

A discrete approach to Rough Parabolic Equations

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Abstract

By combining the formalism of [8] with a discrete approach close to the considerations of [6], we interpret and we solve the rough partial differential equation $dy_t = Ay_t dt + \sum_{i=1}^m f_i(y_t) dx_t^i$ ($t \in [0, T]$) on a compact domain \mathcal{O} of \mathbb{R}^n , where A is a rather general elliptic operator of $L^p(\mathcal{O})$ ($p > 1$), $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$ and x is the generator of a 2-rough path. The (global) existence, uniqueness and continuity of a solution is established under classical regularity assumptions for f_i . Some identification procedures are also provided in order to justify our interpretation of the problem.

Key words: Rough paths theory; Stochastic PDEs; Fractional Brownian motion.

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1 Introduction

The rough paths theory introduced by Lyons in [17] and then refined by several authors (see the recent monograph [12] and the references therein) has led to a very deep understanding of the standard rough systems

$$dy_t^i = \sum_{j=1}^m \sigma_{ij}(y_t) dx_t^j \quad , \quad y_0 = a \in \mathbb{R}^d \quad , \quad t \in [0, T], \quad (1)$$

where $\sigma_{ij} : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth enough vector field and $x : [0, T] \rightarrow \mathbb{R}^m$ is a so-called rough path, that is to say a function allowing the construction of iterated integrals (see Assumption $(X)_\gamma$ for the definition of a 2-rough path and [18] for a rough path of any order). The theory provides for instance a new pathwise interpretation of stochastic systems driven by very general Gaussian processes, as well as fruitful and highly non-trivial continuity results for the Itô solution of (1), i.e., when x is a standard Brownian motion.

One of the new challenges of the rough paths theory now consists in adapting the machinery to infinite-dimensional (rough) equations that involves a non-bounded operator, with, as a final objective, the possibility of new pathwise interpretations for stochastic PDEs. Some progresses have recently been made in this direction, with on the one hand the viscosity-solution approach due to Friz *et al* (see [2, 3, 10, 9]) and on the other hand, the development of a specific algebraic formalism by Gubinelli *et al* (see [14, 15, 8]).

The present paper is a contribution to this global project. It aims at providing, in a concise and self-contained formulation, the analysis of the following rough evolution equation:

$$y_0 = \psi \in L^p(\mathcal{O}) \quad , \quad dy_t = Ay_t dt + \sum_{i=1}^m f_i(y_t) dx_t^i \quad , \quad t \in [0, T], \quad (2)$$

where A is a rather general elliptic operator on a bounded domain \mathcal{O} of \mathbb{R}^n (see Assumptions (A1)-(A2)), $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$ and x generates a m -dimensional 2-rough path (see Assumption $(X)_\gamma$). Although the global form of (2) is quite similar to the equation treated in [8], several differences and notable improvements justify the interest of our study:

- (i) The equation is here analysed on a compact domain \mathcal{O} of \mathbb{R}^n . This allows to simplify the conditions relative to the vector field f_i , which reduce to the classical assumptions of rough paths theory, ie k -times differentiable ($k \in \mathbb{N}^*$) with bounded derivatives (see Assumption $(F)_k$).
- (ii) The conditions on p are less stringent than in [8], where p has to be taken very large. It will here be possible to show the existence and uniqueness of a solution in $L^p(\mathcal{O})$ (for a smooth enough initial condition ψ) as soon as $p > n$ (see Theorem 2.11). In particular, we can go back to the Hilbert framework of [15] for the one-dimensional equation ($n = 1, p = 2$).
- (iii) Last but not least, the arguments we are about to use lead to the existence of a *global* solution for (2), defined on any time interval $[0, T]$. This is a breakthrough with respect to [15, 8], where only local solutions are obtained, on a time interval that depends on the data of the problem, namely x, f and ψ .

In order to reach these three improvements, the strategy will combine elements of the formalism used in [8] with a discrete approach of the equation, close to the machinery developed in [6] for rough standard systems. A first step consists of course in giving some reasonable sense to Equation (2). We have chosen to work with an interpretation à la Davie, derived from the expansion of the ordinary solution (see Definition 2.6), and we have left aside the sewing map at the core of the constructions in [8]. Note however that the expansion under consideration here relies on the operator-valued paths $X^{x,i}, X^{ax,i}, X^{xx,ij}$ which were identified in [8] (see Subsection 2.3), and which plays the role of an infinite-dimensional rough path adapted to the problem. When applying the whole procedure to a differentiable driving path x (resp. a standard Brownian motion), the solution that we retrieve coincides with the classical solution (resp. the Itô solution), as reported in Subsection 2.4. Together with the continuity statement of Theorem 2.12, this identification procedure allows to fully justify our interpretation of (2) (see Corollary 2.13 and Remark 2.14).

Once endowed with this interpretation, our solving method is based on a discrete approach of the problem: as in [6], the solution is obtained as the limit of a discrete scheme the mesh of which tends to 0. Nevertheless, some fundamental differences arise when trying to mimic the strategy of [6]. To begin with, the middle-point argument at the root of the reasoning in the diffusion case (see the proof of [6, Lemma 2.4]) cannot take into account the space-time interactions that occur in the study of PDEs, i.e., the classical estimates (22) and (23). Therefore, the argument must here be replaced with a little bit more complex algorithm described in Appendix A, and which will be used throughout the paper. Let us also mention that the expansion of the vector field $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$ is not as easy to control as in the standard finite-dimensional case, even if one assumes that the functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$ are very smooth. Observe for instance that if $W^{\alpha,p}$ ($\alpha \in (0, 1)$) stands for the fractional Sobolev space likely to accommodate the solution path, and if f_i is assumed to be differentiable, bounded with bounded derivative, then one can only rely on the non-uniform estimate (see [23])

$$\|f_i(\varphi)\|_{W^{\alpha,p}} \leq \|f_i\|_{L^\infty(\mathbb{R})} + \|f_i'\|_{L^\infty(\mathbb{R})} \|\varphi\|_{W^{\alpha,p}} \quad \text{for any } \varphi \in W^{\alpha,p}.$$

Consequently, more subtle patching arguments must be put forward so as to exhibit a global solution. The strategy involves in particular a careful examination of the dependence on the initial condition at each step of the procedure (see for instance the controls (45) and (46)).

The paper is structured as follows: In Section 2, we gather all the elements that allow to understand our interpretation of Equation (2), and we state the three main results of the paper, namely Theorems 2.10-2.12. The three sections that follow are dedicated to the proof of each of these results, with the existence theorem first (Section 3) and then the uniqueness (Section 4) and continuity (Section 5) results. Finally, Appendix A contains the description and the analysis of the algorithm at the root of our machinery, while Appendix B is meant to provide the details relative to the identification procedure in the Brownian case (see Proposition 2.9).

For the sake of clarity, we shall only consider Equation (2) on the generic interval $[0, 1]$. It is however easy to realize that the whole reasoning remains valid on any (fixed) finite interval $[0, T]$ at the price of very minor modifications.

Throughout the paper, we will denote by $\mathcal{C}^{k,b}(\mathbb{R}; \mathbb{R}^l)$ ($k, l \in \mathbb{N}^*$) the set of \mathbb{R}^l -valued functions which are k -times differentiable with bounded derivatives.

Finally, we will use the classical convention for the summation over indexes $x^i y_i = \sum_i x^i y_i$, whenever the underlying index set is obvious from the context.

2 Interpretation of the equation

We first give some precisions about the setting of our study, as far as the operator A , the driving path x and the vector field f_i are concerned (Subsection 2.1). Then we introduce the notation and the tools designed for our analysis (Subsections 2.2 and 2.3), and which enable us to interpret (2) (Subsection 2.4). We finally state the three main results of the paper (Subsection 2.5), and we discuss some possible extensions of the strategy to rougher driving paths (Subsection 2.6).

2.1 Assumptions

As it was announced in the introduction, we mean to tackle the equation $dy_t = Ay_t dt + f_i(y_t) dx_t^i$, $t \in [0, 1]$, in $L^p(\mathcal{O})$, where \mathcal{O} is a bounded domain of \mathbb{R}^n , A is an elliptic operator, $f_i(\varphi)(\xi) := f_i(\varphi(\xi))$ and x is a Hölder path. More precisely, to be in a position to interpret and solve this equation, we will be led to assume that (some of) the following conditions are satisfied:

Assumption (A1): A generates an analytic semigroup of contraction S on any $L^p(\mathcal{O})$. Under this hypothesis, we will denote $S_{ts} := S_{t-s}$ ($s \leq t$), $\mathcal{B}_p := L^p(\mathcal{O})$, $\mathcal{B}_{\alpha,p} := \text{Dom}(A_p^\alpha)$, and we endow the latter space with the graph norm $\|\varphi\|_{\mathcal{B}_{\alpha,p}} := \|A_p^\alpha \varphi\|_{L^p(\mathcal{O})}$. We also assume that for any function $g \in \mathcal{C}^{1,b}(\mathbb{R}; \mathbb{R})$, there exists a constant c_g^1 such that

$$\|g(\varphi)\|_{\mathcal{B}_{1/2,p}} \leq c_g^1 \{1 + \|\varphi\|_{\mathcal{B}_{1/2,p}}\} \quad (3)$$

and for any function $g \in \mathcal{C}^{2,b}(\mathbb{R}; \mathbb{R})$, there exists a constant c_g^2 such that

$$\|g(\varphi)\|_{\mathcal{B}_{\alpha,p}} \leq c_g^2 \{1 + \|\varphi\|_{\mathcal{B}_{\alpha,p}}^2\} \quad \text{if } \alpha \in (1/2, 1) \text{ and } 2\alpha p > n, \quad (4)$$

where, in (3) and (4), $g(\varphi)$ is just understood in the composition sense, i.e., $g(\varphi)(\xi) := g(\varphi(\xi))$.

Assumption (A2): If $2\alpha p > n$, then $\mathcal{B}_{\alpha,p}$ is a Banach algebra continuously included in the space \mathcal{B}_∞ of continuous functions on $\overline{\mathcal{O}}$.

Assumption (X) $_\gamma$: x allows the construction of a 2-rough path

$$(x, \mathbf{x}^2) \in \mathcal{C}_1^\gamma([0, 1]; \mathbb{R}^m) \times \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})$$

for some (fixed) coefficient $\gamma \in (1/3, 1/2)$. In other words, we assume that x is a γ -Hölder path and that there exists a 2-variable path \mathbf{x}^2 (also called a Lévy area) such that for any $0 \leq s \leq u \leq t \leq 1$,

$$|\mathbf{x}_{ts}^2| \leq c |t - s|^{2\gamma} \quad \text{and} \quad \mathbf{x}_{ts}^{2,ij} - \mathbf{x}_{tu}^{2,ij} - \mathbf{x}_{us}^{2,ij} = (x_t^i - x_u^i)(x_u^j - x_s^j).$$

We will then denote

$$\|\mathbf{x}\|_\gamma := \mathcal{N}[x; \mathcal{C}_1^\gamma([0, 1]; \mathbb{R}^m)] + \mathcal{N}[\mathbf{x}^2; \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})],$$

where

$$\mathcal{N}[x; \mathcal{C}_1^\gamma([0, 1]; \mathbb{R}^m)] := \sup_{0 \leq s < t \leq 1} \frac{|x_t - x_s|}{|t - s|^\gamma}, \quad \mathcal{N}[\mathbf{x}^2; \mathcal{C}_2^{2\gamma}([0, 1]; \mathbb{R}^{m,m})] := \sup_{0 \leq s < t \leq 1} \frac{|\mathbf{x}_{ts}^2|}{|t - s|^{2\gamma}}.$$

Assumption (F)_k: f belongs to $\mathcal{C}^{k,b}(\mathbb{R}; \mathbb{R}^m)$.

Before pondering over the plausibility of these conditions, let us precise that we henceforth focus on the mild formulation of Equation (2)

$$y_t = S_t \psi + \int_0^t S_{tu} dx_u^i f_i(y_u) \quad , \quad t \in [0, 1]. \quad (5)$$

This is a standard change of perspective for the study of (stochastic) PDEs (see [5]), which allows to use the regularizing properties of the semigroup. In retrospect, owing to the regularity assumptions on f , it will however be possible to make a link between the mild and strong interpretations of the equation, see Remark 2.14.

Application: Properties (A1)-(A2) are satisfied by any elliptic operator on $L^p((0, 1)^n)$ that can be written as

$$A = - \sum_{i,j=1}^n \partial_{\xi_i} (a_{ij} \cdot \partial_{\xi_j}) + c \quad , \quad \mathcal{D}(A_p) := W^{2,p}((0, 1)^n) \cap W_0^{1,p}((0, 1)^n), \quad (6)$$

where $c \geq 0$ and the functional coefficients a_{ij} are bounded, differentiable with bounded derivatives on $[0, 1]^n$. Indeed, under these assumptions, it is proven in [7] that A generates an analytic semigroup of contraction. Then, thanks to [20], one can identify the domain $\mathcal{D}(A_p^\alpha)$ with the complex interpolation $[L_p, \mathcal{D}(A_p)]_\alpha$ and one can use the result of [22] to assert that $\|\cdot\|_{\mathcal{D}(A_p^\alpha)} \sim \|\cdot\|_{F_{p,2}^{2\alpha}}$, where $F_{p,2}^{2\alpha}$ is the classical Triebel-Lizorkin space described (for instance) in [21]. The results of [21] (resp. [23]) finally enables us to check Condition (A2) (resp. the controls (3) and (4)).

