




A nontrivial solution for a nonautonomous Choquard equation with general nonlinearity

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Abstract. With the help of the monotonicity trick, a nonautonomous Choquard equations with general nonlinearity is studied and a nontrivial solution is obtained.

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1 Introduction and main result

In the paper, we explore nontrivial solutions for the following nonlocal problem

$$-\Delta u + V(x)u = \left(\frac{1}{|x|} * u^2 \right) u + g(u) \quad \text{in } \mathbb{R}^3, \quad (1.1)$$

where $\frac{1}{|x|} * u^2 = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy$, the nonlinearity g satisfies general subcritical growth conditions

(g₁) $g \in C(\mathbb{R}, \mathbb{R})$ is odd;


(g₂) $-\infty < \lim_{s \rightarrow 0^+} \frac{g(s)}{s} = -m < 0$;

(g₃) $\lim_{s \rightarrow +\infty} \frac{g(s)}{s^5} = 0$; and the potential function V verifies

(V₁) $V \in C(\mathbb{R}^3, (-m, 0])$ and $\lim_{|x| \rightarrow \infty} V(x) = 0$;

(V₂) $(\nabla V, x) \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and

$$|(\nabla V, x)|_{\frac{3}{2}} := \left(\int_{\mathbb{R}^3} |(\nabla V, x)|^{\frac{3}{2}} dx \right)^{\frac{2}{3}} < 2S := 2 \inf_{0 \neq u \in D^{1,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} u^6 dx \right)^{\frac{1}{3}}}.$$

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When $V \equiv 0$ and $g(s) = -s$, Eq. (1.1) is simplified to the classical Choquard equation

$$-\Delta u + u = \left(\frac{1}{|x|} * u^2 \right) u \quad \text{in } \mathbb{R}^3. \quad (1.2)$$

Eq. (1.2) appeared at least as early as in 1954, in a work by S. I. Pekar describing the quantum mechanics of a polaron at rest [11]. In 1976, P. Choquard used Eq. (1.2) to describe an electron trapped in its own hole in a certain approximation to Hartree–Fock theory of one component plasma [4]. For more details in the physics aspects, please refer to [7]. Therefore, many scholars have carried out in-depth research on Choquard equations and related problems. For recent results, we refer the readers to [6, 8, 9, 12, 14] and references therein. See also [10] for a broad survey of Choquard equations.

It is important to point out that Liu et al. in [6] considered the following special case of Eq. (1.1)

$$-\Delta u = \left(\frac{1}{|x|} * u^2 \right) u + g(u) \quad \text{in } \mathbb{R}^3. \quad (1.3)$$

Under the assumptions (g_1) – (g_3) , they investigated ground states of Eq. (1.3) by using the Pohožaev manifold method. In the present paper, we study Eq. (1.1) which can be regarded as the perturbation equation of Eq. (1.3). By using the monotonicity trick we obtain the following main result.

Theorem 1.1. *Suppose that (V_1) – (V_2) and (g_1) – (g_3) hold. Then Eq. (1.1) possesses a nontrivial solution.*

Set $K(x) = V(x) + m$ and $f(s) = g(s) + ms$. Then Eq. (1.1) equals to the following equation

$$-\Delta u + K(x)u = \left(\frac{1}{|x|} * u^2 \right) u + f(u) \quad \text{in } \mathbb{R}^3, \quad (1.4)$$

where f satisfies

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd;

(f_2) $\lim_{s \rightarrow 0^+} \frac{f(s)}{s} = 0$;

(f_3) $\lim_{s \rightarrow +\infty} \frac{f(s)}{s^5} = 0$;

and K verifies

(K_1) $K \in C(\mathbb{R}^3, (0, m])$ and $\lim_{|x| \rightarrow \infty} K(x) = m$;

(K_2) $(\nabla K, x) \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $|(\nabla K, x)|_{\frac{3}{2}} < 2S$.

Then we convert to consider the following

Theorem 1.2. *Suppose that (K_1) – (K_2) and (f_1) – (f_3) hold. Then Eq. (1.4) has a nontrivial solution.*

Remark 1.3. If $K(x) \equiv m$, then Theorem 1.2 was proved in [6]. Thus we assume that $K(x) \not\equiv m$.

