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Branching random walks in random environment are diffusive in the regular growth phase

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Abstract

We treat branching random walks in random environment using the framework of Linear Stochastic Evolution. In spatial dimensions three or larger, we establish diffusive behaviour in the entire growth phase. This can be seen through a Central Limit Theorem with respect to the population density as well as through an invariance principle for a path measure we introduce.

Key words: branching random walk, random environment, central limit theorem, invariance principle, diffusivity.

AMS 2000 Subject Classification: Primary 60J80, 60K37, 60F17, 60K35, 82D30.

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1 Introduction

1.1 Background

Branching random walks (and their time–continuous counterpart branching Brownian motion) are treated, with the result of a central limit theorem (CLT), by Watanabe in [Wat67] and [Wat68]. Smith and Wilkinson introduce the notion of random (in time) environment to branching processes [SW69], and in 1972, the book by Athreya and Ney [AN72] appears and gives an excellent overview of the knowledge of the time.

A closely related model, the directed polymers in random environment (DPRE), is studied since the eighties, when the question of diffusivity is treated by Imbrie and Spencer [IS88] as well as Bolthausen [Bol89]. A review can be found in [CSY04].

It took until the new millenium for the time–space random environment known from DPRE to get applied to branching random walks by Birkner, Geiger and Kersting [BGK05]. A CLT in probability is proven in [Yos08a], and improved to an almost sure sense in [Nak11] with the help of Linear Stochastic Evolutions (LSE), which were introduced in [Yos08b] and [Yos10]. Linear stochastic evolutions build a frame to a variety of models, including DPRE. For LSE, the CLT was proven in [Nak09]. Shiozawa treats the time–continuous counterpart, namely branching Brownian motions in random environment [Shi09a, Shi09b].

The present article uses as a blueprint [CY06], which proves a CLT for DPRE, and the larger angle of view allowed by the LSE gives the crucial ingredients to conclude our result, which is a CLT on the event of survival on the entire regular growth phase, but under integrability conditions slightly more restrictive than those from [Nak11]. Compared to the case of DPRE, the necessary notational overhead is unfortunately significantly bigger. Speaking of DPRE, it is possible to extend the results of [CY06] to the case where completely repulsive sites are allowed, using the same conditioning–techniques as here.

A localization–result in the slow growth phase is proven by two of the authors of the present work in [HN11].

1.2 Branching random walks in random environment

We denote the natural numbers by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$. We will need at various places sets of probability measures, which we write as $\mathscr{P}(\cdot)$; for instance,

$$\mathscr{P}(\mathbb{N}_0) := \Big\{ q = (q(k))_{k \in \mathbb{N}_0} \in [0,1]^{\mathbb{N}_0} : \sum_{k \in \mathbb{N}_0} q(k) = 1 \Big\}$$

stands for the set of probability measures on $\mathbb{N}_0$.

We consider particles in $\mathbb{Z}^d$, $d \geq 1$, each performing a simple random walk and branching into independent copies at each time–step.

i) At time $n = 0$, there is one particle born at the origin $x = 0$.

ii) A particle born at site $x \in \mathbb{Z}^d$ at time $n \in \mathbb{N}_0$ is equipped with k eggs with probability $q_{n,x}(k)$, $k \in \mathbb{N}_0$, independently from other particles.

iii) In the next time step, it takes its k eggs to a uniformly chosen nearest–neighbour site and dies. The eggs then are hatched.

The offspring distributions $q_{n,x} = (q_{n,x}(k))_{k\in\mathbb{N}_0}$ are assumed to be i.i.d. in time–space (n,x). This model is called Branching Random Walks in Random Environment (BRWRE). Let $N_{n,y}$ be the number of the particles which occupy the site $y \in \mathbb{Z}^d$ at time n.

For the proofs in this article, a modeling down to the level of individual particles is needed. First, we define namespaces $\mathcal{V}_n$, $n \in \mathbb{N}_0$ for the n–th generation particles and $\mathcal{V}_{\mathbb{N}_0}$ for the particles of all generations together:

$$\mathcal{V}_0 = \{\mathbf{1}\} = \{(1)\},\ \mathcal{V}_{n+1} = \mathcal{V}_n \times \mathbb{N},\ \text{for } n \geq 0,$$
$$\mathcal{V}_{\mathbb{N}_0} = \bigcup_{n\in\mathbb{N}_0} \mathcal{V}_n.$$

Then, we label all particles as follows:

i) At time $n = 0$, there is just one particle which we call $\mathbf{1} = (1) \in \mathcal{V}_0$.

ii) A particle at time n is identified with its genealogical chart $\mathbf{y} = (1, \mathrm{y}_1, \dots, \mathrm{y}_n) \in \mathcal{V}_n$. If the particle $\mathbf{y}$ gives birth to $k_{\mathbf{y}}$ particles at time n, then the children are labeled by $(1, \mathrm{y}_1, \dots, \mathrm{y}_n, 1), \dots, (1, \mathrm{y}_1, \dots, \mathrm{y}_n, k_{\mathbf{y}}) \in \mathcal{V}_{n+1}$.

By using this naming procedure, we define the branching of the particles rigorously. This definition is based on the one in [Yos08a].

Note that the particle with name $\mathbf{x}$ can be located at x anywhere in $\mathbb{Z}^d$. As both informations genealogy and place are usually necessary together, it is convenient to combine them to $\mathbb{x} = (x, \mathbf{x})$; think of x and $\mathbf{x}$ written very closely together.

• *Random environment of offspring distibutions:* Set $\Omega_q = \mathscr{P}(\mathbb{N}_0)^{\mathbb{N}_0\times\mathbb{Z}^d}$. The set $\mathscr{P}(\mathbb{N}_0)$ is equipped with the natural Borel σ–field induced by the one of $[0,1]^{\mathbb{N}_0}$. We denote by $\mathcal{G}_q$ the product σ–field on Ω_q.

We fix a product measure $Q \in \mathscr{P}(\Omega_q, \mathcal{G}_q)$ which describes the i.i.d. offspring distributions assigned to each time–space location.

Each environment $q \in \Omega_q$ is a function $(n,x) \mapsto q_{n,x} = (q_{n,x}(k))_{k\in\mathbb{N}_0}$ from $\mathbb{N}_0 \times \mathbb{Z}^d$ to $\mathscr{P}(\mathbb{N}_0)$. We interpret $q_{n,x}$ as the offspring distribution for each particle which occupies the time–space location (n,x).

• *Spatial motion:* A particle at time–space location (n,x) jumps to some neighbouring location $(n+1,y)$ before it is replaced by its children there. Therefore, the spatial motion should be described by assigning a destination to each particle at each time–space location (n,x). We define the measurable space $(\Omega_X, \mathcal{G}_X)$ as the set $(\mathbb{Z}^d)^{\mathbb{N}_0\times\mathbb{Z}^d\times\mathcal{V}_{\mathbb{N}_0}}$ with the product σ–field, and $\Omega_X \ni X \mapsto X_{n,\mathbb{x}}$ for each $(n,\mathbb{x}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0})$ as the projection. We define $P_X \in \mathscr{P}(\Omega_X, \mathcal{G}_X)$ as the product measure such that

$$P_X(X_{n,\mathbb{x}} = e) = \begin{cases} \frac{1}{2d} & \text{if } |e| = 1, \\ 0 & \text{if } |e| \neq 1 \end{cases}$$

for $e \in \mathbb{Z}^d$ and $(n,\mathbb{x}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathcal{V}_{\mathbb{N}_0})$. Here, we interpret $X_{n,\mathbb{x}}$ as the step at time $n+1$ if the particle $\mathbf{x}$ is located space location x.

• *Offspring realization*: We define the measurable space $(\Omega_K, \mathscr{G}_K)$ as the set ${\mathbb{N}_0}^{\mathbb{N}_0\times\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}}$ with the product σ–field, and $\Omega_K \ni K \mapsto K_{n,\mathbb{x}}$ for each $(n,\mathbb{x}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0})$ as the projection. For each fixed $q \in \Omega_q$, we define $P_K^q \in \mathscr{P}(\Omega_K, \mathscr{G}_K)$ as the product measure such that

$$P_K^q(K_{n,\mathbb{x}} = k) = q_{n,x}(k) \quad \text{for all } (n,\mathbb{x}) = (n,x,\mathbf{x}) \in \mathbb{N}_0 \times \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0} \text{ and } k \in \mathbb{N}_0.$$

We interpret $K_{n,\mathbb{x}}$ as the number of eggs of the particle $\mathbf{x}$ if it is located at time–space location (n,x). One could directly speak of its children as well.

The first steps of such a BRWRE are shown in Figure 1.

Putting everything together, we arrive at the

• *Overall construction:* We define $(\Omega, \mathscr{G})$ by

$$\Omega = \Omega_X \times \Omega_K \times \Omega_q, \quad \mathscr{G} = \mathscr{G}_X \otimes \mathscr{G}_K \otimes \mathscr{G}_q,$$

and with $q \in \Omega_q$,

$$P^q = P_X \otimes P_K^q \otimes \delta_q, \quad P = \int Q(dq) P^q.$$

Now that the BRWRE is completely modeled, we can have a look at where the particles are: for $(n,\mathbb{y}) \in \mathbb{N}_0 \times (\mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0})$, we define

$$N_{n,\mathbb{y}} = \mathbb{1}_{\{\text{the particle } \mathbf{y} \text{ is located at time–space location } (n,y)\}}.$$

This enables the

• *Placement of BRWRE into the framework of Linear Stochastic Evolutions:* We set the starting condition $N_{0,\mathbb{y}} = \mathbb{1}_{\mathbb{y}=(0,\mathbf{1})}$. Then, defining the matrices $(A_n)_n$ via their entries in the manner indicated below, we can describe $N_{n,\mathbb{y}}$ inductively by

$$\begin{aligned} N_{n,\mathbb{y}} &= \sum_{\mathbb{x}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}} N_{n-1,\mathbb{x}} \mathbb{1}_{\{y-x=X_{n-1,\mathbb{x}},\ 1\le \mathbf{y}/\mathbf{x}\le K_{n-1,\mathbb{x}}\}}, \\ &= \sum_{\mathbb{x}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}} N_{n-1,\mathbb{x}} A_{n,\mathbb{x}}^{\ \mathbb{y}} \\ &= (N_0 A_1 \cdots A_n)_{\mathbb{y}},\ \mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}, \end{aligned}$$

where $\mathbf{y}/\mathbf{x}$ is given for $\mathbf{x},\mathbf{y} \in \mathscr{V}_{\mathbb{N}_0}$ as

$$\mathbf{y}/\mathbf{x} = \begin{cases} k & \text{if } \begin{array}{l} \mathbf{x} = (1, \mathrm{x}_1, \dots, \mathrm{x}_n) \in \mathscr{V}_n, \\ \mathbf{y} = (1, \mathrm{x}_1, \dots, \mathrm{x}_n, k) \in \mathscr{V}_{n+1} \end{array} \text{ for some } n \in \mathbb{N}_0, \\ \infty & \text{otherwise,} \end{cases}$$

and where

$$A_{n,\mathbb{x}}^{\ \mathbb{y}} := \mathbb{1}_{\{y-x=X_{n-1,\mathbb{x}},\ 1\le \mathbf{y}/\mathbf{x}\le K_{n-1,\mathbb{x}}\}},\ \mathbb{x},\mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}.$$

One–site- and overall population can be defined respectively as

$$N_{n,y} = \sum_{\mathbf{y}\in\mathscr{V}_{\mathbb{N}_0}} N_{n,(y,\mathbf{y})}, \text{ and } N_n = \sum_{\mathbb{y}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}} N_{n,\mathbb{y}},$$