As far as Condition (X)_γ is concerned, the process that we have in mind in this paper is the fractional Brownian motion B^H with Hurst index $H > 1/3$, for which the (a.s) existence of a Lévy area has been established in [4]. Condition (X)_γ is in fact satisfied by a larger class of Gaussian processes, as reported in [12].

In brief, under the above-stated regularity assumptions, the results that we are about to state and prove can be applied to the stochastic equation

$$dY_t = \left[- \sum_{i,j=1}^n \partial_{\xi_i} (a_{ij} \cdot \partial_{\xi_j} Y_t) + c Y_t \right] dt + \sum_{i=1}^m f_i(Y_t) dB_t^{H,i} \quad , \quad t \in [0, 1] \quad , \quad \xi \in (0, 1)^n.$$

2.2 Hölder spaces

We suppose in this subsection that Assumption (A1) is satisfied. In order to introduce the functional framework of our analysis, let us focus on the following consideration: we know that one of the most appropriate space for the study of rough standard systems is the set of Hölder paths $\{y : [0, 1] \rightarrow \mathbb{R}^d : |y_t - y_s| \leq c |t - s|^\gamma\}$ (see [13]), and this is (among others) due to the convenient expression for the variations of the solution y of (1), namely $y_t - y_s = \int_s^t \sigma_{ij}(y_u) dx_u^j$. Here, if we denote by y the solution of (5) (assume for the moment that x is a differentiable path), it is readily checked that for all $s < t$,

$$y_t - y_s = \int_s^t S_{tu} dx_u^i f_i(y_u) + a_{ts} y_s, \quad \text{where } a_{ts} := S_{ts} - \text{Id}.$$

With this observation in mind, the following notation arises quite naturally:

Notation. For all paths $y : [0, 1] \rightarrow \mathcal{B}_p$ and $z : \mathcal{S}_2 \rightarrow \mathcal{B}_p$, where $\mathcal{S}_2 := \{(t, s) \in [0, 1]^2 : s \leq t\}$, we set, for $s \leq u \leq t \in [0, 1]$,

$$(\delta y)_{ts} := y_t - y_s, \quad (\hat{\delta} y)_{ts} := (\delta y)_{ts} - a_{ts} y_s = y_t - S_{ts} y_s, \quad (7)$$

$$(\hat{\delta} z)_{tus} := z_{ts} - z_{tu} - S_{tu} z_{us}. \quad (8)$$

The (ordinary) system (5) can now be written in the convenient form

$$y_0 = \psi, \quad (\hat{\delta} y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u), \quad s, t \in [0, 1]. \quad (9)$$

To make the notation (7)-(8) even more legitimate in this convolutional context, we let the reader observe the following elementary properties:

Proposition 2.1. *Let $y : [0, 1] \rightarrow \mathcal{B}_p$ and $x : [0, 1] \rightarrow \mathbb{R}$ be differentiable paths. Then it holds:*

- *Telescopic sum: $\hat{\delta}(\hat{\delta} y)_{tus} = 0$ and $(\hat{\delta} y)_{ts} = \sum_{i=0}^{n-1} S_{t t_{i+1}} (\hat{\delta} y)_{t_{i+1} t_i}$ for any partition $\{s = t_0 < t_1 < \dots < t_n = t\}$ of an interval $[s, t]$ of $[0, 1]$.*
- *Chasles relation: if $\mathcal{J}_{ts} := \int_s^t S_{tu} dx_u y_u$ then $\hat{\delta} \mathcal{J} = 0$.*

Like with the standard finite-dimensional systems, the rough-paths treatment of Equation (9) leans on the controlled expansion of the convolutional integral $\int_s^t S_{tu} dx_u^i f_i(y_u)$. To express this control with the highest accuracy, we are naturally led to consider the following semi-norms, that can be seen as adapted versions of the classical Hölder seminorms: if $y : [0, 1] \rightarrow V$, $z : \mathcal{S}_2 \rightarrow V$ and $h : \mathcal{S}_3 \rightarrow V$, where V is any Banach space and $\mathcal{S}_3 := \{(t, u, s) \in [0, 1]^3 : s \leq u \leq t\}$, we denote, for any $\lambda > 0$,

$$\mathcal{N}[y; \mathcal{C}_1^\lambda([a, b]; V)] := \sup_{a \leq s < t \leq b} \frac{\|(\hat{\delta} y)_{ts}\|_V}{|t - s|^\lambda}, \quad \mathcal{N}[y; \mathcal{C}_1^0([a, b]; V)] := \sup_{t \in [a, b]} \|y_t\|_V, \quad (10)$$

$$\mathcal{N}[z; \mathcal{C}_2^\lambda([a, b]; V)] := \sup_{a \leq s < t \leq b} \frac{\|z_{ts}\|_V}{|t-s|^\lambda}, \quad \mathcal{N}[h; \mathcal{C}_3^\lambda([a, b]; V)] := \sup_{a \leq s < u < t \leq b} \frac{\|h_{tus}\|_V}{|t-s|^\lambda}. \quad (11)$$

Then $\mathcal{C}_1^\lambda([a, b]; V)$ stands for the set of paths $y : [0, 1] \rightarrow V$ such that $\mathcal{N}[y; \mathcal{C}_1^\lambda([a, b]; V)] < \infty$, and we define $\mathcal{C}_2^\lambda([a, b]; V)$ and $\mathcal{C}_3^\lambda([a, b]; V)$ along the same lines. With this notation, observe for instance that if $y \in \mathcal{C}_2^\lambda([a, b]; \mathcal{L}(V, W))$ and $z \in \mathcal{C}_2^\beta([a, b]; V)$, the path h defined as $h_{tus} = y_{tu}z_{us}$ ($s \leq u \leq t$) belongs to $\mathcal{C}_3^{\lambda+\beta}([a, b]; W)$.

When $[a, b] = [0, 1]$, we will use the short form $\mathcal{C}_k^\lambda(V) := \mathcal{C}_k^\lambda([a, b]; V)$.

2.3 Infinite-dimensional rough path

By anticipating the proof of Proposition 2.8, we know that, when x is a differentiable path, the expansion of $\int_s^t S_{tu} dx_u^i f_i(y_u)$ puts forward the three following operator-valued paths constructed from x :

$$\int_s^t S_{tu} dx_u^i, \quad \int_s^t a_{tu} dx_u^i, \quad \int_s^t S_{tu} dx_u^i (\delta x^j)_{us}.$$

A priori, these expressions do not make sense for a non-differentiable γ -Hölder (rough)-path x . An integration by parts argument, retrospectively justified by Lemmas 2.3 and 2.4, leads here to the general definition:

Definition 2.2. Under Assumptions (A1) and $(X)_\gamma$, we define the three operator-valued paths $X^{x,i}$, $X^{ax,i}$ and $X^{xx,ij}$ by the formulas

$$X_{ts}^{x,i} := S_{ts}(\delta x^i)_{ts} - \int_s^t AS_{tu}(\delta x^i)_{tu} du, \quad (12)$$

$$X_{ts}^{ax,i} := a_{ts}(\delta x^i)_{ts} - \int_s^t AS_{tu}(\delta x^i)_{tu} du, \quad (13)$$

$$X_{ts}^{xx,ij} := S_{ts} \mathbf{x}_{ts}^{2,ij} - \int_s^t AS_{tu} [\mathbf{x}_{tu}^{2,ij} + (\delta x^i)_{tu} (\delta x^j)_{us}] du. \quad (14)$$

If in addition Assumption $(F)_1$ is satisfied, we set $F_{ij}(\varphi) := f_i'(\varphi) \cdot f_j(\varphi)$ and we associate to every path $y : [0, 1] \rightarrow \mathcal{B}_p$ the two quantities

$$J_{ts}^y := (\hat{\delta} y)_{ts} - X_{ts}^{x,i} f_i(y_s) - X_{ts}^{xx,ij} F_{ij}(y_s), \quad (15)$$

$$K_{ts}^y := (\hat{\delta} y)_{ts} - X_{ts}^{x,i} f_i(y_s). \quad (16)$$

Lemma 2.3. Suppose that x is a m -dimensional differentiable path and let \mathbf{x}^2 be its Lévy area, understood in the classical Lebesgue sense as the iterated integral $\mathbf{x}_{ts}^{2,ij} := \int_s^t dx_u^i (\delta x^j)_{us}$. Then, under Assumption (A1),

$$X_{ts}^{x,i} = \int_s^t S_{tu} dx_u^i, \quad X_{ts}^{ax,i} = \int_s^t a_{tu} dx_u^i, \quad X_{ts}^{xx,ij} = \int_s^t S_{tu} dx_u^i (\delta x^j)_{us}. \quad (17)$$

Proof. As aforementioned, this is just a matter of integration by parts. For instance, one has

$$\begin{aligned}
\int_s^t S_{tu} dx_u^i (\delta x^j)_{us} &= \int_s^t S_{tu} d_u (\mathbf{x}_{us}^{2,ij}) \\
&= \int_s^t S_{tu} d_u \left(-(\delta \mathbf{x}^{2,ij})_{tus} + \mathbf{x}_{ts}^{2,ij} - \mathbf{x}_{tu}^{2,ij} \right) \\
&= \int_s^t S_{tu} d_u \left(-(\delta x^i)_{tu} (\delta x^j)_{us} - \mathbf{x}_{tu}^{2,ij} \right) \\
&= S_{ts} \mathbf{x}_{ts}^{2,ij} - \int_s^t AS_{tu} \left[\mathbf{x}_{tu}^{2,ij} + (\delta x^i)_{tu} (\delta x^j)_{us} \right] du.
\end{aligned}$$

□

Observe now that the three expressions contained in (17) can also be directly interpreted as Itô integrals when x stands for a standard Brownian motion. This interpretation remains consistent with Definition 2.2:

Lemma 2.4. *Suppose that x is m -dimensional Brownian motion defined on a complete filtered probability space (Ω, \mathcal{F}, P) , and let \mathbf{x}^2 be its Lévy area, understood in the Itô sense as the first iterated integral of x . Then, under Assumption (A1), the three identifications of the previous lemma remain valid in this context.*

Proof. It suffices to replace the integration by parts argument with Itô's formula, upon noticing that only Wiener integrals are involved here. For X^{xx} , we know indeed that for any fixed s , the process $u \mapsto \mathbf{x}_{us}^{2,ij} = \int_s^u dx_v^i (\delta x^j)_{vs}$ is a semimartingale and

$$\int_s^t S_{tu} dx_u^i (\delta x^j)_{us} = \int_s^t S_{tu} d_u (\mathbf{x}_{us}^{2,ij}).$$

□

To end up with this subsection, let us highlight the regularity properties that will be at our disposal throughout the study:

Proposition 2.5. *Under Assumptions (A1) and $(X)_\gamma$, one has, for all $\alpha \in (0, 1), \kappa \in [0, \gamma)$,*

$$X^{x,i} \in \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})) \cap \mathcal{C}_2^{\gamma-\kappa}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha+\kappa,p})), \quad (18)$$

$$X^{ax,i} \in \mathcal{C}_2^{\gamma+\alpha}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_p)), \quad (19)$$

$$X^{xx,ij} \in \mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})) \cap \mathcal{C}_2^{2\gamma-\kappa}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha+\kappa,p})). \quad (20)$$

We will denote by $\|X\|_{\gamma,\alpha,\kappa}$ the norm attached to $X := (X^x, X^{ax}, X^{xx})$ through Properties (18)-(20), that is to say

$$\|X\|_{\gamma,\alpha,\kappa} := \sum_{i,j=1}^m \left\{ \mathcal{N}[X^{x,i}; \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p}))] + \dots + \mathcal{N}[X^{xx,ij}; \mathcal{C}_2^{2\gamma-\kappa}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha+\kappa,p}))] \right\}.$$

With this notation, one has $\|X\|_{\gamma,\alpha,\kappa} \leq c_{\gamma,\alpha,\kappa} \|\mathbf{x}\|_{\gamma}$. Moreover, if \tilde{X} stands for the path associated to another trajectory \tilde{x} satisfying $(X)_{\gamma}$, then

$$\|X - \tilde{X}\|_{\gamma,\alpha,\kappa} \leq c_{\gamma,\alpha,\kappa} \{1 + \|\mathbf{x}\|_{\gamma} + \|\tilde{\mathbf{x}}\|_{\gamma}\} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\gamma}. \quad (21)$$