For the rest of this paper, we make the following marks. $H := H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard norm $\|\cdot\|$. $L^s(\mathbb{R}^3)$, $2 \leq s \leq 6$, denotes the usual Lebesgue space with the norm $|\cdot|_s$. C, C_1, C_2, \dots denote different positive constants whose exact value is inessential. For any $u \in H$, we define $u_t(\cdot) := u(t^{-1}\cdot)$ for $t > 0$.

It is widely known that the solutions of Eq. (1.4) correspond to the critical points of the functional defined by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + K(x)u^2) dx - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} F(u) dx, \quad u \in H,$$

where $F(s) = \int_0^s f(t) dt$. Using the Hardy–Littlewood–Sobolev inequality [5], one has

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy \leq C|u|_{\frac{12}{5}}^4 \leq C_1\|u\|^4.$$

Combining with (f_1) – (f_3) and (K_1) we know that I is well defined and I is of C^1 . But it is hard to obtain a bounded (PS) sequence for the functional I under the assumptions (f_1) – (f_3) . In addition, another difficulty we face is the lack of space compactness.

From (K_1) we know that there exists $a > 0$ such that $a\|u\|^2 \leq \int_{\mathbb{R}^3} (|\nabla u|^2 + K(x)u^2) dx$.

2 Preliminaries

In order to prove Theorem 1.2, we cannot directly apply the mountain pass theorem [1]. Instead we use an indirect approach which dated to Struwe [13] and was developed by Jeanjean in [2]. Exactly, we apply the following

Proposition 2.1. *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\mu\}_{\mu \in J}$ of C^1 -functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in J,$$

where $B(u) \geq 0$ for all $u \in X$ and such that either $A(u) \rightarrow +\infty$ or $B(u) \rightarrow +\infty$ as $\|u\|_X \rightarrow +\infty$. We assume that there are two points v_1, v_2 in X such that

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = v_1, \gamma(1) = v_2\},$$

there hold, $\forall \mu \in J$,

$$c_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) > \max\{\Phi_\mu(v_1), \Phi_\mu(v_2)\}.$$

Then for almost every $\mu \in J$, there is a sequence $\{u_n\} \subset X$ such that

- (i) $\{u_n\}$ is bounded in X ,
- (ii) $\Phi_\mu(u_n) \rightarrow c_\mu$ and
- (iii) $\Phi'_\mu(u_n) \rightarrow 0$ in the dual X^* of X .

Moreover, the map $\mu \rightarrow c_\mu$ is non-increasing and continuous from the left.

Define $f^\pm(s) = \max\{\pm f(s), 0\}$, $F_1(s) = \int_0^s f^+(t)dt$ and $F_2(s) = \int_0^s f^-(t)dt$, from (f_1) – (f_3) one has

$$\lim_{s \rightarrow 0} \frac{f^\pm(s)}{s} = 0 \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{f^\pm(s)}{s^5} = 0. \quad (2.1)$$

Set

$$X = H, \quad \|\cdot\|_X = \|\cdot\|, \quad \Phi_\mu = I_\mu, \quad J = [2^{-1}, 1],$$

$$B(u) = \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy + \int_{\mathbb{R}^3} F_1(u) dx$$

and

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + K(x)u^2) dx + \int_{\mathbb{R}^3} F_2(u) dx.$$

Then $A(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$ and

$$I_\mu(u) = A(u) - \mu B(u)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + K(x)u^2) dx + \int_{\mathbb{R}^3} F_2(u) dx$$

$$- \frac{\mu}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy - \mu \int_{\mathbb{R}^3} F_1(u) dx.$$

Specially, $I_1(u) = I(u)$. The following limit equations

$$-\Delta u + mu = \mu \left(\frac{1}{|x|} * u^2 \right) u + l_\mu(u) \quad \text{in } \mathbb{R}^3, \quad (2.2)$$

will play an important role, where $l_\mu(s) = \mu f^+(s) - f^-(s)$. The energy functional of Eq. (2.2) is defined by

$$I_\mu^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + mu^2) dx - \frac{\mu}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} L_\mu(u) dx,$$

where $L_\mu(s) = \int_0^s l_\mu(t) dt$. Set

$$c_\mu^\infty = \inf\{I_\mu^\infty(u) : 0 \neq u \in H, (I_\mu^\infty)'(u) = 0\}.$$

Let ω be a positive ground state solution of Eq. (2.2) with $\mu = 1$. By the proof of [6, Lemma 3.3], one has $c_1^\infty = \max_{t>0} I_1^\infty(\omega(t^{-1}x))$.

