Electronic Journal of Probability

Vol. 16 (2011), Paper no. 89, pages 2452–2480.

Journal URL
http://www.math.washington.edu/~ejpecp/

Functional convergence to stable Lévy motions for iterated random Lipschitz mappings

Sana Louhichi * and Emmanuel Rio†

Abstract

It is known that, in the dependent case, partial sums processes which are elements of $D([0,1])$ (the space of right-continuous functions on $[0,1]$ with left limits) do not always converge weakly in the J_1-topology sense. The purpose of our paper is to study this convergence in $D([0,1])$ equipped with the M_1-topology, which is weaker than the J_1 one. We prove that if the jumps of the partial sum process are associated then a functional limit theorem holds in $D([0,1])$ equipped with the M_1-topology, as soon as the convergence of the finite-dimensional distributions holds. We apply our result to some stochastically monotone Markov chains arising from the family of iterated Lipschitz models..

Key words: Partial sums processes. Skorohod topologies. Functional limit theorem. Association. Tightness. Ottaviani inequality. Stochastically monotone Markov chains. Iterated random Lipschitz mappings. .

AMS 2010 Subject Classification: Primary 60F17; 60J10; Secondary: 60F05; 60G51; 60G52.

Submitted to EJP on April, 15 2011, final version accepted November 16, 2011.

*Corresponding author. Laboratoire Jean Kuntzmann, Tour IRMA IMAG, 51 rue des Mathématiques BP. 53 F-38041 Grenoble Cedex 9. France. E.mail: sana.louhichi@imag.fr

†UMR 8100 CNRS, Université de Versailles Saint-Quentin en Y., Laboratoire de mathématiques, Bât. Fermat, 45 Av. des Etats Unis, 78035 Versailles Cedex. France. E.mail: rio@math.uvsq.fr

1 Introduction

Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of real-valued random variables. Let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$ be the associated partial sums and let $(\xi_n(t))_{t\in[0,1]}$ be the normalized partial sum process,

$$\xi_n(t) = \frac{1}{a_n}(S_n(t) - b_n[nt]),\ \ t \in [0,1],$$

where $(b_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ are some deterministic sequences with $a_n \to \infty$ as $n \to \infty$, $S_n(t) = S_{[nt]}$ and $[x]$ denotes the integer part of x.

The sequence of processes $(\xi_n(t))_{t\in[0,1]}$ obeys a functional limit theorem (FLT) if there exists a proper stochastic process $(Y(t))_{t\in[0,1]}$ with sample paths in the space $D := D([0,1],\mathbb{R})$ of all right-continuous $\mathbb{R}$-valued functions with left limits defined on $[0,1]$, such that $\xi_n(\cdot) \Longrightarrow Y(\cdot)$ in D equipped with some topology.

Appropriate topologies on D were introduced by Skorohod (1956). The first one is the standard J_1-metric defined by

$$d_{J_1}(x_1,x_2) = \inf_{\lambda\in\Lambda}\left\{\|x_1 o\lambda - x_2\| \vee \|\lambda - e\|\right\},$$

where Λ is the set of strictly increasing functions λ on $[0,1]$ such that both λ and its inverse λ^{-1} are continuous, e is the identity map on $[0,1]$ and $\|\cdot\|$ is the uniform metric. Convergence in D equipped with the J_1-topology yields convergence jumps to jumps. In other words, if $\lim_{n\to\infty} d_{J_1}(x_n,x) = 0$ in D and if x has a jump at time t, i.e. $\Delta x(t) := x(t) - x(t^-) \neq 0$, then for n sufficiently large x_n must have a jump with both magnitudes and locations converging to those of the limiting function x (we refer for instance to the book of Whitt (2002), Section 3.3).

We come back to the sequence of the normalized partial sum process $(\xi_n(\cdot))_{n\in\mathbb{N}}$ and their limiting behavior in D equipped with the J_1-topology, when the limit process Y is a special case of Lévy process. Recall that a Lévy process $(Y(t))_{t\geq 0}$ with Lévy measure ν on $\mathbb{R}\setminus\{0\}$ is a stochastic process with sample paths in D such that $Y(0) = 0$ and that Y has stationary and independent increments. The full jump structure of a Lévy process Y is described by the counting random measure μ^Y on $\mathbb{R}^+ \times \mathbb{R}\setminus\{0\}$ associated with its jumps:

$$\mu^Y(\cdot,\cdot) = \sum_{s\geq 0,\,\Delta Y(s)\neq 0} \delta_{(s,\Delta Y(s))},$$

which is a Poisson random measure with intensity measure Leb$\otimes\nu$. The process $(\xi_n(t))_{t\in[0,1]}$ jumps at times $t_k = \dfrac{k}{n}$, $1 \leq k \leq n$. The amount of its kth jump is

$$\triangle\xi_n(t_k) = \xi_n(t_k) - \xi_n(t_k^-) = \sum_{i=1}^{k}\frac{X_i - b_n}{a_n} - \sum_{i=1}^{k-1}\frac{X_i - b_n}{a_n} = \frac{X_k - b_n}{a_n}.$$

Analogously the structure of the jumps of $(\xi_n(t))_{t\in[0,1]}$ is then described by the sequence of point processes μ_n defined by,

$$\mu_n(\cdot,\cdot) = \sum_{k=1}^{n}\delta_{(\frac{k}{n},\frac{X_k - b_n}{a_n})}(\cdot,\cdot).$$

So it is not surprising that a necessary condition for the FLT of the sequence $(\xi_n(t))_{t\in[0,1]}$ to a Lévy process in D equipped with the J_1-topology is the convergence of the sequence of point processes

μ_n towards μ^Y in the space of point measures on $[0,1]\times\mathbb{R}\setminus\{0\}$ equipped with the vague topology. This convergence implies in turn, since μ^Y is a Poisson random measure with intensity Leb $\otimes\,\nu$, that

$$\lim_{n\to\infty}\mathbb{P}(\max_{2\le j\le[nt]}|X_j-b_n|\ge\epsilon a_n\mid|X_1-b_n|\ge\epsilon a_n)=1-e^{-t\nu(|x|>\epsilon)}.$$

We refer to Tyran-Kamińska (2010a)-(2010b), for more details. Hence convergence in D equipped with the J_1-topology does not allow more than one jump in a very little time.

This limit condition does not ensure the convergence of partial sums processes constructed from any dependent random variables $(X_i)_{i\in\mathbb{Z}}$ in D equipped with the J_1-topology. Avram and Taqqu (1992) proved that the J_1-topology is not appropriate for the convergence of a partial sum process constructed from a finite order moving average with at least two non-zero coefficients. But if the coefficients of the moving average are all of the same sign, then the convergence holds in D equipped with the M_1-topology.

The M_1-topology is weaker than the J_1 one. As it was noted, see for instance Avram and Taqqu (1992), Basrak *et al.* (2010), Whitt (2002), these two topologies differ in a neighborhood of a jump of the limit process: in the case of the M_1-topology, several jumps are allowed but the graph of the sequence of processes must be approximately a "monotone staircase". This means that the jumps of $(\xi_n(\cdot))_{t\in[0,1]}$ which are the triangular arrays $(\frac{X_i-b_n}{a_n})_{1\le i\le n}$ evolve in the same directions, or satisfy a kind of positive dependence.

The most popular notion of positive dependence is the association. A sequence $X_1,X_2,\dots$ is associated if for all n, the vector $X^{(n)}=(X_1,X_2,\dots,X_n)$ satisfies the following condition: for any coordinatewise bounded and nondecreasing functions f, g on $\mathbb{R}^n$, $\mathrm{Cov}(f(X^{(n)}),g(X^{(n)}))\ge 0$. In particular, independent random variables and their nondecreasing transformations such that moving average with positive coefficients are all associated, we refer to Esary *et al.* (1967) for more about this notion.

Our first new result, which was announced in Louhichi and Rio (2011), is that, if the jumps of the process $(\xi_n(t))_{t\in[0,1]}$ are associated then a FLT holds in D equipped with the M_1-topology as soon as the convergence of the finite-dimensional distributions holds.

Theorem 1. *Let $X_1,X_2,\dots$ be a strictly stationary sequence of associated real-valued random variables. Let $\alpha\in]0,2]$ be fixed and $(a_n)_{n>0}$ be a nondecreasing sequence of positive reals, regularly varying with index $1/\alpha$, such that $\lim_{n\to\infty}a_n=\infty$. Suppose that there exists a sequence of reals $(b_n)_n$ for which $C:=\sup\{\,|b_k-b_n|\ :\ 0<n\le k\le 2n<\infty\}<\infty$, and that for any k-tuples $(t_1,\dots,t_k)$ with $0\le t_1<t_2<\ldots<t_k\le 1$, the finite-dimensional distribution*

$$(a_n^{-1}(S_n(t_1)-[nt_1]b_n),\dots,a_n^{-1}(S_n(t_k)-[nt_k]b_n))$$

converges in distribution to $(Y_\alpha(t_1),\dots,Y_\alpha(t_k))$ for some stochastic process $(Y_\alpha(t))_{t\in[0,1]}$ with sample paths in D, fulfilling

$$\lim_{x\to+\infty}x^{\alpha/2}\mathbb{P}(|Y_\alpha(t)|\ge x)=0,\ \ \forall\ t\in[0,1]. \tag{1.1}$$

Let ξ_n be the process defined on $[0,1]$ by $\xi_n(t)=a_n^{-1}(S_n(t)-[nt]b_n)$. Suppose that $C=0$ or $\liminf_{n\to\infty}(a_n/n)>0$. Then $\xi_n(\cdot)\Longrightarrow Y_\alpha(\cdot)$ as n tends to infinity, in D equipped with the M_1-topology.

Let us note that for a strictly stationary sequence of associated real-valued random variables with finite second moment and finite series of covariances, the functional convergence towards a Brownian motion was already proved by Newman and Wright (1981).

Our paper is organized as follows. In Section 2, we study an important class of associated Markov chains, called stochastically monotone (we discuss this point in the appendix below). An important example of stochastically monotone Markov chains arises from the family of iterated Lipschitz models. We first recall some key facts about iterated Lipschitz models having a unique in law stationary and heavy-tailed solution with exponent $\alpha > 0$. We next recall (Proposition 2 below), the conditions ensuring the convergence in law of the properly normalized and centered partial sum of this stochastic recursion model to an α-stable law with $\alpha \in]0,2]$. We identify in particular the parameters of this stable limit law. As a consequence of Theorem 1 and Proposition 2 below, we establish Theorem 2 which gives a functional convergence for the partial sum processes constructed from this stochastic recursion model. The limit process is a strictly stable Lévy process if $\alpha \in]0,2[\setminus\{1\}$, a stable Lévy process if $\alpha = 1$ and a Brownian motion otherwise. The proofs are in Section 3. In order to prove Theorem 1, we establish the tightness property in the Skorohod M_1-topology. Our main tools are a well known maximal inequality and an Ottaviani type inequality, which is up to our knowledge new for strictly stationary associated sequences. The proofs of some auxiliary results are in Section 4. The paper ends with an Appendix discussing the association property for the stochastically monotone Markov chains.

2 Limit theorems for iterated Lipschitz models

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. Let Ψ be a real-valued measurable random function on $\mathbb{R} \times \Omega$, so Ψ can be viewed as a random element of the space of Borel-measurable functions from $\mathbb{R}$ to $\mathbb{R}$ and for each $x \in \mathbb{R}$, $\Psi(x) = \Psi(x, \cdot)$ is a random variable. Let $(\Psi_n)_{n\geq 1}$ be a sequence of independent copies of Ψ and X_0 be a real-valued random variable independent of $(\Psi_n)_{n\geq 1}$. Define the recurrence equation,

$$X_n = \Psi_n(X_{n-1}). \tag{2.2}$$

Since Ψ_n is independent of $(X_i)_{i\leq n-1}$, $(X_n)_{n\geq 0}$ defines a Markov chain with initial distribution being the distribution of X_0. Clearly,

$$\mathbb{E}(f(X_n)|X_{n-1} = x) = \mathbb{E}(f(\Psi_n(x))) = \mathbb{E}(f(\Psi(x))).$$

This fact gives a simple criterion for $(X_n)_{n\in\mathbb{N}}$ to be stochastically monotone (and hence associated, by Proposition 5 in the Appendix below).