Proof. Properties (18)-(20) are straightforward consequences of the well-known estimates (see [19])

$$\|S_{ts}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} \leq c_{\kappa} |t-s|^{-\kappa} \|\varphi\|_{\mathcal{B}_{\alpha,p}}, \quad \|AS_{ts}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} \leq c_{\kappa} |t-s|^{-1-\kappa} \|\varphi\|_{\mathcal{B}_{\alpha,p}}, \quad (22)$$

$$\|a_{ts}\varphi\|_{\mathcal{B}_p} \leq c_{\alpha} |t-s|^{\alpha} \|\varphi\|_{\mathcal{B}_{\alpha,p}}. \quad (23)$$

For example, for any $\varphi \in \mathcal{B}_{\alpha,p}$,

$$\begin{aligned} \|X_{ts}^{x,i}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} &\leq \|\mathbf{x}\|_{\gamma} \left\{ |t-s|^{\gamma} \|S_{ts}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} + \int_s^t |t-u|^{\gamma} \|AS_{tu}\varphi\|_{\mathcal{B}_{\alpha+\kappa,p}} du \right\} \\ &\leq c_{\kappa} \|\mathbf{x}\|_{\gamma} \|\varphi\|_{\mathcal{B}_{\alpha,p}} \left\{ |t-s|^{\gamma-\kappa} + \int_s^t |t-u|^{-1+\gamma-\kappa} du \right\} \\ &\leq c_{\gamma,\kappa} \|\mathbf{x}\|_{\gamma} \|\varphi\|_{\mathcal{B}_{\alpha,p}} |t-s|^{\gamma-\kappa}. \end{aligned}$$

The controls of $\|X\|_{\gamma,\alpha,\kappa}$ and $\|X - \tilde{X}\|_{\gamma,\alpha,\kappa}$ can be readily checked from the very definitions (12)-(14). Observe for instance that

$$\begin{aligned} &\left\| \int_s^t AS_{tu}(\delta x^i)_{tu}(\delta x^j)_{us} du - \int_s^t AS_{tu}(\delta \tilde{x}^i)_{tu}(\delta \tilde{x}^j)_{us} du \right\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \\ &\leq \int_s^t \|AS_{tu}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \left\{ |\delta(x^i - \tilde{x}^i)_{tu}| |(\delta x^j)_{us}| + |(\delta \tilde{x}^i)_{tu}| |\delta(x^j - \tilde{x}^j)_{us}| \right\} du \\ &\leq c \{1 + \|\mathbf{x}\|_{\gamma} + \|\tilde{\mathbf{x}}\|_{\gamma}\} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\gamma} \left(\int_s^t |t-u|^{-1+\gamma} |u-s|^{\gamma} du \right) \\ &\leq c |t-s|^{2\gamma} \{1 + \|\mathbf{x}\|_{\gamma} + \|\tilde{\mathbf{x}}\|_{\gamma}\} \|\mathbf{x} - \tilde{\mathbf{x}}\|_{\gamma}. \end{aligned}$$

□

2.4 Interpretation of the equation

Let us now turn to the interpretation of (9) for a generic 2-rough paths $\mathbf{x} = (x, \mathbf{x}^2)$. Like in [6], our approach is based on the Taylor expansion of the ordinary mild equation. We first give the general definition of a solution and then we clarify this definition by considering the two previously-known situations, namely when x is a differentiable path and when x is a standard Brownian motion. Remember that the notation J^{γ} has been introduced in Definition 2.2.

Definition 2.6. Under Assumptions (A1), $(X)_{\gamma}$ and $(F)_1$, for all $\lambda \geq 0$ and $\psi \in \mathcal{B}_{\lambda,p}$, we will call a solution in $\mathcal{B}_{\lambda,p}$ of the equation

$$y_t = S_t \psi + \int_0^t S_{t-u} f_i(y_u) dx_u^i, \quad t \in [0, 1], \quad (24)$$

any path $y : [0, 1] \rightarrow \mathcal{B}_{\lambda,p}$ such that $y_0 = \psi$ and there exists two coefficients $\mu > 1, \varepsilon > 0$ for which

$$J^y \in \mathcal{C}_2^\mu([0, 1]; \mathcal{B}_p) \quad \text{and} \quad J^y \in \mathcal{C}_2^\varepsilon([0, 1]; \mathcal{B}_{\lambda,p}). \quad (25)$$

Remark 2.7. The reader familiar with the strategy of [6] will not be surprised by the condition $J^y \in \mathcal{C}_2^\mu([0, 1]; \mathcal{B}_p)$ for some coefficient $\mu > 1$. The second condition $J^y \in \mathcal{C}_2^\varepsilon([0, 1]; \mathcal{B}_{\lambda,p})$ may be less expected. In fact, due to the property (23), the fractional spaces $\mathcal{B}_{\lambda,p}$ naturally arise from the controlled expansion of $\int_s^t S_{tu} dx_u^i f_i(y_u)$ (observe for instance (27)).

Proposition 2.8. *Suppose that x is a m -dimensional differentiable path, and let \mathbf{x}^2 be its Lévy area, understood in the Lebesgue sense. We suppose that Assumptions (A1) and $(F)_1$ are both satisfied. Then, for all $\eta \in (0, 1)$ and $\psi \in \mathcal{B}_{\eta,p}$, the (ordinary) solution of Equation (24) is also a solution in $\mathcal{B}_{\eta,p}$ in the sense of Definition 2.6.*

Proof. Let y be the ordinary solution of (24), with initial condition $\psi \in \mathcal{B}_{\eta,p}$. Then $y \in \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\eta,p})$ and since $(\delta y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u)$ and f is bounded, one clearly has $y \in \mathcal{C}_1^1([0, 1]; \mathcal{B}_p)$. Now, notice that owing to the identification (17), we get

$$K_{ts}^y = \int_s^t S_{tu} dx_u^i f_i(y_u) - X_{ts}^{x,i} f_i(y_s) = \int_s^t S_{tu} dx_u^i \delta(f_i(y))_{us},$$

and so, due to (23), one has

$$\|K_{ts}^y\|_{\mathcal{B}_p} \leq \|\dot{x}\|_{\infty,[0,1]} \|f'\|_{\infty} \int_s^t \|(\delta y)_{us}\|_{\mathcal{B}_p} du \quad (26)$$

$$\begin{aligned} &\leq c_{x,f} \int_s^t \left\{ \|(\delta y)_{us}\|_{\mathcal{B}_p} + \|a_{us}\|_{\mathcal{L}(\mathcal{B}_{\eta,p}, \mathcal{B}_p)} \|y_s\|_{\mathcal{B}_{\eta,p}} \right\} du \quad (27) \\ &\leq c_{x,f,y} \int_s^t \{|u-s| + |u-s|^\eta\} du \leq c_{x,f,y} |t-s|^{1+\eta}. \end{aligned}$$

To complete the proof, observe that by resorting to the identification (17) once again, we can write $J_{ts}^y = \int_s^t S_{tu} dx_u^i M_{us}^i$, with

$$\begin{aligned} M_{us}^i &= \delta(f_i(y))_{us} - (\delta x^j)_{us} f_i'(y_s) \cdot f_j(y_s) \\ &= \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot (\delta y)_{us} - (\delta x^j)_{us} f_i'(y_s) \cdot f_j(y_s) \\ &= \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot a_{us} y_s \\ &\quad + \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot (\hat{\delta} y)_{us} - (\delta x^j)_{us} f_i'(y_s) \cdot f_j(y_s), \end{aligned}$$

and thus

$$\begin{aligned} M_{us}^i &= \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot a_{us} y_s + \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot K_{us}^y \\ &\quad + \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot X_{us}^{ax,j} f_j(y_s) + \int_0^1 dr [f_i'(y_s + r(\delta y)_{us}) - f_i'(y_s)] \cdot (\delta x^j)_{us} f_j(y_s), \quad (28) \end{aligned}$$

where we have used the trivial relation $X_{us}^{x,i} = X_{us}^{ax,i} + (\delta x^j)_{us}$. From this expression, it is easy to show that $\|M_{us}^i\|_{\mathcal{B}_p} \leq c_y |u - s|^\eta$, which leads to (25) with $\mu = 1 + \eta$, $\varepsilon = 1$. \square

Proposition 2.9. *Suppose that x is a m -dimensional standard Brownian motion defined on a complete filtered probability space (Ω, \mathcal{F}, P) , and let \mathbf{x}^2 be its Lévy area, understood in the Itô sense. Suppose also that Assumptions (A1) and $(F)_2$ are both satisfied. Then, for all $\eta \in (1/2, 1)$ and $\psi \in \mathcal{B}_{\eta,p}$, the Itô solution of Equation (24) is almost surely a solution in $\mathcal{B}_{\eta,p}$ in the sense of Definition 2.6.*

Proof. For the sake of clarity, we have postponed the proof of this result to Appendix B. \square

Together with the forthcoming uniqueness result contained in Theorem 2.11, the above-stated properties allow to identify, in the two reference situations (i.e., when x is a differentiable path and when x is a standard Brownian motion), the solution in the sense of Definition 2.6 with the classical solution. We will then lean on the continuity Theorem 2.12 to fully justify our interpretation of (24) (see Remark 2.14).

2.5 Main results

With the tools and the definitions we have just introduced, we are in a position to state the three main results of this paper, which successively provide the existence, uniqueness and continuity of the solution to (24).

Theorem 2.10. *Under Assumptions (A1), $(X)_\gamma$ and $(F)_2$, for all $\gamma' \in (1 - \gamma, \gamma + 1/2)$ and $\psi \in \mathcal{B}_{\gamma',p}$, Equation (24) admits a solution y in $\mathcal{B}_{\gamma',p}$ in the sense of Definition 2.6, which satisfies*

$$\mathcal{N}[y; \mathcal{C}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma',p})] \leq C(\|\mathbf{x}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma',p}}),$$

for some function $C : (\mathbb{R}^+)^2 \rightarrow \mathbb{R}$ growing with its arguments.

Theorem 2.11. *If $p > n$ and if Assumptions (A1), (A2), $(X)_\gamma$ and $(F)_3$ are all satisfied, then for all $\gamma' \in (1 - \gamma, \gamma + 1/2)$ and $\psi \in \mathcal{B}_{\gamma',p}$, the solution y in $\mathcal{B}_{\gamma',p}$ given by Theorem 2.10 is unique. Moreover, for any*

$$0 < \beta < \inf(3\gamma - 1, \gamma + \gamma' - 1, \gamma - (\gamma' - 1/2)),$$

there exists a constant $c_{x,\psi,f,\beta}$ such that for all n ,

$$\max_{k=0,\dots,2^n} \|y_{t_k^n} - y_{t_k^n}^n\|_{\mathcal{B}_{\gamma',p}} \leq \frac{c_{x,\psi,f,\beta}}{(2^n)^\beta},$$

where y^n stands for the path given by the discrete scheme (35).

Theorem 2.12. *Under the assumptions of Theorem 2.11, the solution of (24) is continuous with respect to the initial condition and the driving rough path. More precisely, if y (resp. \tilde{y}) is the solution in $\mathcal{B}_{\gamma',p}$ associated to (x, \mathbf{x}^2) (resp. $(\tilde{x}, \tilde{\mathbf{x}}^2)$), with initial condition ψ (resp. $\tilde{\psi}$), then*

$$\begin{aligned} & \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma',p})] \\ & \leq C\left(\|\mathbf{x}\|_\gamma, \|\tilde{\mathbf{x}}\|_\gamma, \|\psi\|_{\mathcal{B}_{\gamma',p}}, \|\tilde{\psi}\|_{\mathcal{B}_{\gamma',p}}\right) \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma',p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}, \end{aligned} \quad (29)$$

for some functions $C : (\mathbb{R}^+)^4 \rightarrow \mathbb{R}^+$ growing with its arguments.

Together with the identification result established in Proposition 2.8, these three theorems offer another perspective on the solution of Equation (24), which may be more in accordance with the formalism used in [12] for rough standard systems:

Corollary 2.13. *Under the assumptions of Theorem 2.11, suppose that $\psi \in \mathcal{B}_{\gamma',p}$ and let $(\tilde{x}^n)_n$ be a sequence of differentiable paths such that $\|x - \tilde{x}^n\|_\gamma + \|\mathbf{x}^2 - \tilde{\mathbf{x}}^{2,n}\|_{2\gamma} \rightarrow 0$ as n tends to infinity, where $\tilde{\mathbf{x}}^{2,n}$ stands for the standard Lévy area constructed from \tilde{x}^n . Let \tilde{y}^n be the (ordinary) solution of (24) associated to each \tilde{x}^n . If y is the solution of (24) given by Theorem 2.11, then*

$$\mathcal{N}[y - \tilde{y}^n; \mathcal{C}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y - \tilde{y}^n; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma',p})] \rightarrow 0 \quad (30)$$

as n tends to infinity.

Remark 2.14. Through the latter result, one can see that the exhibited solution y is a solution in the rough paths sense, that is to say a limit of ordinary solutions with respect to some particular topology (compare with [12, Definition 10.17]). In this context, y can legitimately be called a mild solution of (2), as a limit of classical mild solutions. Furthermore, it is worth noticing that given the regularity assumptions on f_i , if we suppose in addition that the initial condition ψ belongs to the domain $\mathcal{D}(A_p)$, then each (ordinary) mild solution \tilde{y}^n is also a strong solution (see [19, Theorem 6.1.6]). Consequently, if $\psi \in \mathcal{D}(A_p)$, y can also be considered as a strong solution of (2), keeping in mind the topology of the underlying convergence result (30).