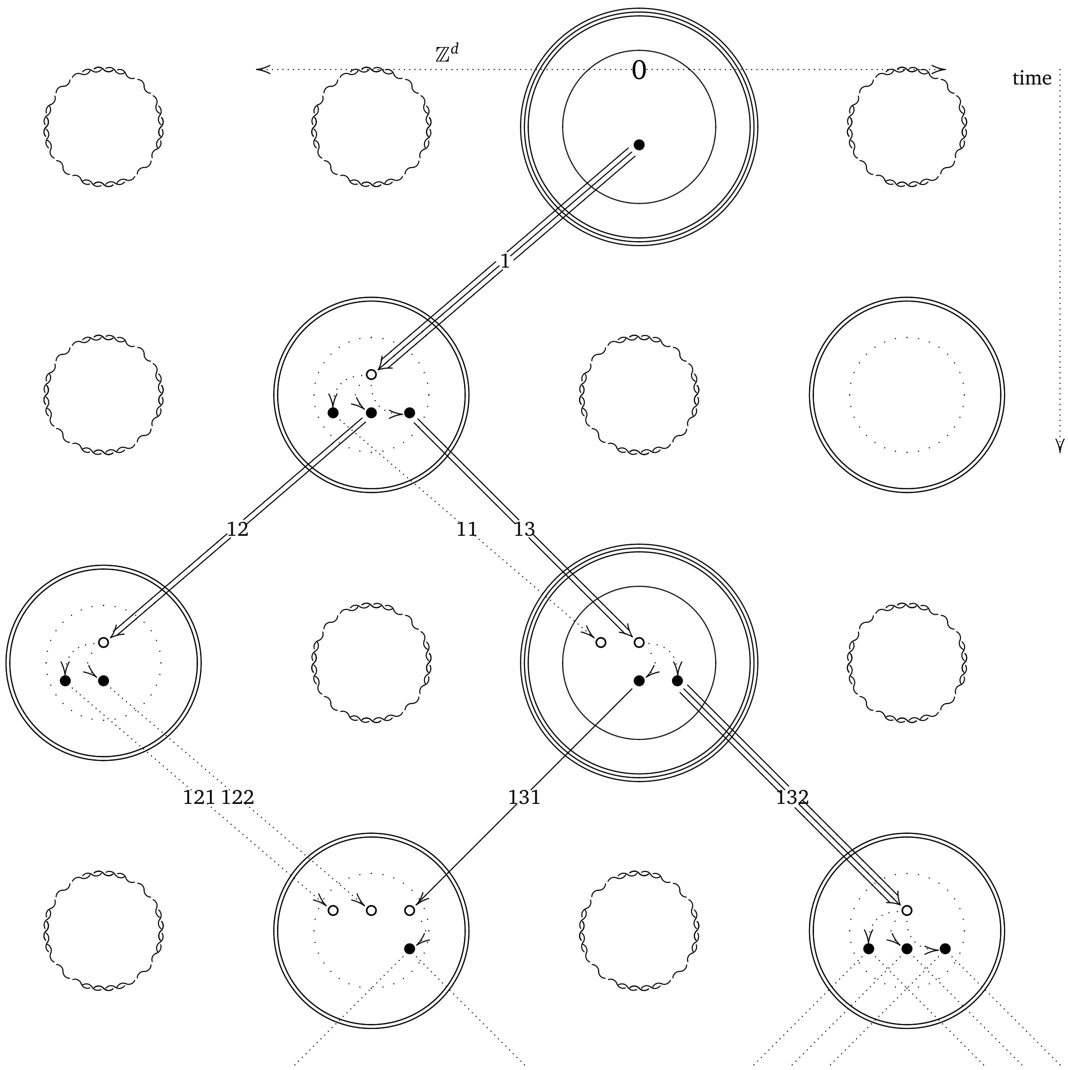


Figure 1: One realization of the first steps and branchings. In this particular example, there are only two types of offspring distibutions, one allowing for one or three eggs, the other one for two or none. This is indicated by the concentrical circles. The curly circles indicate points where the realization of the environment has no influence on the outcome of the random walk. The arrows indicate the movement of the particles, the number of strokes indicating the number of eggs carried. The cones in the lower part of the picture get their full meaning in Remark 2.1.2.

for $n \in \mathbb{N}_0,\ y \in \mathbb{Z}^d$. Other quantities needed later are the moments of the local offspring distributions for $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$,

$$m_{n,x}^{(p)} = \sum_{k \in \mathbb{N}_0} k^p q_{n,x}(k), \quad m^{(p)} = Q(m_{n,x}^{(p)}),\ p \in \mathbb{N}_0, \quad \mathbf{m} = m^{(1)},$$

and the normalized one–site and overall populations

$$\overline{N}_{n,y} = N_{n,y}/\mathbf{m}^n \text{ and } \overline{N}_n = N_n/\mathbf{m}^n, n \in \mathbb{N}_0, y \in \mathbb{Z}^d.$$

It is easy to see that the expectation of the matrix entries, which is an important parameter in the setting of LSE, for $\mathbb{x}, \mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}$computes as

$$a_{\mathbb{x}}^{\mathbb{y}} := P[A_{1,\mathbb{x}}^{\ \mathbb{y}}] = \begin{cases} \frac{1}{2d} \sum_{j \geq k} q(j) & \text{if } |x-y| = 1,\ \mathbf{y}/\mathbf{x} = k,\ k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$q(j) := Q\big(q_{0,0}(j)\big), j \in \mathbb{N}_0.$$

Taking sums, we obtain

$$\sum_{\mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}} a_{\mathbb{x}}^{\mathbb{y}} = \mathbf{m}, \quad \text{for } \mathbb{x} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}.$$

1.3 Preliminaries

In this and the following subsection, we gather properties of BRWRE that are already known. First, we introduce the Markov chain $(\mathbb{S}, P_{\mathbb{S}}^{\mathbb{x}}) = ((S, \mathbf{S}), P_{(S,\mathbf{S})}^{(x,\mathbf{x})})$ on $\mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}$ for $\mathbb{x} = (x, \mathbf{x}) \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}$, independent of $(A_n)_{n \geq 1}$, by

$$P_{\mathbb{S}}^{\mathbb{x}}(\mathbb{S}_0 = \mathbb{x}) = 1,$$

$$P_{\mathbb{S}}\big(\mathbb{S}_{n+1} = \mathbb{y} |\ \mathbb{S}_n = \mathbb{x}\big) = \frac{a_{\mathbb{x}}^{\mathbb{y}}}{\mathbf{m}} = \begin{cases} \frac{\sum_{j \geq k} q(j)}{2d\ \mathbf{m}} & \text{if } |x-y| = 1, \text{ and } \mathbf{y}/\mathbf{x} = k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

where $\mathbb{x}, \mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}$. The filtration of this random walk will be called $\mathscr{F}_n = \sigma(\mathscr{F}_n^1 \times \mathscr{F}_n^2)$, with $\mathscr{F}_n^1 := \sigma(S_1, \ldots, S_n)$, $\mathscr{F}_n^2 := \sigma(\mathbf{S}_1, \ldots, \mathbf{S}_n)$, $n \in \mathbb{N}_0$, and the corresponding sample space $\Omega^1 \times \Omega^2$.

Note that we can regard S and $\mathbf{S}$ as independent Markov chains on $\mathbb{Z}^d$ and $\mathscr{V}_{\mathbb{N}_0}$, respectively, with S the simple random walk on $\mathbb{Z}^d$.

Next, we introduce a process which is essential to the proof of our results:

$$\zeta_0 = 1 \text{ and for } n \geq 1,\ \zeta_n = \zeta_n(\mathbb{S}) = \prod_{m=1}^n \frac{A_{m,\mathbb{S}_{m-1}}^{\ \mathbb{S}_m}}{a_{\mathbb{S}_{m-1}}^{\mathbb{S}_m}}. \tag{1.2}$$

Lemma 1.3.1. *ζ_n is a martingale with respect to the filtration given by*

$$\mathscr{H}_0 := \sigma(\mathbb{S}_0), \quad \mathscr{H}_n := \sigma(A_m, \mathbb{S}_m;\ m \le n),\ n \ge 1.$$

Moreover, we have that

$$N_{n,\mathbb{y}} = \mathbf{m}^n P_{\mathbb{S}}^{(0,1)}(\zeta_n : \mathbb{S}_n = \mathbb{y})\ P\text{–a.s. for } n \in \mathbb{N}_0,\ \mathbb{y} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}.$$

Remark 1.3.2. Summation over all possible sequences of names yields

$$N_{n,y} = \mathbf{m}^n P_{\mathbb{S}}^{(0,1)}(\zeta_n : S_n = y).$$

From this Lemma follows an important result: the next Lemma shows that a phase transition occurs for the growth rate of the total population.

Lemma 1.3.3. *$\overline{N}_n$ is a martingale with respect to $\mathscr{G}_n := \sigma(A_m : m \le n)$. Hence, the limit*

$$\overline{N}_\infty = \lim_{n\to\infty} \overline{N}_n, \text{ exists } P\text{–a.s.} \tag{1.3}$$

and

$$P(\overline{N}_\infty) \in \{0, 1\}.$$

Moreover, $P(\overline{N}_\infty) = 1$ if and only if the limit (1.3) *is convergent in $L^1(P)$.*

The proof of Lemmas 1.3.1 and 1.3.3 can be found in [Nak11].

We refer to the case $P(\overline{N}_\infty) = 1$ as *regular growth phase* and to the other one, $P[\overline{N}_\infty] = 0$, as *slow growth phase*. The regular growth phase means that the growth rate of the total population has the same order as the growth rate of the expectation of the total population $\mathbf{m}^n$; on the other hand, the slow growth phase means that, almost surely, the growth rate of the population is lower than the growth rate of its expectation.

One can also introduce the notions of 'survival' and 'extinction'.

Definition 1.3.4. *The event of survival is the existence of particles at all times:*

$$\{survival\} := \{\forall\ n \in \mathbb{N}_0,\ N_n > 0\}.$$

The extinction event is the complement of survival.

1.4 The result

Definition 1.4.1. *An important quantity of the model is the population density, which can be seen as a probability measure with support on $\mathbb{Z}^d$,*

$$\rho_{n,x} = \rho_n(x) := \frac{N_{n,x}}{N_n} \mathbb{1}_{N_n > 0},\ n \in \mathbb{N}_0,\ x \in \mathbb{Z}^d.$$

Our main result is the following CLT, proven as Corollary 2.2.4 of the invariance principle Theorem 2.2.2.

Theorem 1.4.2. *Assume $d \geq 3$ and regular growth, and the moment conditions $m^{(3)} < \infty$ and $Q\big((m^{(2)}_{n,x})^2\big) < \infty$. Then, for all bounded continuous function $F \in \mathscr{C}_b(\mathbb{R}^d)$, in $P(\,\cdot\,|survival)$–probability,*

$$\lim_{n\to\infty}\sum_{x\in\mathbb{Z}^d}\rho_n(x)F\Big(\frac{x}{\sqrt{n}}\Big) = \int_{\mathbb{R}^d} F(x)\nu(dx),$$

where ν stands for the Gaussian measure with mean 0 and covariance matrix $\frac{1}{d}I$.

Remark 1.4.3. The hypothesis $d \geq 3$ is in fact not necessary because in dimensions one and two, regular growth cannot occur. Instead of a CLT, localized behaviour can be observed, see [HY09, HN11].

It is the following equivalence, recently proven as [CY, Proposition 2.2.2], that enables us to speak easily of $P(\,\cdot\,|survival)$–probability:

Lemma 1.4.4. *If $P(\overline{N}_\infty > 0) > 0$ and $\mathbf{m} < \infty$, then*

$$\{regular\ growth\} := \{\overline{N}_\infty > 0\} = \{survival\},\ P\text{–a.s.}.$$

[CY] handles also the case of slow growth.

2 Proofs

2.1 The path measure

Definition 2.1.1. *We set, on $\mathscr{F}_\infty$,*

$$\mu_n(d\mathbb{S}) := \frac{1}{\overline{N}_n}P_{\mathbb{S}}(\zeta_n d\mathbb{S})\mathbb{1}_{\overline{N}_\infty>0},\ n\in\mathbb{N}_0,$$

where ζ is defined in (1.2).