Lemma 1. *Let $(X_n)_{n\geq 0}$ be the Markov chain defined by (2.2). Suppose moreover that Ψ has nondecreasing paths, i.e. for each $\omega \in \Omega$ the map $x \mapsto \Psi(x, \omega)$ is a.s nondecreasing. Then for any initial distribution, $(X_n)_{n\geq 0}$ is an homogeneous stochastically monotone Markov chain.*

An another formula for X_n is given by iterating (2.2). We have $X_n = W_n(X_0)$, where W_n is the random map from $\mathbb{R}$ to $\mathbb{R}$ defined by $W_0(x) = x$ and,

$$W_n(x) = \Psi_n \circ \Psi_{n-1} \circ \ldots \circ \Psi_1(x). \tag{2.3}$$

Let $(Z_n)_{n\geq 1}$ be another sequence of random mapping from $\mathbb{R}$ to $\mathbb{R}$ defined by $Z_0(x) = x$ and,

$$Z_n(x) = \Psi_1 \circ \ldots \circ \Psi_{n-1} \circ \Psi_n(x).$$

It follows, since the two vectors $(\Psi_1, \dots, \Psi_{n-1}, \Psi_n)$ and $(\Psi_n, \Psi_{n-1}, \dots, \Psi_1)$ are equally distributed, that for any $x \in \mathbb{R}$, $W_n(x)$ and $Z_n(x)$ have the same distribution for any $n \geq 1$.

Based on the a.s. asymptotic behavior of $(Z_n(x))_{n\in\mathbb{N}}$, Letac (1986) gave some criteria ensuring the existence and the uniqueness in law of the solution of the random equation,

$$R = \Psi(R) \text{ in law}, \quad \Psi \text{ and } R \text{ are independent.} \tag{2.4}$$

More precisely Letac's principle is as follows (we refer also to Theorem 2.1 in Goldie (1991)).

Lemma 2. *Assume Ψ has a.s. continuous paths. Suppose that $Z_n(x)$ converges a.s as n tends to infinity to some random variable Z that does not depend on x. Then the law of Z satisfies (2.4) and is the unique such law. Also the sequence $(W_n(x))_{n\geq 1}$ has this law as its limit law for any initial x.*

By adding some contracting-type assumptions on the random map Ψ, one can prove that $(Z_n(x))_{n\geq 1}$ is a Cauchy sequence getting simple sufficient conditions for the existence of Z as defined in Lemma 2. This is the purpose of the following lemma. Its proof is very classical, we refer for instance to Diaconis and Freedman (1999).

Lemma 3. *Suppose that Ψ is a Lipschitz map with finite a.s. Lipschitz random positive constant A. Suppose that,*

$$\mathbb{E}(\ln(A)) < 0, \ \ \mathbb{E}|\ln(A)| < \infty, \ \ \mathbb{E}\ln^+|\Psi(0)| < \infty, \tag{2.5}$$

where $x^+ = \max(x, 0)$, then all the requirements of Lemma 2 are satisfied.

The following result proved in Goldie (1991), see Corollary 2.4 there, gives conditions under which the tails of the stationary solution of (2.4), when it exists, are asymptotic to a power.

Proposition 1. *Let M be a random variable such that $\mathbb{E}|M|^\alpha = 1$, $\mathbb{E}|M|^\alpha \ln^+|M| < \infty$ for some $\alpha > 0$ and the conditional law of $\ln|M|$, given $M \neq 0$, is nonarithmetic. Then $m_\alpha := \mathbb{E}(|M|^\alpha \ln|M|)$ belongs to $]0, \infty[$. Let R be a random variable satisfying (2.4). Suppose that R is independent of the couple (M, Ψ).*

1. *If $M > 0$ a.s. and $\mathbb{E}\left|(\Psi(R)^+)^\alpha - ((MR)^+)^\alpha\right| < \infty$, then*

$$\lim_{t\to\infty} \alpha t^\alpha \mathbb{P}(R > t) = C^+ = \frac{1}{m_\alpha}\mathbb{E}\left((\Psi(R)^+)^\alpha - ((MR)^+)^\alpha\right) \tag{2.6}$$

2. *If $M > 0$ a.s. and $\mathbb{E}\left|(\Psi(R)^-)^\alpha - ((MR)^-)^\alpha\right| < \infty$, then*

$$\lim_{t\to\infty} \alpha t^\alpha \mathbb{P}(R < -t) = C^- = \frac{1}{m_\alpha}\mathbb{E}\left((\Psi(R)^-)^\alpha - ((MR)^-)^\alpha\right). \tag{2.7}$$

Recall that for $x \in \mathbb{R}$, $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.

Examples. Here are some examples of stochastically monotone iterated Markov chains having an unique in law stationary solution R with heavy-tailed distribution.

1. **A random difference equation.** Let (A, B) be a couple of random variables with values in $\mathbb{R}^+ \times \mathbb{R}$. Here the nondecreasing function Ψ is given by

$$\Psi(x) = Ax + B. \tag{2.8}$$

Suppose that $M = A$ satisfies all the requirements of Proposition 1. If $\mathbb{E}|B|^\alpha < \infty$, then the conclusion of Proposition 1 holds with $R = \sum_{k=1}^{\infty} A_1 \dots A_{k-1} B_k$,

$$C^+ = \frac{1}{\mathbb{E}(A^\alpha \ln(A))} \mathbb{E}\left(((B + AR)^+)^\alpha - ((AR)^+)^\alpha \right),$$
$$C^- = \frac{1}{\mathbb{E}(A^\alpha \ln(A))} \mathbb{E}\left(((B + AR)^-)^\alpha - ((AR)^-)^\alpha \right),$$

cf. Theorem 4.1 in Goldie (1991).

2. **Letac's model.** Let (A, B, C) be a random vector with values in $\mathbb{R}^+ \times \mathbb{R}^2$ and define,

$$\Psi(x) = A \max(x, B) + C.$$

Proposition 6.1 of Goldie (1991) gives conditions for the existence of the stationary solution of Letac's model. If $-\infty \le \mathbb{E}(\ln(A)) < 0$, $\mathbb{E}(\ln(1 \vee B)) < \infty$ and $\mathbb{E}(\ln(1 \vee C)) < \infty$, then

$$R = \sup \left(\sum_{k=1}^{\infty} C_k \Pi_{k-1}, (\sum_{k=1}^{m} C_k \Pi_{k-1} + B_m \Pi_m)_{m \ge 1} \right),$$

where $\Pi_m = A_1 A_2 \dots A_m$ is a.s finite and its law is the unique law fulfilling (2.4). Suppose that $M = A$ satisfies all the requirements of Proposition 1. If

$$\mathbb{E}|C|^\alpha < \infty, \ \mathbb{E}(AB^+)^\alpha < \infty$$

then the conclusion of Proposition 1 holds with

$$C^+ = \frac{1}{\mathbb{E}(A^\alpha \ln(A))} \mathbb{E}\left(((C + A \max(R, B))^+)^\alpha - ((AR)^+)^\alpha \right),$$
$$C^- = \frac{1}{\mathbb{E}(A^\alpha \ln(A))} \mathbb{E}\left(((C + A \max(R, B))^-)^\alpha - ((AR)^-)^\alpha \right),$$

cf. Theorem 6.2 in Goldie (1991).

3. **A random extremal equation.** Here Ψ is given, for any $x \in \mathbb{R}$, by

$$\Psi(x) = \max(B, Ax),$$

where $A \ge 0$ a.s. Proposition 5.1 of Goldie (1991) shows that if $\mathbb{E} \ln(A) < 0$ and $\mathbb{E}(\ln(1 \vee B)) < \infty$ then $R = \sup_{k \ge 1} B_k \prod_{i=1}^{k-1} A_i$ is a.s. finite and its law is the unique law such that (2.4) holds. Suppose that $M = A$ satisfies all the requirements of Proposition 1. If $\mathbb{E}(B^+)^\alpha < \infty$, then the conclusion of Proposition 1 holds with

$$C^+ = \frac{1}{\mathbb{E}(A^\alpha \ln(A))} \mathbb{E}\left((B^+ \vee AR^+)^\alpha - ((AR)^+)^\alpha \right),$$
$$C^- = \frac{1}{\mathbb{E}(A^\alpha \ln(A))} \mathbb{E}\left((B^- \wedge AR^-)^\alpha - ((AR)^-)^\alpha \right),$$

cf. Theorem 5.2 in Goldie (1991).

Let, for $x \in \mathbb{R}$, $(X_n^x)_{n\in\mathbb{N}}$ be the Markov chain defined by (2.2) and starting at $X_0 = x$. Convergence in distribution of a suitably normalized partial sum of these Markov chain

$$\frac{1}{a_n}\sum_{i=1}^{n}(X_i^x - b_n)$$

to an α-stable distribution was studied by Mirek (2011) (we refer the reader also to Bartkiewicz *et al.* (2011) for, specially, the case a random difference equation.)

Recall that an infinitely divisible random variable X on $\mathbb{R}$ is called *α-stable* if, for any positive a there exists some real c such that, for any real u,

$$\phi^a(u) = \phi(a^{1/\alpha}u)e^{icu},$$

and it is called *strictly α-stable* if, for any positive a, its characteristic function ϕ satisfies, for any $u \in \mathbb{R}$,

$$\phi^a(u) = \phi(a^{1/\alpha}u),$$

we refer the reader, for instance, to Sato (1999) Chapter 3. The characteristic function of a non-trivial α-stable random variable X is

$$\mathbb{E}(e^{iuX}) = \begin{cases} \exp\left\{-\sigma^\alpha|u|^\alpha\left(1 - i\mathrm{sgn}(u)\beta\tan(\frac{\pi\alpha}{2})\right) + i\mu u\right\} & \text{if } \alpha \in]0,2[\setminus\{1\} \\ \exp\left\{-\sigma|u|\left(1 + i\mathrm{sgn}(u)\beta\frac{2}{\pi}\ln|u|\right) + i\mu u\right\} & \text{if } \alpha = 1 \\ \exp\left\{-\frac{\sigma^2}{2}u^2 + i\mu u\right\} & \text{if } \alpha = 2, \end{cases}$$

where $\mathrm{sgn}(t) = \mathbf{1}_{t>0} - \mathbf{1}_{t<0}$, $\sigma > 0$, $\beta \in [-1,1]$ and $\mu \in \mathbb{R}$. The parameters σ, β and μ are uniquely determined by the distribution of X (Samorodnitsky and Taqqu, 1994). An α-stable random variable with parameters σ, β and μ is denoted by $S_\alpha(\sigma,\beta,\mu)$.

Proposition 2 below is essentially Mirek's result written in the one dimensional case. The parameters β and σ depend on C^+ and C^- (defined respectively by (2.6) and (2.7)) and on Π_∞, where

$$\Pi_\infty = \sum_{k=1}^{\infty}\bar{\psi}_{\theta_k} \times \ldots \times \bar{\psi}_{\theta_1}, \tag{2.9}$$

the iid random variables $\bar{\psi}_{\theta_k}, \ldots, \bar{\psi}_{\theta_1}$ are those defined in (H_1) of Assumption 1.6 in Mirek (2011), that is, if $\bar{\psi}(x) = \lim_{t\to 0} t\Psi(x/t)$ exists for any $x \in \mathbb{R}$ and $\bar{\psi}(x) = Mx$ for any x in the support of the stationary measure ν, then $\bar{\psi}_{\theta_k}, \ldots, \bar{\psi}_{\theta_1}$ are independent copies of M.

Proposition 2. *Suppose that all the requirements of Theorem 1.15 of Mirek (2011) are satisfied. Then there exist real-valued sequences (a_n) and (b_n) such that $a_n^{-1}\sum_{i=1}^n(X_i^x - b_n)$ converges in law to an α-stable random variable. More precisely, denoting by ν the unique solution in law of (2.4), the sequences $(a_n)_n$ and $(b_n)_n$ are given by*

$a_n = n^{1/\alpha}$ *and* $b_n = 0$, *for* $\alpha \in]0,1[$,

$a_n = n$ *and* $b_n = \int n^2x/(n^2+x^2)\nu(dx)$, *for* $\alpha = 1$,

$a_n = n^{1/\alpha}$ *and* $b_n = \int x\nu(dx)$, *for* $\alpha \in]1,2[$

$a_n = (n\ln(n))^{1/2}$ *and* $b_n = \int x\nu(dx)$*, for* $\alpha = 2$.