2.6 Extension to rougher paths

Before we turn to the proof of Theorems 2.10-2.12, let us say a few words about the possibility of extending these results to a rougher path x , or otherwise stated when the Hölder coefficient γ is smaller than $1/3$.

Remember that for standard finite-dimensional rough systems, the results obtained by Davie in [6] have been generalized to any $\gamma \in (0, 1)$ by Friz and Victoir ([11]): essentially, the system (1) can be interpreted and solved provided that (i) the vector field σ_{ij} is smooth enough and (ii) one is able to construct the iterated integrals of x up to the k -th order, where $\frac{1}{k+1} < \gamma \leq \frac{1}{k}$.

As far as Equation (24) is concerned, let us first consider the next step of the procedure, which corresponds to $\frac{1}{4} < \gamma \leq \frac{1}{3}$. For more simplicity, we assume that f_i is infinitely differentiable with bounded derivatives. Suppose for the moment that x is a differentiable path, and let us introduce, on top of (X^x, X^{ax}, X^{xx}) , the two additional operator-valued paths

$$X_{ts}^{axx} = \int_s^t a_{tu} dx_u^i (\delta x^j)_{us} \quad , \quad X_{ts}^{xxx,ijk} = \int_s^t S_{tu} dx_u^i x_{us}^{2,jk}.$$

Let us also define $F_i^1(\varphi) = f_i(\varphi)$, $F_{ij}^2(\varphi) = f'_i(\varphi) \cdot f_j(\varphi)$, $F_{ijk}^3(\varphi) = f''_i(\varphi) \cdot f_j(\varphi) \cdot f_k(\varphi) + f'_i(\varphi) \cdot f'_j(\varphi) \cdot f_k(\varphi)$, and the three intermediate quantities

$$L_{ts}^y = (\hat{\delta}y)_{ts} - X_{ts}^{x,i} F_i^1(y_s) \quad , \quad K_{ts}^y = (\hat{\delta}y)_{ts} - X_{ts}^{x,i} F_i^1(y_s) - X_{ts}^{xx,ij} F_{ij}^2(y_s),$$

$$J_{ts}^y = (\hat{\delta}y)_{ts} - X_{ts}^{x,i} F_i^1(y_s) - X_{ts}^{xx,ij} F_{ij}^2(y_s) - X_{ts}^{xxx,ijk} F_{ijk}^3(y_s).$$

Once endowed with this notation, a Taylor expansion of the (ordinary) equation (24), similar to (28), shows that for all $s < t \in [0, 1]$, one has

$$(\delta y)_{ts} = X_{ts}^{x,i} F_i^1(y_s) + X_{ts}^{xx,ij} F_{ij}^2(y_s) + X_{ts}^{xxx,ijk} F_{ijk}^3(y_s) + \int_s^t S_{tu} dx_u^i y_{us}^{\#,i}, \quad (31)$$

where the 'residual' path $y^\#$ can be decomposed as $y^{\#,i} = y^{\#,i,1} + y^{\#,i,2}$, with

$$\begin{aligned} y_{us}^{\#,i,1} &= \int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot \left[a_{us} y_s + X_{us}^{ax,j} F_j^1(y_s) + X_{us}^{axx,jk} F_{jk}^2(y_s) \right] \\ &\quad + \int_0^1 dr \int_0^1 dr' r f_i''(y_s + r(\delta y)_{us}) \cdot (\delta x^j)_{us} f_j(y_s) \cdot X_{us}^{ax,k} F_k^1(y_s), \end{aligned} \quad (32)$$

$$\begin{aligned} y_{us}^{\#,i,2} &= \\ &\int_0^1 dr f_i'(y_s + r(\delta y)_{us}) \cdot K_{us}^y \\ &\quad + \int_0^1 dr \left[f_i'(y_s + r(\delta y)_{us}) - f_i'(y_s) \right] \cdot \mathbf{x}_{us}^{2,jk} \cdot f_i'(y_s) \cdot f_j'(y_s) \cdot f_k(y_s) \\ &\quad + \int_0^1 dr \int_0^1 dr' r f_i''(y_s + r(\delta y)_{us}) \cdot L_{us}^y \cdot (\delta x^j)_{us} \cdot f_j(y_s) \\ &\quad + \int_0^1 dr \int_0^1 dr' r \left[f_i''(y_s + r(\delta y)_{us}) - f_i''(y_s) \right] \cdot (\delta x^j)_{us} (\delta x^k)_{us} \cdot f_j(y_s) \cdot f_k(y_s). \end{aligned} \quad (33)$$

By looking closely at these expressions, it is not difficult to realize that the arguments displayed in the forthcoming sections 3-5 can be adapted to the decomposition (31) so as to handle the case where $\gamma \in (\frac{1}{4}, \frac{1}{3}]$ (compare for instance (32)-(33) with (40)-(43)). This supposes that the intermediate paths J^y, K^y, L^y should be controlled with the respective topologies $\mathcal{C}_2^{4\gamma}(\mathcal{B}_p) \cap \mathcal{C}_2^\varepsilon(\mathcal{B}_{\gamma',p}), \mathcal{C}_2^{3\gamma}(\mathcal{B}_p), \mathcal{C}_2^{2\gamma}(\mathcal{B}_p)$, and that the space parameter γ' should be picked in the (non-empty) interval $(1 - \gamma, \gamma + 1/2)$, as in Theorems 2.10-2.12. This also supposes, in order to extend the path X^{xxx} , that x allows the construction of a 3-rough path $\mathbf{x} = (x, \mathbf{x}^2 = \iint dx dx, \mathbf{x}^3 = \iiint dx dx dx) \in \mathcal{C}_1^\gamma \times \mathcal{C}_2^{2\gamma} \times \mathcal{C}_2^{3\gamma}$. We know that this assumption covers in particular the case of a fractional Brownian motion with Hurst index $H > 1/4$, see [4].

The situation gets more complicated as soon as $\gamma < 1/4$, since we can no longer pick γ' in the (now empty) interval $(1 - \gamma, \gamma + 1/2)$, and this assumption played a fundamental role in our estimates. Indeed, on the one hand, the condition $\gamma' > 1 - \gamma$ ensures that the order of the terms derived from (32) or (41) is greater than $\gamma + \gamma' > 1$, or otherwise stated that these paths can be considered as residual terms. On the other hand, the condition $\gamma' < \gamma + 1/2$ is used in some estimates like (47) to go from $\mathcal{B}_{\gamma',p}$ to $\mathcal{B}_{1/2,p}$ and thus take profit of the linear control (3) (instead of the quadratic control (4)). Therefore, when $\gamma < 1/4$, some sharpness is to be lost in our estimates and the method under consideration in this paper would only provide us with a local solution, on a time interval linked to x, f and ψ . To overcome this difficulty, it may be useful to modify the path

$(X^x, X^{ax}, X^{xx}, X^{axx}, X^{xxx}, \dots)$ into a more appropriate trajectory, which would for instance includes mixed operators such as

$$X_{ts}^{xa,i}(\varphi_1, \varphi_2) = \int_s^t S_{tu} dx_u^i [a_{us} \varphi_1 \cdot \varphi_2] \quad , \quad \varphi_1, \varphi_2 \in \mathcal{B}. \quad (34)$$

Observe however that the extension of (34) to a generic γ -Hölder path x (with $\gamma < 1$) can no longer be done via an integration by parts argument (as in Lemma 2.3), which considerably increases the difficulty of the study.

3 Existence of a solution

This section is devoted to the proof of Theorem 2.10. Thus, we henceforth suppose that the assumptions of the theorem, namely (A1), $(X)_\gamma$ and $(F)_2$, are all satisfied. Besides, we fix a parameter $\gamma' \in (1 - \gamma, \gamma + 1/2)$ and an initial condition $\psi \in \mathcal{B}_{\gamma', p}$.

3.1 Additional notation

We consider the sequence $(\Pi^n)_n$ of dyadic partitions of $[0, 1]$ (i.e., $t_k^n = \frac{k}{2^n}$) and we define the discrete path y^n following the iteration formula:

$$y_0^n := \psi \quad , \quad y_{t_{k+1}^n}^n := S_{t_{k+1}^n t_k^n} y_{t_k^n}^n + X_{t_{k+1}^n t_k^n}^{x,i} f_i(y_{t_k^n}^n) + X_{t_{k+1}^n t_k^n}^{xx,ij} F_{ij}(y_{t_k^n}^n) \quad , \quad t_k^n \in \Pi^n. \quad (35)$$

The path y^n is then extended on $[0, 1]$ by linear interpolation. For the sake of clarity, we will denote in this section $J^n := J^{y^n}$ and $K^n := K^{y^n}$. Observe that owing to the very definition of y^n , one has $J_{t_{k+1}^n t_k^n}^n = 0$.

In the rest of the paper, we will also appeal to the discrete versions of the generalized Hölder norms introduced in Subsection 2.2. Thus, for any $n \in \mathbb{N}$, we denote $\llbracket a, b \rrbracket_n := [a, b] \cap \Pi^n$ and

$$\mathcal{N}[h; \mathcal{C}_1^\lambda(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_{\alpha,p})] := \sup_{\substack{t_p^n \leq s < t \leq t_q^n \\ s, t \in \Pi^n}} \frac{\|(\hat{\delta}h)_{ts}\|_{\mathcal{B}_{\alpha,p}}}{|t-s|^\lambda},$$

We define the two quantities $\mathcal{N}[:, \mathcal{C}_2^\lambda(\llbracket a, b \rrbracket_n; \mathcal{B}_{\alpha,p})]$ and $\mathcal{N}[:, \mathcal{C}_3^\lambda(\llbracket a, b \rrbracket_n; \mathcal{B}_{\alpha,p})]$ along the same lines.

Now, for any (not necessarily uniform) partition $\tilde{\Pi}$ of $[0, 1]$ made of points of Π^n , we define the path $J^{n, \tilde{\Pi}}$ for all $s < t \in \Pi^n$ as

$$J_{ts}^{n, \tilde{\Pi}} := \begin{cases} 0 & \text{if } (s, t) \cap \tilde{\Pi} = \emptyset \\ (\hat{\delta}J^n)_{tus} & \text{if } (s, t) \cap \tilde{\Pi} = u \\ J_{ts}^n - J_{t\tilde{t}_N}^n - \sum_{k=1}^{N-1} S_{t\tilde{t}_{k+1}} J_{\tilde{t}_{k+1}\tilde{t}_k}^n - S_{t\tilde{t}_1} J_{\tilde{t}_1 s}^n & \text{if } (s, t) \cap \tilde{\Pi} = \{\tilde{t}_1, \dots, \tilde{t}_N\} \end{cases}$$

Remark 3.1. Since $J_{t_{k+1}^n t_k^n}^n = 0$, one has in particular $J_{t_s^n}^{n, \Pi^n} = J_{t_s^n}^n$. Besides, if $\tilde{\Pi}, \hat{\Pi}$ are two partitions of $[0, 1]$ made of points of Π^n and such that $\tilde{\Pi} \cap (s, t) = \{\tilde{t}_1, \dots, \tilde{t}_N\}$ ($N \geq 3$) and $\hat{\Pi} \cap (s, t) = \{\hat{t}_1, \dots, \hat{t}_{k-1}, \hat{t}_{k+1}, \dots, \hat{t}_N\}$ for $1 \leq k \leq N - 1$, then $J_{t_s^n}^{n, \tilde{\Pi}} - J_{t_s^n}^{n, \hat{\Pi}} = S_{t_{\tilde{t}_{k+1}}^n}(\hat{\delta}J^n)_{\tilde{t}_{k+1}^n \tilde{t}_k \tilde{t}_{k-1}}$. With the same notation, if $\hat{\Pi} \cap (s, t) = \{\hat{t}_1, \dots, \hat{t}_{N-1}\}$, then $J_{t_s^n}^{n, \tilde{\Pi}} - J_{t_s^n}^{n, \hat{\Pi}} = (\hat{\delta}J^n)_{t_{\hat{t}_N}^n \hat{t}_{N-1}}$.