Additional notations and definitions comprise the shifted processes: for $m \in \mathbb{N}_0$, $\mathbb{z} \in \mathbb{Z}^d \times \mathscr{V}_{\mathbb{N}_0}$, we define $N^{m,\mathbb{z}}_n = (N^{m,\mathbb{z}}_{n,\mathbb{y}})_{\mathbb{y}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}}$ and $\overline{N}^{m,\mathbb{z}}_n = (\overline{N}^{m,\mathbb{z}}_{n,\mathbb{y}})_{\mathbb{y}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}}$, $n \in \mathbb{N}_0$, respectively by

$$N^{m,\mathbb{z}}_{0,\mathbb{y}} = \mathbb{1}_{\mathbb{y}=\mathbb{z}}, \qquad N^{m,\mathbb{z}}_{n+1,\mathbb{y}} = \sum_{\mathbb{x}\in\mathbb{Z}^d\times\mathscr{V}_{\mathbb{N}_0}} N^{m,\mathbb{z}}_{n,\mathbb{x}} A^{\mathbb{y}}_{m+n+1,\mathbb{x}}, \text{ and}$$
$$\overline{N}^{m,\mathbb{z}}_{n,\mathbb{y}} = N^{m,\mathbb{z}}_{n,\mathbb{y}}/\mathbf{m}^n.$$

Using this, we can, with $m \leq n$, express μ_n on a finite time–horizon as

$$\mu_n(\mathbb{S}_{[0,m]} = \mathbb{x}_{[0,m]}) = \zeta_m(\mathbb{x}_{[0,m]})\frac{\overline{N}^{m,\mathbb{x}_m}_{n-m}}{\overline{N}_n}P_{\mathbb{S}}(\mathbb{S}_{[0,m]} = \mathbb{x}_{[0,m]})\mathbb{1}_{\overline{N}_\infty>0}; \tag{2.1}$$

in particular,

$$\frac{N_{n,x}}{N_n}\mathbb{1}_{\overline{N}_\infty>0} = \sum_{\mathbb{x}_{[0,n]}:x_n=x}\mu_n(\mathbb{S}_{[0,n]} = \mathbb{x}_{[0,n]}).$$

Note that for $B \in \mathscr{F}_\infty$, the limit

$$\mu_\infty(B) = \lim_{n\to\infty} \mu_n(B)$$

exists P–a.s. because of the martingale limit theorem for $P_{\mathbb{S}}(\zeta_n : B)$, which is indeed a positive martingale with respect to the filtration $(\mathscr{G}_n)_n$, as can be easily checked, and for $\overline{N}_n$, see Lemma 1.3.3.

Remark 2.1.2. We can write, for $B \in \mathscr{F}_n^1$,

$$\mu_\infty(B \times \Omega^2) = \frac{1}{\overline{N}_\infty} \sum_{\mathbb{x}_n} P_{\mathbb{S}}(\zeta_n : (B \times \Omega^2) \cap \{\mathbb{S}_n = \mathbb{x}_n\}) \overline{N}_\infty^{n,\mathbb{x}_n} \mathbb{1}_{\overline{N}_\infty > 0}.$$

The reader who cares to return to the lower part of Figure 1 will be rewarded with an intuitive picture of how we can let run our BRW up to time $n = 3$ and plug in there the shifted processes, indicated by the dotted cones.

Definition 2.1.3. *We define the environmental measure conditional on survival, or under the assumptions of Lemma* 1.4.4 *equivalently, regular growth, by*

$$\widetilde{P}(\cdot) = P\left(\,\cdot\, \middle| \overline{N}_\infty > 0\right) = \frac{P(\cdot \cap \overline{N}_\infty > 0)}{P(\overline{N}_\infty > 0)}.$$

Lemma 2.1.4. *Assume regular growth. Then,*

$$\widetilde{P}\mu_\infty(\,\cdot \times \Omega^2) \text{ is a probability measure on } \mathscr{F}_\infty^1, \tag{2.2}$$

and

$$\widetilde{P}\mu_\infty(\,\cdot \times \Omega^2) \ll P_S \text{ on } \mathscr{F}_\infty^1, \tag{2.3}$$

where P_S denotes the measure of a simple random walk.

In order to prove this Lemma, we need the following observation:

Lemma 2.1.5. *Suppose $(B_m)_{m\geq 1} \subset \mathscr{F}_\infty^1$ are such that $\lim_{m\to\infty} P_{\mathbb{S}}(B_m \times \Omega^2) = 0$. Then*

$$0 = \lim_{m\to\infty} \sup_n \widetilde{P}\mu_n(B_m \times \Omega^2) = \lim_{m\to\infty} \widetilde{P}\mu_\infty(B_m \times \Omega^2).$$

Proof. We first prove the first equality. For $\delta > 0$,

$$P\big(\mu_n(B_m \times \Omega^2)\big) \leq P\big(\mu_n(B_m \times \Omega^2) : \overline{N}_n \geq \delta\big) + P\big(\mathbb{1}_{\overline{N}_\infty > 0} : \overline{N}_n \leq \delta\big).$$

We can estimate

$$\begin{aligned}
\sup_n P\big(\mu_n(B_m \times \Omega^2) : \overline{N}_n \geq \delta\big) &\leq \delta^{-1} \sup_n P\big(\overline{N}_n \mu_n(B_m \times \Omega^2)\big) \\
&= \delta^{-1} \sup_n P\Big(\overline{N}_n \frac{P_{\mathbb{S}}(\zeta_n : B_m \times \Omega^2)}{\overline{N}_n} \mathbb{1}_{\overline{N}_\infty > 0}\Big) \\
&\leq \delta^{-1} \sup_n P_{\mathbb{S}}\big(P(\zeta_n) : B_m \times \Omega^2\big) \\
&= \delta^{-1} P_{\mathbb{S}}(B_m \times \Omega^2) \xrightarrow[m\to\infty]{} 0.
\end{aligned}$$

On the other hand, as $\overline{N}_n^{-1}$ converges $\widetilde{P}$–a.s., their distributions are tight, and

$$\lim_{\delta\to 0}\sup_n \widetilde{P}(\overline{N}_n \le \delta) = 0.$$

The second equality follows directly by an application of dominated convergence. □

Proof of Lemma 2.1.4. The statement (2.2) is in some sense an affirmation of well–definiteness. The proof consists in verifying that $\widetilde{P}\mu_\infty$ is finitely additive, that $\widetilde{P}\mu_\infty(\Omega^1\times\Omega^2) = 1$, and that $\mathscr{F}_\infty \ni B_n \times \Omega^2 \searrow \emptyset$ implies $\widetilde{P}\mu_\infty(B_n\times\Omega^2)\to 0$. The first two are quite obvious and the third one is a trivial application of the preceding Lemma 2.1.5, as is the absolute continuity (2.3). □

In the following Proposition, we introduce the variational norm

$$\|\nu-\nu'\|_{\mathscr{E}} := \sup\{\nu(B)-\nu'(B),\ B\in\mathscr{E}\},$$

where ν and ν' are probability measures on $\mathscr{E}$. This norm will be applied to $\mu_{n+m}(\cdot\times\Omega^2)$ and $\mu_\infty(\cdot\times\Omega^2)$, which are indeed, $\widetilde{P}$–a.s., probability measures on $\mathscr{F}_r^1$ because of the finiteness of $\mathscr{F}_r^1$, for all $r, m, n \in \mathbb{N}_0$.

Proposition 2.1.6. *In the regular growth phase,*

$$\lim_{m\to\infty}\sup_n \widetilde{P}\big(\|\mu_{m+n}(\cdot\times\Omega^2)-\mu_\infty(\cdot\times\Omega^2)\|_{\mathscr{F}_n^1}\big) = 0.$$

Proof. From (2.1) and its analogue for μ_∞, for $n, m \ge 0$,

$$\begin{aligned}
&\overline{N}_\infty\|\mu_{m+n}(\cdot\times\Omega^2)-\mu_\infty(\cdot\times\Omega^2)\|_{\mathscr{F}_n^1}\\
&= \overline{N}_\infty \sup_{B=B^1\times\Omega^2\in\mathscr{F}_n^1\otimes\mathscr{F}_n^2}\Big\{P_{\mathbb{S}}\Big(\zeta_n\frac{\overline{N}_m^{n,\mathbb{S}_n}}{\overline{N}_{n+m}}\mathbb{1}_B-\zeta_n\frac{\overline{N}_\infty^{n,\mathbb{S}_n}}{\overline{N}_\infty}\mathbb{1}_B\Big)\mathbb{1}_{\overline{N}_\infty>0}\Big\}\\
&\le \overline{N}_\infty P_{\mathbb{S}}\left(\zeta_n\left|\frac{\overline{N}_m^{n,\mathbb{S}_n}}{\overline{N}_{n+m}}-\frac{\overline{N}_\infty^{n,\mathbb{S}_n}}{\overline{N}_\infty}\right|\right)\mathbb{1}_{\overline{N}_\infty>0}\\
&= \mathbb{1}_{\overline{N}_\infty>0}\overline{N}_{n+m}^{-1}P_{\mathbb{S}}\big(\zeta_n|\overline{N}_\infty\overline{N}_m^{n,\mathbb{S}_n}-\overline{N}_{n+m}\overline{N}_\infty^{n,\mathbb{S}_n}|\big)\\
&\le \mathbb{1}_{\overline{N}_\infty>0}\overline{N}_{n+m}^{-1}P_{\mathbb{S}}\Big(\zeta_n\big[|\overline{N}_\infty\overline{N}_m^{n,\mathbb{S}_n}-\overline{N}_{n+m}\overline{N}_m^{n,\mathbb{S}_n}|+|\overline{N}_{n+m}\overline{N}_m^{n,\mathbb{S}_n}-\overline{N}_{n+m}\overline{N}_\infty^{n,\mathbb{S}_n}|\big]\Big)\\
&\le \frac{|\overline{N}_\infty-\overline{N}_{n+m}|}{\overline{N}_{n+m}}P_{\mathbb{S}}\big(\zeta_n\overline{N}_m^{n,\mathbb{S}_n}\big)+P_{\mathbb{S}}\big(\zeta_n|\overline{N}_m^{n,\mathbb{S}_n}-\overline{N}_\infty^{n,\mathbb{S}_n}|\big).
\end{aligned}$$

Note that in the first of the right–hand terms, the denominator is cancelled out with $P_{\mathbb{S}}\big(\zeta_n\overline{N}_m^{n,\mathbb{S}_n}\big)$; so, as $\overline{N}_n$ converges in $L^1(P)$, the P–expectation of the first term vanishes as $m\to\infty$, and the second one yields

$$\begin{aligned}
PP_{\mathbb{S}}\Big(\zeta_n\big|\overline{N}_m^{n,\mathbb{S}_n}-\overline{N}_\infty^{n,\mathbb{S}_n}\big|\Big) &= PP_{\mathbb{S}}\Big(\zeta_n P\Big(\big|\overline{N}_m^{n,\mathbb{S}_n}-\overline{N}_\infty^{n,\mathbb{S}_n}\big|\ \Big|\ \mathscr{G}_n\Big)\Big)\\
&= PP_{\mathbb{S}}\Big(\zeta_n\big\|\overline{N}_m-\overline{N}_\infty\big\|_{L^1(P)}\Big) = \|\overline{N}_m-\overline{N}_\infty\|_{L^1(P)} \xrightarrow[m\to\infty]{} 0.
\end{aligned}$$

This proves

$$\sup_n P\big(\overline{N}_\infty \|\mu_{m+n} - \mu_\infty\|_{\mathscr{F}_n^1}\big) \xrightarrow[m\to\infty]{} 0.$$

Now, we use the same trick with the Chebychev–inequality that gives us a $\overline{N}_\infty$ in front of the norm as in Lemma 2.1.4:

$$\begin{aligned}\widetilde{P}\big(\|\mu_{m+n} - \mu_\infty\|_{\mathscr{F}_n^1}\big) &= \widetilde{P}\big(\|\mu_{m+n} - \mu\|_{\mathscr{F}_n^1}(\mathbb{1}_{\overline{N}_\infty > \delta} + \mathbb{1}_{\overline{N}_\infty \le \delta})\big) \\ &\le \delta^{-1}\widetilde{P}\big(\overline{N}_\infty \|\mu_{m+n} - \mu_\infty\|_{\mathscr{F}_n^1}\big) + 2\widetilde{P}\big(\overline{N}_\infty \le \delta\big)\end{aligned}$$

tends to 0 with $\delta \to 0,\ m \to \infty$ if we control δ and m appropriately, independently of n. □

2.2 The main statements

Definition 2.2.1. *For $n \ge 1$, the rescaling of the path S is defined by*

$$S_t^{(n)} = S_{nt}/\sqrt{n}, \quad 0 \le t \le 1,$$

with $(S_t)_{t\ge 0}$ the linear interpolation of $(S_n)_{n\in\mathbb{N}}$. We write $S^{(n)}$ for $(S_t^{(n)})_{t\ge 0}$.
Furthermore, we will denote by $\mathbb{W} = \{w \in \mathscr{C}([0,1] \to \mathbb{R}^d); w(0) = 0\}$ the d–dimensional Wiener–space, equipped with the topology induced by the supremum–norm. The probability space $(\mathbb{W}, \mathscr{F}^{\mathbb{W}}, P^{\mathbb{W}})$ features the Borel–σ–algebra $F^{\mathbb{W}}$ and $P^{\mathbb{W}}$ the Wiener–measure. We will be using $W = (W_t)_{t\ge 0}$ a Wiener-process on this probability-space.