Furthermore the parameters β, σ *and* μ *of the corresponding stable limit laws are given by*

$$\beta = (C^+ - C^-)/(C^+ + C^-),$$

and for $\alpha \in]0,2[\setminus\{1\}$, $\mu = 0$,

$$\sigma^\alpha = \frac{(C^+ + C^-)\Gamma(2-\alpha)}{\alpha(1-\alpha)} \cos(\frac{\pi}{2}\alpha)\mathbb{E}\left((\Pi_\infty + 1)^\alpha - \Pi_\infty^\alpha\right),$$

for $\alpha = 1$,

$$\mu = \mathbb{E}\left(\Pi_\infty \ln(\Pi_\infty) - (\Pi_\infty + 1)\ln(\Pi_\infty + 1)\right) + \int_1^\infty \frac{\sin r}{r^2} dr + \int_0^1 \frac{\sin r - r}{r^2} dr,\ \sigma = \frac{\pi}{2}(C^+ + C^-),$$

and finally, for $\alpha = 2$, $\mu = 0$ *and* $\sigma^2 = (1 + 2\mathbb{E}(\Pi_\infty))(C^+ + C^-)/2$.

Our task is to get a functional limit theorem for the iterated Lipschitz models fulfilling the assumptions of Proposition 2. This follows by combining Proposition 2 and Theorem 1. The limit process is either a Brownian motion, a stable or a strictly stable Lévy process. Recall that a Lévy process $(X(t))_{t\geq 0}$ is called stable or strictly stable if the distribution of X_1 is, respectively, stable or strictly stable. We refer the reader to Definition 13.2 of Sato (1999). More precisely, we have the following functional limit theorem.

Theorem 2. *Suppose that the random map* Ψ *is a.s. nondecreasing and Lipschitz with finite a.s. Lipschitz random positive constant* A *fulfilling (2.5). Let* ν *be the unique solution in law of (2.4). Let* $(X_n^\nu)_{n\in\mathbb{N}}$ *be the Markov chain defined by (2.2) and starting at* X_0 *with distribution* ν*. Suppose that for* $0 < \alpha \leq 2$*, the requirements and then the conclusions of Proposition 2 hold i.e.*

$$\frac{1}{a_n}\sum_{i=1}^{n}(X_i^x - b_n) \Longrightarrow S_\alpha(\sigma,\beta,\mu),\ \text{in distribution as } n \text{ tends to infinity,} \tag{2.10}$$

for any $x \in \mathbb{R}$*, where* a_n *and* b_n *are introduced in Proposition 2 and* $\mu = 0$ *when* $\alpha \neq 1$*. Then the sequence of processes*

$$\left\{\frac{1}{a_n}\sum_{i=1}^{[nt]}(X_i^\nu - b_n),\ t \in [0,1]\right\}$$

converges in $D([0,1])$ *equipped with the* M_1*-topology to a Lévy process* $(Z(t))_{t\in[0,1]}$*, where*

1. *If* $\alpha \in]0,2[\setminus\{1\}$*, then* $(Z(t))_{t\geq 0}$ *is a strictly stable Lévy process and, for any fixed* $t \in [0,1]$*, the random variable* $Z(t)$ *is distributed as* $t^{1/\alpha}S_\alpha(\sigma,\beta,0)$ *or as* $S_\alpha(t^{1/\alpha}\sigma,\beta,0)$.
2. *If* $\alpha = 1$*, then* $(Z(t))_{t\geq 0}$ *is a stable Lévy process and for any fixed* $t \in [0,1]$ *the random variable* $Z(t)$ *is distributed as* $tS_1(\sigma,\beta,\mu) + t\frac{2}{\pi}\beta\sigma\log(t)$*, which is the distribution of* $S_1(t\sigma,\beta,t\mu)$.
3. *If* $\alpha = 2$*, then* $(Z(t))_{t\geq 0}$ *is a centered Brownian motion with variance* σ^2.

Remark. In the case $\alpha = 2$, the limit is a continuous process. Since M_1-convergence to a continuous limit implies uniform convergence (Whitt (2002), Chapter 12, Section 4), the ordinary invariance principle has been established.

3 Proofs

3.1 Proof of Theorem 1

In order to prove the convergence in D of the sequence of processes $(\xi_n(\cdot))_n$, we have only to establish the tightness property since the convergence of the finite-dimensional distributions is assumed to hold. For this, we need first some notation. Throughout the sequel, for any real numbers y_1, y_2 and y_3,

$$\|y_2 - [y_1, y_3]\| = \inf_{t \in [y_1, y_3]} |y_2 - t|$$

denotes the distance between y_2 and the segment $[y_1, y_3]$. For each cadlag function ξ in the space $D([0,1])$, let

$$\omega(\xi, \delta) = \sup_{t \in [0,1]} \sup\{\|\xi(t_2) - [\xi(t_1), \xi(t_3)]\| : (t-\delta) \vee 0 \le t_1 < t_2 < t_3 \le (t+\delta) \wedge 1\}.$$

According to Skorohod (1956) (we refer also to Whitt (2002), Chapter 12 and to Louhichi and Rio (2011)), we have to prove that,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}(\omega(\xi_n, \delta) > \varepsilon) = 0, \tag{3.11}$$

since this last limit provides the tightness property in the Skorohod M_1-topology and consequently proves Theorem 1. The first tool to prove (3.11) is the following lemma, which will allow us to control $M_n^* = \sup_{0 \le n_1 < n_2 < n_3 \le n} M(n_1, n_2, n_3)$ in the case of associated sequences, where for n_1, n_2 and n_3 natural integers, $M(n_1, n_2, n_3)$ denotes the distance from S_{n_2} to the segment $[S_{n_1}, S_{n_3}]$, that is,

$$M(n_1, n_2, n_3) = \begin{cases} 0 & \text{if } S_{n_2} \in [S_{n_1}, S_{n_3}] \\ |S_{n_1} - S_{n_2}| \wedge |S_{n_3} - S_{n_2}| & \text{if } S_{n_2} \notin [S_{n_1}, S_{n_3}]. \end{cases}$$

Lemma 4. *Let $(X_i)_{i \in \mathbb{Z}}$ be a sequence of associated random variables. Then, for any positive ε,*

$$\mathbb{P}(M_n^* \ge \varepsilon) \le \mathbb{P}^2\big(\max_{0 \le k \le n} |S_k| \ge \varepsilon/2\big).$$

Proof of Lemma 4. We first claim that

$$M_n^* \le \min\Big(\sup_{0 \le l \le m \le n} (S_m - S_l), \sup_{0 \le l \le m \le n} (S_l - S_m)\Big). \tag{3.12}$$

In fact suppose that $S_{n_2} > S_{n_1}$ and $S_{n_2} > S_{n_3}$. Clearly either $S_{n_3} \le S_{n_1}$ or $S_{n_3} > S_{n_1}$. In the first case, $M(n_1, n_2, n_3) = S_{n_2} - S_{n_1} = \min(S_{n_2} - S_{n_1}, S_{n_2} - S_{n_3})$. In the second case $M(n_1, n_2, n_3) = S_{n_2} - S_{n_3} = \min(S_{n_2} - S_{n_3}, S_{n_2} - S_{n_1})$. In both cases $M(n_1, n_2, n_3)$ is less than the right hand side of (3.12) for all $n_1 < n_2 < n_3$. This proves (3.12) when $S_{n_2} > S_{n_1}$ and $S_{n_2} > S_{n_3}$. The other cases being similar, they will be omitted.

Now define

$$W_n^+ = \sup_{0 \le l \le m \le n} (S_m - S_l), \quad W_n^- = \sup_{0 \le l \le m \le n} (S_l - S_m).$$

The association property gives, since W_n^+ and W_n^- are respectively a nondecreasing and a nonincreasing function of $(X_1, \ldots, X_n)$,

$$\mathbb{P}(W_n^+ \geq \varepsilon, W_n^- \geq \varepsilon) \leq \mathbb{P}(W_n^+ \geq \varepsilon)\mathbb{P}(W_n^- \geq \varepsilon). \tag{3.13}$$

Now

$$W_n^+ \leq 2 \max_{0 \leq l \leq n} |S_l|, \;\; W_n^- \leq 2 \max_{0 \leq l \leq n} |S_l|. \tag{3.14}$$

Therefore, collecting (3.12), (4.41) and (3.14), we get that

$$\mathbb{P}(M_n^* \geq \varepsilon) \leq \mathbb{P}(W_n^+ \geq \varepsilon)\mathbb{P}(W_n^- \geq \varepsilon) \leq \mathbb{P}^2\big(\max_{0 \leq k \leq n} |S_k| \geq \varepsilon/2\big),$$

which completes the proof of Lemma 4. □

According to Lemma 4, we have to control $\mathbb{P}\big(\max_{0 \leq k \leq n} |S_k| \geq \varepsilon/2\big)$. For this we need an Ottaviani type inequality (cf. Proposition 3 below). The following lemma is the main ingredient to establish an Ottaviani type inequality for a strictly stationary associated sequence. It is a key tool for bounding the median of $S_n^* := \max(0, S_1, \ldots, S_n)$.

Lemma 5. *Let $(X_j)_{j \in \mathbb{Z}}$ be a strictly stationary associated sequence of real-valued random variables. For any positive N, set $Z_N = S_{2^N}^* - \min(0, S_{2^N})$. Then, for any u in $[0, 1[$,*

$$Q_{Z_N}(u) \leq Q_{|X_1|}(u) + 2 \sum_{L=0}^{N-1} Q_{|S_{2^L}|}(u(1-u)/2),$$

recall that, for any real-valued random variable X, Q_X denotes the quantile function of X, which is the cadlag inverse of the tail function H_X of the random variable X.

Proof of Lemma 5. We prove this lemma by induction on N. Throughout the proof $a \vee b = \max(a, b)$ and $a^+ = \max(a, 0)$. Let x and y be nonnegative reals:

$$(S_{2^{N+1}}^* > x + y) \subset (S_{2^N} > x/2) \cup (S_{2^{N+1}} - S_{2^N} > x/2) \cup (S_{2^N} \vee S_{2^{N+1}} \leq x, S_{2^{N+1}}^* > x + y).$$

Next

$$(S_{2^N} \vee S_{2^{N+1}} \leq x, S_{2^{N+1}}^* > x + y) \subset (S_{2^N}^* - S_{2^N}^+ > y) \cup (\max_{k \in [2^N, 2^{N+1}]} S_k - (S_{2^N} \vee S_{2^{N+1}}) > y).$$

Now, from the stationarity of the sequence $(X_i)_i$, we have that

$$\mathbb{P}(\max_{k \in [2^N, 2^{N+1}]} S_k - (S_{2^N} \vee S_{2^{N+1}}) > y) = \mathbb{P}(S_{2^N}^* - S_{2^N}^+ > y)$$

and

$$\mathbb{P}(S_{2^N} > x/2) = \mathbb{P}(S_{2^{N+1}} - S_{2^N} > x/2).$$

Both the above facts imply that

$$\mathbb{P}(S_{2^{N+1}}^* > x + y) \leq 2\mathbb{P}(S_{2^N} > x/2) + 2\mathbb{P}(S_{2^N}^* - \max(0, S_{2^N}) > y). \tag{3.15}$$

We now bound up the second term on right hand in the above inequality. Clearly

$$(S^*_{2^N} - S^+_{2^N} > y) = (S^*_{2^N} > y) \cap (S^*_{2^N} - S_{2^N} > y).$$

Next $S^*_{2^N}$ is a nondecreasing function of $(X_i)_{1\le i\le 2^N}$ and $S^*_{2^N} - S_{2^N}$ is a nonincreasing function of $(X_i)_{1\le i\le 2^N}$. Hence, by the association property,

$$\mathbb{P}(S^*_{2^N} > y \text{ and } S^*_{2^N} - S_{2^N} > y) \le \mathbb{P}(S^*_{2^N} > y)\mathbb{P}(S^*_{2^N} - S_{2^N} > y).$$

Consider now the stationary and associated sequence $(Y_i)_{i\in\mathbb{Z}}$ defined by $Y_i = -X_{-i}$. Set $T_n = Y_1 + \ldots + Y_n$ and $T^*_n = \max(0, T_1, \ldots, T_n)$. Then

$$S^*_{2^N} - S_{2^N} = \max(0, -X_{2^N}, \ldots, -X_{2^N} - \ldots - X_1) \overset{\text{Law}}{=} T^*_{2^N}$$

due to the fact that $(Y_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence. Hence, from (3.15) and the above inequality,

$$\mathbb{P}(S^*_{2^{N+1}} > x + y) \le 2\mathbb{P}(S_{2^N} > x/2) + 2\mathbb{P}(S^*_{2^N} > y)\mathbb{P}(T^*_{2^N} > y). \tag{3.16}$$

