{C}_t$, the following integrability property holds

$$\mathbb{E}\big[\sup_{s\in[t,T]}\exp\big(-\gamma X_s^{t,\pi}\big)\big] \;<\; \infty\,. \tag{6.15}$$

Using this integrability property, the process $J(.)$ can be proved to satisfy the following dynamic programming principle: $J(.)$ is the largest $\mathbb{G}$-adapted process such that $\exp(-\gamma X^{\pi})J(.)$ is a submartingale for any $\pi \in \mathscr{C}$ with $J(T) = \exp(-\gamma \xi)$, to get this property it is sufficient to mimick the proof of Proposition 4.1.

We now show the following characterization of the dynamic value function:

Theorem 6.5. *(Characterization of the dynamic value function)*
There exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is the unique solution *in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (3.3).*
Also, the optimal strategy $\hat{\pi} \in \mathscr{C}$ for $J(0)$ is characterized by the fact that $\hat{\pi}_t$ attains the essential infimum in (3.3), $dt \otimes d\mathbb{P} - a.e.$

Proof of Theorem 6.5

Step 1: There exists a rcll version of $J(.)$ which will be denoted by J. Let us give the proof, which is very simple here because we have an existence result. More precisely, by Proposition 6.3, there exists $\hat{\pi} \in \mathscr{C}$ which is optimal for $J(0)$. Hence, by the optimality criterium (similar to that of Proposition 4.1 with $\mathscr{D}$ replaced by $\mathscr{C}$), we have $J(t) = \exp(\gamma X_t^{\hat{\pi}})\mathbb{E}[\exp(-\gamma(X_T^{\hat{\pi}} + \xi))|\mathscr{G}_t]$ for any $t \in [0, T]$ (in other words, $\hat{\pi}$ is also optimal for $J(t)$). By classical results on the conditional expectation, there exists a rcll version denoted by J.

Step 2: Let us prove that there exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is a solution in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (3.3).
Note first that since $0 \in \mathscr{C}$, the process J satisfies $0 \le J_t \le 1$, $\forall\, t \in [0, T]$ a.s. From the Doob-Meyer decomposition, since the process J is a bounded rcll submartingale, there exist $Z \in L^2(W)$, $U \in L^2(M)$ and A a nondecreasing process with $A_0 = 0$ such that

$$dJ_t \quad = \quad Z_t dW_t + U_t dM_t + dA_t\,.$$

Since for any $\pi \in \mathscr{C}$ the process $\exp(-\gamma X^{\pi})J(.)$ is a submartingale, one derives that

$$dA_t \quad \ge \quad \operatorname*{ess\,sup}_{\pi \in \mathscr{C}} \Big\{ \gamma \pi_t (\mu_t J_t + \sigma_t Z_t) + \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(J_t + U_t) - \frac{\gamma^2}{2} |\pi_t \sigma_t|^2 J_t \Big\} dt\,.$$

Since, by Proposition 6.3, there exists an optimal strategy $\hat{\pi} \in \mathscr{C}$, the optimality criterion (similar to that of Proposition 4.1 with $\mathscr{D}$ replaced by $\mathscr{C}$) gives

$$dA_t \quad = \quad \Big\{ \gamma \hat{\pi}_t (\mu_t J_t + \sigma_t Z_t) + \lambda_t \big(1 - e^{-\gamma \hat{\pi}_t \beta_t}\big)(J_t + U_t) - \frac{\gamma^2}{2} |\hat{\pi}_t \sigma_t|^2 J_t \Big\} dt\,,$$

which implies

$$dA_t \quad = \quad \operatorname*{ess\,sup}_{\pi \in \mathscr{C}} \Big\{ \gamma \pi_t (\mu_t J_t + \sigma_t Z_t) + \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(J_t + U_t) - \frac{\gamma^2}{2} |\pi_t \sigma_t|^2 J_t \Big\} dt\,.$$

Hence, (J, Z, U) is solution of BSDE (3.3).

Using similar arguments as in the proof of Theorem 4.1, one can derive that (J, Z, U) is the *maximal solution* in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (3.3).