Theorem 2.2.2. *Assume $d \ge 3$ and regular growth, and the technical assumptions $m^{(3)} < \infty$, $P\big((m_{0,0}^{(2)})^2\big) < \infty$. Then, for all $F \in \mathscr{C}_b(\mathbb{W})$,*

$$\lim_{n\to\infty} \mu_n\big(F(S^{(n)})\big) = P^{\mathbb{W}}\big(F(W/\sqrt{d})\big), \tag{2.4}$$

$$\lim_{n\to\infty} \mu_\infty\big(F(S^{(n)})\big) = P^{\mathbb{W}}\big(F(W/\sqrt{d})\big), \tag{2.5}$$

in $\widetilde{P}$–probability.

Remark 2.2.3. This is equivalent to $L^p(\widetilde{P})$–convergence for any finite p.

This Theorem implies the following CLT:

Corollary 2.2.4. *Under the same assumptions as in the Theorem, for all $F \in \mathscr{C}_b(\mathbb{R}^d)$,*

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} F\Big(\frac{x}{\sqrt{n}}\Big)\frac{\overline{N}_{n,x}}{\overline{N}_n} = \int_{\mathbb{R}^d} F(x)d\nu(x), \text{ in } \widetilde{P}\text{–probability,}$$

where ν designs the Gaussian measure with mean 0 and covariance matrix $\frac{1}{d}I$.

2.3 Some easier analogue of the main Theorem

The following Proposition is not needed for the proof of our result. We literally propose it nevertheless to the reader's attention because the proof is much easier than the one of Theorem 2.2.2, while the proceeding is the same. Basically, it can be done with the one–dimensional tools we have at hand from subsection 2.1 and without the technical hassles in Lemmas 2.4.2, 2.4.8, and 2.4.13. We will try to break it down to small parts as much as we can, and refer to these parts in the proof of Theorem 2.2.2.

Proposition 2.3.1. *Assume regular growth. Then,*

$$\lim_{n\to\infty} \widetilde{P}\mu_n(S^{(n)} \in \cdot) = P^{\mathbb{W}}(W/\sqrt{d} \in \cdot), \text{ weakly,} \tag{2.6}$$

$$\lim_{n\to\infty} \widetilde{P}\mu_\infty(S^{(n)} \in \cdot) = P^{\mathbb{W}}(W/\sqrt{d} \in \cdot), \text{ weakly.} \tag{2.7}$$

The following notation will prove useful.

Definition 2.3.2. *We define, for $w \in \mathbb{W}$,*

$$\overline{F}(w) = F(w) - P^{\mathbb{W}}\big(F(\frac{W}{\sqrt{d}})\big),\ F \in \mathscr{C}_b(\mathbb{W})$$

and

$$BL(\mathbb{W}) = \{F : \mathbb{W} \to \mathbb{R};\ \|F\|_{BL} := \|F\| + \|F\|_L < \infty\}$$

the set of bounded Lipschitz–functionals on $\mathbb{W}$. The two norms are defined respectively by

$$\|F\| := \sup_{w\in\mathbb{W}} |F(w)|,$$
$$\|F\|_L := \sup\left\{\frac{F(w) - F(\widetilde{w})}{\|w - \widetilde{w}\|} :\ w \neq \widetilde{w} \in \mathbb{W}\right\}.$$

Proof of Proposition 2.3.1. The second statement is easier to prove. We attack it first, and use it later to manage the first one.

Two ingredients from outside this article will help us to prove (2.7). First, (2.7) is equivalent to

$$\lim_{m\to\infty} \widetilde{P}\mu_\infty(\overline{F}(S^{(m)})) = 0 \text{ for all } F \in BL(\mathbb{W}), \tag{2.8}$$

e.g., [Dud89, Theorem 11.3.3].

To prove (2.8), we make use of the following result for the simple random walk (S, P_S), see [AW00]: If $(n_k)_{k\geq 1} \subset \mathbb{Z}_+$ is an increasing sequence such that $\inf_{k\geq 1} n_{k+1}/n_k > 1$, then for any $F \in BL(\mathbb{W})$,

$$\lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} \overline{F}(S^{(n_k)}) = 0,\ P_S\text{–a.s..} \tag{2.9}$$

One of the key ideas of the proof is that in the last line, due to (2.3), we can replace 'P_S–a.s.' by '$\widetilde{P}\mu(\cdot \times \Omega^2)$–a.s.', and the statement still holds.

This enables us to prove (2.8) by contradiction. Assume that (2.8) does not hold. Then there is some subsequence $a_{m_l} = \widetilde{P}\mu_\infty(\overline{F}(S^{(m_l)})) > c > 0$ (or $< c < 0$). It has bounded domain, so has a convergent subsequence $a_{m_{l_k}}$ which can be chosen such that $n_k := m_{l_k}$ satisfies the above $\inf_{k\geq 1} n_{k+1}/n_k > 1$. To this n_k, we apply (2.9) and integrate with respect to $\widetilde{P}\mu_\infty$. By dominated convergence, we can switch integration and limit and get

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}\widetilde{P}\mu_\infty\big(\overline{F}(S^{(n_k)})\big)=0.$$

But this is a contradition to the assumption that all the $\widetilde{P}\mu_\infty(\overline{F}(S^{(n_k)})) = \widetilde{P}\mu_\infty(\overline{F}(S^{(m_{l_k})})) > c$ (or $< c$). So we conclude that (2.8) does hold, indeed.

Now, it remains to prove (2.6) with the help of (2.7). We need to show the analogue of (2.8):

$$\lim_{n\to\infty}\widetilde{P}\mu_n\big(\overline{F}(S^{(n)})\big)=0 \text{ for all } F\in BL(\mathbb{W}). \tag{2.10}$$

For $0\leq k\leq n$, we add some telescopic terms:

$$\begin{aligned}\widetilde{P}\mu_n\big(\overline{F}(S^{(n)})\big) = \widetilde{P}\mu_n&\big(\overline{F}(S^{(n)})-\overline{F}(S^{(n-k)})\big)\\ &+\widetilde{P}\mu_n\big(\overline{F}(S^{(n-k)})\big)-\widetilde{P}\mu_\infty\big(\overline{F}(S^{(n-k)})\big)\\ &+\widetilde{P}\mu_\infty\big(\overline{F}(S^{(n-k)})\big)\end{aligned} \tag{2.11}$$

We apply what we just proved, i.e. (2.8), and conclude that the last line vanishes for fixed k and $n\to\infty$. The middle one does the same due to Proposition 2.1.6. As for the first line, we note that $\overline{F}$ is uniformly continuous and that

$$\sup_{S\in\Omega^1}\max_{0\leq t\leq 1}\left|S_t^{(n)}-S_t^{(n-k)}\right| = O(k/\sqrt{n}).$$

Hence, (2.10) holds, so that we conclude (2.6) and thus the Proposition. □

2.4 The real work

In order to prove the statement of Theorem 2.2.2 'in probability', we take the path via 'L^2'. While the proceeding is basically the same as in the last section, the notation becomes much more complicated. As a start, we take a copy of our path $\mathbb{S}$:

Definition 2.4.1. *Let $(\widetilde{\mathbb{S}}, P_{\widetilde{\mathbb{S}}})$ be an independent copy of $(\mathbb{S}, P_{\mathbb{S}})$ defined on the probability space $(\widetilde{\Omega} = \widetilde{\Omega}^1\times\widetilde{\Omega}^2, \widetilde{\mathscr{F}})$ for $i = 1,2,3,4$. Similarily, we write $\widetilde{\zeta} = \zeta(\widetilde{\mathbb{S}})$, $P_{\mathbb{S}\widetilde{\mathbb{S}}}$, and $P_{S\widetilde{S}}$ for the simultaneous product measures and so on.*

Lemma 2.4.2. *For all $B\in\mathscr{F}^1_\infty\otimes\widetilde{\mathscr{F}}^1_\infty$, with the notation $\overline{B} = B\times\Omega^2\times\widetilde{\Omega}^2$, the following limit exists P–a.s. in the regular growth phase:*

$$\mu_\infty^{(2)}(B)=\lim_{n\to\infty}\mu_n^{\otimes 2}(\overline{B}), \tag{2.12}$$

where we define

$$\mu_n^{\otimes 2}(\overline{B})=\frac{1}{\overline{N}_n^2}P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\overline{B}}\big)\mathbb{1}_{\overline{N}_\infty>0},$$

Moreover, we have that for all $n \in \mathbb{N}$, P–a.s. on $\{\overline{N}_\infty > 0\}$,

$$\begin{aligned}
&\mu_\infty^{(2)}\Big((S,\widetilde{S})_{[0,n]} = \{(x_k,\widetilde{x}_k)\}_{k=1}^n\Big)\\
&= \frac{1}{\overline{N}_\infty^2}\sum_{\mathbf{x}_n,\widetilde{\mathbf{x}}_n\in\mathscr{V}_{\mathbb{N}_0}} \overline{N}_\infty^{n,(x_n,\mathbf{x}_n)}\overline{N}_\infty^{n,(\widetilde{x}_n,\widetilde{\mathbf{x}}_n)}\cdot \qquad (2.13)\\
&\qquad \cdot P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_n\widetilde{\zeta}_n : (S,\widetilde{S})_{[0,n]} = \{(x_k,\widetilde{x}_k)\}_{k=1}^n, (\mathbf{S}_n,\widetilde{\mathbf{S}}_n) = (\mathbf{x}_n,\widetilde{\mathbf{x}}_n)\Big).
\end{aligned}$$

For the proof, we need a few Definitions and Lemmas.