3 The proof of Theorem 1.2

The following lemma is to verify the assumptions of Proposition 2.1.

Lemma 3.1. *Suppose that (K_1) and (f_1) – (f_3) hold. Then there exist $v_1, v_2 \in H$ such that for any $\mu \in J$, $c_\mu > \max\{I_\mu(v_1), I_\mu(v_2)\}$.*

Proof. From (f_1) – (f_3) it follows that

$$F(s) \leq \frac{a}{4}|s|^2 + C|s|^6 \quad \text{for all } s \in \mathbb{R}.$$

Combining with the Hardy–Littlewood–Sobolev and Sobolev inequality, for any $u \in H$ and $\mu \in J$ one has

$$\begin{aligned} I_\mu(u) &\geq I(u) \\ &\geq \frac{a}{4}\|u\|^2 - \frac{C_1}{4}\|u\|^4 - C\|u\|^6 \end{aligned}$$

which implies that there exist $\alpha, \rho > 0$ such that $I_\mu(u) \geq \alpha$ for all $\mu \in J$ and $\|u\| = \rho$. Let $\omega \in H$ be a positive ground state solution of Eq. (2.2) with $\mu = 1$. For any $\mu \in J$, one has

$$\begin{aligned} I_\mu(\omega_t) &\leq I_{\frac{1}{2}}^\infty(\omega_t) \\ &= \frac{t}{2} \int_{\mathbb{R}^3} |\nabla \omega|^2 dx + \frac{mt^3}{2} \int_{\mathbb{R}^3} \omega^2 dx - \frac{t^5}{8} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|} dx dy - \frac{t^3}{2} \int_{\mathbb{R}^3} L_{\frac{1}{2}}(\omega) dx. \end{aligned}$$

Combining with

$$\|\omega_t\|^2 = t \int_{\mathbb{R}^3} |\nabla \omega|^2 dx + t^3 \int_{\mathbb{R}^3} |\omega|^2 dx,$$

there exists $t_0 > 0$ such that $\|\omega_{t_0}\| > \rho$ and $I_\mu(\omega_{t_0}) < 0$. Set $v_1 = 0$ and $v_2 = \omega_{t_0}$. Thus for any $\gamma \in \Gamma$, $\max_{t \in [0,1]} I_\mu(\gamma(t)) \geq \alpha > 0$. So $c_\mu \geq \alpha > \max\{I_\mu(v_1), I_\mu(v_2)\}$. \square

Lemma 3.2. *Suppose that (K_1) and (f_1) – (f_3) hold. Then there exists $\delta \in [\frac{1}{2}, 1)$ such that for any $\mu \in [\delta, 1]$, $c_\mu < c_\mu^\infty$.*

Proof. According to the proof of Lemma 3.1, for any $\mu \in J$, there exists $t_\mu \in (0, t_0)$ such that $I_\mu(\omega(t_\mu^{-1}x)) = \max_{t \in (0,1]} I_\mu(\omega((t_0t)^{-1}x)) \geq c_\mu$. Set $\theta = \inf_{\mu \in J} t_\mu$. We claim $\theta > 0$. Otherwise, there exists $\mu_n \in J$ such that $t_{\mu_n} \rightarrow 0$ and then

$$c_1 \leq c_{\mu_n} \leq I_{\mu_n}(\omega(t_{\mu_n}^{-1}x)) \rightarrow 0.$$

It is a contradiction. Note that $K(x) \leq m$ and $K(x) \not\equiv m$. Define

$$\delta = \max \left\{ \frac{1}{2}, 1 - \frac{\theta^3 \min_{s \in [\theta, t_0]} \int_{\mathbb{R}^3} [m - K(sx)] \omega^2 dx}{2t_0^3 \int_{\mathbb{R}^3} F_1(\omega) dx + \frac{t_0^5}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|} dx dy} \right\}.$$