Step 3: Let us show that (J,Z,U) is the unique solution of BSDE (3.3). Let $(\bar{J},\bar{Z},\bar{U})$ be a solution of BSDE (3.3). By a measurable selection theorem, we know that there exists at least a strategy $\bar{\pi}\in\mathscr{C}$ such that $dt\otimes d\mathbb{P}-a.e.$

$$\operatorname*{ess\,inf}_{\pi\in\mathscr{C}}\Big\{\frac{\gamma^2}{2}|\pi_t\sigma_t|^2\bar{J}_t-\gamma\pi_t(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t)-\lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big\}$$
$$=\frac{\gamma^2}{2}|\bar{\pi}_t\sigma_t|^2\bar{J}_t-\gamma\bar{\pi}_t(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t)-\lambda_t\big(1-e^{-\gamma\bar{\pi}_t\beta_t}\big)(\bar{J}_t+\bar{U}_t).$$

Hence, BSDE (3.3) can be written under the form

$$d\bar{J}_t \;=\; \Big\{\gamma\bar{\pi}_t(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t)+\lambda_t\big(1-e^{-\gamma\bar{\pi}_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)-\frac{\gamma^2}{2}|\bar{\pi}_t\sigma_t|^2\bar{J}_t\Big\}dt+\bar{Z}_tdW_t+\bar{U}_tdM_t\,.$$

Let us introduce $B_t=\exp(-\gamma X_t^{\bar{\pi}})$. Itô's formula and rule product give

$$d(B_t\bar{J}_t) \;=\; (B_t\bar{Z}_t-\gamma\sigma_t\bar{\pi}_tB_t\bar{J}_t)dW_t+\big[(e^{-\gamma\beta_t\bar{\pi}_t}-1)B_{t^-}\bar{J}_t+e^{-\gamma\beta_t\bar{\pi}_t}B_{t^-}\bar{U}_t\big]dM_t\,.$$

By Assumption 6.2 and since $\bar{J}$ is bounded, one can derive that the local martingale $B\bar{J}$ satisfies $\mathbb{E}[\sup_{0\le t\le T}|B_t\bar{J}_t|]<\infty$. Hence, $B\bar{J}$ is a martingale and we get

$$\bar{J}_t \;=\; \mathbb{E}\Big[\frac{B_T}{B_t}e^{-\gamma\xi}\Big|\mathscr{G}_t\Big] \;=\; \mathbb{E}\big[\exp(-\gamma(X_T^{t,\bar{\pi}}+\xi))\big|\mathscr{G}_t\big]\,.$$

Hence,

$$\bar{J}_t \;\ge\; \operatorname*{ess\,inf}_{\pi\in\mathscr{C}}\mathbb{E}\big[\exp(-\gamma(X_T^{t,\pi}+\xi))\big|\mathscr{G}_t\big] \;=\; J_t\,.$$

Now, by step 2, J is the maximal solution of BSDE (3.3). This yields that for any $t\in[0,T]$, $J_t\le\bar{J}_t$, a.s. Hence, $J_t=\bar{J}_t,\ \forall\,t\in[0,T]$ a.s. and $\bar{\pi}$ is optimal. □

6.2 The non compact convex case

In this part, the set of admissible strategies is given by $\mathscr{D}$, where D is a convex set with $0\in D$. Under some exponential integrability conditions on the coefficients, by applying the result of the previous section, we derive that for each $k\in\mathbb{N}$, the process J^k is the unique solution of a BSDE. More precisely, we make the following integrability assumption.

Assumption 6.3. β is uniformly bounded, $\mathbb{E}[\int_0^T\lambda_tdt]<\infty$ and for any $p>0$ we have

$$\mathbb{E}\Big[\exp\Big(p\int_0^T|\mu_t|dt\Big)\Big]+\mathbb{E}\Big[\exp\Big(p\int_0^T|\sigma_t|^2dt\Big)\Big] \;<\; \infty\,.$$

First, recall that the value function associated with $\mathscr{D}'$ is equal to J, the one which is associated with $\mathscr{D}$. Using martingale inequalities and Assumption 6.3, one can show that all the bounded strategies belong to $\mathscr{D}'$ (but not necessarily in $\mathscr{D}$). By the same arguments as in the proof of Theorem 5.3, we have

Proposition 6.4. *(Characterization of the dynamic value function)*
The value function J is characterized as the nonincreasing limit *of the sequence $(J^k)_{k\in\mathbb{N}}$ as k tends to $+\infty$, where for each k, J^k is* the unique solution *of BSDE (3.3) with $\mathscr{C}$ equal to the set of all strategies taking their values in $[-k,k]\cap D$.*

