A Correction to "Hitting Properties of a Random String"

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The purpose of this errata is to correct the proof of Corollary 4, page 12 of [MT02]. The corollary is true as stated, but there is a mistake in the last paragraph of the proof. For easy reading, we repeat the statement of the corollary and the first part of its proof. The notation comes from [MT02].

Theorem 1 (Corollary 4 of [MT02]). *Suppose* $(u_t(x) : t \geq 0,\ x \in \mathbf{R})$ *is a solution to (1.1) of [MT02] and* $(\tilde{u}_t(x) : t \geq 0, x \in \mathbf{T})$ *is a solution to (1.1) on the circle. For any compact set* $A \subseteq (0,\infty) \times (0,1)$ *the laws of the fields* $(u_t(x) : (t,x) \in A)$ *and* $(\tilde{u}_t(x) : (t,x) \in A)$ *are mutually absolutely continuous.*

Proof. We may suppose that the initial functions $u_0 = f \in \mathcal{E}_{exp}$ and $\tilde{u}_0 = g \in C(\mathbf{T})$ are deterministic. The case where u_0 and $\tilde{u}_0$ are random then follows by using the Markov property at time zero. We also suppose that they are defined on the same probability space and the noise driving $(\tilde{u}_t(x))$ is the restriction to the circle of the noise W driving $(u_t(x))$.

We use a standard symmetry trick to extend the solution $(\tilde{u}_t(x))$ over the real line. We may extend the solution to $(\tilde{u}_t^{(\mathrm{per})}(x) : t \geq 0,\ x \in \mathbf{R})$ by making it periodic with period one. We also extend the noise to a noise $W^{(\mathrm{per})}(dx\,dt)$ over the whole line by making it periodic. Note that $\tilde{u}_t^{(\mathrm{per})}(x) = \tilde{u}_t(x)$ and $W^{(\mathrm{per})}(dx\,dt) = W(dx\,dt)$ for $t \geq 0,\ x \in \mathbf{T}$. Then $(\tilde{u}_t^{(\mathrm{per})}(x))$ satisfies (1.2) of [MT02] over the whole line, with the Green's function for the whole line but with the periodic noise $W^{(\mathrm{per})}(dx\,dt)$.

We again take a $\mathcal{C}^\infty$ function $\psi_t(x)$ that equals 1 on A and still has compact support inside $(0,\infty) \times (0,1)$. Define

$$\begin{aligned} v_t(x) &= u_t(x) + \psi_t(x) \int G_t(x-y) \left(g^{(\mathrm{per})}(y) - f(y)dy \right) \\ &\quad + \psi_t(x) \int_0^t \int G_{t-s}(x-y) \left(W^{(\mathrm{per})}(dy\,ds) - W(dy\,ds) \right). \end{aligned}$$

Then using the representation (1.2) of [MT02] we see that $v_t(x) = \tilde{u}_t(x)$ for $(t,x) \in A$. Also $v_0 = f$ and it is straightforward to check that $(v_t(x))$ is a solution to (3.1) in [MT02] with

$$
\begin{aligned}
h_t(x) \quad &= \quad \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial^2 x}\right)\left(\psi_t(x)\int G_t(x-y)\left(g^{(\mathrm{per})}(y) - f(y)\right)dy\right) \\
&\qquad + \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial^2 x}\right)\left(\psi_t(x)\int_0^t \int G_{t-s}(x-y)\right. \qquad (1) \\
&\qquad\qquad \left.\left(W^{(\mathrm{per})}(dy\,ds) - W(dy\,ds)\right)\right). \\
&=: \quad h_{\mathrm{deterministic}}(t,x) + h_{\mathrm{random}}(t,x)
\end{aligned}
$$

Note that $h_t(x)$ has compact support. We claim that $h_t(x)$ is also smooth. The only term in $h_t(x)$ for which this is not clear is the stochastic integral

$$
I(t,x) = \int_0^t \int G_{t-s}(x-y)\left(W^{(\mathrm{per})}(dy\,ds) - W(dy\,ds)\right).
$$

However, since $W^{(\mathrm{per})}(dy\,ds) - W(dy\,ds) = 0$ for $y \in (0,1)$, the function $I(t,x)$ solves the deterministic heat equation in the region $[0,\infty)\times(0,1)$, with zero initial conditions and continuous random boundary values $I(t,1)$ and $I(t,0)$. Hence it is smooth in this region, and since $\psi_t(x)$ is also supported in this region, the claim follows.