3.2 Preliminary results on J^n

We fix $t_p^n < t_q^n \in \Pi^n$ and we apply the algorithm described in Appendix A to the uniform partition $\{t_p^n, t_{p+1}^n, \dots, t_q^n\}$. Set $N := q - p$, and so, for any $k \in \{p, \dots, q\}$, $t_k^n = t_p^n + \frac{(k-p)(t_q^n - t_p^n)}{N}$. We also denote by $(\Pi^{n,m})_{m \in \{1, \dots, N-1\}}$ the (decreasing) sequence of partitions of $[t_p^n, t_q^n]$ deduced from the algorithm, and $\Pi^{n,0} := \{t_p^n, t_{p+1}^n, \dots, t_q^n\}$. Finally, set $J_{t_q^n t_p^n}^{n,m} := J_{t_q^n t_p^n}^{n, \Pi^{n,m}}$. With this notation in hand, one has

$$J_{t_q^n t_p^n}^n = \sum_{m=0}^{N-1} \left\{ J_{t_q^n t_p^n}^{n,m} - J_{t_q^n t_p^n}^{n,m+1} \right\}.$$

Once endowed with this decomposition, we can show the following result, which turns out to be the starting point of our reasoning:

Lemma 3.2. *For all $\mu > 1$ and $\kappa > 0$, there exists a constant $c = c_{\mu, \kappa}$ such that*

$$\|J_{t_q^n t_p^n}^n\|_{\mathcal{B}_{\gamma', p}} \leq c \left\{ \left| t_q^n - t_p^n \right|^\kappa + \left| t_q^n - t_p^n \right|^{\mu - \gamma'} \right\} \left\{ \mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^\kappa(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^\mu(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_p)] \right\}, \quad (36)$$

and

$$\|J_{t_q^n t_p^n}^n\|_{\mathcal{B}_p} \leq c \left| t_q^n - t_p^n \right|^\mu \mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^\mu(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_p)]. \quad (37)$$

Proof. We use the notation of Appendix A. By referring to Remark 3.1, one easily deduces

$$\begin{aligned} & \sum_{m=0}^{N-1} \left\{ J_{t_q^n t_p^n}^{n,m} - J_{t_q^n t_p^n}^{n,m+1} \right\} \\ &= \sum_{r=1}^{M-1} \left\{ (\hat{\delta}J^n)_{t_q^n t_{p+k_{A_{r-1}+1}}^n t_{p+k_{A_{r-1}+1}}^-} + \sum_{m=A_{r-1}+2}^{A_r} S_{t_q^n t_{p+k_m^+}}^{t_{p+k_m^+} t_{p+k_m^-}} (\hat{\delta}J^n)_{t_{p+k_m^+} t_{p+k_m^-} t_{p+k_m^-}} \right\} \\ & \quad + (\hat{\delta}J^n)_{t_q^n t_{p+k_{A_{M-1}+1}}^n t_{p+k_{A_{M-1}+1}}^-} + \mathbf{1}_{\{A_{M-1}+1 < N-1\}} (\hat{\delta}J^n)_{t_q^n t_{p+k_{N-1}}^n t_p^n}. \end{aligned}$$

Then, if $C_n := \mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^\kappa(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^\mu(\llbracket t_p^n, t_q^n \rrbracket_n; \mathcal{B}_p)]$, one has

$$\begin{aligned}
& \sum_{m=0}^{N-1} \|J_{t_q^n t_p^n}^{n,m} - J_{t_q^n t_p^n}^{n,m+1}\|_{\mathcal{B}_{\gamma', p}} \\
& \leq 2C_n \left| t_q^n - t_p^n \right|^\kappa \\
& \quad + C_n \sum_{r=0}^{M-1} \left\{ \left| t_q^n - t_{p+k_{A_{r-1}+1}^-}^n \right|^\kappa + \sum_{m=A_{r-1}+2}^{A_r} \left| t_q^n - t_{p+k_m^+}^n \right|^{-\gamma'} \left| t_{p+k_m^+}^n - t_{p+k_m^-}^n \right|^\mu \right\} \\
& \leq C_n \left\{ \left| t_q^n - t_p^n \right|^\kappa + \left| t_q^n - t_p^n \right|^{\mu-\gamma'} \right\} \\
& \quad \left\{ 2 + \sum_{r=0}^{M-1} \left\{ \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\kappa + \frac{1}{N^\mu} \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} \left| k_m^+ - k_m^- \right|^\mu \right\} \right\} \\
& \leq c_{\kappa, \mu, \gamma'} \left\{ \left| t_q^n - t_p^n \right|^\kappa + \left| t_q^n - t_p^n \right|^{\mu-\gamma'} \right\} C_n,
\end{aligned}$$

thanks to Proposition 6.2. The second control (37) can be obtained with the same arguments, upon noticing that (65) entails in particular

$$\sum_{r=1}^{M-1} \left\{ \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\mu + \frac{1}{N^\mu} \sum_{m=A_{r-1}+2}^{A_r} \left| k_m^+ - k_m^- \right|^\mu \right\} \leq c_\mu < \infty.$$

□

Lemma 3.3. For every path $y : [0, 1] \rightarrow \mathcal{B}_p$ and all $s < u < t \in [0, 1]$,

$$(\hat{\delta}J^y)_{tus} = X_{tu}^{x,i} \delta(f_i(y))_{us} - X_{tu}^{x,i} (\delta x^j)_{us} F_{ij}(y_s) + X_{tu}^{xx,ij} \delta(F_{ij}(y))_{us} \quad (38)$$

and also

$$(\hat{\delta}J^y)_{tus} = I_{tus} + II_{tus} + III_{tus} + IV_{tus}, \quad (39)$$

with

$$I_{tus} := X_{tu}^{x,i} \left(\int_s^1 dr f'_i(y_s + r(\delta y)_{us}) \cdot K_{us}^y \right), \quad (40)$$

$$II_{tus} := X_{tu}^{x,i} \left(\int_s^1 dr f'_i(y_s + r(\delta y)_{us}) \cdot \{a_{us} y_s + X_{us}^{ax,j} f_j(y_s)\} \right), \quad (41)$$

$$III_{tus} := X_{tu}^{x,i} \left(\int_0^1 dr [f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s)] \cdot (\delta x^j)_{us} f_j(y_s) \right), \quad (42)$$

$$IV_{tus} := X_{tus}^{xx,ij} \delta(F_{ij}(y))_{us}. \quad (43)$$

Proof. Those are only straightforward expansions. For (38), we use the fact that if $m_{ts} := g_{ts}h_s$, then $(\hat{\delta}m)_{tus} = (\hat{\delta}g)_{tus}h_s - g_{tu}(\delta h)_{us}$, together with the algebraic relations

$$(\hat{\delta}X^{x,i})_{tus} = 0 \quad , \quad (\hat{\delta}X^{xx,ij})_{tus} = X_{tu}^{x,i}(\delta x^j)_{us} \quad \text{for all } s \leq u \leq t,$$

that can be readily checked from the expressions (12) and (14). The expansion of $\delta(f_i(y))_{us} - (\delta x^j)_{us}F_{ij}(y_s)$ which then leads to (39) has already been elaborated on in the proof of Proposition 2.8, see (28). \square

3.3 Existence of a solution

Thanks to the above preliminary results, we are first able to control J^n on successive time intervals independent of n :

Proposition 3.4. *Suppose that μ, ε satisfy*

$$3\gamma > \mu > 1 \quad , \quad \gamma + \gamma' > \mu > 1 \quad , \quad \gamma - (\gamma' - \frac{1}{2}) > \varepsilon > 0. \quad (44)$$

Then there exists a time $T_0 = T_0(x, f, \gamma, \gamma', \mu, \varepsilon) > 0$, $T_0 \in \Pi^n$, such that for any k ,

$$\mathcal{N}[J^n; \mathcal{C}_2^\varepsilon(\llbracket kT_0, (k+1)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_{\gamma', p})] \leq 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma', p}} \quad (45)$$

and

$$\mathcal{N}[J^n; \mathcal{C}_2^\mu(\llbracket kT_0, (k+1)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_p)] \leq 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma', p}}. \quad (46)$$

Proof. This is an iteration procedure over the points of the partition, for which we first focus on the case $k = 0$ in (45) and (46). Assume that both estimates hold true on $\llbracket 0, t_q^n \rrbracket_n$. Then, for any $t \in \llbracket 0, t_q^n \rrbracket_n$, one has, thanks to (18), (20) and (3),

$$\|y_t^n\|_{\mathcal{B}_{\gamma', p}} \leq \|J_{t0}^n\|_{\mathcal{B}_{\gamma', p}} + \|S_{t0}\psi\|_{\mathcal{B}_{\gamma', p}} + c_x^0 t^{\gamma - (\gamma' - \frac{1}{2})} \left\{ \|f_i(\psi)\|_{\mathcal{B}_{1/2, p}} + \|F_{ij}(\psi)\|_{\mathcal{B}_{1/2, p}} \right\} \quad (47)$$

$$\begin{aligned} &\leq \|J_{t0}^n\|_{\mathcal{B}_{\gamma', p}} + \|S_{t0}\psi\|_{\mathcal{B}_{\gamma', p}} + c_{x,f}^1 t^{\gamma - (\gamma' - \frac{1}{2})} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma', p}} \right\} \\ &\leq c_{x,f}^2 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma', p}} \right\}, \end{aligned} \quad (48)$$

and so $\mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, t_q^n \rrbracket_n, \mathcal{B}_{\gamma', p})] \leq c_{x,f}^2 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma', p}} \right\}$. Besides, if $s < t \in \llbracket 0, t_q^n \rrbracket_n$,

$$\begin{aligned} \|(\hat{\delta}y^n)_{ts}\|_{\mathcal{B}_p} &\leq \|J_{ts}^n\|_{\mathcal{B}_p} + \|X_{ts}^{x,i}f_i(y_s^n)\|_{\mathcal{B}_p} + \|X_{ts}^{xx,ij}F_{ij}(y_s^n)\|_{\mathcal{B}_p} \\ &\leq |t-s|^\gamma c_{x,f}^3 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma', p}} \right\}, \end{aligned}$$

and hence

$$\mathcal{N}[y^n; \hat{\mathcal{C}}_1^\gamma(\llbracket 0, t_q^n \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^3 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma', p}} \right\}. \quad (49)$$

One can also rely on the estimate

$$\|K_{ts}^n\|_{\mathcal{B}_p} \leq \|J_{ts}^n\|_{\mathcal{B}_p} + \|X_{ts}^{xx,ij}F_{ij}(y_s^n)\|_{\mathcal{B}_p} \leq c_{x,f}^4 |t-s|^{2\gamma} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma', p}} \right\}. \quad (50)$$

Now, from the decomposition (39), we easily deduce, for all $0 \leq s < u < t \in \llbracket 0, t_{q+1}^n \rrbracket_n$,

$$\|(\hat{\delta}J^n)_{tus}\|_{\mathcal{B}_p} \leq c_{x,f}^5 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left\{ |t-s|^{3\gamma} + |t-s|^{\gamma+\gamma'} \right\}.$$

Indeed, one has for instance

$$\begin{aligned} & \left\| \left[f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s) \right] \cdot (\delta x^j)_{us} f_j(y_s) \right\|_{\mathcal{B}_p} \\ & \leq c_{x,f} |u-s|^\gamma \|(\delta y)_{us}\|_{\mathcal{B}_p} \\ & \leq c_{x,f} |u-s|^\gamma \left\{ \|(\delta y)_{us}\|_{\mathcal{B}_p} + \|a_{us} y_s\|_{\mathcal{B}_p} \right\} \\ & \leq c_{x,f} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left\{ |u-s|^{2\gamma} + |u-s|^{\gamma+\gamma'} \right\} \leq c_{x,f} |u-s|^{2\gamma} \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}, \end{aligned}$$

where the constant $c_{x,f}$ may of course vary from line to line. Consequently,

$$\mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^\mu(\llbracket 0, t_{q+1}^n \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^5 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left\{ T_0^{3\gamma-\mu} + T_0^{\gamma+\gamma'-\mu} \right\}.$$

On the other hand, it is readily checked from (38) that

$$\|(\hat{\delta}J^n)_{tus}\|_{\mathcal{B}_{\gamma',p}} \leq c_{x,f}^6 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} |t-s|^{\gamma-(\gamma'-\frac{1}{2})},$$

and therefore

$$\mathcal{N}[\hat{\delta}J^n; \mathcal{C}_3^{\gamma-(\gamma'-\frac{1}{2})}(\llbracket 0, t_{q+1}^n \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^6 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}.$$

By using the estimate (36), we get

$$\mathcal{N}[J^n; \mathcal{C}_2^\varepsilon(\llbracket 0, t_{q+1}^n \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^7 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} \left(T_0^{3\gamma-\mu} + T_0^{\gamma+\gamma'-\mu} + T_0^{\gamma-(\gamma'-\frac{1}{2})-\varepsilon} \right).$$

It only remains to pick T_0 such that

$$c_{x,f}^7 \left(T_0^{3\gamma-\mu} + T_0^{\gamma+\gamma'-\mu} + T_0^{\gamma-(\gamma'-\frac{1}{2})-\varepsilon} \right) \leq 1.$$

We can follow the same lines to show (46) from the estimate (37).