7 Generalizations

In this section, we give some generalizations of the previous results. The proofs are not given, but they are identical to the proofs of the case with a default time and a stock. In all this section, elements of $\mathbb{R}^n$, $n \geq 1$, are identified to column vectors, the superscript $'$ stands for the transposition, $||.||$ the square norm, $\mathbb{1}$ the vector of $\mathbb{R}^n$ such that each component of this vector is equal to 1. Let U and V two vectors of $\mathbb{R}^n$, $U * V$ denotes the vector such that $(U * V)_i = U_i V_i$ for each $i \in \{1, \dots, n\}$. Let $X \in \mathbb{R}^n$, $diag(X)$ is the matrix such that $diag(X)_{ij} = X_i$ if $i = j$ else $diag(X)_{ij} = 0$.

7.1 Several default times and several stocks

We consider a market defined on the complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with two stochastic processes: an n-dimensional Brownian motion W and a p-dimensional jump process $N = (N^i, 1 \leq i \leq p)$ with $N_t^i = \mathbb{1}_{\tau^i \leq t}$, where $(\tau^i)_{1 \leq i \leq p}$ are p default times. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these processes. This filtration is supposed to be right-continuous and W is a $\mathbb{G}$-Brownian motion. We make the following assumptions on the default times:

Assumption 7.1. (i) The defaults do not appear simultaneously: $\mathbb{P}(\tau^i = \tau^j) = 0$ for $i \neq j$.

(ii) Each default can appear at any time: $\mathbb{P}(\tau^i > t) > 0$.

We denote by M^j the compensated martingale of N^j and Λ^j its compensator for each $j \in \{1, \dots, p\}$. We assume that Λ^j is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a process λ^j such that $\Lambda_t^j = \int_0^t \lambda_s^j ds$.

We consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at any time, and n risky assets, whose price processes $(S^i)_{1 \leq i \leq n}$ admit p discontinuities at times $(\tau^j)_{1 \leq j \leq p}$. Throughout the sequel, we consider that the price process $S := (S^i)_{1 \leq i \leq n}$ evolves according to the equation

$$dS_t \quad = \quad diag(S_{t^-})(\mu_t dt + \sigma_t dW_t + \beta_t dN_t),$$

with the classical assumptions:

Assumption 7.2.

(i) μ, σ, β and λ are uniformly bounded $\mathbb{G}$-predictable processes such that σ is nonsingular for any $t \in [0, T]$,

(ii) there exist d coefficients $\theta^1, \dots, \theta^d$ that are $\mathbb{G}$-predictable processes such that

$$\mu_t^i + \sum_{j=1}^{p} \lambda_t^j \beta_t^{i,j} \quad = \quad \sum_{j=1}^{d} \sigma_t^{i,j} \theta_t^j, \quad \forall\, t \in [0, T] \quad a.s., \ 1 \leq i \leq n,$$

we suppose that θ^j is bounded,

(iii) the process β satisfies $\beta_{\tau_j}^{i,j} > -1$ a.s. for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, p\}$.

Using the same techniques as in the previous sections, all the results stated in the previous sections can be generalized to this framework. In particular, we have:

Theorem 7.1. *There exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is the maximal solution in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the BSDE*

$$\begin{cases} -dJ_t = \underset{\pi\in\mathscr{D}}{\operatorname{ess\,inf}}\Big\{\frac{\gamma^2}{2}||\pi'_t\sigma_t||^2 J_t - \gamma\pi'_t(\mu_t J_t + \sigma_t Z_t) - \big(\mathbb{1} - e^{-\gamma\pi'_t\beta_t}\big)(\lambda_t J_t + \lambda_t * U_t)\Big\}dt \\ \qquad\qquad - Z_t dW_t - U_t dM_t\,, \\ \quad J_T = \exp(-\gamma\xi)\,. \end{cases}$$

7.2 Poisson jumps

We consider a market defined on the complete probability space $(\Omega, \mathscr{G}, \mathbb{P})$ equipped with two independent processes: a unidimensional Brownian motion W and a real-valued Poisson point process p defined on $[0, T] \times \mathbb{R}\backslash\{0\}$, we denote by $N_p(ds, dx)$ the associated counting measure, such that its compensator is $\hat{N}_p(ds, dx) = n(dx)ds$ and the Levy measure $n(dx)$ is positive and satisfies $n(\{0\}) = 0$ and $\int_{\mathbb{R}\backslash\{0\}}(1 \wedge |x|)^2 n(dx) < \infty$. We denote by $\mathbb{G} = \{\mathscr{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by the two processes W and N_p. We denote by $\tilde{N}_p(ds, dx)$ ($\tilde{N}_p(ds, dx) = N_p(ds, dx) - \hat{N}_p(ds, dx)$) the compensated measure, which is a martingale random measure.