Now, proceeding in the same way for the sequence (Y_i) and noticing that the transformation which turns the sequence $(X_i)_{i\in\mathbb{Z}}$ into the sequence $(Y_i)_{i\in\mathbb{Z}}$ is involutive, we get that

$$\mathbb{P}(T^*_{2^{N+1}} > x + y) \le 2\mathbb{P}(T_{2^N} > x/2) + 2\mathbb{P}(T^*_{2^N} > y)\mathbb{P}(S^*_{2^N} > y). \tag{3.17}$$

Let

$$p_N(z) = \mathbb{P}(T^*_{2^N} > z) + \mathbb{P}(S^*_{2^N} > z).$$

Since T_n has the same distribution as $-S_n$, adding the two above inequalities, we get that

$$p_{N+1}(x+y) \le 2\mathbb{P}(|S_{2^N}| > x/2) + 4\mathbb{P}(S^*_{2^N} > y)\mathbb{P}(T^*_{2^N} > y) \le 2\mathbb{P}(|S_{2^N}| > x/2) + (p_N(y))^2. \tag{3.18}$$

Choosing $y = y_N = p_N^{-1}(u)$ and $x = x_N = 2Q_{|S_{2^N}|}(u(1-u)/2)$ in (3.18) and noting that, for this choice of (x, y),

$$p_N(y_N) \le u \quad\text{and}\quad \mathbb{P}(|S_{2^N}| > x_N/2) \le u(1-u)/2,$$

we then infer that

$$p_{N+1}(x_N + y_N) \le u,$$

which means that $p^{-1}_{N+1}(u) \le x_N + y_N$. Hence, for any u in $]0,1[$,

$$p^{-1}_{N+1}(u) \le p_N^{-1}(u) + 2Q_{|S_{2^N}|}(u(1-u)/2). \tag{3.19}$$

Proceeding by induction and noting that $p_0(x) = \mathbb{P}(|X_1| > x)$, which ensures that $p_0^{-1}(u) \le Q_{|X_1|}(u)$, we infer from (3.19) that, for any positive N

$$p_N^{-1}(u) \le Q_{|X_1|}(u) + 2\sum_{L=0}^{N-1} Q_{|S_{2^L}|}(u(1-u)/2).$$

Finally, since T^*_n has the same distribution as $S^*_n - S_n$,

$$p_N(z) = \mathbb{P}(S^*_{2^N} - S_{2^N} > z) + \mathbb{P}(S^*_{2^N} > z) \ge \mathbb{P}(\max(S^*_{2^N} - S_{2^N}, S^*_{2^N}) > z),$$

whence $Q_{Z_N}(u) \le p_N^{-1}(u)$, which completes the proof of Lemma 5 . □

In order to get an Ottaviani type inequality, we now apply Lemma 5 under the assumptions of Theorem 1. Our result is the following.

Proposition 3. *Let $(X_j)_{j\in\mathbb{Z}}$ be a strictly stationary associated sequence of real-valued random variables. Let Z_N be defined as in Lemma 5 and $\beta_N = a_{2^N}^{-1} Q_{Z_N}(1/2)$. Then for any positive x,*

$$(a) \qquad \mathbb{P}(S^*_{2^N} \geq x + a_{2^N}\beta_N) \leq 2\mathbb{P}(S_{2^N} \geq x).$$

Assume furthermore that, for some α in $]0,2]$ and some sequence $(a_n)_{n>0}$ with the same properties as in Theorem 1, $a_n^{-1}S_n$ converges in distribution to a finite random variable Y_α as $n \to \infty$. Then

$$(b) \qquad \limsup_{N\to\infty} \beta_N \leq 2(2^{1/\alpha}-1)^{-1} Q_{|Y_\alpha|}((1/8)-0).$$

Proof of Proposition 3. To prove (a), we apply inequalities (21) and (22) in Newman (1982), page 365, with $\lambda_1 = x$, $\lambda_2 = x + y_N$ and $y_N = a_{2^N}\beta_N$, yielding

$$\mathbb{P}(S^*_{2^N} \geq x + y_N) \leq \mathbb{P}(S_{2^N} \geq x) + \mathbb{P}(S^*_{2^N} \geq x + y_N)\mathbb{P}(S^*_{2^N} - S_{2^N} > y_N). \tag{3.20}$$

Next, since $S^*_{2^N} - S_{2^N} \leq Z_N$,

$$\mathbb{P}(S^*_{2^N} - S_{2^N} > y_N) \leq \mathbb{P}(Z_N > Q_{Z_N}(1/2)) \leq 1/2,$$

which together with (3.20), implies (a) of Proposition 3.

We now prove (b). Let x be any continuity point of $H_{|Y_\alpha|}$ such that $x > Q_{|Y_\alpha|}((1/8)-0)$. The real x can be arbitrarily near from $Q_{|Y_\alpha|}((1/8)-0)$, since the set of discontinuities of $H_{|Y_\alpha|}$ is countable. Moreover, there exists some $z < 1/8$ such that $Q_{|Y_\alpha|}(z) \leq x$. Now, by the assumption of convergence in distribution,

$$\lim_{N\to\infty} \mathbb{P}(a_{2^N}^{-1}|S_{2^N}| > x) = H_{|Y_\alpha|}(x) \leq z < 1/8.$$

Hence, there exists some positive integer N_0 such that, for any $N \geq N_0$,

$$\mathbb{P}(a_{2^N}^{-1}|S_{2^N}| > x) \leq 1/8.$$

Thus, for $N \geq N_0$,

$$Q_{a_{2^N}^{-1}|S_{2^N}|}(1/8) = a_{2^N}^{-1} Q_{|S_{2^N}|}(1/8) \leq x.$$

Since x can be arbitrarily near from $Q_{|Y_\alpha|}((1/8)-0)$, it ensures that

$$\limsup_{N\to\infty} a_{2^N}^{-1} Q_{|S_{2^N}|}(1/8) \leq Q_{|Y_\alpha|}((1/8)-0).$$

Hence, by the Toeplitz lemma,

$$\limsup_{N\to\infty} \Big(\sum_{L=0}^{N-1} a_{2^L}\Big)^{-1} \sum_{L=0}^{N-1} Q_{|S_{2^L}|}(1/8) \leq Q_{|Y_\alpha|}((1/8)-0). \tag{3.21}$$

To complete the proof, we will need the following elementary lemma.

Lemma 6. *Let $(a_n)_{n>0}$ be a sequence of positive reals regularly varying with index $1/\alpha$. Then*

$$\limsup_{N\to\infty} a_{2^N}^{-1} \sum_{L=0}^{N-1} a_{2^L} \leq (2^{1/\alpha}-1)^{-1}.$$

Proof of Lemma 6. We have, since the sequence (a_n) is regularly varying with index $1/\alpha$:

$$\lim_{L\to\infty} \frac{a_{2^{L-1}}}{a_{2^L}} = 2^{-1/\alpha}.$$

We deduce that for any $1 > \beta > 2^{-1/\alpha}$ there exists a positive integer L_0 such that,

$$\forall\ L > L_0,\ \frac{a_{2^{L-1}}}{a_{2^L}} \le \beta.$$

Let $k \ge L_0$. Clearly,

$$\frac{a_{2^k}}{a_{2^N}} = \prod_{L=k+1}^{N} \frac{a_{2^{L-1}}}{a_{2^L}} \le \beta^{N-k}.$$

Consequently,

$$\sum_{k=L_0}^{N-1} \frac{a_{2^k}}{a_{2^N}} \le \sum_{k=L_0}^{N-1} \beta^{N-k} \le \frac{\beta}{1-\beta}.$$

Hence,

$$\limsup_{N\to\infty} \sum_{L=0}^{N-1} \frac{a_{2^L}}{a_{2^N}} \le \limsup_{N\to\infty} \sum_{L=0}^{L_0-1} \frac{a_{2^L}}{a_{2^N}} + \limsup_{N\to\infty} \sum_{L=L_0}^{N-1} \frac{a_{2^L}}{a_{2^N}}.$$

We then conclude, since $\lim_{N\to\infty} a_N = 0$, that

$$\limsup_{N\to\infty} \sum_{L=0}^{N-1} \frac{a_{2^L}}{a_{2^N}} \le \frac{\beta}{1-\beta},$$

for any $2^{-1/\alpha} < \beta < 1$. We conclude the proof of Lemma 6 from the last limit letting β tending to $2^{-1/\alpha}$. □

Now, starting from (3.21) and applying Lemmas 6 and 5, we get (b), which completes the proof of Proposition 3. □

End of the Proof of (3.11). We have now all the ingredients for the proof of (3.11). For a fixed n, let us consider the stationary sequence $(X'_i)_i$ defined by $X'_i = X_i - b_n$. Set $S'_m = X'_1 + X'_2 + \ldots + X'_m$ and $S'_n(t) = S'_{[nt]}$. Then $\xi_n(t) = a_n^{-1} S'_{[nt]}$. Let $k \ge 3$ be any integer and $\delta_k = 1/k$. It is enough to prove that, for any positive ε,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(\omega(\xi_n, \delta_k) > \varepsilon) = 0. \tag{3.22}$$

We start by noting that

$$\begin{aligned}
\mathbb{P}(w(\xi_n,\delta_k) > \varepsilon) &\le \sum_{j=0}^{k-3} \mathbb{P}\Big(\sup_{j/k \le t_1 < t_2 < t_3 \le (j+3)/k} \|S'_n(t_2) - [S'_n(t_1), S'_n(t_3)]\| > a_n \varepsilon \Big) \\
&\le \sum_{j=0}^{k-3} \mathbb{P}\Big(\sup_{[jn/k] \le n_1 < n_2 < n_3 \le [(j+3)n/k]} M'(n_1,n_2,n_3) > a_n \varepsilon \Big),
\end{aligned}$$

where $M'(n_1,n_2,n_3)$ is defined from the random variables S'_m exactly as $M(n_1,n_2,n_3)$ from the random variables S_m just before the proof of Lemma 4. Next, from the stationarity of the sequence $(X'_i)_{i>0}$,

$$\sup_{[jn/k]\le n_1<n_2<n_3\le[(j+3)n/k]} M'(n_1,n_2,n_3) \stackrel{\text{Law}}{=} \sup_{0\le n_1<n_2<n_3\le[(j+3)n/k]-[jn/k]} M'(n_1,n_2,n_3),$$

which ensures that

$$\mathbb{P}(w(\xi_n,\delta_k)>\varepsilon)\le(k-2)\mathbb{P}\Big(\sup_{0\le n_1<n_2<n_3\le 1+[3n/k]} M'(n_1,n_2,n_3)>a_n\varepsilon\Big).$$

Hence, by Lemma 4 applied to the sequence $(X'_i)_{i>0}$,

$$\mathbb{P}(w(\xi_n,\delta_k)>\varepsilon)\le(k-2)\mathbb{P}^2\big(\max_{0\le j\le 1+[3n/k]}|S'_j|>a_n\varepsilon/2\big). \tag{3.23}$$

For $n\ge k$, let the positive integer N be defined by $2^{N-1}\le[3n/k]<2^N$. Set

$$Z'_L=\max(0,S'_1,\dots,S'_{2^L})-\min(0,S'_{2^L}) \text{ and } Z''_L=\max(0,-S'_1,\dots,-S'_{2^L})-\min(0,-S'_{2^L}).$$

Applying now (a) of Proposition 3 to the sequences $(X'_i)_i$ and $(-X'_i)_i$, we get that

$$\mathbb{P}\big(\max_{0\le j\le 1+[3n/k]}|S'_j|>a_n\varepsilon/2\big)\le 2\mathbb{P}\big(|S'_{2^N}|\ge a_n\varepsilon/2-\max(Q_{Z'_N}(1/2),Q_{Z''_N}(1/2))\big). \tag{3.24}$$

We now prove that, for k large enough, under the assumptions of Theorem 1

$$A_N:=\max(Q_{Z'_N}(1/2),Q_{Z''_N}(1/2))\le a_n\varepsilon/4. \tag{3.25}$$

By Lemma 5 applied successively to the sequences (X'_i) and $(-X'_i)$,

$$A_N\le Q_{|X'_1|}(1/2)+2\sum_{L=0}^{N-1}Q_{|S'_{2^L}|}(1/8). \tag{3.26}$$

Now let $\tilde S_k=S_k-kb_k$, then

$$|S'_{2^L}|\le|\tilde S_{2^L}|+2^L|b_n-b_{2^L}|.$$

Set $\log_2(u)=\log(u)/\log(2)$. Let $M=[\log_2(n)]$. Then

$$|b_n-b_{2^L}|\le\sum_{K=L}^{M-1}|b_{2^{K+1}}-b_{2^K}|+|b_n-b_{2^M}|\le C(M-L+1)$$

