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CRITICALITY OF ONE TERM $2n$-ORDER SELF-ADJOINT DIFFERENTIAL EQUATIONS

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ABSTRACT. We analyse the criticality (the existence of linearly dependent principal solutions at ∞ and $-\infty$) of the one term $2n$-order differential equation $(ry^{(n)})^{(n)} = 0$. Using the structure of the principal and the non-principal system of solutions, we find the equivalent conditions of subcriticality and at least p-criticality of this equation.

1. INTRODUCTION

In this paper, we deal with the differential equation

$$\left[r(x)y^{(n)}(x)\right]^{(n)} = 0, \tag{1.1}$$

where $r(x) > 0$, $x \in \mathbb{R}$, and $r^{-1} \in L_{loc}(\mathbb{R})$. Eq. (1.1) appears as a base for perturbations in the oscillation theory, i.e., the studied equations are regarded as perturbations of Eq. (1.1). It is shown that certain properties of Eq. (1.1) are preserved or lost by perturbations. Therefore, it is useful to know as much as possible about Eq. (1.1) (to analyse its properties), see, e.g., [1, 3, 4, 5, 6, 7, 9, 10, 16]. Typical example of this approach is the investigation of the self-adjoint differential equation

$$\sum_{k=0}^{n}(-1)^k \left[r_k(x)y^{(k)}(x)\right]^{(k)} = 0, \tag{1.2}$$

where it is assumed that one term is dominant (in a certain sense) and Eq. (1.2) is viewed as a perturbation of this term, which is in fact Eq. (1.1), see, e.g., [11, 15].

Our paper is motivated by results presented in [10], where the principal and non-principal systems of solutions of Eq. (1.1) are studied and then used for the conjugacy criterion of the two term equation

$$(-1)^n \left[r(x)y^{(n)}(x)\right]^{(n)} + s(x)y(x) = 0, \tag{1.3}$$

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which is viewed as a perturbation of Eq. (1.1). Similar use of Eq. (1.1) for the investigation of Eq. (1.3) can be found, e.g., in [3, 4]. Note that the method based on the concept of the principal system was introduced in [3]. We are also motivated by [13, 14], where the difference equation $\Delta^n(r_k\Delta^n y_k)=0$ is investigated.

The paper is organized as follows. For the reader's convenience, we recall necessary preliminaries and state the background for our work including the concept of p-criticality in Section 2. Then, in Section 3, we prove the equivalent conditions of subcriticality and at least p-criticality of Eq. (1.1).

2. Preliminaries

Let us consider Eq. (1.2) with $r_n(x)>0$, $x\in\mathbb{R}$, and $r_0,\dots,r_{n-1},r_n^{-1}\in L_{loc}(\mathbb{R})$. We say that points $x_1,x_2\in\mathbb{R}$ are *conjugate* relative to Eq. (1.2) if there exists a non-trivial solution y of this equation for which

$$y^{(i)}\left(x_1\right)=0=y^{(i)}\left(x_2\right),\quad i\in\{0,\dots,n-1\}.$$

Eq. (1.2) is *conjugate* on an interval $I\subseteq\mathbb{R}$ if I contains a pair of points, which are conjugate relative to Eq. (1.2); in the opposite case, Eq. (1.2) is *disconjugate* on I.

Since the most comfortable and the most widely used definition of (non-)principal solutions of Eq. (1.2) is via linear Hamiltonian systems, we recall this notion. See, e.g., [2, 18]. The substitution

$$u^{[y]}=\begin{pmatrix} y\\ y'\\ \vdots\\ y^{(n-1)}\end{pmatrix},\quad v^{[y]}=\begin{pmatrix}\sum_{k=1}^{n}(-1)^{k-1}\left(r_k y^{(k)}\right)^{(k-1)}\\ \vdots\\ -(r_n y^{(n)})'+r_{n-1}y^{(n-1)}\\ r_n y^{(n)}\end{pmatrix}$$

transforms Eq. (1.2) to the linear Hamiltonian system

$$u'(x)=Au(x)+B(x)v(x),\qquad v'(x)=C(x)u(x)-A^Tv(x), \tag{2.1}$$

where the $n\times n$ matrices $A, B(x)$, and $C(x)$ are given by the formulas

$$\begin{gathered}A=(a_{ij})_{i,j=1}^{n},\quad a_{ij}=\begin{cases}1, & j=i+1,\ i=1,\dots,n-1,\\ 0, & \text{elsewhere},\end{cases}\\ B(x)=\operatorname{diag}\left\{0,\dots,0,\frac{1}{r_n(x)}\right\},\quad C(x)=\operatorname{diag}\left\{r_0(x),\dots,r_{n-1}(x)\right\}.\end{gathered} \tag{2.2}$$

We will say that the solution $\left(u^{[y]},v^{[y]}\right)$ of system (2.1) is *generated* by the solution y of Eq. (1.2).