The financial market consists of one risk-free asset, whose price process is assumed to be equal to 1, and one single risky asset, whose price process is denoted by S. In particular, the stock price process satisfies

$$dS_t \;=\; S_{t^-}\Big(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}\backslash\{0\}} \beta_t(x) N_p(dt, dx)\Big)\,.$$

μ, σ and β are assumed to be uniformly bounded $\mathbb{G}$-predictable processes. Moreover, the process σ (resp. β) satisfies $\sigma_t > 0$ (resp. $\beta_t(x) > -1$ a.s.). This framework corresponds to that considered in [28].

Using the same techniques as in the previous sections, all the results stated in the previous sections can be generalized to this framework. In particular, we have:

Theorem 7.2. *There exist $Z \in L^2(W)$ and $U \in L^2(\tilde{N}_p)$ such that (J, Z, U) is the maximal solution in $\mathscr{S}^{+,\infty} \times L^2(W) \times L^2(\tilde{N}_p)$ of the BSDE*

$$\begin{cases} -dJ_t = \underset{\pi\in\mathscr{D}}{\operatorname{ess\,inf}}\Big\{\frac{\gamma^2}{2}|\pi_t\sigma_t|^2 J_t - \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) - \int_{\mathbb{R}\backslash\{0\}}\big(1 - e^{-\gamma\pi_t x}\big)(J_t + U_t(x))n(dx)\Big\}dt \\ \qquad\qquad - Z_t dW_t - \int_{\mathbb{R}\backslash\{0\}} U_t(x)\tilde{N}_p(dt, dx)\,, \\ \quad J_T = \exp(-\gamma\xi)\,. \end{cases}$$

Appendix

A Additional comments on the set of admissibility

We first make some comments concerning the admissibility set $\mathscr{D}^{'}$.

We first stress on that the integrability condition $\mathbb{E}[\exp(-\gamma X_s^{t,\pi})] < \infty$ is required to ensure the dynamic programming principle in the form of Proposition 4.1 (see Remark 4.1). Note that if, instead of the p-integrability condition (4.4), we only had the 1-integrability condition, then the property of closedness by binding would fail. Also, a uniform integrability property as in (4.4) is required to obtain the characterization of the dynamic value function via a BSDE (see Remark 4.5). The set $\mathscr{D}'$ thus appears as a relatively large suitable set which ensures the dynamics results and the characterization of the dynamic value function via a BSDE provided in this paper.

Also, the p-exponential integrability condition (4.4) is not so surprising. Indeed, it is well-known that the exponential utility maximization problem is related to quadratic BSDEs (see for example Rouge and El Karoui [33]) and that this type of p-exponential integrability condition is often made to solve quadratic BSDEs (see for example Briand and Hu [7]).

Moreover, in the case where the coefficients are bounded or satisfy some exponential integrability assumption (see Assumption 6.3), by using martingale inequalities, one can show that all the bounded strategies belong to $\mathscr{D}^{'}$ (but not necessarily to $\mathscr{D}$). This, with Remark 4.6, yields that the value function J associated with $\mathscr{D}$ is not only equal to that associated with $\mathscr{D}^{'}$, but also to that associated with the set consisting in all the bounded strategies. These properties are interesting from a financial point of view.

Let us now make some additional remarks concerning the particular case of *non constrained strategies* that is, the case where $D = \mathbb{R}$.

Recall that in their paper, Delbaen *et al.* [8] consider the set of strategies Θ_2 defined by

$$\Theta_2 \quad := \quad \Big\{\pi\,,\, \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi+\xi)\big)\big] < +\infty \text{ and } X^\pi \text{ is a } \mathbb{Q}\text{-martingale for all } \mathbb{Q}\in\mathbb{P}_f\Big\},$$

where $\mathbb{P}_f$ is the set of absolutely continuous local martingale measures $\mathbb{Q}$ such that its entropy $H(\mathbb{P}|\mathbb{Q})$ is finite.