Therefrom

$$|S'_{2^L}|\le|\tilde S_{2^L}|+C2^L(\log_2(n)-L+1),$$

Moreover $Q_{|X'_1|}(1/2)\le Q_{|X_1|}(1/2)+|b_n|$. Consequently, starting from (3.26) and using the above upper bounds,

$$A_N\le Q_{|X_1|}(1/2)+2\sum_{L=0}^{N-1}Q_{|\tilde S_{2^L}|}(1/8)+|b_n|+2C2^N(\log_2(n)+1-N)+4C.$$

Now, using the convergence in distribution of $a_{2^L}^{-1}\tilde{S}_{2^L}$ to $Y_\alpha := Y_\alpha(1)$ and proceeding as in the proof of (b) of Proposition 3, we get that

$$\limsup_{N\to\infty} a_{2^N}^{-1}\Big(Q_{|X_1|}(1/2) + 2\sum_{L=0}^{N-1} Q_{|\tilde{S}_{2^L}|}(1/8)\Big) \le 2Q_{|Y_\alpha|}((1/8)-0),$$

which ensures that there exists some positive constant c such that, for any n and any $k \le n$,

$$A_N \le c a_{2^N} + C\log_2(2n) + 2C2^N\log_2(k) + 4C.$$

(note that $|b_n| \le C\log_2(2n) + |b_1|$). From the above inequality and the assumption that $\liminf_n(a_n/n) > 0$ if $C \neq 0$, for any positive ε there exists some positive integer $k_0 > 6$ such that, for $k \ge k_0$ and $n \ge k$, (3.25) holds true.

Next, from (3.25), for $k \ge k_0$ and n large enough, since the sequence $(a_n)_{n>0}$ is regularly varying with index $1/\alpha$ and $k2^N \le 6n$,

$$\mathbb{P}\big(\max_{0\le j\le 1+[3n/k]} |S_j'| > a_n\varepsilon/2\big) \le 2\mathbb{P}(|S_{2^N}'| > a_n\varepsilon/4) \le 2\mathbb{P}(|S_{2^N}'| > a_{2^N}(k/6)^{1/\alpha}\varepsilon/6).$$

Collecting the above inequality and (3.23), we get that, for $k \ge k_0$ and $n \ge k$,

$$\mathbb{P}(w(\xi_n,\delta_k) > \varepsilon) \le 4k\mathbb{P}^2\big(a_{2^N}^{-1}|S_{2^N}'| > (k/6)^{1/\alpha}\varepsilon/6\big). \tag{3.27}$$

Next

$$|S_{2^N}'| \le |\tilde{S}_{2^N}| + 2^N|b_n - b_{2^N}| \le |\tilde{S}_{2^N}| + C2^N\log_2(k),$$

since $n2^{-N} \le k/3$. Now, under the assumptions of Theorem 1, there exists some positive integer $k_1(\ge k_0)$ such that, for $k \ge k_1$ and n large enough, $C2^N\log_2(k) \le a_{2^N}(k/6)^{1/\alpha}\varepsilon/12$ (recall that $C = 0$ or $\liminf_n(a_n/n) > 0$). Then, from (3.27),

$$\mathbb{P}(w(\xi_n,\delta_k) > \varepsilon) \le 4k\mathbb{P}^2\big(a_{2^N}^{-1}|\tilde{S}_{2^N}| > (k/6)^{1/\alpha}\varepsilon/12\big). \tag{3.28}$$

Now $a_{2^N}^{-1}\tilde{S}_{2^N}$ converges in distribution to Y_α, and therefrom, for $k \ge k_1$,

$$\limsup_{n\to\infty} \mathbb{P}\big(a_{2^N}^{-1}|\tilde{S}_{2^N}| > (k/6)^{1/\alpha}\varepsilon/12\big) \le \mathbb{P}(|Y_\alpha| \ge (k/6)^{1/\alpha}\varepsilon/12),$$

which, combined with (3.28), ensures that

$$\limsup_{n\to\infty} \mathbb{P}(w(\xi_n,\delta_k) > \varepsilon) \le 4k\mathbb{P}^2(|Y_\alpha| \ge (k/6)^{1/\alpha}\varepsilon/12). \tag{3.29}$$

The convergence result (3.22) follows then from (3.29) and the tail condition (1.1). Theorem 1 is then proved. □

3.2 Proof of Proposition 2

Define for $v \in \{-1,1\}$, $h_v(x) = \mathbb{E}\left(\exp\left(ixv\Pi_\infty\right)\right)$, where Π_∞ is the random variable defined by (2.9). Let φ_α be the characteristic function of the α-stable limit law of Proposition 2. Our purpose is to give the expressions of the parameters of this α-stable limit law using Theorem 1.15 and its proof in Mirek (2011).

1. If $\alpha \in]0,1[$, then $\varphi_\alpha(uv) = \exp(u^\alpha C_\alpha(v))$, for any $u > 0$ and $v \in \{-1,1\}$. The constants $C_\alpha(v)$ are given by (6.39) of Mirek (2011), $C_\alpha(v) = \int (e^{ivx} - 1)h_v(x)\Lambda(dx)$, where Λ is the unique Radon measure on $\mathbb{R} \setminus \{0\}$ for which,

$$\lim_{y\to 0} \frac{1}{|y|^\alpha} \int_{-\infty}^{+\infty} f(yx)\nu(dx) = \int f(x)\Lambda(dx),$$

for any bounded and continuous function f that vanishes in a neighborhood of 0, (cf. Theorem 4.3 in Mirek (2011)). In particular, Λ satisfies the following polar decomposition (cf. Equation (4.5) in Mirek (2011))

$$\int f(x)\Lambda(dx) = C^+ \int_0^\infty f(r)\frac{dr}{r^{\alpha+1}} + C^- \int_0^\infty f(-r)\frac{dr}{r^{\alpha+1}},$$

for functions f as above. Denoting by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively the real and the imaginary parts of a complex number z, we infer that

$$\begin{aligned}\mathrm{Re}(C_\alpha(1)) &= (C^+ + C^-)\int_0^\infty \mathrm{Re}((e^{ix} - 1)h_1(x))\frac{dx}{x^{\alpha+1}} \\ &= -\frac{(C^+ + C^-)}{\alpha(1-\alpha)}\mathbb{E}\left((\Pi_\infty + 1)^\alpha - \Pi_\infty^\alpha\right)\Gamma(2-\alpha)\cos(\frac{\pi}{2}\alpha).\end{aligned}$$

The last bound follows, since for $\alpha \in]0,2[\setminus\{1\}$,

$$\begin{aligned}\int_0^\infty \frac{1-\cos ax}{x^{1+\alpha}}dx &= \frac{a^\alpha}{\alpha}\int_0^\infty \frac{\sin x}{x^\alpha}dx \\ \int_0^\infty \frac{\sin x}{x^\alpha}dx &= \frac{\Gamma(2-\alpha)}{1-\alpha}\cos(\frac{\pi}{2}\alpha).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Im}(C_\alpha(1)) &= (C^+ - C^-)\int_0^\infty \mathrm{Im}((e^{ix} - 1)h_1(x))\frac{dx}{x^{\alpha+1}} \\ &= (C^+ - C^-)\frac{1}{\alpha}\mathbb{E}\left((\Pi_\infty + 1)^\alpha - \Pi_\infty^\alpha\right)\Gamma(1-\alpha)\sin(\frac{\pi}{2}\alpha).\end{aligned}$$

The two last equalities together with the equation $C_\alpha(1) = -\sigma^\alpha + i\sigma^\alpha\beta\tan(\pi\alpha/2)$, give

$$\mathrm{Im}(C_\alpha(1)) = \sigma^\alpha\beta\tan(\pi\alpha/2), \;\; \mathrm{Re}(C_\alpha(1)) = -\sigma^\alpha.$$

Hence, $\beta\tan(\pi\alpha/2) = -\frac{\mathrm{Im}(C_\alpha(1))}{\mathrm{Re}(C_\alpha(1))}$. This proves the result, since $\Gamma(2-\alpha) = (1-\alpha)\Gamma(1-\alpha)$.

2. If $\alpha \in]1,2[$, then $\varphi_\alpha(uv) = \exp(u^\alpha C_\alpha(v))$, for any $u > 0$ and $v \in \{-1,1\}$. The constants $C_\alpha(v)$ are given by (6.43) of Mirek (2011), $C_\alpha(v) = \int \left((e^{ivx} - 1)h_v(x) - ivx\right)\Lambda(dx)$. As before, we obtain,

$$\begin{aligned}\mathrm{Re}(C_\alpha(1)) &= (C^+ + C^-)\int_0^\infty \mathrm{Re}((e^{ix} - 1)h_1(x))\frac{dx}{x^{\alpha+1}} \\ &= -\frac{C^+ + C^-}{\alpha(1-\alpha)}\mathbb{E}\left((\Pi_\infty + 1)^\alpha - \Pi_\infty^\alpha\right)\Gamma(2-\alpha)\cos(\frac{\pi}{2}\alpha),\end{aligned}$$

and,

$$\mathrm{Im}(C_\alpha(1)) \quad = \quad (C^+ - C^-)\int_0^\infty \mathrm{Im}((e^{ix}-1)h_1(x) - ix)\frac{dx}{x^{\alpha+1}}.$$

We deduce, since for $\gamma \in]2,3[$

$$\int_0^\infty \frac{\sin x - x}{x^\gamma}dx = \frac{1}{(\gamma-1)(2-\gamma)}\int_0^\infty \frac{\sin x}{x^{\gamma-2}}dx = -\frac{\Gamma(3-\gamma)}{(\gamma-1)(2-\gamma)}\cos(\frac{\pi}{2}\gamma),$$

that

$$\mathrm{Im}(C_\alpha(1)) \quad = \quad \frac{C^+ - C^-}{\alpha(1-\alpha)}\mathbb{E}\left((\Pi_\infty+1)^\alpha - \Pi_\infty^\alpha\right)\Gamma(2-\alpha)\sin(\frac{\pi}{2}\alpha).$$

Noting that $C_\alpha(1) = -\sigma^\alpha + i\sigma^\alpha\beta\tan(\pi\alpha/2)$, we have then

$$\sigma^\alpha = -\mathrm{Re}(C_\alpha(1)), \quad \beta\tan(\pi\alpha/2) = -\frac{\mathrm{Im}(C_\alpha(1))}{\mathrm{Re}(C_\alpha(1))},$$

which gives the expressions of the parameters β and σ.