Definition 2.4.3. *For $B \in \mathscr{F}_\infty^1 \otimes \widetilde{\mathscr{F}}_\infty^1$, define the processes $(X_n)_{n\in\mathbb{N}_0}$ and $(Y_n)_{n\in\mathbb{N}_0}$ which depend on B as*

$$X_0 = X_1 := 0,\ X_n = X_n(B) := P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n \mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}}\big);$$

$$Y_0 := P_{\mathbb{S}\widetilde{\mathbb{S}}}(\overline{B}),\ Y_1 = Y_1(B) := P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_1\widetilde{\zeta}_1\mathbb{1}_{\overline{B}}\big),\ Y_n = Y_n(B) := P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}=\widetilde{\mathbb{S}}_{n-1}\}}\big).$$

Lemma 2.4.4. *Y_n converges to 0 P–almost surely, independently of B.*

Proof. A consequence of the construction of the BRWRE is that $\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1},\mathbb{S}_n=\widetilde{\mathbb{S}}_n\}} = 0$, $P\otimes P_{\mathbb{S}\widetilde{\mathbb{S}}}$–a.s., so that we have

$$\begin{aligned}
0 \le P\big(Y_n\big) &\le PP_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\mathbb{S}_{n-1}=\widetilde{\mathbb{S}}_{n-1}}\big)\\
&= PP_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\mathbb{S}_{[0,n-1]}=\widetilde{\mathbb{S}}_{[0,n-1]}}\big) \qquad (2.14)\\
&= PP_{\mathbb{S}\widetilde{\mathbb{S}}}\Bigg(\prod_{k=1}^{n-1}\frac{(A_{k,\mathbb{S}_{k-1}}^{\mathbb{S}_k})^2}{(a_{\mathbb{S}_{k-1}}^{\mathbb{S}_k})^2}\mathbb{1}_{\mathbb{S}_{[0,n-1]}=\widetilde{\mathbb{S}}_{[0,n-1]}}P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\frac{P(A_{n,\mathbb{S}_{n-1}}^{\mathbb{S}_n}A_{n,\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}|\mathscr{G}_{n-1})}{a_{\mathbb{S}_{n-1}}^{\mathbb{S}_n}a_{\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}\Big|\mathscr{F}_{n-1},\widetilde{\mathscr{F}}_{n-1}\Big)\Bigg)\\
&= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\prod_{k=1}^{n-1}\frac{1}{a_{\mathbb{S}_{k-1}}^{\mathbb{S}_k}} : \mathbb{S}_{[0,n-1]} = \widetilde{\mathbb{S}}_{[0,n-1]}\Big)\sum_{\mathbb{x},\,\mathbb{y}}\frac{P\big(A_{1,(0,\mathbf{1})}^{\mathbb{x}}A_{1,(0,\mathbf{1})}^{\mathbb{y}}\big)}{a_{\mathbb{x}}^{\mathbb{y}}}\frac{a_{\mathbb{x}}^{\mathbb{y}}}{\mathbf{m}^2}\\
&= \frac{m^{(2)}}{\mathbf{m}^2}\frac{1}{\mathbf{m}^{n-1}}
\end{aligned}$$

We made use of the fact that in the third line, because the $A_{k,\mathbb{S}_{k-1}}^{\mathbb{S}_k}$'s are indicators, we can erase the square. Also erasable is the condition in the inner P–expectation. After that, the outmost P–expectation can be taken into the first fraction, cancelling out one of the $a_{\mathbb{S}_{k-1}}^{\mathbb{S}_k}$'s. To what remains, we apply the definition of the expectation, using (1.1). This technique is hinted in the second part of the fifth line, and applied similarly to the first part.

The assertion now follows from the Borel–Cantelli lemma. □

Lemma 2.4.5. *X_n is a submartingale with respect to $\mathscr{G}_n$.*

Proof. We start calculating

$$\begin{aligned}
P(X_n|\mathscr{G}_{n-1}) &= P\big(P_{\mathbb{S}\widetilde{\mathbb{S}}}(\zeta_n\widetilde{\zeta}_n\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}})\big|\mathscr{G}_{n-1}\big)\\
&= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Bigg(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}}P\Big(\frac{A_{n,\mathbb{S}_{n-1}}^{\mathbb{S}_n}}{a_{\mathbb{S}_{n-1}}^{\mathbb{S}_n}}\frac{A_{n,\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}{a_{\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}\Big)\Bigg). \qquad (2.15)
\end{aligned}$$

We do not use the following definition again, but we should like to point out its similarity to W to be defined later. The inner P–expectation computes as

$$w(\mathbb{x},\widetilde{\mathbb{x}},\mathbb{y},\widetilde{\mathbb{y}}) := \frac{P(A_{1,\mathbb{x}}^{\mathbb{y}} A_{1,\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}})}{a_{\mathbb{x}}^{\mathbb{y}} a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}}} = \begin{cases} 0 & \text{if } a_{\mathbb{x}}^{\mathbb{y}} a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} = 0, \\ 1 & \text{if } x \neq \widetilde{x},\ a_{\mathbb{x}}^{\mathbb{y}} a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0, \\ \frac{P(\sum_{i\geq k} q_{0,0}(i) \sum_{j \geq l} q_{0,0}(j))}{\sum_{i\geq k} q(i) \sum_{j\geq l} q(j)} & \text{if } \begin{matrix} x = \widetilde{x},\ \mathbf{x} \neq \widetilde{\mathbf{x}}, \\ \mathbf{y}/\mathbf{x} = k,\ \widetilde{\mathbf{y}}/\widetilde{\mathbf{x}} = l, \end{matrix} a_{\mathbb{x}}^{\mathbb{y}} a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0, \\ 0 & \text{if } \mathbb{x} = \widetilde{\mathbb{x}},\ y \neq \widetilde{y},\ a_{\mathbb{x}}^{\mathbb{y}} a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0, \\ \left[\frac{1}{2d} \sum_{j \geq \min\{k,l\}} q(j)\right]^{-1} & \text{if } \begin{matrix} \mathbb{x} = \widetilde{\mathbb{x}},\ y = \widetilde{y}, \\ \mathbf{y}/\mathbf{x} = k,\ \widetilde{\mathbf{y}}/\widetilde{\mathbf{x}} = l, \end{matrix} a_{\mathbb{x}}^{\mathbb{y}} a_{\widetilde{\mathbb{x}}}^{\widetilde{\mathbb{y}}} \neq 0. \end{cases}$$

Using this, we note that, under the condition $\{\mathbf{S}_{n-1} \neq \widetilde{\mathbf{S}}_{n-1}\}$, $w(\mathbb{S}_{n-1}, \widetilde{\mathbb{S}}_{n-1}, \mathbb{S}_n, \widetilde{\mathbb{S}}_n)$ depends only on $S_{n-1} - \widetilde{S}_{n-1}$, $\mathbf{S}_n/\mathbf{S}_{n-1}$ and $\widetilde{\mathbf{S}}_n/\widetilde{\mathbf{S}}_{n-1}$. Thus, we pursue

$$(2.15) = P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(\zeta_{n-1}\widetilde{\zeta}_{n-1} \mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}}(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}} + \alpha \mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}})\big), \tag{2.16}$$

where $\alpha = P(m_{0,0}^2)/\mathbf{m}^2 > 1$. This last equality is obtained by introducing a $P_{\mathbb{S}\widetilde{\mathbb{S}}}(\ \cdot\ |\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1})$–conditional expectation, and remarking that the event $\overline{B}$ depends only on the random walk–part while the corresponding above fraction depends only on the children–part, and the two are thus independent. The calculus reads as follows:

$$\begin{aligned} &P_{\mathbb{S}\widetilde{\mathbb{S}}}\Bigg(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\{S_{n-1}=\widetilde{S}_{n-1}, \mathbf{S}_{n-1}\neq\widetilde{\mathbf{S}}_{n-1}\}} \\ &\qquad P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\frac{P(\sum_{i\geq \mathbf{S}_n/\mathbf{S}_{n-1}} q_{0,0}(i) \sum_{j\geq\widetilde{\mathbf{S}}_n/\widetilde{\mathbf{S}}_{n-1}} q_{0,0}(j))}{\sum_{i\geq \mathbf{S}_n/\mathbf{S}_{n-1}} q(i) \sum_{j\geq\widetilde{\mathbf{S}}_n/\widetilde{\mathbf{S}}_{n-1}} q(j)} : \overline{B}\Big|\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1}\Big)\Bigg) \\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\{S_{n-1}=\widetilde{S}_{n-1}, \mathbf{S}_{n-1}\neq\widetilde{\mathbf{S}}_{n-1}\}} \\ &\qquad \sum_{\mathbf{x},\mathbf{y}} \frac{P(\sum_{i\geq \mathbf{x}/\mathbf{S}_{n-1}} q_{0,0}(i) \sum_{j\geq\mathbf{y}/\widetilde{\mathbf{S}}_{n-1}} q_{0,0}(j))}{\mathbf{m}^2} P_{\mathbb{S}\widetilde{\mathbb{S}}}(\overline{B}|\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1})\Big) \\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(P_{\mathbb{S}\widetilde{\mathbb{S}}}(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\{S_{n-1}=\widetilde{S}_{n-1}, \mathbf{S}_{n-1}\neq\widetilde{\mathbf{S}}_{n-1}, \overline{B}\}} \frac{P(m_{0,0}^2)}{\mathbf{m}^2}|\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1})\big) \end{aligned}$$

The BRWRE has, due to the strict construction of the ancestry, the feature that

$$\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}\}} \geq \zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}\}}.$$

So, we continue (2.16) and finish the proof of the submartingale property by

$$(2.16) \geq P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}\cap\{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}\}}\Big) = X_{n-1}.$$

□

Notation 2.4.6. For some sequence $(a_n)_{n\geq 0}$, we set $\Delta a_n := a_n - a_{n-1}$ for $n \geq 1$.

This notation is convenient when we treat the Doob–decomposition of the process X_n from Definition 2.4.3, i.e.

Definition 2.4.7.

$$X_n = X_n(B) =: M_n + \widehat{A}_n, \tag{2.17}$$

with M_n a martingale, $M_0 = \widehat{A}_0 = 0$, and $\widehat{A}_n$ the increasing process defined by its increments $\Delta\widehat{A}_n := P(\Delta X_n|\mathscr{G}_{n-1})$. By $\langle M\rangle_n$, we denote the quadratic variation of $(M_n)_n$, defined by $\Delta\langle M\rangle_n := P\big((\Delta M_n)^2|\mathscr{G}_{n-1}\big)$. Passing to the limit, we define

$$\widehat{A}_\infty := \lim_{n\to\infty}\widehat{A}_n, \quad \langle M\rangle_\infty := \lim_{n\to\infty}\langle M\rangle_n. \tag{2.18}$$

The next Lemma deals with the two processes $\widehat{A}_n$ and M_n:

Lemma 2.4.8.

$$\widehat{A}_\infty < \infty \text{ and } \langle M\rangle_\infty < \infty, \quad P\text{–a.s.}.$$

Now take a sequence of events $(B_m)_{m\in\mathbb{N}_0}$ verifying $P^{\otimes 2}_{\mathbb{S}\widetilde{\mathbb{S}}}(\overline{B}_m) \searrow 0$ with $m\to\infty$. If we replace B by B_m and define $X^m_n := X_n(B_m)$ together with its Doob–decomposition $M^m_n + \widehat{A}^m_n$, $m,n\in\mathbb{N}_0$, and the corresponding limits as in (2.18)*, we have*

$$\widehat{A}^m_\infty \xrightarrow[m\to\infty]{} 0 \text{ and } \langle M^m\rangle_\infty \xrightarrow[m\to\infty]{} 0, \quad P\text{–a.s.}. \tag{2.19}$$

The proof is lengthy and will be postponed a little bit. But with this Lemma at hand, we can catch up on the

Proof of Lemma 2.4.2. Applying the 'B'–version of the last Lemma, we get that X_n converges, and by the convergence of $\overline{N}_n^{-2}$, $\mu_n^{\otimes 2} = \overline{N}_n^{-2}(X_n + Y_n)\mathbb{1}_{\overline{N}_\infty > 0}$ as well, $\widetilde{P}$–a.s. On the event of extinction, the statement is trivial, and we conclude (2.12).

The second statement (2.13) follows immediately from the definition. □

In order to prove Lemma 2.4.8, we also need the so called *replica overlap*, which is the probability of two particles to meet at the same place:

$$\mathscr{R}_n := \mathbb{1}_{N_n>0}\sum_x \frac{N_{n,x}}{N_n}.$$

This replica overlap can be related to the event of survival via a Corollary of the following general result for martingales [Yos10, Proposition 2.1.2].