In general, there is no existence result on the set $\mathscr{D}$ neither on $\mathscr{D}^{'}$ whereas there is one on the set Θ_2. Recall that this existence result has been proven by [8]. More precisely, under the assumption that the price process is locally bounded, using the dual approach, these authors show the existence of an optimal strategy on the set Θ_2.

We have the following important point: the value function associated with Θ_2 coincides with that associated with $\mathscr{D}$. More precisely,

Lemma A.1. *The value function $V(0,\xi)$ associated with $\mathscr{D}$ (with $D = \mathbb{R}$) defined by*

$$V(0,\xi) \quad = \quad -\inf_{\pi\in\mathscr{D}} \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi+\xi)\big)\big]\,, \tag{A.1}$$

is equal to the one associated with Θ_2.

Proof. This property follows from the results of [8]. More precisely, let $V^2(0,\xi)$ be the value function associated with Θ_2. Let us introduce Θ_3 the set of strategies such that the associated wealth process is bounded and let $V^3(0,\xi)$ be the value function associated with Θ_3. By the results of [8],

$V^2(0,\xi) = V^3(0,\xi)$. Since $\Theta_3 \subset \mathscr{D}$, we have $V(0,\xi) \geq V^3(0,\xi)$. And by a classical localization argument (quite similar to the one used in Appendix B), one can show that $V(0,\xi) = V^3(0,\xi)$. Hence, $V(0,\xi) = V^2(0,\xi)$. □

In Section 4.1, we have seen that for the sets $\mathscr{D}$ and $\mathscr{D}^{'}$, the dynamic extension of the optimization problem was easy. However, in the case of the set Θ_2, things are not so clear. Actually, this is partly linked to the fact that the sets $\mathscr{D}$ and $\mathscr{D}^{'}$ are closed by *binding* whereas Θ_2 is not because of the integrability condition $\mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))] < +\infty$. One could naturally think of considering the space $\Theta_2^{'} := \{\pi,\ X^\pi \text{ is a } \mathbb{Q}-\text{martingale for all } \mathbb{Q} \in \mathbb{P}_f\}$ (instead of Θ_2) but this set is not really appropriate here: in particular it does not allow to have the dynamic programming principle (in the form of Proposition 4.1) because in this case, the Lebesgue theorem cannot be applied (see Remark 4.1).

Recall that the property of closedness by binding of the set $\mathscr{D}^{'}$ can be verified by using the assumption of p-integrability (4.4) and Cauchy-Schwarz inequality (see Appendix C for details). We stress on that the weaker integrability condition $\mathbb{E}[\exp(-\gamma X_T^\pi)] < +\infty$ is not sufficient to ensure this property.

B Proof of Lemma 4.1

We have to prove that the dynamic value function $J(.)$ associated with $\mathscr{D}$ coincides a.s. with the one associated with $\mathscr{D}'$.

Fix $t \in [0,T]$. Put $J^{'}(t) := \operatorname{ess\,inf}_{\pi \in \mathscr{D}_t^{'}} \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi))|\mathscr{G}_t]$, where $\mathscr{D}_t^{'}$ is the set of the restrictions to $[t,T]$ of the strategies of $\mathscr{D}'$. Since $\mathscr{D}_t \subset \mathscr{D}_t^{'}$, it follows that $J^{'}(t) \leq J(t)$ a.s. To prove the other inequality, it is sufficient to show that for any $\pi \in \mathscr{D}_t^{'}$, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathscr{D}_t$ such that $\pi^n \to \pi,\ dt \otimes d\mathbb{P}$ a.s. Let us define π^n by

$$\pi_s^n \quad = \quad \pi_s \mathbb{1}_{s \leq \tau_n}, \quad \forall\, s \in [t,T],$$

where τ_n is the stopping time defined by $\tau_n = \inf\{s \geq t, |X_s^{t,\pi}| \geq n\}$.