3. If $\alpha = 1$, then $\varphi_\alpha(uv) = \exp(uC_1(v) + iuv\tau(u))$ for any $u > 0$ and $v \in \{-1,1\}$ where $C_1(v)$ is given by (6.41) of Mirek (2011), $C_1(v) = \int\left((e^{ivx}-1)h_v(x) - i\frac{xv}{1+x^2}\right)\Lambda(dx)$, and

$$\begin{aligned}\tau(u) &= \int\left(\frac{x}{1+u^2x^2} - \frac{x}{1+x^2}\right)\Lambda(dx)\\ &= (C^+ - C^-)\int_0^\infty\left(\frac{x}{1+u^2x^2} - \frac{x}{1+x^2}\right)\frac{dx}{x^2} = -(C^+ - C^-)\ln(u).\end{aligned}$$

We have, since $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$,

$$\mathrm{Re}(C_1(1)) = (C^+ + C^-)\int_0^\infty \mathrm{Re}((e^{ix}-1)h_1(x))\frac{dx}{x^2} = -\frac{\pi}{2}(C^+ + C^-)$$

and

$$\begin{aligned}\mathrm{Im}(C_1(1)) + \tau(u) &= (C^+ - C^-)\int_0^\infty\left(\mathrm{Im}((e^{ix}-1)h_1(x)) - \frac{x}{1+x^2}\right)\frac{dx}{x^2} + \tau(u)\\ &= \int_0^\infty\left(\mathbb{E}(\sin(\Pi_\infty+1)x) - \mathbb{E}(\sin(\Pi_\infty x)) - \frac{x}{1+x^2}\right)\frac{dx}{x^2} + \tau(u).\end{aligned}$$

Now, recall that, for any $a > 0$

$$\int_0^\infty (\sin(ax) - axI_{x\in]0,1]})\frac{dx}{x^2} = -a\ln(a) + ca$$

with $c=\int_1^\infty r^{-2}\sin r\,dr+\int_0^1 r^{-2}(\sin r-r)dr$ (Sato (1999), page 84). Hence,

$$\begin{aligned}&\int_0^\infty\left(\mathbb{E}\sin((\Pi_\infty+1)x)-\mathbb{E}\sin(\Pi_\infty x)-\frac{x}{1+x^2}\right)\frac{dx}{x^2}\\&=\mathbb{E}(\Pi_\infty\ln(\Pi_\infty)-(\Pi_\infty+1)\ln(\Pi_\infty+1))+c+\int_0^\infty\left(xI_{x\in]0,1]}-\frac{x}{1+x^2}\right)\frac{dx}{x^2}\\&=\mathbb{E}(\Pi_\infty\ln(\Pi_\infty)-(\Pi_\infty+1)\ln(\Pi_\infty+1))+c.\end{aligned}$$

This completes the proof in the case $\alpha=1$, since

$$\sigma=-\mathrm{Re}(C_1(1)),\;\; u\mathrm{Im}(C_1(1))+u\tau(u)=-\sigma u\beta\frac{2}{\pi}\ln(u)+\mu u.$$

4. If $\alpha=2$, then $\varphi_2(uv)=\exp(u^2C_2(v))$, for any positive u and $v\in\{-1,1\}$, where $C_2(v)$ is given by (6.45) of Mirek (2011), that is $C_2(v)=-(1+2\mathbb{E}(\Pi_\infty))(C^++C^-)/4$.

□

3.3 Proof of Theorem 2

In order to prove Theorem 2, we shall apply Theorem 1. Since the random map Ψ is a.s. nondecreasing, Lemma 1 and Proposition 5 of the appendix allow to deduce that the Markov chain $(X_n^\nu)_{n\geq 0}$ is homogeneous and associated. We can also check that $(X_n^\nu)_{n\geq 0}$ is strictly stationary. According to Theorem 1, we have then to check first the convergence of the finite-dimensional distributions. For this, we shall use the following proposition.

Proposition 4. *Let $(X_n^x)_{n\in\mathbb{N}}$ be the Markov chain defined by (2.2) and starting at x. Suppose that the random map Ψ is Lipschitz with finite a.s. Lipschitz random positive constant A fulfilling (2.5). Let ν be the unique solution in law of (2.4). Let, for $0<\alpha\leq 2$, (a_n) and (b_n) be as defined in Theorem 1. Suppose moreover that $\lim_{n\to\infty}b_n/a_n=0$ and that,*

$$\lim_{n\to\infty}\int\mathbb{E}\left(f\left(\frac{1}{a_n}\sum_{i=1}^n(X_i^x-b_n)\right)\right)\nu(dx)=\mathbb{E}(f(Z)),\tag{3.30}$$

for any bounded and Lipschitz function f and some random variable Z.

1. *Suppose that $\liminf_{n\to\infty}\frac{a_n}{n}$ is either 0 or ∞ and*

$$C=0\;\; if\;\; \liminf_{n\to\infty}\frac{a_n}{n}=0,\tag{3.31}$$

recall that $C:=\sup\{|b_k-b_n|\;:\;0<n\leq k\leq 2n<\infty\}$. Let $(Z(t))_{t\geq 0}$ be a process with stationary and independent increments such that for any fixed t, the random variable $Z(t)$ is distributed as $t^{1/\alpha}Z$. Define, for any $t\in[0,1]$, $S_n^x(t)=\sum_{i=1}^{[nt]}X_i^x$. Then the finite-dimensional distributions of the process $\{a_n^{-1}(S_n^x(t)-[nt]b_n),\; t\in[0,1]\}$ converge in distribution to those of the process $(Z(t))_{0\leq t\leq 1}$.

2. *Suppose now that,*

$$\lim_{n\to\infty}\frac{a_n}{n} =: a \in]0,\infty[,\ \ \lim_{n\to\infty}(b_{[nt]}-b_n) =: C_t,\ \ \forall\ t\in]0,1]. \tag{3.32}$$

Let $(Z(t))_{t\geq 0}$ *be a process with stationary and independent increments such that for any fixed* t, *the random variable* $Z(t)$ *is distributed as* $t\left(Z+\frac{C_t}{a}\right)$. *Then the finite-dimensional distributions of the process* $\{a_n^{-1}(S_n^x(t)-[nt]b_n),\ t\in[0,1]\}$ *converge in distribution to those of the process* $(Z(t))_{0\leq t\leq 1}$.

We prove Proposition 4 in the subsection below and we continue the proof of Theorem 2. Let us note first that the convergence (3.30) follows by integrating over ν the limit convergence (2.10) and by using the dominated convergence theorem. Our purpose now is to check that the sequences (a_n) and (b_n) of Theorem 2 satisfy all the requirements of Theorem 1 and Proposition 4. Clearly, the sequence (a_n) is nondecreasing, tending to infinity with n and regularly varying with index $1/\alpha$ and $\lim_{n\to\infty} b_n/a_n = 0$. If $\alpha\in]0,2]\setminus\{1\}$, then $\liminf_{n\to\infty}\frac{a_n}{n}$ is either 0 or ∞ and b_n is either 0 or $\int x\nu(dx)$, consequently $C=0$ and Condition (3.31) is then satisfied. According to Proposition 2, the first conclusion of Proposition 4 is then satisfied with $Z(t)\sim t^{1/\alpha}S_\alpha(\sigma,\beta,0)$ which is distributed as $S_\alpha(t^{1/\alpha}\sigma,\beta,0)$, by property 1.2.3 of Samorodnitsky and Taqqu (1994). It remains then to check (3.32) i.e. the case when $\alpha=1$. For this we need the following lemma.

Lemma 7. *Let* ν *be some probability law on* $\mathbb{R}$ *satisfying* $\sup\{t\nu(|x|>t): t>0\} = C_0 < \infty$. *Let*

$$b_n = \int_{\mathbb{R}} \frac{n^2 x}{x^2+n^2}d\nu(x).$$

Then the sequence $(b_n)_n$ *satisfies the condition*

$$(a) \qquad \sup\{|b_k - b_n| : n\leq k\leq 2n\} < \infty.$$

Assume furthermore that, for some nonnegative real constants C^+ *and* C^-,

$$\lim_{t\to+\infty} t\nu(x>t) = C^+ \text{ and } \lim_{t\to+\infty} t\nu(x<-t) = C^-.$$

Then, for any u *in* $]0,1]$,

$$(b) \qquad \lim_{n\to\infty}(b_{[nu]}-b_n) = (C^+-C^-)\ln(u).$$

We shall prove this lemma in Subsection 4.2 below and we continue the proof of Theorem 2. Using Proposition 1 we deduce that the requirements of Lemma 7 are satisfied by the measure ν which is the unique solution of (2.4). Hence the limits (3.32) are satisfied with

$$a_n = n,\ a = 1,\ C_t = (C^+-C^-)\ln(t),\ t\in]0,1].$$

From this we get that $C_t = \beta\sigma\frac{2}{\pi}\ln(t)$ for any positive t. According to Proposition 2, the second conclusion of Proposition 4 holds with $Z(t)\sim t(S_1(\sigma,\beta,\mu)+C_t)$, which is distributed as $S_1(t\sigma,\beta,t\mu)$, by properties 1.2.3 and 1.2.2 of Samorodnitsky and Taqqu (1994). To finish the proof of Theorem 2, we have to check, according to Theorem 1, that

$$\lim_{x\to\infty} x^{\alpha/2}\mathbb{P}(|Z|\geq x) = 0, \tag{3.33}$$

for a random variable Z distributed as $S_\alpha(\sigma,\beta,\mu)$. When $\alpha\in]0,2[$, Property 1.2.15 page 16 of Samorodnitsky and Taqqu (1994) shows that

$$\lim_{x\to\infty} x^\alpha \mathbb{P}(Z>x)=C_\alpha\frac{1+\beta}{2}\sigma^\alpha,\ \lim_{x\to\infty} x^\alpha \mathbb{P}(Z<-x)=C_\alpha\frac{1-\beta}{2}\sigma^\alpha,$$

with

$$C_\alpha=\left(\int_0^\infty x^{-\alpha}\sin(x)dx\right)^{-1}=\frac{2}{\pi}\mathbb{I}_{\alpha=1}+\frac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}\mathbb{I}_{\alpha\neq 1}.$$

Those limits prove the tail condition (3.33) when $\alpha\in]0,2[$. When $\alpha=2$, (3.33) is satisfied since $S_2(\sigma,\beta,0)$ is a Gaussian random variable. Hence (3.33) holds true which implies (1.1). The proof of Theorem 2 is then complete by using Theorem 1. □

4 Proofs of some auxiliary results

4.1 Proof of Proposition 4

We first need the following technical lemma.

Lemma 8. *Let $(X_n^x)_{n\in\mathbb{N}}$ be an homogenous Markov chain on $\mathbb{R}$ starting at x. Let $S_n^x(t)=\sum_{i=1}^{[nt]}X_i^x$ for $t\in[0,1]$. Suppose that:*

1. *There exists a probability measure ν_x such that, for any $0\le s<t\le 1$ and for any continuous Lipschitz function g with values in [0,1] one has, for some real sequences (a_n) strictly positive and (b_n),*
$$\lim_{n\to\infty}\mathbb{E}\left|h_n(X_{[ns]}^x)-\int h_n(y)\nu_x(dy)\right|=0, \tag{4.34}$$
where
$$h_n(y)=h_n(y,s,t)=\mathbb{E}\left(g\left(a_n^{-1}\sum_{l=1}^{[nt]-[ns]}(X_l^y-b_n)\right)\right).$$
2. *For any $0<s<t\le 1$*
$$\lim_{n\to\infty}\sum_{l=1+[n(t-s)]}^{[nt]-[ns]}\int \mathbb{P}\left(a_n^{-1}\left|X_l^y-b_n\right|\ge\epsilon\right)\nu_x(dy)=0. \tag{4.35}$$
3. *For any bounded and Lipschitz function f and any fixed $t\in[0,1]$,*
$$\lim_{n\to\infty}\int\mathbb{E}\left(f\left(a_n^{-1}\sum_{i=1}^{[nt]}(X_i^y-b_n)\right)\right)\nu_x(dy)=\mathbb{E}(f(Z(t))), \tag{4.36}$$
for some process $(Z(t))_{0\le t\le 1}$ with stationary and independent increments.

Then the finite-dimensional distributions of the process $\{a_n^{-1}(S_n^x(t)-[nt]b_n),\ t\in[0,1]\}$ converge in distribution to those of the process $(Z(t))_{0\le t\le 1}$.