It is now easy to realize that the same reasoning (with the same constants) can be applied on the interval $[T_0, 2T_0]$ by replacing ψ with $y_{T_0}^n$, and then on the interval $[2T_0, 3T_0]$, etc. \square

Corollary 3.5. *With the notation of Proposition 3.4, there exists a constant $c_{x,f}$ such that for any k ,*

$$\mathcal{N}[J^n; \mathcal{C}_2^\mu(\llbracket kT_0, (k+2)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_p)] \leq \left\{ 1 + \|y_{(k+1)T_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\} + c_{x,f} \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}, \quad (51)$$

$$\mathcal{N}[J^n; \mathcal{C}_2^\varepsilon(\llbracket kT_0, (k+2)T_0 \wedge 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq \left\{ 1 + \|y_{(k+1)T_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\} + c_{x,f} \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (52)$$

Proof. If $kT_0 \leq s < (k+1)T_0 \leq t < (k+2)T_0$,

$$J_{ts}^n = J_{t,(k+1)T_0}^n - S_{t,(k+1)T_0} J_{(k+1)T_0,s}^n - (\hat{\delta}J^n)_{t,(k+1)T_0,s}.$$

We already know that

$$\|J_{t,(k+1)T_0}^n\|_{\mathcal{B}_p} + \|J_{(k+1)T_0,s}^n\|_{\mathcal{B}_p} \leq |t-s|^\mu \left\{ 2 + \|y_{(k+1)T_0}^n\|_{\mathcal{B}_{\gamma',p}} + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}.$$

By using the decomposition (39), together with the estimates (48), (49) and (50), we get $\|(\hat{\delta}J^n)_{t,(k+1)T_0,s}\|_{\mathcal{B}_p} \leq c_x |t-s|^\mu \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\}$, which yields (51). (52) can be shown with the same arguments. \square

Proof of Theorem 2.10. With the same estimates as in (48), we first deduce from Proposition 3.4

$$\mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket kT_0, (k+1)T_0 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^1 \left\{ 1 + \|y_{kT_0}^n\|_{\mathcal{B}_{\gamma',p}} \right\},$$

where the constant $c_{x,f}^1$ does not depend on k . As T_0 is independent of y^n , this leads to

$$\mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^2 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (53)$$

From this uniform control, we get, by repeating the argument of Corollary 3.5,

$$\mathcal{N}[J^n; \mathcal{C}_2^\mu(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^4 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}, \quad (54)$$

and then

$$\mathcal{N}[y^n; \mathcal{C}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \leq c_{x,f}^5 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \quad (55)$$

Now remember that y^n is extended on $[0, 1]$ by linear interpolation, and so

$$\begin{aligned} \mathcal{N}[y^n; \mathcal{C}_1^\gamma([0, 1]; \mathcal{B}_p)] &\leq 3\mathcal{N}[y^n; \mathcal{C}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \\ &\leq 3\mathcal{N}[y^n; \mathcal{C}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + c_{\gamma'} \mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \\ &\leq c_{x,f}^6 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}. \end{aligned}$$

Thus, we are in a position to apply the Ascoli Theorem and to assert the existence of a subsequence y^{n_k} of y^n that converges to an element y in $\mathcal{C}_1^0([0, 1]; \mathcal{B}_p)$. It remains to check that y is a solution of (24). To do so, let $s < t \in [0, 1]$ and consider two sequences $s_{n_k} < t_{n_k} \in \Pi^{n_k}$ such that s_{n_k} decreases to s and t_{n_k} increases to t . Then

$$\|J_{ts}^y\|_{\mathcal{B}_p} \leq \|J_{ts}^y - J_{ts}^{y^{n_k}}\|_{\mathcal{B}_p} + \|J_{ts}^{y^{n_k}} - J_{t_{n_k}s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} + \|J_{t_{n_k}s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p}. \quad (56)$$

On the one hand,

$$\|J_{ts}^y - J_{ts}^{y^{n_k}}\|_{\mathcal{B}_p} \leq c_{x,f} \mathcal{N}[y - y^{n_k}; \mathcal{C}_1^0([0, 1]; \mathcal{B}_p)] \rightarrow 0,$$

while on the other hand

$$\begin{aligned} \|J_{ts}^{y^{n_k}} - J_{t_{n_k}s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} &\leq c_f \left\{ \|X_{ts}^{x,i} - X_{t_{n_k}s_{n_k}}^{x,i}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} + \|X_{ts}^{xx,ij} - X_{t_{n_k}s_{n_k}}^{xx,ij}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \right\} \\ &\quad + c_{x,f} \left\{ \|y_t^{n_k} - y_{t_{n_k}}^{n_k}\|_{\mathcal{B}_p} + \|y_{s_{n_k}}^{n_k} - y_s^{n_k}\|_{\mathcal{B}_p} \right\}, \end{aligned}$$

from which we easily deduce, with the uniform controls (53) and (55) in mind,

$$\|J_{ts}^{y^{n_k}} - J_{t_{n_k}s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} \rightarrow 0.$$

Finally, owing to (54),

$$\|J_{t_{n_k} s_{n_k}}^{y^{n_k}}\|_{\mathcal{B}_p} \leq c_{x,f}^7 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\} |t - s|^\mu.$$

Going back to (56), this proves that $J^y \in \mathcal{C}_3^\mu([0, 1]; \mathcal{B}_p)$. Then we follow the same lines starting with the estimate $\mathcal{N}[J^n; \mathcal{C}_3^\varepsilon(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{x,f}^4 \left\{ 1 + \|\psi\|_{\mathcal{B}_{\gamma',p}} \right\}$ to get $J^y \in \mathcal{C}_3^\varepsilon([0, 1]; \mathcal{B}_{\gamma',p})$, and so y is indeed a solution of (24) in $\mathcal{B}_{\gamma',p}$. \square

4 Uniqueness of the solution

In this section, we mean to prove Theorem 2.11. Accordingly, we assume that $p > n$ and that Conditions (A1), (A2), $(X)_\gamma$ and $(F)_3$ are all checked. Let y be a solution of (24) in $\mathcal{B}_{\gamma',p}$, for some (fixed) parameter $\gamma' \in (1 - \gamma, 1/2 + \gamma)$, with initial condition $\psi \in \mathcal{B}_{\gamma',p}$, and let y^n be the path described by the scheme (35), with the same initial condition ψ .

We introduce, for all $s < t \in \Pi^n$, the quantity

$$\mathcal{N}[y - y^n; \mathcal{Q}(\llbracket s, t \rrbracket_n)] :=$$

$$\mathcal{N}[y - y^n; \mathcal{C}_1^{\gamma'}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y - y^n; \mathcal{C}_1^0(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[K^y - K^{y^n}; \mathcal{C}_2^{2\gamma}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)].$$

Besides, we fix $\mu > 1$, $\varepsilon > 0$ such that $\|J_{ts}^y\|_{\mathcal{B}_p} \leq c |t - s|^\mu$ and $\|J_{ts}^y\|_{\mathcal{B}_{\gamma',p}} \leq c |t - s|^\varepsilon$.

The proof of Theorem 2.11 is based on the two following preliminary results, which aim at controlling, as in the previous section, the residual term J :

Lemma 4.1. *For all $\tilde{\mu} > 1$ and $\kappa > 0$, there exists two constants $c_y, c_{\tilde{\mu}}$ such that if $s < t \in \Pi^n$,*

$$\begin{aligned} \|J_{ts}^y - J_{ts}^{y^n}\|_{\mathcal{B}_{\gamma',p}} &\leq c_y \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\} + c_{\tilde{\mu}} \left\{ |t - s|^\kappa + |t - s|^{\tilde{\mu}-\gamma'} \right\} \\ &\quad \left\{ \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^\kappa(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \right\}. \\ \|J_{ts}^y - J_{ts}^{y^n}\|_{\mathcal{B}_p} &\leq \frac{c_y |t - s|}{(2^n)^{\mu-1}} + c_{\tilde{\mu}} |t - s|^{\tilde{\mu}} \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)]. \end{aligned}$$

Proof. Going back to the notation of Subsection 3.1, we decompose $J_{ts}^y - J_{ts}^{y^n}$ as

$$J_{ts}^y - J_{ts}^{y^n} = \left[J_{ts}^{y, \Pi^n} - J_{ts}^{y^n, \Pi^n} \right] + R_{ts}^{y, \Pi^n},$$

with, if $s = t_k^n$ and $t = t_l^n$,

$$R_{ts}^{y, \Pi^n} := J_{t_{l-1}^n}^y + \sum_{i=k}^{l-2} S_{t_{i+1}^n} J_{t_{i+1}^n t_i^n}^y.$$

To handle the term into brackets, we use the same arguments as in the proof of Lemma 3.2, which yield here

$$\begin{aligned} \|J_{ts}^{y, \Pi^n} - J_{ts}^{y^n, \Pi^n}\|_{\mathcal{B}_{\gamma',p}} &\leq c_{\tilde{\mu}, \gamma'} \left\{ |t - s|^\kappa + |t - s|^{\tilde{\mu}-\gamma'} \right\} \\ &\quad \left\{ \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^\kappa(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma',p})] + \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \right\} \end{aligned}$$

and

$$\|J_{ts}^{y, \Pi^n} - J_{ts}^{y^n, \Pi^n}\|_{\mathcal{B}_p} \leq c_{\mu, \gamma'} |t - s|^{\tilde{\mu}} \mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)].$$

Then it suffices to observe that

$$\|R_{ts}^{y, \Pi^n}\|_{\mathcal{B}_p} \leq \frac{c_y}{(2^n)^{\mu-1}} \left\{ |t - t_{l-1}^n| + \sum_{i=k}^{l-2} |t_{i+1}^n - t_i^n| \right\} \leq \frac{c_y |t - s|}{(2^n)^{\mu-1}}$$

and

$$\|R_{ts}^{y, \Pi^n}\|_{\mathcal{B}_{\gamma', p}} \leq \frac{c_y}{(2^n)^\varepsilon} + \sum_{i=k}^{l-2} |t - t_{i+1}^n|^{-\gamma'} \frac{c_y}{(2^n)^\mu} \leq c_{y, \gamma'} \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}. \quad (57)$$

□

Lemma 4.2. Set $\tilde{\mu} := \inf(\gamma + \gamma', 3\gamma)$. Then for all $s < t \in \Pi^n$,

$$\mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \leq c_{y, x, f, \psi} \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket s, t \rrbracket_n)], \quad (58)$$

$$\mathcal{N}[\hat{\delta}(J^y - J^{y^n}); \mathcal{C}_3^\gamma(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma', p})] \leq c_{y, x, f, \psi} \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket s, t \rrbracket_n)]. \quad (59)$$

Proof. (58) is a consequence of the decomposition (39). Indeed, one has for instance, if $\mathcal{N}_y := \mathcal{N}[y; \mathcal{C}_1^\gamma([0, 1]; \mathcal{B}_p)] + \mathcal{N}[y; \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\gamma', p})]$,

$$\begin{aligned} & \|f'_i(y_s + r(\delta y)_{us}) - f'_i(y_s) - f'_i(y_s^n + r(\delta y^n)_{us}) + f'_i(y_s^n)\|_{\mathcal{B}_p} \\ & \leq \|r \int_0^1 dr' [f''_i(y_s + rr'(\delta y)_{us}) - f''_i(y_s^n + rr'(\delta y^n)_{us})] (\delta y)_{us}\|_{\mathcal{B}_p} \\ & \quad + \|r \int_0^1 dr' f''_i(y_s^n + rr'(\delta y^n)_{us}) \delta(y - y^n)_{us}\|_{\mathcal{B}_p} \\ & \leq c_f \mathcal{N}_y |u - s|^\gamma \int_0^1 dr' \|(y_s + rr'(\delta y)_{us}) - (y_s^n + rr'(\delta y^n)_{us})\|_{\mathcal{B}_\infty} \\ & \quad + c_f |u - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)] \\ & \leq c_{f, \mathcal{N}_y} |u - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)], \end{aligned}$$

where we have used the continuous inclusion $\mathcal{B}_{\gamma', p} \subset \mathcal{B}_\infty$. As for (59), it suffices to observe, with the expression (38) in mind, that for instance, due to the assumption (A2) and the control (4), one has

$$\begin{aligned} & \|X_{tu}^{x, i}(f_i(y_u) - f_i(y_u^n))\|_{\mathcal{B}_{\gamma', p}} \\ & \leq c_x |t - s|^\gamma \|f_i(y_u) - f_i(y_u^n)\|_{\mathcal{B}_{\gamma', p}} \\ & \leq c_x |t - s|^\gamma \left\| \int_0^1 dr f'_i(y_u + r(y_u^n - y_u))(y_u^n - y_u) \right\|_{\mathcal{B}_{\gamma', p}} \\ & \leq c_x |t - s|^\gamma \|y_u^n - y_u\|_{\mathcal{B}_{\gamma', p}} \left\| \int_0^1 dr f'_i(y_u + r(y_u^n - y_u)) \right\|_{\mathcal{B}_{\gamma', p}} \\ & \leq c_{x, f, \mathcal{N}_y, \mathcal{N}_{y^n}} |t - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)] \\ & \leq c_{x, f, \mathcal{N}_y, \psi} |t - s|^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(I)], \end{aligned} \quad (60)$$

where, to get the last estimate, we have appealed to the uniform control $\mathcal{N}_{y^n} \leq c_{x,f,\psi}$ established in the proof of Theorem 2.10. \square