Then for any $\mu \in [\delta, 1]$, we get

$$\begin{aligned} c_\mu^\infty &\geq c_1^\infty \\ &\geq I_1^\infty(\omega(t_\mu^{-1}x)) \\ &= I_\mu(\omega(t_\mu^{-1}x)) + \frac{t_\mu^3}{2} \int_{\mathbb{R}^3} [m - K(t_\mu x)] \omega^2 dx - (1 - \mu)t_\mu^3 \int_{\mathbb{R}^3} F_1(\omega) dx \\ &\quad - \left(\frac{1}{4} - \frac{\mu}{4} \right) t_\mu^5 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|} dx dy \\ &> c_\mu + \frac{\theta^3}{2} \min_{s \in [\theta, t_0]} \int_{\mathbb{R}^3} [m - K(sx)] \omega^2 dx - (1 - \mu)t_0^3 \int_{\mathbb{R}^3} F_1(\omega) dx \\ &\quad - \left(\frac{1}{4} - \frac{\mu}{4} \right) t_0^5 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|} dx dy \\ &\geq c_\mu. \end{aligned} \quad \square$$

Lemma 3.3. Fix $\mu \in [\delta, 1]$. Suppose that (K_1) and (f_1) – (f_3) hold and that $\{u_n\} \subset H$ is a bounded $(PS)_{c_\mu}$ sequence for I_μ . Then there exists $u \in H$, $k \in \mathbb{N}$, $v_i \in H \setminus \{0\}$, $y_{n,i} \in \mathbb{R}^3$ for $1 \leq i \leq k$ such that up to a subsequence,

$$(i) |y_{n,i}| \rightarrow \infty, |y_{n,i} - y_{n,j}| \rightarrow \infty, i \neq j, \text{ for } 1 \leq i, j \leq k,$$

$$(ii) (I_\mu)'(u) = 0 \text{ and } (I_\mu^\infty)'(v_i) = 0 \text{ for } 1 \leq i \leq k,$$

$$(iii) u_n - u - \sum_{i=1}^k v_i(\cdot - y_{n,i}) \rightarrow 0 \text{ in } H,$$

$$(iv) c_\mu = I_\mu(u) + \sum_{i=1}^k I_\mu^\infty(v_i) + o(1),$$

where we agree that in the case $k = 0$ the above holds without $v_i, y_{n,i}$.

Proof. The proof is in the spirit of [3]. Obviously, there exists $u \in H$ such that up to a subsequence $u_n \rightharpoonup u$ in H , $u_n \rightarrow u$ in $L_{\text{loc}}^p(\mathbb{R}^3)$ with $2 \leq p < 6$ and $u_n(x) \rightarrow u(x)$ a.e. in \mathbb{R}^3 . For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, one has

$$0 = \langle I_\mu'(u_n), \varphi \rangle + o(1) = \langle I_\mu'(u), \varphi \rangle.$$

Set $u_{n,1} = u_n - u$. If $u_n \rightarrow 0$ in H , we are done. So we can assume that $\{u_{n,1}\}$ does not converge strongly to 0 in H . Thus up to a subsequence $u_{n,1} \rightharpoonup 0$ in H , $u_{n,1} \rightarrow 0$ in $L_{\text{loc}}^p(\mathbb{R}^3)$ and $u_{n,1}(x) \rightarrow 0$ a.e. in \mathbb{R}^3 . Then we have

$$\|u_{n,1}\|^2 = \|u_n\|^2 - \|u\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} K(x)u_{n,1}^2 dx = \int_{\mathbb{R}^3} K(x)u_n^2 dx - \int_{\mathbb{R}^3} K(x)u^2 dx + o(1),$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{n,1}^2(x)u_{n,1}^2(y)}{|x-y|} dx dy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy + o(1)$$

and

$$\int_{\mathbb{R}^3} L_\mu(u_{n,1}) dx = \int_{\mathbb{R}^3} L_\mu(u_n) dx - \int_{\mathbb{R}^3} L_\mu(u) dx + o(1).$$