It is clear that for each $n \in \mathbb{N}$, $\pi^n \in \mathscr{D}_t$. Thus, $\exp(-\gamma X_T^{t,\pi^n}) = \exp(-\gamma X_{T\wedge\tau_n}^{t,\pi}) \xrightarrow{a.s.} \exp(-\gamma X_T^{t,\pi})$ as $n \to +\infty$. By definition of $\mathscr{D}_t^{'}$, $\mathbb{E}[\sup_{s\in[t,T]} \exp(-\gamma X_s^{t,\pi})] < \infty$. Hence, by the Lebesgue theorem, $\mathbb{E}[\exp(-\gamma(X_T^{t,\pi^n} + \xi))|\mathscr{G}_t] \to \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi))|\mathscr{G}_t]$ a.s. as $n \to +\infty$. Therefore, we have $J(t) \leq J^{'}(t)$ a.s.

C Proof of the closedness by binding of $\mathscr{D}'$

Lemma C.1. *Let π^1, π^2 be two admissible strategies of $\mathscr{D}^{'}$ and $s \in [0,T]$. The strategy π^3 defined by*

$$\pi_t^3 \quad = \quad \begin{cases} \pi_t^1 & \text{if } t \leq s\,, \\ \pi_t^2 & \text{if } t > s\,, \end{cases}$$

belongs to $\mathscr{D}^{'}$.

Proof. For any $u \in [0,T]$, we have for any $p > 1$

(i) if $u > s$, then

$$\mathbb{E}\Big[\sup_{r\in[u,T]} \exp\big(-\gamma p X_r^{u,\pi^3}\big)\Big] = \mathbb{E}\Big[\sup_{r\in[u,T]} \exp\big(-\gamma p X_r^{u,\pi^2}\big)\Big] < \infty,$$

(ii) if $u \le s$, then

$$\mathbb{E}\Big[\sup_{r\in[u,T]} \exp\big(-\gamma p X_r^{u,\pi^3}\big)\Big] \le \mathbb{E}\Big[\sup_{r\in[u,T]} \exp\big(-\gamma p X_r^{u,\pi^1}\big)\Big] + \mathbb{E}\Big[\sup_{r\in[s,T]} \exp\big(-\gamma p (X_s^{u,\pi^1} + X_r^{s,\pi^2})\big)\Big].$$

By Cauchy-Schwarz inequality,

$$\mathbb{E}\Big[\sup_{r\in[s,T]} \exp\big(-\gamma p (X_s^{u,\pi^1} + X_r^{s,\pi^2})\big)\Big] \le \mathbb{E}\Big[\sup_{r\in[u,T]} \exp\big(-2\gamma p X_r^{u,\pi^1}\big)\Big]^{1/2} \times \mathbb{E}\Big[\sup_{r\in[s,T]} \exp\big(-2\gamma p X_r^{s,\pi^2}\big)\Big]^{1/2}.$$

Hence, $\mathbb{E}[\sup_{r\in[u,T]} \exp(-\gamma p X_r^{u,\pi^3})] < \infty$.

□

D Proof of the existence of a rcll modification of J

The proof is not so simple since we do not know if there exists an optimal strategy in $\mathcal{D}$. Let $\mathbb{D} = [0, T] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Since $J(.)$ is a submartingale, the mapping $t \to J(t,\omega)$ defined on $\mathbb{D}$ has for almost every $\omega \in \Omega$ and for any t of $[0, T[$ a finite right limit

$$J(t^+,\omega) = \lim_{s\in\mathbb{D}, s\downarrow t} J(s,\omega)$$

(see Proposition 1.3.14 in [21] or Chapter 6 in [9]). It is possible to define $J(t^+,\omega)$ for any $(t,\omega) \in [0, T] \times \Omega$ by $J(T^+,\omega) := J(T,\omega)$ and

$$J(t^+,\omega) := \limsup_{s\in\mathbb{D}, s\downarrow t} J(s,\omega), \quad t \in [0, T[.$$

From the right-continuity of the filtration $\mathbb{G}$, the process $J(.^+)$ is $\mathbb{G}$-adapted. It is possible to show that $J(.^+)$ is a $\mathbb{G}$-submartingale and even that the process $\exp(-\gamma X^\pi)J(.^+)$ is a $\mathbb{G}$-submartingale for any $\pi \in \mathcal{D}$. Indeed, from Proposition 4.1, for any $s \le t$ and for each sequence of rational numbers $(t_n)_{n\in\mathbb{N}}$ converging down to t, we have

$$\mathbb{E}\big[\exp(-\gamma X_{t_n}^\pi) J(t_n)\big|\mathcal{G}_s\big] \ge \exp(-\gamma X_s^\pi) J(s) \quad a.s.$$