Proof of Lemma 8. Let $0 = t_{-1} < t_0 < \ldots < t_m < t_{m+1} = 1$ be fixed. Our aim is to check that the $(m+1)$-tuple $a_n^{-1}(S_n^x(t_0) - [nt_0]b_n, S_n^x(t_1) - S_n^x(t_0) - ([nt_1] - [nt_0])b_n, \ldots, S_n^x(t_m) - S_n^x(t_{m-1}) - ([nt_m] - [nt_{m-1}])b_n)$ converges in distribution to $(Z(t_0), \ldots, Z(t_m) - Z(t_{m-1}))$. We claim that it suffices to prove this result for $m = 1$. For this, let f and g be two continuous and Lipschitz functions with values in $[0,1]$. Fix $0 < s < t < 1$. We have, using the Markov property,

$$A_n := \mathbb{E}\left(f\left(a_n^{-1}(S_n^x(s) - [ns]b_n)\right) g\left(a_n^{-1}(S_n^x(t) - S_n^x(s) - ([nt] - [ns])b_n)\right)\right)$$
$$= \mathbb{E}\left(f\left(a_n^{-1}(S_n^x(s) - [ns]b_n)\right) h_n(X_{[ns]}^x)\right), \tag{4.37}$$

where

$$h_n(y) = \mathbb{E}\left(g\left(a_n^{-1}\sum_{i=[ns]+1}^{[nt]}(X_i^x - b_n)\right)\middle| X_{[ns]}^x = y\right).$$

The homogeneity property of the chain gives

$$h_n(y) = \mathbb{E}\left(g\left(a_n^{-1}\sum_{l=1}^{[nt]-[ns]}(X_l^y - b_n)\right)\right). \tag{4.38}$$

Recall that A_n is defined by (4.37), whence

$$\left|A_n - \mathbb{E}\left(f\left(a_n^{-1}(S_n^x(s) - [ns]b_n)\right)\right)\mathbb{E}(g(Z(t-s)))\right|$$
$$\le \mathbb{E}\left|h_n(X_{[ns]}^x) - \mathbb{E}(g(Z(t-s)))\right|. \tag{4.39}$$

We first control the right hand side of (4.39):

$$\mathbb{E}\left|h_n(X_{[ns]}^x) - \mathbb{E}(g(Z(t-s)))\right|$$
$$\le \mathbb{E}\left|h_n(X_{[ns]}^x) - \int h_n(y)\nu_x(dy)\right| + \left|\int \left(h_n(y) - \mathbb{E}(g(Z(t-s)))\right)\nu_x(dy)\right|. \tag{4.40}$$

Our task is to control, using (4.35) and (4.36), the second term in the right hand side of (4.40). We have, for any positive ϵ, letting $\mathrm{Lip}(g) = \sup_{x\in\mathbb{R}, y\in\mathbb{R}^*} |(g(x+y) - g(x))/y|$,

$$\left|\mathbb{E}\left(g\left(a_n^{-1}\sum_{l=1}^{[nt]-[ns]}(X_l^y - b_n)\right)\right) - \mathbb{E}\left(g\left(a_n^{-1}\sum_{l=1}^{[n(t-s)]}(X_l^y - b_n)\right)\right)\right|$$
$$\le \epsilon\mathrm{Lip}(g) + \mathbb{P}\left(a_n^{-1}\left|\sum_{l=1+[n(t-s)]}^{[nt]-[ns]}(X_l^y - b_n)\right| \ge \epsilon\right)$$
$$\le \epsilon\mathrm{Lip}(g) + \sum_{l=1+[n(t-s)]}^{[nt]-[ns]}\mathbb{P}\left(a_n^{-1}\left|X_l^y - b_n\right| \ge \epsilon/2\right), \tag{4.41}$$

since $0 \le [nt] - [ns] - [n(t-s)] \le 2$. Clearly

$$\begin{aligned}
&\left|\int \left(h_n(y) - \mathbb{E}(g(Z(t-s)))\right) \nu_x(dy)\right| \\
&\le \left|\int \left(h_n(y) - \mathbb{E}\left(g\left(a_n^{-1} \sum_{l=1}^{[n(t-s)]} (X_l^y - b_n)\right)\right)\right) \nu_x(dy)\right| \\
&+ \left|\int \mathbb{E}\left(g\left(a_n^{-1} \sum_{l=1}^{[n(t-s)]} (X_l^y - b_n)\right)\right) \nu_x(dy) - \mathbb{E}(g(Z(t-s)))\right|.
\end{aligned}$$

We now get, integrating (4.41) over ν_x and taking in mind the definition of h_n,

$$\begin{aligned}
&\left|\int \left(h_n(y) - \mathbb{E}(g(Z(t-s)))\right) \nu_x(dy)\right| \\
&\le \epsilon \mathrm{Lip}(g) + \sum_{l=1+[n(t-s)]}^{[nt]-[ns]} \int \mathbb{P}\left(a_n^{-1}\left|X_l^y - b_n\right| \ge \epsilon/2\right) \nu_x(dy) \\
&+ \left|\int \mathbb{E}\left(g\left(a_n^{-1} \sum_{l=1}^{[n(t-s)]} (X_l^y - b_n)\right)\right) \nu_x(dy) - \mathbb{E}(g(Z(t-s)))\right|.
\end{aligned}$$

The last bound together with (4.35) and (4.36) gives then

$$\lim_{n\to\infty} \left|\int \left(h_n(y) - \mathbb{E}(g(Z(t-s)))\right) \nu_x(dy)\right| = 0.$$

Now this last limit, (4.40) and (4.34) yield

$$\lim_{n\to\infty} \mathbb{E}\left|h_n(X_{[ns]}^x) - \mathbb{E}(g(Z(t-s)))\right| = 0, \tag{4.42}$$

for any $0 < s < t < 1$. We get, letting $s = 0$ in (4.34) and using (4.36), that, for any $t \in]0,1]$ (the function g being arbitrary),

$$\lim_{n\to\infty} \mathbb{E}\left(f\left(a_n^{-1}(S_n^x(t) - [nt]b_n)\right)\right) = \mathbb{E}(f(Z(t))).$$

Consequently

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left(f\left(a_n^{-1}(S_n^x(s) - [ns]b_n)\right)\right)\mathbb{E}(g(Z(t-s))) &= \mathbb{E}(f(Z(s)))\mathbb{E}(g(Z(t-s))) \\
&= \mathbb{E}(f(Z(s))g(Z(t) - Z(s))), \quad (4.43)
\end{aligned}$$

since the process $(Z(t))_{t\ge 0}$ is assumed to be with stationary and independent increments. We finally conclude, combining (4.39), (4.43) and (4.42), that

$$\lim_{n\to\infty} A_n = \mathbb{E}(f(Z(s))g(Z(t) - Z(s))).$$

The proof of Lemma 8 is then complete. □

End of the proof of Proposition 4. Let ν be the unique solution in law of (2.4). Our purpose is to check all the requirements of Lemma 8 from that of Proposition 4 with $\nu_x = \nu$.

Proof of (4.34). Recall that

$$h_n(y) = \mathbb{E}\left(g\left(a_n^{-1}\sum_{l=1}^{[nt]-[ns]}(X_l^y - b_n)\right)\right),$$

where g is a Lipschitz function with values in $[0,1]$. Recall also that $X_n^x = W_n(x)$ where W_n is defined by (2.3). Since for each n the random map Ψ_n is a.s. A_n-Lipschitz,

$$|W_n(x) - W_n(y)| \leq A_1 \dots A_n |x - y|.$$

Hence, for any reals y, z and any positive M,

$$\begin{aligned}
\left|h_n(y) - h_n(z)\right| &\leq \mathbb{E}\left(1 \wedge \text{Lip (g)}\frac{1}{a_n}\sum_{l=1}^{n}|W_l(y) - W_l(z)|\right) \\
&\leq \mathbb{E}\left(1 \wedge \text{Lip (g)}\frac{1}{a_n}|y - z|\sum_{l=1}^{n}A_1 \dots A_l\right) \\
&\leq \mathbb{E}\left(1 \wedge \text{Lip (g)}\frac{M}{a_n}\sum_{l=1}^{n}A_1 \dots A_l\right) + \mathbb{1}_{|y-z|\geq M}. \qquad (4.44)
\end{aligned}$$

Now

$$\begin{aligned}
\mathbb{E}\left|h_n(X_{[ns]}^x) - \int h_n(y)\nu(dy)\right| &= \int\left|h_n(z) - \int h_n(y)\nu(dy)\right|\mathbb{P}_{X_{[ns]}^x}(dz) \\
&\leq \int\int\left|h_n(z) - h_n(y)\right|\nu(dy)\mathbb{P}_{X_{[ns]}^x}(dz).
\end{aligned}$$

The last bound together with (4.44) gives, for any $M > 0$,

$$\mathbb{E}\left|h_n(X_{[ns]}^x) - \int h_n(y)\nu(dy)\right| \leq \mathbb{E}\left(1 \wedge \text{Lip (g)}\frac{M}{a_n}\sum_{l=1}^{n}A_1 \dots A_l\right) + \int \mathbb{P}(|X_{[ns]}^x - y| \geq M)\nu(dy).$$

We deduce from the first two conditions in (2.5) that $\sum_{l\geq 1} A_1 \dots A_l < \infty$ a.s. Since a_n tends to infinity with n, it follows that

$$\lim_{n\to\infty}\mathbb{E}\left(1 \wedge \text{Lip (g)}\frac{M}{a_n}\sum_{l=1}^{n}A_1 \dots A_l\right) = 0.$$

Now we use the conclusion of Lemma 2 that guarantees the convergence in law of X_n^x for any x to conclude that,

$$\lim_{M\to\infty}\lim_{n\to\infty}\int \mathbb{P}(|X_{[ns]}^x - y| \geq M)\nu(dy) = 0.$$

The two last limits prove (4.34).

Proof of (4.35). Recall that ν is the unique solution in law of (2.4). We have, since for each n X_n^ν is distributed according to ν,

$$\sum_{l=1+[n(t-s)]}^{[nt]-[ns]} \int \mathbb{P}\left(a_n^{-1}\left|X_l^y - b_n\right| \geq \epsilon\right) \nu(dy) \leq 2\mathbb{P}\left(a_n^{-1}\left|X_1^\nu - b_n\right| \geq \epsilon\right),$$

which tends to 0 as n goes to infinity since $\lim_{n\to\infty} b_n/a_n = 0$ and $\lim_{n\to\infty} a_n = \infty$.

Proof of (4.36) assuming (3.30). Clearly,

$$\mathbb{E}\left(f\left(a_n^{-1}\sum_{i=1}^{[nt]}(X_i^y - b_n)\right)\right) = E\left(f\left(\frac{a_{[nt]}}{a_n}a_{[nt]}^{-1}\sum_{i=1}^{[nt]}(X_i^y - b_{[nt]}) + \frac{[nt]}{a_n}(b_{[nt]} - b_n)\right)\right). \quad (4.45)$$

We shall first control the term $\frac{[nt]}{a_n}(b_{[nt]} - b_n)$ by using Condition (3.31). We have, for any positive integer L,

$$\frac{[nt]}{a_n}|b_{[nt]} - b_n| \leq \frac{n}{a_n}\sum_{i=0}^{L-1}|b_{2^{i+1}[nt]} - b_{2^i[nt]}| + \frac{n}{a_n}|b_{2^L[nt]} - b_n|.$$

Choose $L = L_n = \left[\log_2\left(\frac{n}{[nt]}\right)\right]$ so that $2^L[nt] \leq n < 2^{L+1}[nt]$. Hence

$$|b_{2^L[nt]} - b_n| \leq \sup\{|b_k - b_n|,\ 0 < n \leq k \leq 2n < \infty\} = C.$$

Clearly, for any $0 \leq i \leq L-1$,

$$|b_{2^{i+1}[nt]} - b_{2^i[nt]}| \leq C.$$

Consequently,

$$\frac{[nt]}{a_n}|b_{[nt]} - b_n| \leq \frac{n}{a_n}(L_n + 1)C,$$

which converges to 0 as n tends to infinity by (3.31), since $\limsup_{n\to\infty} L_n \leq \log_2(1/t)$. Hence the left hand side of (4.45) converges, under the initial distribution ν, to $\mathbb{E}(f(t^{1/\alpha}Z))$ by (3.30), since a_n is regularly varying with index $1/\alpha$. This proves (4.36) with $Z(t)$ distributed as $t^{1/\alpha}Z$ for any $t \in [0,1]$. We suppose now that Condition (3.32), instead of (3.31), holds. We deduce first that the sequence (a_n) is regularly varying with index 1 and we complete the proof by combining (4.45), (3.30) and (3.32) as before.

Hence the requirements of Lemma 8 are fulfilled, which implies Proposition 4. □

4.2 Proof of Lemma 7

We first prove (a). By definition

$$b_k - b_n = \int_{\mathbb{R}} \frac{(k^2 - n^2)x^3}{(x^2 + n^2)(x^2 + k^2)} d\nu(x),$$

whence

$$|b_k - b_n| \le \int_{\mathbb{R}} \frac{k^2|x|^3}{(x^2+n^2)(x^2+k^2)} d\nu(x) \le \frac{1}{2n} \int_{\mathbb{R}} \frac{k^2x^2}{x^2+k^2} d\nu(x),$$

since $x^2 + n^2 \ge 2n|x|$. Set $H(t) = \nu(|x| > t)$. Noting that

$$\frac{k^2x^2}{x^2+k^2} = \int_0^{|x|} \frac{2t}{(1+(t/k)^2)^2} dt$$

and applying the Fubini-Tonelli Theorem, we get that

$$\frac{1}{2n} \int_{\mathbb{R}} \frac{k^2x^2}{x^2+k^2} d\nu(x) = \frac{1}{n} \int_0^\infty \frac{tH(t)}{(1+(t/k)^2)^2} dt \le \frac{C_0 k}{n} \int_0^\infty \frac{1}{(1+y^2)^2} dy,$$

using the change of variables $y = t/k$ and the assumption that $tH(t) \le C_0$. Since $(k/n) \le 2$, it implies (a).