Therefore,

$$I_\mu(u_{n,1}) = I_\mu(u_n) - I_\mu(u) + o(1).$$

Define

$$\beta_1 = \limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^3} \int_{B_1(z)} u_{n,1}^2 dx.$$

If $\beta_1 = 0$, one sees $u_{n,1} \rightarrow 0$ in $L^p(\mathbb{R}^3)$ with $2 < p < 6$ from Lion's lemma [15]. Then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy \rightarrow \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy$$

and

$$\int_{\mathbb{R}^3} l_\mu(u_n) u_n dx \rightarrow \int_{\mathbb{R}^3} l_\mu(u) u dx.$$

Therefore,

$$0 = \langle I_\mu'(u_n), u_n \rangle + o(1) \geq \langle I_\mu'(u), u \rangle = 0,$$

which infers $u_n \rightarrow u$ in H . It is a contradiction. If $\beta_1 > 0$, we may assume the existence of $y_{n,1} \in \mathbb{R}^3$ such that $\int_{B_1(y_{n,1})} u_{n,1}^2 dx > \frac{\beta_1}{2}$. Set $w_{n,1} = u_n(\cdot + y_{n,1})$, there exists $v_1 \in H$ such that up to a subsequence $w_{n,1} \rightharpoonup v_1$ in H , $w_{n,1} \rightarrow v_1$ in $L_{loc}^p(\mathbb{R}^3)$ with $2 \leq p < 6$ and $w_{n,1}(x) \rightarrow v_1(x)$ a.e. in \mathbb{R}^3 . From

$$\int_{B_1(0)} v_1^2 dx = \lim_{n \rightarrow \infty} \int_{B_1(0)} w_{n,1}^2 dx = \lim_{n \rightarrow \infty} \int_{B_1(y_{n,1})} u_{n,1}^2 dx = \lim_{n \rightarrow \infty} \int_{B_1(y_{n,1})} (u_{n,1}^2 + u^2) dx \geq \frac{\beta_1}{2},$$

we know $v_1 \neq 0$. Since $u_{n,1} \rightharpoonup 0$ in H , $\{y_{n,1}\}$ is unbounded in \mathbb{R}^3 and, up to a subsequence, we can assume that $|y_{n,1}| \rightarrow \infty$. Thus

$$\begin{aligned} 0 &= \langle I'_\mu(u_n), \varphi(\cdot - y_{n,1}) \rangle + o(1) \\ &= \langle (I_\mu^\infty)'(v_1), \varphi \rangle. \end{aligned}$$

Set $u_{n,2} = u_n - u - v_1(\cdot - y_{n,1})$. If $u_{n,2} \rightarrow 0$ in H , we are done. So we can assume that $\{u_{n,2}\}$ does not converge strongly to 0 in H . Thus up to a subsequence $u_{n,2} \rightharpoonup 0$ in H , $u_{n,2} \rightarrow 0$ in $L_{loc}^p(\mathbb{R}^3)$ and $u_{n,2}(x) \rightarrow 0$ a.e. in \mathbb{R}^3 . Thus we have

$$\begin{aligned} \|u_{n,2}\|^2 &= \|u_n\|^2 - \|u\|^2 - \|v_1\|^2 + o(1), \\ \int_{\mathbb{R}^3} K(x) u_{n,2}^2 dx &= \int_{\mathbb{R}^3} K(x) u_n^2 dx - \int_{\mathbb{R}^3} K(x) u^2 dx - \int_{\mathbb{R}^3} m v_1^2 dx + o(1), \\ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{n,2}^2(x) u_{n,2}^2(y)}{|x-y|} dx dy &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy \\ &\quad - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v_1^2(x) v_1^2(y)}{|x-y|} dx dy + o(1) \end{aligned}$$

and

$$\int_{\mathbb{R}^3} L_\mu(u_{n,2}) dx = \int_{\mathbb{R}^3} L_\mu(u_n) dx - \int_{\mathbb{R}^3} L_\mu(u) dx - \int_{\mathbb{R}^3} L_\mu(v_1) dx + o(1).$$