Along with system (2.1), we consider the matrix system

$$U'(x)=AU(x)+B(x)V(x),\qquad V'(x)=C(x)U(x)-A^TV(x), \tag{2.3}$$

where A, $B(x)$, $C(x)$ are as in (2.2). We will also say that the matrix solution (U, V) of system (2.3) is *generated* by the solutions $y_1, \ldots, y_n$ of Eq. (1.2) if its columns are generated by $y_1, \ldots, y_n$. A solution (U, V) of system (2.3) is called *isotropic* if $U^T V - V^T U \equiv 0$. (In the literature, there is, instead of 'isotropic' from [2], also used 'prepared', 'self-conjugate', or 'self-conjoined' – see [9] and the references cited therein.)

An isotropic solution (U, V) of system (2.3) is said to be *principal* at ∞ if U is non-singular on $[a, \infty)$ for some $a \in \mathbb{R}$ and there exists a solution $(\tilde{U}, \tilde{V})$, linearly independent of (U, V), such that $\tilde{U}$ is non-singular on $[a, \infty)$ and it is valid

$$\lim_{x \to \infty} \tilde{U}^{-1}(x) U(x) = 0.$$

A solution $(\tilde{U}, \tilde{V})$, linearly independent of the principal solution, is called *non-principal* at ∞. Note that isotropic solutions (U, V), $(\tilde{U}, \tilde{V})$ are linearly independent if and only if the constant matrix $U^T(x)\tilde{V}(x) - V^T(x)\tilde{U}(x)$ is non-singular. Another characterization of the principal solution is that it is an isotropic solution (U, V) for which U is non-singular on an interval $[a, \infty)$ and

$$\lim_{x \to \infty} \left(\int_a^x U^{-1}(s) B(s) U^{T-1}(s) \, \mathrm{d}s \right)^{-1} = 0.$$

We remark that the principal solution is determined uniquely up to the right multiple by a constant non-singular matrix. Concerning Eq. (1.2), a system of solutions $y_1, \ldots, y_n$ form a *principal* (*non-principal*) system of solutions at ∞ if the corresponding solution

$$(U, V) = \left(u^{[y_1]}, \ldots, u^{[y_n]}, v^{[y_1]}, \ldots, v^{[y_n]} \right)$$

of system (2.3) is principal (non-principal) at ∞. The definition of the principal and non-principal solutions at $-\infty$ is analogous.

Now we can introduce the concept of p-criticality for Eq. (1.2), hence for Eq. (1.1). For the discrete counterpart, we refer to [8]; for the concept of a critical operator, see [12]. Suppose that Eq. (1.2) is disconjugate on $\mathbb{R}$. Let y_i and $\tilde{y}_i$ be the principal systems of solutions of Eq. (1.2) at ∞ and $-\infty$, respectively. We consider the linear space

$$\mathcal{H} = \text{Lin}\, \{y_1, \ldots, y_n\} \cap \text{Lin}\, \{\tilde{y}_1, \ldots, \tilde{y}_n\}.$$

Eq. (1.2) is called *p-critical* (on $\mathbb{R}$) if $\dim \mathcal{H} = p \in \{1, \ldots, n\}$, and Eq. (1.2) is called *subcritical* (on $\mathbb{R}$) if $\dim \mathcal{H} = 0$. Note that Eq. (1.2) is called *supercritical* (on $\mathbb{R}$) in the case, when it is conjugate on $\mathbb{R}$.

To formulate an important tool for our results, we have to mention the notion of the Markov system of solutions and another approach to disconjugacy, which deal with the linear differential equation

$$y^{(n)}(x) + q_{n-1}(x) y^{(n-1)}(x) + \cdots + q_0(x) y(x) = 0. \tag{2.4}$$

A system of solutions $y_1, \dots, y_n$ of Eq. (2.4) forms the *Markov system* of solutions on an interval $I \subseteq \mathbb{R}$ if the Wronskians

$$W(y_1, \dots, y_k) = \begin{vmatrix} y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix}, \qquad k \in \{1, \dots, n\},$$

are positive on I. The following definition of disconjugacy of Eq. (2.4) has been introduced by Z. Nehari (see [17]), so this concept is denoted as N-disconjugacy. We say that Eq. (2.4) is *N-disconjugate* on an interval $I \subseteq \mathbb{R}$ if no non-trivial solution y of Eq. (2.4) has more than $n-1$ zeros on I (multiplicity counted).