Remark. It remains to show that the stochastic exponential

$$
\exp\left(\int_0^T h_t(x)\cdot W(dx\,dt) - \frac{1}{2}\int_0^T |h_t(x)|^2 dxdt\right)
$$

is a true martingale. Here is where we depart from the proof in [MT02]. We appealed incorrectly to Lemma 2 of [MT02], the hypotheses of which were not true. We now correct this mistake by showing that the Novikov condition applies over short time intervals, and then iterating the argument to get the result for long time intervals.

To verify the exponential moment required by Novikov's condition, we note that the integrand $h_t(x)$ is Gaussian. Hence $|h_t(x)|^2$ will have some finite exponential moments. We will apply Borell's inequality to see that $\sup_{t,x}|h_t(x)|^2$ also has some exponential moments. This will be enough to verify Novikov's criterion over a sufficiently small time interval $[0,\delta]$. By iterating this argument the stochastic exponential will remain a true martingale over $[0,T]$. Here are the details.

The following lemma deals with absolute continuity of Wiener measure when a drift is added. It is an easy generalization of Theorem 3.1.1, page 216, of Gyöngy and Pardoux [GP93] (see also [GP92]), and we will not prove it here. Gyöngy and Pardoux do not deal with the case in which $h_t(x)$ is a vector.

Lemma 1. *Assume that $D \subset \mathbf{R}^m$ is a bounded open set. Suppose that $(\dot{W}(t,x) : x \in \mathbf{R}^n)$ is a vector of independent white noises. Let $\mathcal{F}_t$ be the σ-field generated by $(W(S) : S \subset [0,t] \times \mathbf{R}^m)$, where the sets S must be Borel and bounded. Suppose $h_t : \Omega \times \mathbf{R}^m \to \mathbf{R}^n$ is a jointly measurable function adapted to the filtration $\mathcal{F}_t$, satisfying*

$$\int_0^T \int_D |h(t,x)|^2 dx dt < \infty \qquad \text{a.s.}$$

Define the measure $\tilde{P}$ by $d\tilde{P} = Z dP$, where

$$Z = \exp\left(\int_0^T \int_D h(t,x) \cdot W(dxdt) - \frac{1}{2} \int_0^T \int_D |h(t,x)|^2 dx dt \right).$$

Assume that $\tilde{P}$ is a probability measure. Then, under $\tilde{P}$, we have that $\tilde{W}$ is a vector of independent space-time white noises on $[0,T] \times D$, where $\tilde{W}$ is defined by

$$\tilde{W}(C) = W(C) - \int_0^T \int_D \mathbf{1}_C(t,x) h(t,x) dx dt$$

for C a measurable subset of $[0,T] \times D$.

Remark. In our case, D is just the closure of $\cup_{t>0}\{x : h_t(x) > 0\}$, which is compact because $h_t(x)$ has compact support.

If $EZ = 1$, then restricted to $[0,T] \times D$, we have that W and $\tilde{W}$ are mutually absolutely continuous. Therefore, we need to show that upon setting $g(t,x) = h_t(x)$ and $D = (\varepsilon, 1-\varepsilon)$ for ε sufficiently small, we get $EZ = 1$.

We claim that $EZ = 1$ if Novikov's criterion holds, namely

$$E\left[\exp\left(\frac{1}{2} \int_0^T \int_D |h_t(x)|^2 dx dt \right) \right] < \infty. \tag{2}$$

Indeed, suppose that (2) holds. Then

$$M_t = \int_0^t \int_D h_s(x) \cdot W(dxds)$$

is a continuous $(\mathcal{F}_t)$-martingale with quadratic variation

$$\langle M\rangle_t = \int_0^t \int_D |h_s(x)|^2 dxds.$$

With this definition of M, Proposition 5.12 of Karatzas and Shreve [KS91] page 198 states that (2) implies $EZ = 1$.

We formulate our problem more generally in the following lemma.

Lemma 2. *Let $D \subset \mathbf{R}$ be a compact set, let $T > 0$, and let $\Lambda = [0, T] \times D$. Suppose that $h_t(x) = \alpha(t, x) + \beta(t, x)$, where $\alpha(t, x)$ is a bounded deterministic function, and $\beta(t, x)$ is a mean-zero, almost surely bounded Gaussian field given by*

$$\beta(t, x) := \int_0^t \int_{\mathbf{R}} \eta(s, t, x, y) \cdot W(dyds)$$

Suppose that α, β are both (almost surely) supported on $(t, x) \in \Lambda$. Also assume that

$$\sigma_\Lambda := \sup_{(t,x)\in\Lambda} E[\beta^2(t, x)] = \sup_{(t,x)\in\Lambda} \int_0^t \int_{\mathbf{R}} |\eta(s, t, x, y)|^2 dyds < \infty.$$

Then (2) holds, so $EZ = 1$.