Therefore,

$$I_\mu(u_{n,2}) = I_\mu(u_n) - I_\mu(u) - I_\mu^\infty(v_1) + o(1).$$

Define

$$\beta_2 = \limsup_{n \rightarrow \infty} \sup_{z \in \mathbb{R}^3} \int_{B_1(z)} u_{n,2}^2 dx.$$

We replaced $u_{n,1}$ by $u_{n,2}$ and repeat the above arguments. If $\beta_2 = 0$, then $u_{n,2} \rightarrow 0$. It is a contradiction. If $\beta_2 > 0$, we may assume the existence of $y_{n,2} \in \mathbb{R}^3$ such that $\int_{B_1(y_{n,2})} u_{n,2}^2 dx > \frac{\beta_2}{2}$. Set $w_{n,2} = u_n(\cdot + y_{n,2})$, there exists $v_2 \in H$ such that up to a subsequence $w_{n,2} \rightharpoonup v_2$ in H , $w_{n,2} \rightarrow v_2$ in $L_{loc}^p(\mathbb{R}^3)$ with $2 \leq p < 6$ and $w_{n,2}(x) \rightarrow v_2(x)$ a.e. in \mathbb{R}^3 . From

$$\int_{B_1(0)} v_2^2 dx = \lim_{n \rightarrow \infty} \int_{B_1(0)} w_{n,2}^2 dx = \lim_{n \rightarrow \infty} \int_{B_1(y_{n,2})} u_{n,2}^2 dx = \lim_{n \rightarrow \infty} \int_{B_1(y_{n,2})} u_{n,2}^2 dx \geq \frac{\beta_2}{2},$$

we know $v_2 \neq 0$. Since $u_{n,2} \rightharpoonup 0$ in H , $\{y_{n,2}\}$ is unbounded in \mathbb{R}^3 and, up to a subsequence, we can assume that $|y_{n,2}| \rightarrow \infty$ and $|y_{n,2} - y_{n,1}| \rightarrow \infty$. Similarly, $(I_\mu^\infty)'(v_2) = 0$. Set

$$u_{n,3} = u_n - u - v_1(\cdot - y_{n,1}) - v_2(\cdot - y_{n,2}).$$

Again we repeat the above arguments, then there exists $k \in \mathbb{N}$, $v_i \in H \setminus \{0\}$, $y_{n,i} \in \mathbb{R}^3$ for $1 \leq i \leq k$ such that up to a subsequence, $|y_{n,i}| \rightarrow \infty$, $|y_{n,i} - y_{n,j}| \rightarrow \infty$, $i \neq j$, for $1 \leq i, j \leq k$, $(I_\mu^\infty)'(v_i) = 0$ for $1 \leq i \leq k$,

$$u_{n,k+1} = u_n - u - \sum_{i=1}^k v_i(\cdot - y_{n,i})$$

and

$$c_\mu = I_\mu(u_{n,k+1}) + I_\mu(u) + \sum_{i=1}^k I_\mu^\infty(v_i) + o(1).$$

Note that there exists $\alpha > 0$ such that $\|v\| \geq \alpha$ for any $v \in \{v \in H : v \neq 0 \text{ and } (I_\mu^\infty)'(v) = 0\}$. The iterations must stop after steps because $\{u_n\}$ is bounded in H . \square

For almost every $\mu \in [\delta, 1]$, by Proposition 2.1 there is a sequence $\{u_n\} \subset H$ such that

$$\begin{aligned} (i) \quad & \{u_n\} \text{ is bounded in } H, \\ (ii) \quad & I_\mu(u_n) \rightarrow c_\mu, \\ (iii) \quad & I'_\mu(u_n) \rightarrow 0 \text{ in the dual } H^* \text{ of } H. \end{aligned} \tag{3.1}$$

Moreover, the map $\mu \rightarrow c_\mu$ is non-increasing and continuous from the left.