Let n tend to $+\infty$. By the Lebesgue theorem, we have that for any $s \le t$,

$$\mathbb{E}\big[\exp(-\gamma X_t^\pi) J(t^+)\big|\mathcal{G}_s\big] \ge \exp(-\gamma X_s^\pi) J(s) \quad a.s. \tag{D.1}$$

This implies that for any $s \leq t$, $\mathbb{E}[\exp(-\gamma X_t^\pi)J(t^+)|\mathcal{G}_s] \geq \exp(-\gamma X_s^\pi)J(s^+)$ a.s., which gives the submartingale property of the process $\exp(-\gamma X^\pi)J(.^+)$. Using the right-continuity of the filtration $\mathbb{G}$ and inequality (D.1) applied to $\pi = 0$ and $s = t$, we get

$$J(t^+) \;=\; \mathbb{E}\big[J(t^+)\big|\mathcal{G}_t\big] \;\geq\; J(t) \quad a.s.$$

On the other hand, by the characterization of $J(.)$ (see Proposition 4.1), and since the process $\exp(-\gamma X^\pi)J(.^+)$ is a $\mathbb{G}$-submartingale for any $\pi \in \mathcal{D}$, we have that for any $t \in [0, T]$,

$$J(t^+) \;\leq\; J(t) \quad a.s.$$

Thus, for any $t \in [0, T]$,

$$J(t^+) \;=\; J(t) \quad a.s.$$

Furthermore, the process $J(.^+)$ is rcll. The result follows by taking $J_t = J(t^+)$.

E Proof of equality (4.13)

For any $\pi \in \mathcal{D}$, we define the strategy $\pi_t^k = \pi_t \mathbb{1}_{|\pi_t| \leq k}$ for each $k \in \mathbb{N}$. The strategy π^k is uniformly bounded but not necessarily admissible. For that we define for each $(k, n) \in \mathbb{N} \times \mathbb{N}$ the stopping time

$$\tau_{k,n} \;:=\; \inf\{t \geq 0, |X_t^{\pi^k}| \geq n\},$$

and the strategy $\pi_t^{k,n} := \pi_t^k \mathbb{1}_{t \leq \tau_{k,n}}$. By construction, it is clear that the strategy $\pi^{k,n} \in \mathcal{D}^k$ for each (k, n). Since $\pi_t = \lim_k \lim_n \pi_t^{k,n}$ $dt \otimes d\mathbb{P}$ a.s., the following equality

$$\operatorname*{ess\,inf}_{\pi \in \bar{\mathcal{D}}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t - \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t (1 - e^{-\gamma \pi_t \beta_t})(\bar{J}_t + \bar{U}_t) \Big\} \;=\;$$
$$\operatorname*{ess\,inf}_{\pi \in \mathcal{D}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t \gamma \pi_t - (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t (1 - e^{-\gamma \pi_t \beta_t})(\bar{J}_t + \bar{U}_t) \Big\}$$

holds $dt \otimes d\mathbb{P}$ a.s.

Acknowledgement: The authors would like to thank the anonymous referees and the associate editor for their useful remarks which helped to improve the article.

References

[1] V.-E. Benes. Existence of Optimal Strategies Based on Specified Information for a Class of Stochastic Decision Problems. *SIAM Journal of Control and Optimization* **8** (1970), 179-188. MR0300726

[2] T. Bielecki, M. Jeanblanc and M. Rutkowski. Hedging of defaultable claims. *Lectures notes in mathematical finance, Springer* (2004). MR2105196

[3] T. Bielecki, M. Jeanblanc and M. Rutkowski. Modeling and valuation of credit risk. *Lectures notes in Mathematics, Springer* **1856** (2004), 27-126. MR2113721

[4] T. Bielecki and M. Rutkowski. Credit risk: modeling, valuation and hedging. *Springer Finance* (2004). MR1869476

[5] J.-M. Bismut. Conjugate Convex Functions in Optimal Stochastic Control. *Journal of Mathematical Analysis* **44** (1973), 384-404. MR0329726