End of the proof of Theorem 1, Corollary 4 of [MT02]. Note that $h_t(x)$, as defined in (1), satisfies the conditions of Lemma 2. Indeed, the support of $\psi_t(x)$ is bounded away from $t = 0$ and $f \in \mathcal{E}_{exp}$, so it follows that $h_{\text{deterministic}}$ is a bounded deterministic function. Note that $W^{(\text{per})}(dy\, ds) - W(dy\, ds) = 0$ on $y \in (0, 1)$, and $h_{\text{random}}(t, x) = 0$ unless $x \in (\varepsilon, 1 - \varepsilon)$. Thus, in the integral used to define $h_{\text{random}}(t, x)$ in (1), $|x - y| > \varepsilon$. Furthermore, the term $G(t - s, x - y)$ which appears in that integral, along with its first and second derivatives, is bounded and decreases exponentially in $|y|$, uniformly in t, s, since $t, s \in [0, T]$.

We claim that $h_{\text{random}}(t, x)$ has finite maximal variance. Indeed, since $h_{\text{random}}(t, x)$ is supported on A, we find that

$$\begin{aligned} \sup_{(t,x)} E\left[|h_{\text{random}}(t, x)|^2\right] \quad &\leq \quad \sup_{(t,x)\in A} E\Bigg[\bigg|\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial^2 x}\right)\bigg(\psi_t(x) \int_0^t \int G_{t-s}(x - y) \\ &\qquad \cdot \Big(W^{(\text{per})}(dy\, ds) - W(dy\, ds)\Big)\bigg)\bigg|^2\Bigg] \end{aligned} \tag{3}$$

Applying the partial derivative operators on the right of (3), we would use the product rule to get a number of nonrandom terms multiplied by

$W^{(\mathrm{per})} - W$. But since we are taking the supremum over $(t, x) \in A$, and since $W^{(\mathrm{per})}(dy\, ds) - W(dy\, ds) = 0$ for $y \in (0, 1)$, we find that $|x - y|$ is bounded away from 0 in the region of integration. But $\psi_t(x)$ and its derivatives are bounded, and the same is true of $G_{t-s}(x - y)$ away from the singularity at $x = y$. Furthermore, the nonrandom terms have exponential decay in y. Using these observations, the reader can easily show that $h_{\mathrm{random}}(t, x)$ has finite maximal variance, so $\sigma_\Lambda < \infty$, and $h_{\mathrm{random}}(t, x)$ is almost surely bounded. Thus, Corollary 4 of [MT02] will be proved once we have established Lemma 2.

Proof of Lemma 2. Our strategy is to express Z as a product involving short time intervals, and to show that the terms have conditional expectation 1. Let

$$Z_{t,t+\delta} = \exp\left(\int_t^{t+\delta}\int_D h_s(x) \cdot W(dxds) - \frac{1}{2}\int_t^{t+\delta}\int_D |h_s(x)|^2 dxds\right)$$

Recall that $h_t(x)$ has (deterministic) compact support contained in $(t, x) \in [0, T] \times (\varepsilon, 1 - \varepsilon)$ for some $\varepsilon > 0$.

Lemma 3. *Suppose that*

$$\delta < \min\left(\frac{1}{12\sigma_\Lambda^2}, T\right). \tag{4}$$

For $t \in [0, T - \delta]$, we have

$$E\left[Z_{t,t+\delta}\middle|\mathcal{F}_t\right] = 1 \tag{5}$$

where $\mathcal{F}_t$ is the σ-field generated by $W(S)$ for $S \subset [0, t] \times \mathbf{R}$.

Proof. Let

$$H_{t,t+\delta} = E\left[\exp\left(\frac{1}{2}\int_t^{t+\delta}\int_D |h_s(x)|^2 dxds\right)\middle|\mathcal{F}_t\right].$$

Conditioning on $\mathcal{F}_t$ and using (2), we see that (5) follows from

$$H_{t,t+\delta} < \infty. \tag{6}$$

We claim that it suffices to prove (6) for $t = 0$. Indeed, we can write

$$\begin{aligned} h_{t+r}(x) &= \alpha(t + r, x) + \int_0^t \int_{\mathbf{R}} \eta(s, t + r, x, y) \cdot W(dyds) \qquad (7) \\ &\quad + \int_t^{t+r}\int_{\mathbf{R}} \eta(s, t + r, x, y) \cdot W(dyds) \\ &=: Q_1(r, x) + Q_2(r, x) + Q_3(r, x). \end{aligned}$$

The reader can check that the conditions of Lemma 2 imply that $Q_2(r,x)$ is almost surely bounded, and that $Q_3(r,x)$ is a Gaussian field satisfying the conditions required of $\beta(t,x)$ of Lemma 2. Therefore, conditioning on $\mathcal{F}_t$ and shifting the variable t, we see that if (6) is true for $t=0$, then it is true in general.