Lemma 3.4. Fix $\mu \in [\delta, 1]$. Suppose that (K_1) and (f_1) – (f_3) hold and that $\{u_n\} \subset H$ satisfies (3.1). Then there exists $u \in H$ such that $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$.

Proof. We assume $k \geq 1$ in Lemma 3.3. Then

$$\|u_n - u - \sum_{i=1}^k v_i(\cdot - y_{n,i})\| \rightarrow 0$$

and

$$c_\mu = I_\mu(u) + \sum_{i=1}^k I_\mu^\infty(v_i) + o(1),$$

where $I'_\mu(u) = 0$ and $(I_\mu^\infty)'(v_i) = 0$ for $1 \leq i \leq k$. Because $I_\mu(u) \geq 0$ and $I_\mu^\infty(v_i) \geq c_\mu^\infty$ for $1 \leq i \leq k$, we have $c_\mu \geq c_\mu^\infty$. It is a contradiction. Thus $k = 0$ and $\|u_n - u\| \rightarrow 0$. Therefore, $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$. \square

Lemma 3.5. Suppose that (K_1) – (K_2) and (f_1) – (f_3) hold. Then there exists $u \in H$ such that $I(u) = c_1$ and $I'(u) = 0$.

Proof. Choosing $\mu_n \in [\delta, 1]$ and $\mu_n \nearrow 1$, then Lemma 3.4 implies that there exists a sequence $\{u_{\mu_n} := u_n\} \subset H$ such that

$$\begin{aligned} c_{\mu_n} &= I_{\mu_n}(u_n) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + K(x)u_n^2) dx - \frac{\mu_n}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} L_{\mu_n}(u_n) dx \end{aligned} \tag{3.2}$$

and the following Pohožaev identity

$$\begin{aligned} 0 &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} K(x)u_n^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla K(x), x)u_n^2 dx \\ &\quad - \frac{5\mu_n}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy - 3 \int_{\mathbb{R}^3} L_{\mu_n}(u_n) dx. \end{aligned} \tag{3.3}$$

(3.2) $\times 3 - (3.3)$ implies

$$\begin{aligned} 3c_{\mu_n} &= \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} (\nabla K(x), x) u_n^2 dx + \frac{\mu_n}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dx dy \\ &\geq \left(1 - \frac{|(\nabla K, x)|_{\frac{3}{2}}}{2S}\right) \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dx dy \end{aligned}$$

which implies that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx, \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dx dy \leq C, \quad \forall n \in \mathbb{N}^*.$$

Combining with (2.1) and the Sobolev inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} K(x) u_n^2 dx &= \mu_n \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dx dy + \int_{\mathbb{R}^3} l_{\mu_n}(u_n) u_n dx \\ &\leq C + \frac{\min_{x \in \mathbb{R}^3} K(x)}{2} \int_{\mathbb{R}^3} u_n^2 dx + C \int_{\mathbb{R}^3} u_n^6 dx \end{aligned}$$

which implies

$$\int_{\mathbb{R}^3} u_n^2 dx \leq C \quad \forall n \in \mathbb{N}^*.$$

Then $\{u_n\}$ is bounded in H . Recall that $\mu_n \nearrow 1$,

$$I(u_n) = c_{\mu_n} + \frac{\mu_n - 1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dx dy + (\mu_n - 1) \int_{\mathbb{R}^3} F_1(u_n) dx$$

and

$$\|I'(u_n)\|_* = \sup_{\|\varphi\|_E=1} \left| (\mu_n - 1) \int_{\mathbb{R}^3} \frac{u_n(x) \varphi(x) u_n^2(y)}{|x-y|} dx dy + (\mu_n - 1) \int_{\mathbb{R}^3} f^+(u_n) \varphi dx \right|,$$

where $\|\cdot\|_*$ denotes the norm in H^* . So $I(u_n) \rightarrow c_1$ and $\|I'(u_n)\|_* \rightarrow 0$. According to Lemma 3.3, we get that there exists $u \in H$ such that $I(u) = c_1$ and $I'(u) = 0$.

Hence, we complete the proof. □

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