[6] C. Blanchet-Scalliet and M. Jeanblanc. Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics* **8** (2004), 145-159. MR2022983

[7] P. Briand and Y. Hu. BSDE with quadratic growth and unbounded terminal value. *Probability Theory Related Fields* **136** (2006), 604-618. MR2257138

[8] F. Delbaen, P. Grandits, T. Rheinlander, D. Samperi, M. Schweizer and C. Stricker. Exponential Hedging and Entropic Penalties. *Mathematical Finance* **12** (2002), 99-123. MR1891730

[9] C. Dellacherie and P.-A. Meyer. Probabilités et potentiel, Théorie des martingales. *Hermann* (1980). MR0566768

[10] D. Duffie and L. Epstein. Stochastic differential utility. *Econometrica* **60** (1992), 353-394. MR1162620

[11] I. Ekeland and R. Temam. Convex analysis and variational problems. *North-Holland-Elsevier* (1976). MR0463994

[12] N. El Karoui. Les aspects probabilistes du contrôle stochastique. *Ecole d'Eté de Probabilités de Saint-Flour IX, Springer-Verlag* (1979). MR0637471

[13] N. El Karoui and M.-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control and Optimization* **33** (1995), 29-66. MR1311659

[14] N. El Karoui, S. Peng and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical finance* **7** (1997), 1-71. MR1434407

[15] M. Elliott, M. Jeanblanc and M. Yor. On models of default risk. *Mathematical Finance* **10** (2000), 179-195. MR1802597

[16] M. Jeanblanc, M. Yor and M. Chesney. Mathematical methods for financial markets. *Springer Finance*. MR2568861

[17] M. Jeanblanc, A. Matoussi and A. Ngoupeyou. Robust utility maximization from terminal wealth and consumption considering a discontinuous filtration. *Preprint* (2010).

[18] Y. Hu, P. Imkeller and M. Muller. Utility maximization in incomplete markets. *The Annals of Applied Probability* **15** (2005), 1691-1712. MR2152241

[19] I. Karatzas, J.-P. Lehoczky and S. Shreve. Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM Journal on Control and Optimization* **25** (1987), 1557-1586. MR0912456

[20] I. Karatzas, J.P. Lehoczky, S. Shreve and G. Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization* **29** (1991), 702-730. MR1089152

[21] I. Karatzas and S. Shreve. Brownian motion and stochastic calculus. *Springer Verlag* (1991). MR1121940

[22] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Probability* **28** (2000), 558-602. MR1782267

[23] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability* **9** (1999), 904-950. MR1722287

[24] S. Kusuoka. A remark on default risk models. *Advances in Mathematical Economics* **1** (1999), 69-82. MR1722700

[25] T. Lim and M.-C. Quenez. Portfolio optimization in a default model under full/partial information. *Preprint* (2010). arXiv:1003.6002

[26] S. Lukas. On pricing and hedging defaultable contingent claims. *Thesis, Humbolt University* (2001).

[27] R.-C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* **3** (1971), 373-413. MR0456373

[28] M.-A. Morlais. Utility maximization in a jump market model. *Stochastics and Stochastics Reports* **81** (2009), 1-27. MR2489997

[29] J. Neveu. Discrete-parameter martingales. *English translation, North-Holland, Amsterdam and American Elsevier, New York* (1975). MR0402915

[30] E. Pardoux. Generalized discontinuous backward stochastic differential equation. *In: N, El Karoui, L. Mazliak (Eds.), Backward Stochastic Differential Equations, in: Pitman Res. Notes Math. Ser., vol. 364, Longman, Harlow* (1997), 207-219. MR1752684

[31] S. Peng. A general stochastic maximum principle for optimal control problems. *SIAM Journal on control and optimization* **28** (1990), 966-979. MR1051633

[32] P. Protter. Stochastic integration and differential equations. *Springer Verlag* (1990). MR1037262

[33] R. Rouge and N. El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance* **10** (2000), 259-276. MR1802922

[34] M. Royer. Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic processes and their applications* **116** (2006), 1358-1376. MR2260739

[35] W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability* **11** (2001), 694-734. MR1865021

[36] S.-H. Tang and X. Li. Necessary conditions for optimal control of stochastic systems with random jumps. *SIAM Journal on Control and Optimization* **32** (1994), 1447-1475. MR1288257