Now we turn to the proof of (6) for $t=0$. Note that

$$\begin{aligned} \int_0^\delta \int_{\mathbf{R}} |h_s(x)|^2 dx ds &\leq 2\int_0^\delta \int_{\mathbf{R}} |\alpha(s,x)|^2 dx ds \\ &\quad +2\int_0^\delta \int_{\mathbf{R}} |\beta(s,x)|^2 dx ds \\ &=: 2K+2L. \end{aligned}$$

where $\beta(s,x)$ was defined in Lemma 2.

Now

$$\begin{aligned} &E\left[\exp\left(\int_0^\delta \int_{\mathbf{R}} |h_s(x)|^2 dx ds\right)\right] \\ &= E\left[\exp\left(2K+2L\right)\right] \\ &= \int_0^\infty P\left(\exp\left(2K+2L\right) > \lambda\right) d\lambda \\ &= \int_{-\infty}^\infty P\left(\exp\left(2K+2L\right) > e^z\right) de^z \\ &\leq 1 + \int_0^\infty P(2K+2L > z) e^z dz \\ &\leq 1 + \int_0^\infty \Big(P(K > z/6) + P(L > z/6)\Big) e^z dz \\ &=: 1 + A_1 + A_2. \end{aligned}$$

Our goal is to show that $A_1, A_2 < \infty$. But K is a constant, so $A_1 < \infty$.

Now we deal with A_2. We need Borell's inequality for Gaussian processes; see Adler [Adl90], page 43, Theorem 2.1.

Theorem 2 (Borell's inequality). *Let $\{X_t\}_{t\in\Lambda}$ be a centered Gaussian process with sample paths bounded almost surely. Let $\|X\|_\infty = \sup_{t\in\Lambda} |X_t|$. Then $E\|X\|_\infty < \infty$, and for all $\lambda > 0$,*

$$P(\|X\|_\infty > \lambda) \leq 2\exp\left(-\frac{(\lambda - E[\|X\|_\infty])^2}{2\sigma_\Lambda^2}\right)$$

where $\sigma_\Lambda^2 = \sup_{t\in\Lambda} E[X_t^2]$.

We intend to apply Borell's inequality with $X = \beta(t,x)$ and $\Lambda = [0,T] \times (\varepsilon, 1-\varepsilon)$. Recall that $\beta(t,x)$ is almost surely bounded, and so by Borell's inequality $E\|\beta\|_\infty < \infty$. Also,

$$\int_0^\delta \int_0^1 |\beta(t,x)|^2 dxdt \leq \delta \|\beta\|_\infty^2.$$

Therefore,

$$\begin{aligned} P(L > z/6) &= P\left(\int_0^\delta \int_0^1 |\beta(t,x)|^2 dxdt > z/6\right) \\ &\leq P\left(\delta\|\beta\|_\infty^2 > z/6\right) \\ &\leq P\left(\|\beta\|_\infty > \sqrt{\frac{z}{6\delta}}\right) \end{aligned}$$

Using Borell's inequality, we see that if (4) holds, then

$$-\frac{[(z/(6\delta))^{1/2} - E\|\beta\|_\infty]^2}{2\sigma_\Lambda^2} + z \leq -cz + C\sqrt{z}$$

for some $C > 0$, $c > 1$, and

$$\int_0^\infty P\left(L > z/6\right) e^z dz \leq \int_0^\infty 2\exp\left(-\frac{\left[(z/(6\delta))^{1/2} - E\|\beta\|_\infty\right]^2}{2\sigma_\Lambda^2}\right) e^z dz < \infty.$$

Thus, $A_2 < \infty$, which proves Lemma 3. ∎

Now we use Lemma 3 to finish the proof of Corollary 4 of Mueller and Tribe [MT02]. We need to prove $EZ = 1$. Choose a partition

$$0 = t_0 < t_1 < \cdots < t_n = T$$

such that $t_k - t_{k-1} \leq \delta$ for all $k = 1, \ldots, n$. Recall that $Z = Z_{t_0,t_1} \cdots Z_{t_{n-1},t_n}$. By (5) of Lemma 3, we have

$$E\left[Z_{t_n,t_{n-1}} \middle| \mathcal{F}_{t_{n-1}}\right] = 1$$

and therefore

$$E\left[Z \middle| \mathcal{F}_{t_{n-1}}\right] = Z_{0,t_{n-1}}.$$

Iterating this argument, we find that $EZ = Z_{0,0} = 1$, which proves (6).

This finishes the proof Lemma 2, and thus establishes Corollary 4 of [MT02]. ∎

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