Proof of Theorem 2.11. Let $T_1 \leq 1 \in \Pi^n$. Write

$$\hat{\delta}(y - y^n)_{ts} = X_{ts}^{x,i} [f_i(y_s) - f_i(y_s^n)] + X_{ts}^{xx,ij} [F_{ij}(y_s) - F_{ij}(y_s^n)] + [J_{ts}^y - J_{ts}^{y^n}],$$

and use the two previous lemmas to deduce first

$$\mathcal{N}[y - y^n; \mathcal{C}_1^\gamma(\llbracket 0, T_1 \rrbracket_n; \mathcal{B}_p)] \leq c_{y,x,f,\psi} T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] + \frac{c_y}{(2^n)^{\mu-1}}$$

and secondly

$$\begin{aligned} \mathcal{N}[y - y^n; \mathcal{C}_1^0(\llbracket 0, T_1 \rrbracket_n; \mathcal{B}_{\gamma',p})] &\leq c_{y,x,f,\psi} T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] \\ &\quad + c_y \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}. \end{aligned}$$

Then, since $K_{ts}^y - K_{ts}^{y^n} = X_{ts}^{xx,ij} [F_{ij}(y_s) - F_{ij}(y_s^n)] + [J_{ts}^y - J_{ts}^{y^n}]$, one has

$$\mathcal{N}[K^y - K^{y^n}; \mathcal{C}_2^{2\gamma}(\llbracket 0, T_1 \rrbracket_n; \mathcal{B}_p)] \leq c_{y,x,f,\psi} T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] + \frac{c_y}{(2^n)^{\mu-1}}$$

and we have thus proved that

$$\mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] \leq c_{y,x,f,\psi}^1 T_1^\gamma \mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] + c_y^1 \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}.$$

Choose T_1 such that $c_{y,x,f,\psi}^1 T_1^\gamma = \frac{1}{2}$ to obtain

$$\mathcal{N}[y - y^n; \mathcal{Q}(\llbracket 0, T_1 \rrbracket_n)] \leq 2c_y^1 \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}.$$

By using the same arguments on $\llbracket kT_1, (k+1)T_1 \rrbracket_n$, we get

$$\mathcal{N}[y - y^n; \mathcal{Q}(\llbracket kT_1, (k+1)T_1 \rrbracket_n)] \leq 2c_y^1 \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\} + c_{x,f} \|y_{kT_1} - y_{kT_1}^n\|_{\mathcal{B}_{\gamma',p}},$$

and it is now easy to establish that

$$\mathcal{N}[y - y^n; \mathcal{C}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y - y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma',p})] \leq c_{y,x,f,\psi} \left\{ \frac{1}{(2^n)^\varepsilon} + \frac{1}{(2^n)^{\mu-1}} \right\}. \quad (61)$$

This inequality clearly proves the uniqueness of the solution and therefore, it enables us to identify y with the solution constructed in Section 3. This identification allows in turn to choose μ and ε as in Proposition 3.4 and to assert that $\mathcal{N}_y \leq c_{x,f,\psi}$, which completes the proof. \square

5 Continuity of the solution

It remains to prove Theorem 2.12. In accordance with the statement of this result, we suppose that $p > n$ and that Assumptions (A1), (A2), $(X)_\gamma$ and $(F)_3$ are all satisfied. We fix $\gamma' \in (1-\gamma, \gamma+1/2)$ and the two initial conditions $\psi, \tilde{\psi} \in \mathcal{B}_{\gamma', p}$. We denote by $X = (X^x, X^{ax}, X^{xx})$ (resp. $\tilde{X} = (\tilde{X}^x, \tilde{X}^{ax}, \tilde{X}^{xx})$) the path constructed from (x, \mathbf{x}^2) (resp. $(\tilde{x}, \tilde{\mathbf{x}}^2)$) through Definition 2.2. With this notation, we define y^n as the path described by the scheme (35) and \tilde{y}^n as the path obtained by replacing (ψ, X^x, X^{xx}) with $(\tilde{\psi}, \tilde{X}^x, \tilde{X}^{xx})$ in the latter scheme.

Besides, we define \tilde{J} and \tilde{K} by replacing (X^x, X^{xx}) with $(\tilde{X}^x, \tilde{X}^{xx})$ in Formulas (15) and (16). For the sake of clarity, we also set $J^n := J^{y^n}$, $K^n := K^{y^n}$, $\tilde{J}^n := \tilde{J}^{\tilde{y}^n}$, $\tilde{K}^n = \tilde{K}^{\tilde{y}^n}$, and as in the previous section, we introduce the intermediate quantity

$$\begin{aligned} & \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)] \\ := & \mathcal{N}[y^n - \tilde{y}^n; \mathcal{C}_1^\gamma(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y^n - \tilde{y}^n; \mathcal{C}_1^0(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[K^n - \tilde{K}^n; \mathcal{C}_2^{2\gamma}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)]. \end{aligned}$$

Remember that owing to the results of Section 3, we can rely on the uniform control

$$\mathcal{N}[y^n; \mathcal{C}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[K^n; \mathcal{C}_2^{2\gamma}(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] \leq c_{x, \psi},$$

with an equivalent result for \tilde{y}^n . The proof of Theorem 2.12 now leans on the two following lemmas:

Lemma 5.1. *For all $\tilde{\mu} > 1$ and $\kappa > 0$, there exists a constant $c = c_{\tilde{\mu}, \kappa}$ such that if $s < t \in \Pi^n$,*

$$\begin{aligned} \|J_{ts}^n - \tilde{J}_{ts}^n\|_{\mathcal{B}_{\gamma', p}} & \leq c \left\{ |t-s|^\kappa + |t-s|^{\tilde{\mu}-\gamma'} \right\} \\ & \left\{ \mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^\kappa(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma', p})] + \mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \right\} \end{aligned}$$

and

$$\|J_{ts}^n - \tilde{J}_{ts}^n\|_{\mathcal{B}_p} \leq c |t-s|^{\tilde{\mu}} \mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)].$$

Proof. It suffices to follow the lines of the proof of Lemma 3.2. □

Lemma 5.2. *Set $\tilde{\mu} := \inf(\gamma + \gamma', 3\gamma)$. Then for all $s < t \in \Pi^n$,*

$$\mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^{\tilde{\mu}}(\llbracket s, t \rrbracket_n; \mathcal{B}_p)] \leq c_{x, \tilde{x}, \psi, \tilde{\psi}} \left\{ \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)] + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\} \quad (62)$$

and

$$\mathcal{N}[\hat{\delta}(J^n - \tilde{J}^n); \mathcal{C}_3^\gamma(\llbracket s, t \rrbracket_n; \mathcal{B}_{\gamma', p})] \leq c_{x, \tilde{x}, \psi, \tilde{\psi}} \left\{ \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)] + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}. \quad (63)$$

Proof. This is the same type of arguments as in the proof of Lemma 4.2. For (62), resort to the

decomposition (39) and notice for instance that

$$\begin{aligned}
& \|X_{tu}^{x,i} \left(\int_0^1 dr f'_i(y_s^n + r(\delta y^n)_{us}) \cdot K_{us}^n \right) - \tilde{X}_{tu}^{x,i} \left(\int_0^1 dr f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us}) \cdot \tilde{K}_{us}^n \right)\|_{\mathcal{B}_p} \\
& \leq c \|X_{tu}^{x,i} - \tilde{X}_{tu}^{x,i}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \|K_{us}^n\|_{\mathcal{B}_p} + \|\tilde{X}_{tu}^{x,i}\|_{\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)} \\
& \quad \left\| \int_0^1 dr f'_i(y_s^n + r(\delta y^n)_{us}) \cdot K_{us}^n - \int_0^1 dr f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us}) \cdot \tilde{K}_{us}^n \right\|_{\mathcal{B}_p} \\
& \leq c_{x, \tilde{x}, \psi} |t-s|^{3\gamma} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma + c_{\tilde{x}} |t-u|^\gamma \\
& \quad \left\{ \left\| \int_0^1 dr [f'_i(y_s^n + r(\delta y^n)_{us}) - f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us})] \cdot K_{us}^n \right\|_{\mathcal{B}_p} \right. \\
& \quad \left. + \left\| \int_0^1 dr f'_i(\tilde{y}_s^n + r(\delta \tilde{y}^n)_{us}) \cdot [K_{us}^n - \tilde{K}_{us}^n] \right\|_{\mathcal{B}_p} \right\} \\
& \leq c_{x, \tilde{x}, \psi}^1 |t-s|^{3\gamma} \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma + c_{x, \tilde{x}, \psi}^2 |t-s|^{3\gamma} \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket s, t \rrbracket_n)],
\end{aligned}$$

where we have used the continuous inclusion $\mathcal{B}_{\gamma', p} \subset \mathcal{B}_\infty$. (63) can be proved likewise, with the same kind of estimates as in the proof of (60). \square

Proof of Theorem 2.12. By following the same procedure as in the proof of Theorem 2.11, we first deduce

$$\begin{aligned}
& \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] \\
& \leq c_{x, \tilde{x}, \psi, \tilde{\psi}}^1 \left\{ T_2^\gamma \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] + \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}.
\end{aligned}$$

Indeed, one has for instance, if $0 \leq s < t \leq T_2$,

$$\begin{aligned}
\|X_{ts}^{x,i} [f_i(y_s^n) - f_i(\tilde{y}_s^n)]\|_{\mathcal{B}_p} & \leq c_x |t-s|^\gamma \|y_s^n - \tilde{y}_s^n\|_{\mathcal{B}_p} \\
& \leq c_x |t-s|^\gamma \left\{ \|\hat{\delta}(y^n - \tilde{y}^n)_{s0}\|_{\mathcal{B}_p} + \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} \right\} \\
& \leq c_x |t-s|^\gamma \left\{ T_2^\gamma \mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] + \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} \right\}.
\end{aligned}$$

Then we take T_2 such that $c_{x, \tilde{x}, \psi, \tilde{\psi}}^1 T_2^\gamma = \frac{1}{2}$ so as to retrieve

$$\mathcal{N}[y^n - \tilde{y}^n; \tilde{Q}(\llbracket 0, T_2 \rrbracket_n)] \leq 2c_{x, \tilde{x}, \psi, \tilde{\psi}}^1 \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}.$$

Repeating the procedure on $[T_2, 2T_2]$, $[2T_2, 3T_2]$, ..., leads to the uniform control

$$\begin{aligned}
& \mathcal{N}[y^n - \tilde{y}^n; \mathcal{C}_1^\gamma(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_p)] + \mathcal{N}[y^n - \tilde{y}^n; \mathcal{C}_1^0(\llbracket 0, 1 \rrbracket_n; \mathcal{B}_{\gamma', p})] \\
& \leq c_{x, \tilde{x}, \psi, \tilde{\psi}} \left\{ \|\psi - \tilde{\psi}\|_{\mathcal{B}_{\gamma', p}} + \|\mathbf{x} - \tilde{\mathbf{x}}\|_\gamma \right\}. \quad (64)
\end{aligned}$$

To conclude with, let us introduce, for all $s < t \in [0, 1]$, two sequences $s_n < t_n \in \Pi^n$ such that s_n decreases to s and t_n increases to t , and write (for instance) successively

$$\|\hat{\delta}(y - \tilde{y})_{ts}\|_{\mathcal{B}_p} \leq \|\hat{\delta}(y - \tilde{y})_{tt_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(y - \tilde{y})_{t_n s_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(y - \tilde{y})_{s_n s}\|_{\mathcal{B}_p},$$

$$\|\hat{\delta}(y - \tilde{y})_{t_n s_n}\|_{\mathcal{B}_p} \leq \|\hat{\delta}(y - y^n)_{t_n s_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(y^n - \tilde{y}^n)_{t_n s_n}\|_{\mathcal{B}_p} + \|\hat{\delta}(\tilde{y} - \tilde{y}^n)_{t_n s_n}\|_{\mathcal{B}_p}.$$

The control (64), together with the approximation result (61), then provides (29). □

6 Appendix A: a useful algorithm

We give here the description and an analysis of the algorithm used in the proofs of Lemmas 3.2, 4.1 and 5.1.

Consider a generic partition $\{0, 1, 2, \dots, N\}$. We remove the inner points of this partition ($\{1, 2, \dots, N-1\}$) one by one according to the following procedure (see Figure 1):

- At step 1, we successively remove, from the right to the left, every two points, starting from N (excluded) until 0 (also excluded). Then, still at step 1, we take off the point of the (updated) partition between 0 (excluded) and the last removed point, if such a middle point exists.
- We repeat the procedure with the remaining points (steps 2,3,...) until the partition is empty.

We denote by:

- M the number of steps necessary to empty the partition.
- $(k_m)_{m \in \{1, \dots, N-1\}}$ the sequence of successively removed points.
- k_m^+ the point of the partition (at 'time' m of the algorithm) that follows k_m (when reading from the left to the right), k_m^- the point that precedes it.
- A_r the total number of points that have been taken off at the end of step r . We also set $A_0 := 0$.

Lemma 6.1. *For every $r \in \{0, 1, \dots, M\}$,*

$$0 \leq A_r - N \left(1 - \frac{1}{2^r}\right) \leq 1.$$

In particular, $\left|A_r - A_{r-1} - \frac{N}{2^r}\right| \leq 1$ and $2^{M-1} \leq N \leq 2^{M+1}$.