Proposition 2.4.9. *Let $(\mathsf{Y}_n)_{n\in\mathbb{N}_0}$ be a mean–zero martingale on a probability space with measure $\mathbb{E}$ and filtration $(\mathscr{I}_n)_{n\in\mathbb{N}_0}$ such that $-1 \le \Delta\mathsf{Y}_n$, $\mathbb{E}$–a.s. and*

$$\mathsf{X}_n := \prod_{m=1}^n (1+\Delta\mathsf{Y}_m).$$

Then,

$$\{\mathsf{X}_\infty > 0\} \supseteq \{\mathsf{X}_n > 0 \text{ for all } n \ge 0\} \cap \left\{\sum_{N=1}^\infty \mathbb{E}\big[(\Delta\mathsf{Y}_n)^2\big|\mathscr{I}_{n-1}\big] < \infty\right\}, \ \mathbb{E}\text{–a.s.}, \tag{2.20}$$

holds if Y_n is square–integrable and $\mathbb{E}\left[(\Delta Y_n)^2\middle|\mathscr{I}_{n-1}\right]$ is uniformly bounded.
The opposite inclusion is provided by Y_n being cube–integrable and

$$\mathbb{E}\left[(\Delta \mathsf{Y}_n)^3\middle|\mathscr{I}_{n-1}\right] \le const \cdot \mathbb{E}\left[(\Delta \mathsf{Y}_n)^2\middle|\mathscr{I}_{n-1}\right].$$

Corollary 2.4.10. *Suppose $P(\overline{N}_\infty > 0) > 0$ and $m^{(3)} < \infty$. Then*

$$\{\overline{N}_\infty > 0\} = \{survival\} \cap \Big\{\sum_{n\ge 0}\mathscr{R}_n < \infty\Big\}, \ P\text{–a.s.}.$$

For proving this Corollary, we start with some notation.
Notation 2.4.11. Define

$$U_{n+1,x} := \frac{\mathbb{1}_{N_n>0}}{\mathbf{m}N_n}\sum_{\substack{\mathbf{x}\in\mathscr{V}_{\mathbb{N}_0}:\\ N_{n,(x,\mathbf{x})}=1}} K_{n,(x,\mathbf{x})} \ge 0.$$

It is imporant to note that the sum in this definition is taken over exactly $N_{n,x}$ random variables. Also define

$$U_{n+1} := \sum_{x\in\mathbb{Z}^d} U_{n+1,x} = \frac{N_{n+1}}{\mathbf{m}N_n}\mathbb{1}_{N_n>0} = \frac{\overline{N}_{n+1}}{\overline{N}_n}\mathbb{1}_{\overline{N}_n>0}.$$

The $(U_{n+1,x})_{x\in\mathbb{Z}^d}$ are independent under $P(\cdot|\mathscr{G}_n)$. It is not difficult to see that, on the event $\{N_n > 0\}$,

$$P(U_{n+1,x}|\mathscr{G}_n) = \rho_n(x), \text{ and hence } P(U_{n+1}|\mathscr{G}_n) = 1.$$

Also, with $\widetilde{c}_i = \frac{m^{(i)}}{\mathbf{m}^i}$, $i = 2, 3$,

$$\begin{aligned}\alpha\rho(x)^2 = \frac{1}{\mathbf{m}^2N_n^2}N_{n,x}^2 Q(m_{n,x}^2) &\le P\left(U_{n+1,x}^2\middle|\mathscr{G}_n\right)\\ &= \frac{1}{\mathbf{m}^2N_n^2}P\left(\Big(\sum_{\substack{\mathbf{x}\in\mathscr{V}_{\mathbb{N}_0}:\\ N_{n,(x,\mathbf{x})}=1}} K_{n,(x,\mathbf{x})}\Big)^2\middle|\ \mathscr{G}_n\right) \le \frac{N_{n,x}^2 m^{(2)}}{\mathbf{m}^2N_n^2} = \widetilde{c}_2\rho_n(x)^2,\end{aligned}$$

$$P\left(U_{n+1,x}^3\middle|\mathscr{G}_n\right) \le \frac{m^{(3)}}{\mathbf{m}^3}\rho_n(x)^3 = \widetilde{c}_3\rho_n(x)^3, \text{ again on the event } \{N_n > 0\}.$$

Proof of Corollary 2.4.10. We need to verify the prerequisites of Proposition 2.4.9 which we apply to $\mathsf{X}_n := \overline{N}_n$ and

$$\Delta\mathsf{Y}_n := \frac{\overline{N}_n}{\overline{N}_{n-1}}\mathbb{1}_{\overline{N}_{n-1}>0} - \mathbb{1}_{\overline{N}_{n-1}>0} = \sum_x [U_{n,x} - \rho_{n,x}] \ge 1.$$

The second moments compute as

$$\begin{aligned}P\left((\Delta\mathsf{Y}_n)^2\middle|\mathscr{G}_n\right) &= P\left((\sum_x[U_{n,x} - \rho_{n-1,x}])^2\middle|\mathscr{G}_n\right)\\ &= \sum_{x,y} P\left((U_{n,x} - \rho_{n-1,x})(U_{n,y} - \rho_{n-1,y})\middle|\mathscr{G}_{n-1}\right) = \sum_x P\left((U_{n,x} - \rho_{n-1,x})^2\middle|\mathscr{G}_{n-1}\right)\\ &= \sum_x \left[P\left(U_{n,x}^2\middle|\mathscr{G}_{n-1}\right) - \rho_{n-1,x}^2\right].\end{aligned}$$

Using the observations after Notation 2.4.11, we hence get

$$\Big(\frac{Q(m_{n,x}^2)}{\mathbf{m}^2}-1\Big)\sum_x \rho_{n-1,x}^2 \le P\Big(\big(\Delta \mathrm{Y}_n\big)^2\Big|\mathscr{G}_n\Big) \le \Big(\frac{m^{(2)}}{\mathbf{m}^2}-1\Big)\sum_x \rho_{n-1,x}^2.$$

Similar observations lead to estimate for the third moment:

$$\begin{aligned} P\big((\Delta \mathrm{Y}_n)^3\big|\mathscr{G}_{n-1}\big) &= P\Big(\big(\sum_x [U_{n,x}-\rho_{n-1,x}]\big)^3\Big|\mathscr{G}_{n-1}\Big)\\ &= \sum_x P\Big((U_{n,x}-\rho_{n-1,x})^3\Big|\mathscr{G}_{n-1}\Big)\\ &\le 3\sum_x P\Big(U_{n,x}^3+\rho_{n-1,x}^3\Big|\mathscr{G}_{n-1}\Big) \le \Big(\frac{m^{(3)}}{\mathbf{m}^3}-1\Big)\sum_x \rho_{n-1,x}^3. \end{aligned}$$

This proves that all hypotheses of Proposition 2.4.9 are fulfilled and in fact equality holds for (2.20). □

Proof of Lemma 2.4.8. We make a slight abuse of notation writing $B_{(m)}$ as templates for both the cases B and B_m, and so on for similar cases of notation. We can make use of (2.16) and, splitting two times 1 into complementary indicators, get

$$\begin{aligned} \Delta\widehat{A}_n^{(m)} &= P\big(\Delta X_n^{(m)}\big|\mathscr{G}_{n-1}\big)\\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}_{(m)}}\Big[\big(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}+\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\big)\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}}\\ &\qquad +\big(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}+\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\big)\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}=\widetilde{\mathbb{S}}_{n-2}}\\ &\qquad -\big(\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}+\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\big)\mathbb{1}_{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}}\Big]\Big). \end{aligned} \tag{2.21}$$

In the last term, $\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}$ is implied by the following indicator, while in the second term, $\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}$ is 0 due to the fact that $\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}=\widetilde{\mathbb{S}}_{n-2}}=0$, $P\otimes P_{\mathbb{S}\widetilde{\mathbb{S}}}$–a.s.. Thus, we can continue

$$\begin{aligned} (2.21) &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}_{(m)}}\Big[(\alpha-1)\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}\neq\widetilde{\mathbb{S}}_{n-2}}\\ &\qquad +\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\mathbb{1}_{\mathbb{S}_{n-2}=\widetilde{\mathbb{S}}_{n-2}}\Big]\Big)\\ &\le \alpha P_{\mathbb{S}\widetilde{\mathbb{S}}}\Big(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}\mathbb{1}_{\overline{B}_{(m)}}\Big). \end{aligned}$$

The sum

$$Z^{\widehat{A}} := \sum_n \zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}}$$

is $P_{\mathbb{S}\widetilde{\mathbb{S}}}$–integrable, thanks to Corollary 2.4.10 together with Lemma 1.3.1 and Lemma 1.3.3. So, summation over all $n\in\mathbb{N}$ yields

$$\widehat{A}_n^{(m)} \nearrow \widehat{A}_\infty^{(m)} \le P_{\mathbb{S}\widetilde{\mathbb{S}}}\big(Z^{\widehat{A}}:B_{(m)}\big)\begin{cases} <\infty & \text{for } B_{(m)}=B\\ \xrightarrow[m\to\infty]{} 0 & \text{for } B_{(m)}=B_m, \end{cases} \tag{2.22}$$

P–almost surely.

Now, the same sort of estimates will be carried out for M_n, but involves much more work.

First, we note that $\Delta M_n^{(m)}$ can be written as

$$\begin{aligned}\Delta M_n^{(m)} &= X_n^{(m)} - P\big(X_n^{(m)}\big|\mathscr{G}_{n-1}\big)\\ &= P_{\mathbb{S}\widetilde{\mathbb{S}}}\Bigg(\zeta_{n-1}\widetilde{\zeta}_{n-1}\mathbb{1}_{\overline{B}_{(m)}}\Bigg[\frac{A_{n,\mathbb{S}_{n-1}}^{\mathbb{S}_n}}{a_{\mathbb{S}_{n-1}}^{\mathbb{S}_n}}\frac{A_{n,\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}{a_{\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}\mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\\ &\qquad - \mathbb{1}_{\mathbb{S}_{n-1}\neq\widetilde{\mathbb{S}}_{n-1}}\big(\alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}} + \mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}\big)\Bigg]\Bigg).\end{aligned}$$

Definition 2.4.12. *For convenience, we define*

$$\varphi_n(\mathbb{S},\widetilde{\mathbb{S}}) := \frac{A_{n,\mathbb{S}_{n-1}}^{\mathbb{S}_n}}{a_{\mathbb{S}_{n-1}}^{\mathbb{S}_n}}\frac{A_{n,\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}}{a_{\widetilde{\mathbb{S}}_{n-1}}^{\widetilde{\mathbb{S}}_n}} - \alpha\mathbb{1}_{S_{n-1}=\widetilde{S}_{n-1}} - \mathbb{1}_{S_{n-1}\neq\widetilde{S}_{n-1}}.$$

This is the point where we cannot maintain our easy notation of $\mathbb{S}$ and $\widetilde{\mathbb{S}}$, for we need four independent random walks $\mathbb{S}^{[1]}$, $\mathbb{S}^{[2]}$, $\mathbb{S}^{[3]}$, $\mathbb{S}^{[4]}$. The probability spaces and other notations are adjusted accordingly, refer to Definition 2.4.1. We compute

$$\begin{aligned}\Delta\langle M^{(m)}\rangle_n &= P\big((\Delta M_n^{(m)})^2\big|\mathscr{G}_{n-1}\big)\\ &= P_{\mathbb{S}}^{\otimes 4}\Big(\zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{\overline{B}_{(m)}\times\overline{B}_{(m)}}\mathbb{1}_{\mathbb{S}_{n-1}^{[1]}\neq\mathbb{S}_{n-1}^{[2]}}\mathbb{1}_{\mathbb{S}_{n-1}^{[3]}\neq\mathbb{S}_{n-1}^{[4]}}\\ &\qquad P\big(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})\big)\Big).\end{aligned}\tag{2.23}$$