To prove (b), we separate the integral on $\mathbb{R}$ defining b_n in two parts. So, let

$$b_n^+ = \int_{\mathbb{R}^+} \frac{n^2 x}{x^2+n^2} d\nu(x) \text{ and } b_n^- = \int_{\mathbb{R}^-} \frac{-n^2 x}{x^2+n^2} d\nu(x).$$

With these definitions, $b_n = b_n^+ - b_n^-$. Consequently we have to prove that

$$\lim_{n\to\infty} (b_{[nu]}^+ - b_n^+) = C^+ \ln(u) \ \text{ and } \ \lim_{n\to\infty} (b_{[nu]}^- - b_n^-) = C^- \ln(u). \tag{4.46}$$

Since the proof of these two assertions uses exactly the same arguments, we will only prove the first part of (4.46). Let $H_+(t) = \nu(x > t)$. Set $k = [nu]$. With these notations, exactly as in the proof of (a),

$$b_k^+ - b_n^+ = \int_0^\infty \frac{x^3(k^2-n^2)}{(x^2+n^2)(x^2+k^2)} d\nu(x).$$

Set

$$\varphi_n(x) = \frac{x^3(k^2-n^2)}{(x^2+n^2)(x^2+k^2)}.$$

Since $\varphi(0) = 0$, exactly as in the proof of (a)

$$b_k^+ - b_n^+ = \int_0^\infty \varphi_n'(t) H_+(t) dt.$$

Next $tH_+(t) = C_+ + \varepsilon(t)$, for some function ε converging to 0 as t tends to ∞. It follows that

$$b_k^+ - b_n^+ = C_+ \int_0^\infty \varphi_n'(t) \frac{dt}{t} + I_n \ \text{ with } \ I_n = \int_0^\infty \varepsilon(t) \varphi_n'(t) \frac{dt}{t}.$$

Some elementary computations show that

$$\left| \frac{\varphi_n'(t)}{t} \right| \le \frac{8t}{(1+(t/n)^2)(k^2+t^2)}.$$

Whence, using the change of variables $y = (t/n)^2$,

$$|I_n| \le 4\int_0^\infty \frac{|\varepsilon(ny^{1/2})|}{(y+1)(y+([nu]/n)^2)}dy.$$

Since $([nu]/n)$ converges to the positive real u as n tends to ∞, the upper bound tends to 0 by the dominated convergence theorem (note that the function ε is uniformly bounded on $\mathbb{R}^+$). It remains to prove that

$$\lim_{n\to\infty}\int_0^\infty \varphi_n'(t)\frac{dt}{t} = \ln(u). \tag{4.47}$$

To prove (4.47) we integrate by parts again (here $k = [nu]$):

$$\int_0^\infty \varphi_n'(t)\frac{dt}{t} = \int_0^\infty \varphi_n(t)\frac{dt}{t^2} = \int_0^\infty \frac{t(k^2-n^2)}{(t^2+n^2)(t^2+k^2)}dt = \ln\Big(\frac{[nu]}{n}\Big),$$

which implies (4.47). Hence (4.46) holds true, which implies the second part of Lemma 7. □

5 Appendix: Stochastically monotone Markov chains

Recall that a real-valued Markov chain $(X_n)_{n\in\mathbb{N}}$ is stochastically monotone if for any $n \in \mathbb{N}$, the function $x \mapsto \mathbb{E}(f(X_{n+1})|X_n = x)$ is nondecreasing for every fixed nondecreasing real-valued function f defined on $\mathbb{R}$. The purpose of the proposition below is to study the association property for stochastically monotone Markov chains.

Proposition 5. *If $(X_n)_{n\in\mathbb{N}}$ is a real-valued and stochastically monotone Markov chain then, for any initial distribution, $(X_n)_{n\in\mathbb{N}}$ is a sequence of associated random variables.*

Proposition 5 is not new. It was stated in Daley (1968) without proof and proved for monotone Markov processes in Liggett (1985) (cf. Corollary 2.21 there). We give its proof for the sake of clarity. For this, we need first the following lemma.

Lemma 9. *Let $(X_n)_{n\in\mathbb{N}}$ be a real-valued and stochastically monotone Markov chain then, for any $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, the function $x \mapsto \mathbb{E}(f(X_{n+1},\dots,X_{n+k})|X_n = x)$ is nondecreasing for every fixed nondecreasing real-valued function f defined on $\mathbb{R}^k$.*

Proof of Lemma 9. The proof is done by induction on k. For $k = 1$ the property follows from the definition of stochastically monotone Markov chain. Now, let f be a nondecreasing function on $\mathbb{R}^k$. Define,

$$h_n(y) = \mathbb{E}(f(X_{n+1},\dots,X_{n+k})|X_{n+1} = y) = \mathbb{E}(f(y,X_{n+2},\dots,X_{n+k})|X_{n+1} = y).$$

By the induction assumption, the function h_n is nondecreasing . From this we conclude, since the Markov chain is stochastically monotone, that the function $x \mapsto \mathbb{E}(h_n(X_{n+1})|X_n = x)$ is also nondecreasing. This fact completes the proof of Lemma 9 since $\mathbb{E}(f(X_{n+1},\dots,X_{n+k})|X_n = x) = \mathbb{E}(h_n(X_{n+1})|X_n = x)$. □

From Lemma 9, we now derive Proposition 5.

Proof of Proposition 5. Let $(X_n)_{n\in\mathbb{N}}$ be a real-valued and a stochastically monotone Markov chain with initial distribution μ. Denote by Cov_μ and $\mathbb{E}_\mu$, respectively, the covariance and the expectation when the initial distribution is μ. Our purpose is to check that, for any $n \in \mathbb{N}$,

$$\mathrm{Cov}_\mu(f(X_0,\ldots,X_n), g(X_0,\ldots,X_n)) \geq 0,$$

for any coordinatewise nondecreasing real-valued functions f and g defined on $\mathbb{R}^{n+1}$, whenever this covariance is well defined. The proof is done by induction on n. The induction property holds for $n=0$ since a real-valued random variable is associated. Now we have,

$$\begin{aligned}&\mathbb{E}_\mu\left(f(X_0,\ldots,X_n)g(X_0,\ldots,X_n)\right) = \mathbb{E}_\mu\left(\mathbb{E}\left(f(X_0,\ldots,X_n)g(X_0,\ldots,X_n)|X_0\right)\right)\\ &= \mathbb{E}_\mu(h_n(X_0)),\end{aligned} \tag{5.48}$$

where $h_n(x) = \mathbb{E}\left(f(x,X_1,\ldots,X_n)g(x,X_1,\ldots,X_n)|X_0=x\right)$, which can be written as $h_n(x) = \mathbb{E}_{\mu_1}\left(\tilde{h}(X_1)\right)$, where μ_1 is the distribution of X_1 given $X_0 = x$, and $\tilde{h}$ is defined by

$$\tilde{h}(y) = \mathbb{E}(f(x,y,X_2,\ldots,X_n)g(x,y,X_2,\ldots,X_n)|X_1=y).$$

The induction assumption gives,

$$\tilde{h}(y) \geq \mathbb{E}(f(x,y,X_2,\ldots,X_n)|X_1=y)\mathbb{E}(g(x,y,X_2,\ldots,X_n)|X_1=y). \tag{5.49}$$

In order to control the right-hand side of the last inequality we note, taking into account Lemma 9, that the two functions

$$y \mapsto \mathbb{E}(f(x,y,X_2,\ldots,X_n)|X_1=y),\;\; y \mapsto \mathbb{E}(g(x,y,X_2,\ldots,X_n)|X_1=y)$$

are both nondecreasing. We then obtain, taking the expectation over μ_1 in Inequality (5.49) and using again the induction assumption,

$$h_n(x) \geq \mathbb{E}\left(f(x,X_1,\ldots,X_n)|X_0=x\right)\mathbb{E}\left(g(x,X_1,\ldots,X_n)|X_0=x\right). \tag{5.50}$$

We conclude, combining (5.48) and (5.50), that

$$\mathbb{E}_\mu\left(f(X_0,\ldots,X_n)g(X_0,\ldots,X_n)\right) \geq \mathbb{E}_\mu(F_n(X_0)G_n(X_0)),$$

where $F_n(x) = \mathbb{E}\left(f(x,X_1,\ldots,X_n)|X_0=x\right)$ and $G_n(x) = \mathbb{E}\left(g(x,X_1,\ldots,X_n)|X_0=x\right)$. Now, from Lemma 9, the two functions $x \mapsto F_n(x)$ and $x \mapsto G_n(x)$ are nondecreasing. Since a real-valued random variable is always associated, this gives

$$\mathbb{E}_\mu(F_n(X_0)G_n(X_0)) \geq \mathbb{E}_\mu(F_n(X_0))\mathbb{E}_\mu(G_n(X_0)).$$

The last inequality completes the proof, since

$$\mathbb{E}_\mu(F_n(X_0)) = \mathbb{E}_\mu\left(f(X_0,\ldots,X_n)\right),\;\; \mathbb{E}_\mu(G_n(X_0)) = \mathbb{E}_\mu\left(g(X_0,\ldots,X_n)\right).$$

□

References

[1] Avram, F., Taqqu, M.S. (1992). Weak convergence of sums of moving averages in the α-stable domain of attraction. Ann. Probab. 20, 483-503. MR1143432

[2] Bartkiewicz, K., Jakubowski, A., Mikosch, Th., Wintenberger, O. (2011). Stable limits for sums of dependent infinite variance random variables. Probab. Theory Relat. Fields. 150, 337-372.

[3] Basrak, B., Krizmanic, D., Segers, J. (2010). A functional limit theorem for partial sums of dependent random variables with infinite variance. To appear in Annals of Probability.

[4] Daley, D. J. (1968). Stochastically Monotone Markov Chains. Z. Wahrscheinlichkeitstheorie verw. Geb. 10, 305-317. MR0242270

[5] Diaconis, P., Freedman, D. (1999). Iterated random functions. SIAM Rev. Society for Industrial and Applied Mathematics Vol. 41, No. 1, 45-76. MR1669737

[6] Esary, J., Proschan, F. and Walkup, D. (1967). Association of random variables with applications. Ann. Math. Stat. 38, 1466–1476. MR0217826

[7] Goldie, Ch. M. (1991). Implicit Renewal Theory and Tails of Solutions of Random Equations. Ann. Appl. Probab. Volume 1, Number 1, 126-166. MR1097468

[8] Letac, G. (1986). A contraction principle for certain Markov chains and its applications. Random matrices and their applications: proceedings (Brunswick, Maine, 1984), 263–273. Contemp. Math., 50, Amer. Math. Soc., Providence, RI. MR0841098

[9] Liggett, T.M. (1985). Interacting Particle Systems. Springer-Verlag, New York. MR0776231

[10] Louhichi, S. and Rio, E. (2011). Convergence du processus de sommes partielles vers un processus de Lévy pour les suites associées. C. R. Acad. Sci. Paris, Ser. I 349, 89-91. MR2755704

[11] Mirek, M. (2011). Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps. Probab. Theory Relat. Fields. 151, 705-734.

[12] Newman, C.M., Wright, A. L. (1981). An invariance principle for certain dependent sequences. Ann. Probab. 9, 671-675. MR0624694

[13] Newman, C.M., Wright, A. L. (1982). Associated random variables and martingale inequalities. Z. Wahrscheinlichkeitstheorie verw. Gebiete 59, 361-371. MR0721632

[14] Samorodnitsky, G. and Taqqu, Murad S. (1994). Stable non-Gaussian random processes: stochastic models with infinite variance. Stochastic Modeling. Chapman & Hall, New York. MR1280932

[15] Sato, K.I. (1999). Lévy processes and infinitely divisible distributions. Cambridge studies in advanced mathematics, 68. MR1739520

[16] Skorohod, A.V. (1956). Limit theorems for stochastic processes. Theory Probab. Appl. 1, 261-290. MR0084897

[17] Tyran-Kamińska, M. (2010a). Weak convergence to Lévy stable processes in dynamical systems. Stochastics and Dynamics. 10, 263-289. MR2652889

[18] Tyran-Kamińska, M. (2010b). Convergence to Lévy stable processes under some weak dependence conditions. Stochastic Proc. Appl. 120, 1629-1650. MR2673968

[19] Whitt, W. (2002). Stochastic-process limits. Springer Series in Operations Research. Springer-Verlag. New-York. MR1876437