Proof. This stems from a straightforward iteration procedure based on the formula $A_{r+1} = A_r + \left\lfloor \frac{N - A_r + 1}{2} \right\rfloor$, $r \in \{0, 1, \dots, M-1\}$, where $\lfloor \cdot \rfloor$ stands for the integer part. □

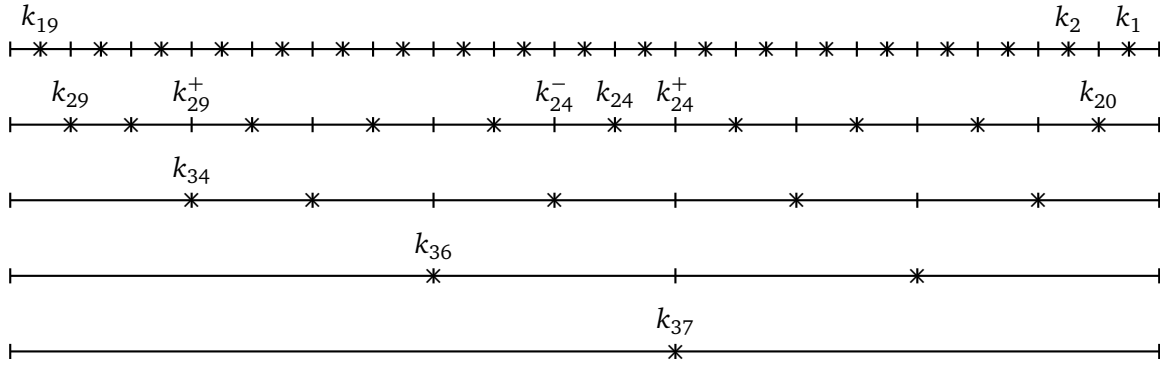


Figure 1: The algorithm for $N = 38$. Each line corresponds to one step. Thus, $M = 5$, $A_1 = 19$, $A_2 = 29$, $A_3 = 34$, $A_4 = 36$.

Proposition 6.2. *Suppose that $\mu > 1$, $0 < \gamma' < 1$ and $\kappa > 0$. Then*

$$\sum_{r=1}^{M-1} \left\{ \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\kappa + \frac{1}{N^\mu} \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \right\} \leq c_{\kappa, \mu, \gamma'}, \quad (65)$$

for some finite constant $c_{\kappa, \mu, \gamma'}$ independent of N .

Proof. Actually, we use the following explicit expressions: at step r ($r \in \{1, \dots, M-1\}$), if $N - A_{r-1}$ is even, one has, for every $m \in \llbracket A_{r-1} + 1, A_r - 1 \rrbracket$,

$$k_m^+ = N - 2^r(m - A_{r-1}) + 2^r, \quad (66)$$

$$k_m^- = N - 2^r(m - A_{r-1}), \quad (67)$$

and $k_{A_r}^+ = N - 2^r(A_r - A_{r-1}) + 2^r$, $k_{A_r}^- = 0$, while if $N - A_{r-1}$ is odd, Formulas (66) and (67) remain true for $m \in \llbracket A_{r-1} + 1, A_r - 1 \rrbracket$, but $k_{A_r}^- = 0$, $k_{A_r}^+ = k_{A_{r-1}}^+ = N - 2^r(A_r - A_{r-1} - 1) + 2^r$. From these expressions, we first deduce

$$\sum_{r=1}^{M-1} \left| 1 - \frac{k_{A_{r-1}+1}^-}{N} \right|^\kappa = \frac{1}{N^\kappa} \sum_{r=1}^{M-1} (2^r)^\kappa \leq c_\kappa^1 \left(\frac{2^M}{N} \right)^\kappa \leq c_\kappa^2,$$

according to Lemma 6.1. Then, if $N - A_{r-1}$ is even, one has

$$\begin{aligned}
& \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \\
&= \sum_{m=A_{r-1}+2}^{A_r-1} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu + \left| 1 - \frac{k_{A_r}^+}{N} \right|^{-\gamma'} |k_{A_r}^+|^\mu \\
&= \frac{2^{r(\mu-\gamma')}}{N^{-\gamma'}} \sum_{m=1}^{A_r-A_{r-1}-2} m^{-\gamma'} + \left| 1 - \frac{k_{A_r}^+}{N} \right|^{-\gamma'} |k_{A_r}^+|^\mu \\
&\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 2)^{1-\gamma'} \\
&\quad + \frac{(2^r)^{-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 1)^{-\gamma'} (N - 2^r(A_r - A_{r-1} - 1))^\mu \\
&\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 2)^{1-\gamma'} + \frac{(2^r)^{-\gamma'}}{N^{-\gamma'}} (N - 2^r(A_r - A_{r-1} - 1))^\mu.
\end{aligned}$$

since $A_r - A_{r-1} \geq 2$. In the same way, if $N - A_{r-1}$ is odd, one has

$$\begin{aligned}
& \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \\
&\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 2)^{1-\gamma'} \\
&\quad + \frac{(2^r)^{-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 2)^{-\gamma'} (N - 2^r(A_r - A_{r-1} - 2))^\mu \\
&\leq c_{\gamma'}^3 \frac{(2^r)^{\mu-\gamma'}}{N^{-\gamma'}} (A_r - A_{r-1} - 2)^{1-\gamma'} + \frac{(2^r)^{-\gamma'}}{N^{-\gamma'}} (N - 2^r(A_r - A_{r-1} - 2))^\mu.
\end{aligned}$$

since, in that case, $A_r - A_{r-1} \geq 3$. Thanks to Lemma 6.1, we now easily deduce

$$\frac{1}{N^\mu} \sum_{r=1}^{M-1} \sum_{m=A_{r-1}+2}^{A_r} \left| 1 - \frac{k_m^+}{N} \right|^{-\gamma'} |k_m^+ - k_m^-|^\mu \leq \frac{c_{\gamma'}^3}{N^{\mu-1}} \sum_{r=1}^{M-1} (2^r)^{\mu-1} + \frac{c_\mu^4}{N^{\mu-\gamma'}} \sum_{r=1}^{M-1} (2^r)^{\mu-\gamma'} \leq c_{\mu,\gamma'}.$$

□

7 Appendix B

This section is devoted to the proof of Proposition 2.9. To this end, we will resort to the two following lemmas, respectively borrowed from [15] and [1]:

Lemma 7.1. *Fix a time $T > 0$. For all $\alpha, \beta \geq 0$, $p, q \geq 1$, there exists a constant c such that for every $R \in \mathcal{C}_2([0, T]; \mathcal{B}_{\alpha,p})$,*

$$\mathcal{N}[R; \mathcal{C}_2^\beta([0, T]; \mathcal{B}_{\alpha,p})] \leq c \left\{ U_{\beta+\frac{2}{q}, \alpha, p}(R) + \mathcal{N}[\hat{\delta}R; \mathcal{C}_3^\beta([0, T]; \mathcal{B}_{\alpha,p})] \right\},$$

where

$$U_{\beta,q,\alpha,p}(R) = \left[\int_{0 \leq u < v \leq T} \left(\frac{\|R_{vu}\|_{\mathcal{B}_{\alpha,p}}}{|v-u|^\beta} \right)^q dudv \right]^{1/q}.$$

Lemma 7.2. *For every $p \geq 2$, the Burkholder-Davies-Gundy inequality holds in \mathcal{B}_p . In other words, for every $T > 0$, if B is a one-dimensional Brownian motion defined on complete filtered probability space (Ω, \mathcal{F}, P) and H is an adapted process with values in $L^2([0, T]; \mathcal{B}_p)$, then for any $q \geq 2$, there exists a constant c independent of H such that*

$$E \left[\sup_{0 \leq t \leq T} \left\| \int_0^t H_u dB_u \right\|_{\mathcal{B}_p}^q \right] \leq c E \left[\left(\int_0^T \|H_u\|_{\mathcal{B}_p}^2 du \right)^{q/2} \right]. \quad (68)$$

Proof of Proposition 2.9. On the whole, this is the same identification procedure as in the proof of Proposition 2.8. The only difference lies in the fact that the direct estimates of the integrals under consideration will here be replaced with a joint use of Lemmas 7.1 and 7.2.

We denote by y the (Itô) solution of (24), with initial condition $\psi \in \mathcal{B}_{\eta,p}$. Let us fix $\gamma \in (1/3, 1/2)$ such that $\gamma + \eta > 1$ and $2\gamma > \eta$. If one refers to [16, Theorem 1], one can assert that $y \in \mathcal{C}_1^0([0, 1]; \mathcal{B}_{\eta,p})$ a.s, and one even knows that $\sup_{t \in [0,1]} E \left[\|y_t\|_{\mathcal{B}_{\eta,p}}^q \right] < \infty$ for every $q \in \mathbb{N}$. Then, since $(\hat{\delta}y)_{ts} = \int_s^t S_{tu} dx_u^i f_i(y_u)$, one has, thanks to Lemma 7.2,

$$\begin{aligned} E \left[\left\| \int_s^t S_{tu} dx_u^i f_i(y_u) \right\|_{\mathcal{B}_p}^q \right] &\leq c E \left[\left(\int_s^t \|S_{tu} f_i(y_u)\|_{\mathcal{B}_p}^2 du \right)^{q/2} \right] \\ &\leq c |t-s|^{q/2-1} \int_s^t E \left[\|S_{tu} f_i(y_u)\|_{\mathcal{B}_p}^q \right] du \\ &\leq c |t-s|^{q/2}, \end{aligned} \quad (69)$$

and consequently, with the notation of Lemma 7.1,

$$\begin{aligned} E \left[U_{\gamma+\frac{2}{q},q,0,p}(\hat{\delta}y) \right] &\leq \left(\iint_{0 \leq u < v \leq 1} \frac{E \left[\|(\hat{\delta}y)_{vu}\|_{\mathcal{B}_p}^q \right]}{|v-u|^{\gamma q+2}} dudv \right)^{1/q} \\ &\leq \left(\iint_{0 \leq u < v \leq 1} |v-u|^{q(\frac{1}{2}-\gamma)-2} dudv \right)^{1/q} < \infty \end{aligned}$$

by picking $q > 1/(\frac{1}{2} - \gamma)$. Together with the result of Lemma 7.1, this yields $y \in \mathcal{C}_1^\gamma([0, 1]; \mathcal{B}_p)$ a.s.

As far as K^y is concerned, we already know that $\hat{\delta}K^y = X^{x,i} \delta(f_i(y))$, which leads to $\hat{\delta}K^y \in \mathcal{C}_3^{2\gamma}([0, 1]; \mathcal{B}_p)$ a.s. Then, from the expression $K_{ts}^y = \int_s^t S_{tu} dx_u^i \delta(f_i(y))_{us}$, we deduce, as in (69), $E \left[\|K_{ts}^y\|_{\mathcal{B}_p}^q \right] \leq c |t-s|^q$, and accordingly, thanks to Lemma 7.1, $K^y \in \mathcal{C}_2^{2\gamma}([0, 1]; \mathcal{B}_p)$ a.s.

Finally, for J^y , we first lean on the decomposition (39) of $\hat{\delta}J^y$ to assert that $\hat{\delta}J^y \in \mathcal{C}_3^{\gamma+\eta}([0, 1]; \mathcal{B}_p)$ a.s. Then we appeal to the expression of J^y that we have exhibited in the proof of Proposition 2.8, namely $J_{ts}^y = \int_s^t S_{tu} dx_u^i M_{us}^i$ with M^i given by (28), to show that $E \left[\|J_{ts}^y\|_{\mathcal{B}_p}^q \right] \leq c |t-s|^{q(\frac{1}{2}+\eta)}$. Together with Lemma 7.1, these results clearly provide the expected regularity, i.e., $J^y \in \mathcal{C}_2^\mu([0, 1]; \mathcal{B}_p)$ a.s, with $\mu = \gamma + \eta > 1$.

The control of the regularity of J^y as a process with values in $\mathcal{B}_{\eta,p}$ stems from the same reasoning. Indeed, we first deduce from (38) that $\hat{\delta}J^y \in \mathcal{C}_3^\gamma([0, 1]; \mathcal{B}_{\eta,p})$ a.s, since for instance $\|X_{tu}^{x,i} f_i(y_u)\|_{\mathcal{B}_{\eta,p}} \leq c_{x,f,y} |t-u|^\gamma$ and

$$\|X_{tu}^{x,i}(\delta x^j)_{us} F_{ij}(y_s)\|_{\mathcal{B}_{\eta,p}} \leq c_x |t-s|^{2\gamma-(\eta-\frac{1}{2})} \|F_{ij}(y_u)\|_{\mathcal{B}_{1/2,p}} \leq c_{x,f,y} |t-s|^\gamma.$$

We can then write J^y as $J_{ts}^y = \int_s^t S_{tu} dx_u^i \delta(f_i(y))_{us} - X_{ts}^{xx,ij} F_{ij}(y_s)$ to easily derive $E \left[\|J_{ts}^y\|_{\mathcal{B}_{\eta,p}}^q \right] \leq c_{x,f,y} |t-s|^{q/2}$, and hence $J^y \in \mathcal{C}_2^\gamma([0, 1]; \mathcal{B}_{\eta,p})$ a.s.

□

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