We note that if $S_{n-1}^{[i]} \neq S_{n-1}^{[j]}$ for $i = 1,2$ and $j = 3,4$, then $\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})$ and $\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})$ are independent, and that under $\{\mathbb{S}_{n-1}^{[1]} \neq \mathbb{S}_{n-1}^{[2]}\}$, it holds that $P_{\mathbf{S}^{[1]},\mathbf{S}^{[2]}}\big(P(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]}))\big) = 0$, where $P_{\mathbf{S}^{[1]},\mathbf{S}^{[2]}}$ is the probability measure with respect to $(\mathbf{S}^{[1]},\mathbf{S}^{[2]})$. From these observations, we get

$$\begin{aligned}(2.23) \le \sum_{i=1,2;j=3,4} P_{\mathbb{S}}^{\otimes 4}\Big(&\zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{\overline{B}_{(m)}\times\overline{B}_{(m)}}\mathbb{1}_{\mathbb{S}_{n-1}^{[1]}\neq\mathbb{S}_{n-1}^{[2]}}\mathbb{1}_{\mathbb{S}_{n-1}^{[3]}\neq\mathbb{S}_{n-1}^{[4]}}\\ &P\big(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})\big) : S_{n-1}^{i} = S_{n-1}^{j}\Big).\end{aligned}\tag{2.24}$$

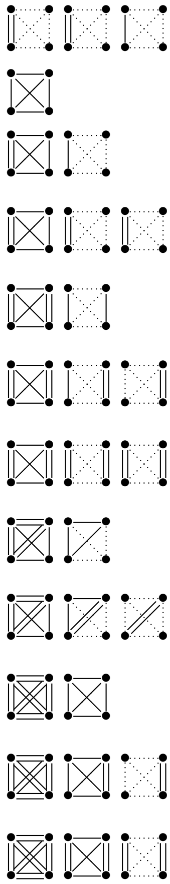
It is clear that

$$P\big(\varphi_n(\mathbb{S}^{[1]},\mathbb{S}^{[2]})\varphi_n(\mathbb{S}^{[3]},\mathbb{S}^{[4]})\big) \le P\left(\frac{A_{n,\mathbb{S}_{n-1}^{[1]}}^{\mathbb{S}_n^{[1]}} A_{n,\mathbb{S}_{n-1}^{[2]}}^{\mathbb{S}_n^{[2]}} A_{n,\mathbb{S}_{n-1}^{[3]}}^{\mathbb{S}_n^{[3]}} A_{n,\mathbb{S}_{n-1}^{[4]}}^{\mathbb{S}_n^{[4]}}}{a_{\mathbb{S}_{n-1}^{[1]}}^{\mathbb{S}_n^{[1]}} a_{\mathbb{S}_{n-1}^{[2]}}^{\mathbb{S}_n^{[2]}} a_{\mathbb{S}_{n-1}^{[3]}}^{\mathbb{S}_n^{[3]}} a_{\mathbb{S}_{n-1}^{[4]}}^{\mathbb{S}_n^{[4]}}}\right).$$

We define $W(X,Y)$ for $X = \big(\mathbb{x}^{[1]},\mathbb{x}^{[2]},\mathbb{x}^{[3]},\mathbb{x}^{[4]}\big)$, $Y = \big(\mathbb{y}^{[1]},\mathbb{y}^{[2]},\mathbb{y}^{[3]},\mathbb{y}^{[4]}\big)$ by

$$W(X,Y) = P\left(A_{1,\mathbb{x}^{[1]}}^{\mathbb{y}^{[1]}}A_{1,\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}}A_{1,\mathbb{x}^{[3]}}^{\mathbb{y}^{[3]}}A_{1,\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}}\right).$$

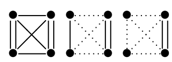
$W(X,Y)$ is zero whenever $\{a4\} := a^{\mathbb{S}_n^{[1]}}_{\mathbb{S}_{n-1}^{[1]}} a^{\mathbb{S}_n^{[2]}}_{\mathbb{S}_{n-1}^{[2]}} a^{\mathbb{S}_n^{[3]}}_{\mathbb{S}_{n-1}^{[3]}} a^{\mathbb{S}_n^{[4]}}_{\mathbb{S}_{n-1}^{[4]}}$ is zero; we hence care only about cases where $\{a4\} \neq 0$. Also, remember that (2.23) restricts to the event $\{\mathbb{S}_{n-1}^{[1]} \neq \mathbb{S}_{n-1}^{[2]},\ \mathbb{S}_{n-1}^{[3]} \neq \mathbb{S}_{n-1}^{[4]}\}$. Such cases can be separated as follows, with the definition $k^{[j]} := \mathbf{y}^{[j]}/\mathbf{x}^{[j]}$ for $j = 1,2,3,4$:



0	$x^{[1]} = x^{[3]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]},\ y^{[1]} \neq y^{[3]},\ \{a4\} \neq 0$
1	$x^{[j]} \neq x^{(\ell)}\ \forall\ j,\ell \in \{1,2,3,4\}:\ j \neq \ell,\ \{a4\} \neq 0$
2	$x^{[1]} = x^{[3]} \neq x^{[2]} \neq x^{[4]} \neq x^{[1]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \{a4\} \neq 0$
3	$x^{[1]} = x^{[3]} \neq x^{[2]} \neq x^{[4]} \neq x^{[1]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]},\ y^{[1]} = y^{[3]},\ \{a4\} \neq 0$
4	$x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} \neq \mathbf{x}^{[4]},\ \{a4\} \neq 0$
5	$x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0$
6	$x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[1]} = y^{[3]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0$
7	$x^{[1]} = x^{[3]} = x^{[2]} \neq x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]} \neq \mathbf{x}^{[2]},\ \{a4\} \neq 0$
8	$x^{[1]} = x^{[3]} = x^{[2]} \neq x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]} = \mathbf{x}^{[2]},\ y^{[3]} = y^{[2]},\ \{a4\} \neq 0$
9	$x^{[1]} = x^{[3]} = x^{[2]} = x^{[4]},\ \mathbf{x}^{[j]} \neq \mathbf{x}^{(\ell)}\ \forall\ j,\ell \in \{1,2,3,4\}:\ j \neq \ell,\ \{a4\} \neq 0$
10	$x^{[1]} = x^{[3]} = x^{[2]} = x^{[4]},\ \mathbf{x}^{[2]} \neq \mathbf{x}^{[3]} \neq \mathbf{x}^{[1]} \neq \mathbf{x}^{[2]} = \mathbf{x}^{[4]}\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0$
11	$x^{[1]} = x^{[3]} = x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} = \mathbf{x}^{[3]} \neq \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[1]} = y^{[3]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0$

Cases that can be obtained by symmetry are not listed here. Case 0 yields $W(X,Y) = 0$ for it is impossible in the BRWRE–Model: particles with the same name at the same place are blown by the wind to the same site, so their children cannot be born at different sites.

The notation with the small squares is solely for the ease of understanding; all information is fully contained in the written part. For how to read it, let us take as an example case number 5:



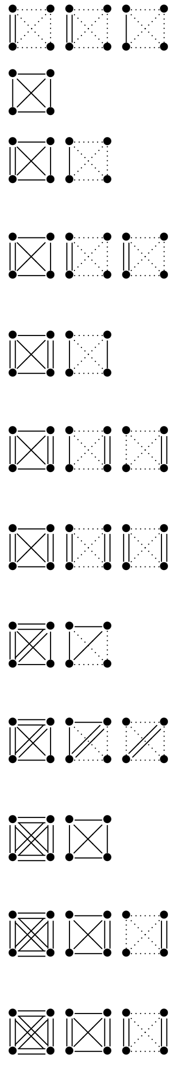
5 $\quad x^{[1]} = x^{[3]} \neq x^{[2]} = x^{[4]},\ \mathbf{x}^{[1]} \neq \mathbf{x}^{[3]},\ \mathbf{x}^{[2]} = \mathbf{x}^{[4]},\ y^{[2]} = y^{[4]},\ \{a4\} \neq 0$

The first square corresponds to the 'x'–part, the second one to the '$\mathbf{x}$'–part, and the last one to the 'y'–part of the restriction. Each • corresponds to an index $j = 1,\ldots,4$, read left–right, top–down. The two left bullets of the first square are connected with a double stroke, read: equality sign, just as the two left ones. Indeed, $x^{[1]} = x^{[3]}$ and $x^{[2]} = x^{[4]}$. All other connections are single–stroked, and are supposed to be read as inequalties. The second square conveys hence the information that $\mathbf{x}^{[1]} \neq \mathbf{x}^{[3]}$ and $\mathbf{x}^{[2]} = \mathbf{x}^{[4]}$. The other dotted connections indicate that both the cases of equality and inequality are comprised. Lastly, the third square stands for all $y^{[j]},\ j \in \{1,\ldots,4\}$ with $y^{[2]} = y^{[4]}$.

If one changes the mapping of bullet–position and index, one gets all the symmetries immediately. A missing square has the same meaning as a square with only dotted lines would have.

Now, we can compute $W(X,Y)$, which equals in the respective cases to:

$$\begin{cases}
0 & 0\\
a_{\mathbb{x}^{[1]}}^{\mathbb{y}^{[1]}} a_{\mathbb{x}^{[3]}}^{\mathbb{y}^{[3]}} a_{\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}} a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 1\\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\ge k^{[1]}} q_{00}(i) \sum_{i\ge k^{[3]}} q_{00}(i)\Big) a_{\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}} a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 2\\
\left(\frac{1}{2d}\right) P\Big(\sum_{i\ge \max\{k^{[1]},k^{[3]}\}} q_{00}(i)\Big) a_{\mathbb{x}^{[2]}}^{\mathbb{y}^{[2]}} a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 3\\
\left(\frac{1}{2d}\right)^4 P\Big(\sum_{i\ge k^{[1]}} q_{00}(i) \sum_{i\ge k^{[3]}} q_{00}(i)\Big) P\Big(\sum_{i\ge k^{[2]}} q_{00}(i) \sum_{i\ge k^{[4]}} q_{00}(i)\Big) & 4\\
\left(\frac{1}{2d}\right)^3 P\Big(\sum_{i\ge k^{[1]}} q_{00}(i) \sum_{i\ge k^{[3]}} q_{00}(i)\Big) P\Big(\sum_{i\ge \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 5\\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\ge \max\{k^{[1]},k^{[3]}\}} q_{00}(i)\Big) P\Big(\sum_{i\ge \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 6\\
\left(\frac{1}{2d}\right)^3 P\Big(\sum_{i\ge k^{[1]}} q_{00}(i) \sum_{i\ge k^{[3]}} q_{00}(i) \sum_{i\ge k^{[2]}} q_{00}(i)\Big) a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 7\\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\ge k^{[1]}} q_{00}(i) \sum_{i\ge \max\{k^{[3]},k^{[2]}\}} q_{00}(i)\Big) a_{\mathbb{x}^{[4]}}^{\mathbb{y}^{[4]}} & 8\\
\left(\frac{1}{2d}\right)^4 P\Big(\sum_{i\ge k^{[1]}} q_{00}(i) \sum_{i\ge k^{[3]}} q_{00}(i) \sum_{i\ge k^{[2]}} q_{00}(i) \sum_{i\ge k^{[4]}} q_{00}(i)\Big) & 9\\
\left(\frac{1}{2d}\right)^3 P\Big(\sum_{i\ge k^{[1]}} q_{00}(i) \sum_{i\ge k^{[3]}} q_{00}(i) \sum_{i\ge \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 10\\
\left(\frac{1}{2d}\right)^2 P\Big(\sum_{i\ge \max\{k^{[1]},k^{[3]}\}} q_{00}(i) \quad \sum_{i\ge \max\{k^{[2]},k^{[4]}\}} q_{00}(i)\Big) & 11
\end{cases}$$



The number of *different* points in the first square corresponds to the number of separate expectations (there are expectations hidden in the $a_{\mathbb{x}^{[j]}}^{\mathbb{y}^{[j]}}$'s). The equalities in the second square that are written down are important inasmuch as they decide about which sums become united to one sum running over $i \ge \max\{\cdots\}$. The third square decides if in fact the case is at all possible. The exponent of the fraction corresponds to the number of summation marks (there are summation marks hidden in the $a_{\mathbb{x}^{[j]}}^{\mathbb{y}^{[j]}}$'s, but fractions, as well, so these $a_{\mathbb{x}^{[j]}}^{\mathbb{y}^{[j]}}$'s do not contribute to the exponent of the fraction).

Now, we can continue with $\Delta\langle M^{(m)}\rangle_n$. To get from (2.24) to the following line, one can apply the same trick with insterted conditional expectations as in the succession of equalities (2.14), and pick the worst case, which is case 11. We continue (2.24) and find

$$\Delta\langle M^{(m)}\rangle_n \le \sum_{i=1,2;j=3,4} C P_{\mathbb{S}}^{\otimes 4}\Big(\zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{S_{n-1}^{[i]}=S_{n-1}^{[j]}} : \overline{B}_{(m)}\times\overline{B}_{(m)}\Big),$$

where $C = c\, Q\big((m_{0,0}^{(2)})^2\big)/\mathbf{m}^4 < \infty$ and c is a constant depending only on d.

$$Z_{i,j}^M := \sum_n \zeta_{n-1}^{[1]}\zeta_{n-1}^{[2]}\zeta_{n-1}^{[3]}\zeta_{n-1}^{[4]}\mathbb{1}_{S_{n-1}^{[i]}=S_{n-1}^{[j]}}$$

serves the same aim as $Z^{\widehat{A}}$ in (2.22), and is $P_{\mathbb{S}}^{\otimes 4}$–integrable for the same reasons as for $Z^{\widehat{A}}$. So, in the same manner, we conclude

$$\langle M^{(m)}\rangle_\infty = \sum_n \Delta\langle M^{(m)}\rangle_n \le \sum_{i=1,2;j=3,4} P_{\mathbb{S}}^{\otimes 4}(Z_{i,j}^M : \overline{B}_{(m)}\times\overline{B}_{(m)}) \begin{cases} < \infty & \text{for } B_{(m)} = B \\ \xrightarrow[m\to\infty]{} 0 & \text{for } B_{(m)} = B_m, \end{cases}$$

P–almost surely. This finishes the proof of Lemma 2.4.8. □

Lemma 2.4.13.

$$\widetilde{P}\mu_\infty^{(2)} \text{ is a probability measure on } \mathscr{F}_\infty^1 \otimes \widetilde{\mathscr{F}}_\infty^1. \tag{2.25}$$

$$\widetilde{P}\mu_\infty^{(2)} \ll P_{S\widetilde{S}}(\cdot\times(\Omega^2\times\widetilde{\Omega}^2)) \text{ on } \mathscr{F}_\infty^1 \otimes \widetilde{\mathscr{F}}_\infty^1. \tag{2.26}$$

Proof. As in the proof of (2.2), (2.25) and (2.26) boil down to proving that

$$\lim_{m\to\infty} \widetilde{P}\mu_\infty^{(2)}(\overline{B}_m) = 0,$$

for $\{B_m\} \subset (\mathscr{F}^1)^{\otimes 2}$ with $\lim_{m\to\infty} P_{S\widetilde{S}}(\overline{B}_m) = 0$. We show, in a way similar to the very end of the proof of Lemma 2.4.2,

$$\lim_{m\to\infty} \mu_\infty^{(2)}(B_m) = \lim_{m\to\infty}\lim_{n\to\infty} \mu_n^{\otimes 2}(\overline{B}_m) = 0 \text{ in } \widetilde{P}\text{–probability},$$

by proving that

$$\lim_{m\to\infty} \sup_n X_n^m = 0 \text{ in } \widetilde{P}\text{–probability}, \tag{2.27}$$

where $X_n^m = X_n(B_m)$ defined for B_m. Let also

$$X_n^m =: M_n^m + \widehat{A}_n^m$$

be the submartingale decomposition as in (2.17) and as hinted in Lemma 2.4.8. Now, we can apply the 'B_m'–version of Lemma 2.4.8. $\widehat{A}_n^m$ is taken care of by the first statement of (2.19), and for M_n^m, the second statement and a little calculus will yield

$$\lim_{m\nearrow\infty} \sup_n |M_n^m| = 0 \text{ in } \widetilde{P}\text{–probability}. \tag{2.28}$$

In fact, for $\ell \in \mathbb{R}$, let $\tau_\ell^m = \inf\{n \ge 0 : \langle M^m\rangle_{n+1} > \ell\}$. Then,

$$P\Big(\sup_n \big|M_n^m\big| \ge \varepsilon,\ \overline{N}_\infty > 0\Big) \le P\Big(\langle M^m\rangle_\infty > \ell,\ \overline{N}_\infty > 0\Big) + P\Big(\sup_n \big|M_n^m\big| \ge \varepsilon,\ \tau_\ell^m = \infty\Big).$$

Clearly, the first term on the right–hand–side vanishes as $m \nearrow \infty$ because of (2.19), and so does the second term as can be seen from the following application of Doob's inequality (for instance [Dur91, p.248]):

$$P\Big(\sup_n \big|M_n^m\big| \geq \varepsilon,\ \tau_\ell^m = \infty\Big) \leq P\Big(\sup_n \Big|M_{n\wedge\tau_\ell^m}^m\Big| \geq \varepsilon\Big)$$
$$\leq 4\varepsilon^{-2} P\Big(\langle M^m\rangle_{\tau_\ell^m}\Big) \leq 4\varepsilon^{-2} P\Big(\langle M^m\rangle_\infty \wedge \ell\Big).$$

Since ℓ is arbitrary, (2.28) follows and hence we conclude (2.27). □

Proof of Theorem 2.2.2. We are going to make use of the experience gathered in proving Proposition 2.3.1. In a manner very similar to the proof of (2.7), for (2.5), we need to show an analogue of (2.8) with the help of an analogue of (2.9). To be more concrete, we show

$$\lim_{n\to\infty} \widetilde{P}\Big(\big[\mu_\infty(\overline{F}(S^{(n)}))\big]^2\Big) = 0, \tag{2.29}$$

which implies

$$\widetilde{P}\big(|\mu_\infty(\overline{F}(S^{(n)}))|\big) \xrightarrow[n\to\infty]{} 0, \tag{2.30}$$

and hence the convergence in probability. Indeed, using the same replacement argument, but with (2.26) instead of (2.3), we get

$$\lim_{n\to\infty} \widetilde{P}\mu_\infty^{(2)}\Big(G\big(S^{(n)}, \widetilde{S}^{(n)}\big)\Big) = (P^{\mathbb{W}})^{\otimes 2}\Big(G\big(\cdot/\sqrt{d}, \tilde{\cdot}/\sqrt{d}\big)\Big)$$

for any $G \in \mathscr{C}_b(\mathbb{W}\times\mathbb{W})$. In particular, we can take $G(w,\widetilde{w}) = \overline{F}(w)\overline{F}(\widetilde{w})$, and get (2.29), and hence (2.5). The proof of (2.4) works with the same telescopic technique seen in (2.11) used in the proof of (2.6):

$$\begin{aligned}\widetilde{P}\big(|\mu_n(\overline{F}(S^{(n)}))|\big) = {} & \widetilde{P}\big(|\mu_n(\overline{F}(S^{(n)} - \overline{F}(S^{(n-k)}))\big)\\ & + \widetilde{P}\big(|\mu_n(\overline{F}(S^{(n-k)})) - \mu_\infty(\overline{F}(S^{(n-k)}))|\big)\\ & + \widetilde{P}\big(|\mu_\infty(\overline{F}(S^{(n-k)}))|\big).\end{aligned}$$

Note that the L^2–techniques in this paragraph that lead to (2.30) are needed only for the treatment of the last line; the other two can be dealt with with the same arguments than after (2.11). □

References

[AN72] K.B. Athreya and P.E. Ney. *Branching processes.* Die Grundlehren der mathematischen Wissenschaften. Band 196. Berlin-Heidelberg-New York: Springer-Verlag. XI., 1972.

[AW00] Mohamed Atlagh and Michel Weber. The almost sure central limit theorem. (Le théorème central limite presque sûr.). *Expo. Math.*, 18(2):97–126, 2000.

[BGK05] Matthias Birkner, Jochen Geiger, and Götz Kersting. Branching processes in random environment – a view on critical and subcritical cases. In *Interacting stochastic systems*, pages 269–291. Springer, Berlin, 2005.

[Bol89] Erwin Bolthausen. A note on the diffusion of directed polymers in a random environment. *Commun. Math. Phys.*, 123(4):529–534, 1989.

[CSY04] Francis Comets, Tokuzo Shiga, and Nobuo Yoshida. Probabilistic analysis of directed polymers in a random environment: a review. Funaki, Tadahisa (ed.) et al., Stochastic analysis on large scale interacting systems. Papers from the 11th international conference, Hayama, Japan, July 17–26, 2002 and the international conference on stochastic analysis and statistical mechanics, Kyoto, Japan, July 29–30, 2002. Tokyo: Mathematical Society of Japan. Advanced Studies in Pure Mathematics 39, 115-142, 2004.

[CY] Francis Comets and Nobuo Yoshida. Branching Random Walks in Space-Time Random Environment: Survival Probability, Global and Local Growth Rates. to appear in J. Theor. Probab.

[CY06] Francis Comets and Nobuo Yoshida. Directed polymers in random environment are diffusive at weak disorder. *Ann. Probab.*, 34(5):1746–1770, 2006.

[Dud89] Richard M. Dudley. *Real analysis and probability.* Wadsworth & Brooks/Cole Mathematics Series. Pacific Grove, CA: Wadsworth & Brooks/Cole Advanced Books & Software. xi, 436 p., 1989.

[Dur91] Richard Durrett. *Probability. Theory and examples.* Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991.

[HN11] Hadrian Heil and Makoto Nakashima. A Remark on Localization for Branching Random Walks in Random Environment. *Electron. Commun. Prob.*, 16:323–336, 2011.

[HY09] Yueyun Hu and Nobuo Yoshida. Localization for branching random walks in random environment. *Stochastic Processes Appl.*, 119(5):1632–1651, 2009.

[IS88] J. Z. Imbrie and T. Spencer. Diffusion of directed polymers in a random environment. *J. Stat. Phys.*, 52(3-4):609–626, 1988.

[Nak09] Makoto Nakashima. Central limit theorem for linear stochastic evolutions. *J. Math. Kyoto Univ.*, 49(1):201–224, 2009.

[Nak11] Makoto Nakashima. Almost sure central limit theorem for branching random walks in random environment. *Ann. Appl. Probab.*, 21(1):351–373, 2011.

[Shi09a] Yuichi Shiozawa. Central Limit Theorem for Branching Brownian motions in Random Environment. *J. Stat. Phys.*, 136:145–163, 2009.

[Shi09b] Yuichi Shiozawa. Localization for branching Brownian motions in random environment. *Tohoku Math. J.*, 61(4):483–497, 2009.

[SW69] W.L. Smith and W.E. Wilkinson. On branching processes in random environments. *Ann. Math. Stat.*, 40:814–827, 1969.

[Wat67] Shinzo Watanabe. Limit theorem for a class of branching processes. Markov Processes potential Theory. Proc. Sympos. Univ. Wisconsin, Madison 1967, 205-232, 1967.

[Wat68] Shinzo Watanabe. A limit theorem of branching processes and continuous state branching processes. *J. Math. Kyoto Univ.*, 8:141–167, 1968.

[Yos08a] Nobuo Yoshida. Central limit theorem for branching random walks in random environment. *Ann. Appl. Probab.*, 18(4):1619–1635, 2008.

[Yos08b] Nobuo Yoshida. Phase transitions for the growth rate of linear stochastic evolutions. *J. Stat. Phys.*, 133(6):1033–1058, 2008.

[Yos10] Nobuo Yoshida. Localization for Linear Stochastic Evolutions. *J. Stat. Phys.*, 138(4/5):960–977, 2010.