3. Results

This section is devoted to the study of criticality of Eq. (1.1). First, we recall the types of solutions of Eq. (1.1). A solution y is called *polynomial* if $y^{(n)} \equiv 0$, and a solution y is *non-polynomial* if $y^{(n)}(x) \neq 0$ for some $x \in \mathbb{R}$. Eq. (1.1) possesses n linearly independent polynomial solutions

$$y_i^{[p]}(x) = x^{i-1}, \quad i \in \{1, \dots, n\}, \tag{3.1}$$

and n linearly independent non-polynomial solutions

$$y_i^{[n]}(x) = \int_0^x (x-t)^{n-1} t^{i-1} r^{-1}(t)\, \mathrm{d}t, \quad i \in \{1, \dots, n\}, \tag{3.2}$$

where $x \in \mathbb{R}$. It is seen that Eq. (1.1) is disconjugate and N-disconjugate on $\mathbb{R}$.

Now we recall two known results. Their proofs may be found in [2, Chapter 3] (and their formulations are taken from [10]).

Lemma 1. *Eq.* (2.4) *is N-disconjugate on an interval* $I = (a, \infty) \subseteq \mathbb{R}$ *if and only if there exists the Markov system of solutions of Eq.* (2.4) *on* I. *This system can be found in such a way that it satisfies the additional conditions*

$$y_i(x) > 0, \quad x \in (a, \infty), \qquad i \in \{1, \dots, n\},$$

$$\lim_{x\to\infty} \frac{y_k(x)}{y_{k+1}(x)} = 0, \qquad k \in \{1, \dots, n-1\}.$$

Lemma 2. *Let* $y_1, \dots, y_{2n}$ *be the linearly independent solutions of Eq.* (1.2) *with the property that*

$$\lim_{x\to\infty} \frac{y_k(x)}{y_{k+1}(x)} = 0, \qquad k \in \{1, \dots, 2n-1\}.$$

Then $y_1, \dots, y_n$ *form the principal system of solutions at* ∞ *and* $y_{n+1}, \dots, y_{2n}$ *form the non-principal system of solutions at* ∞ *of Eq.* (1.2).

For simplicity, we will use the following notation. For arbitrary non-zero functions f, g defined on an interval $[a, \infty)$, we write

$$f \prec g \quad \text{as} \quad x \to \infty \qquad \Longleftrightarrow \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

If $f_1 \prec \cdots \prec f_l$ as $x \to \infty$ for some functions $f_i : [a, \infty) \to \mathbb{R}$, $i \in \{1, \dots, l\}$, $a \in \mathbb{R}$, we say that the system of f_i is *ordered* at ∞. Now we can formulate the next auxiliary result.

Lemma 3. *Eq.* (1.1) *possesses a system of solutions* $y_j, \tilde{y}_j$, $j \in \{1, \dots, n\}$, *such that*

$$y_1 \prec \cdots \prec y_n \prec \tilde{y}_1 \prec \cdots \prec \tilde{y}_n \quad \textit{as} \quad x \to \infty. \tag{3.3}$$

If (3.3) *holds, the solutions* y_j *form the principal system of solutions at* ∞*, while* $\tilde{y}_j$ *form the non-principal system of solutions at* ∞*.*

Proof. Since Eq. (1.1) is N-disconjugate on $\mathbb{R}$, applying Lemma 1, we obtain a system of solutions $y_1, \dots, y_n, \tilde{y}_1, \dots, \tilde{y}_n$ of Eq. (1.1) satisfying (3.3). In this case, Lemma 2 says that y_j form the principal system and $\tilde{y}_j$ form the non-principal system of solutions at ∞. $\square$

Remark 1. All previous arguments can be used in the case when $x \to -\infty$. Especially, the analogous statement of Lemma 3 holds for the ordered system of solutions at $-\infty$. We add that the discrete version of Lemma 3 is mentioned, e.g., in [14].

Let $\mathcal{V}^+$ and $\mathcal{V}^-$ denote the subspaces of the solution space of Eq. (1.1) generated by the principal system of solutions at ∞ and $-\infty$, respectively. We summarize results about the polynomial principal solutions of Eq. (1.1) in Theorem 1.

Theorem 1. *Let* $m \in \{0, 1, \dots, n-1\}$ *be arbitrarily given; and let* $q := n-m-1$.

(i)

$$\textit{If} \int_0^\infty x^{2q} r^{-1}(x)\,\mathrm{d}x = \infty, \quad \textit{then } \{1, \dots, x^m\} \subseteq \mathcal{V}^+.$$

(ii)

$$\textit{If} \int_{-\infty}^0 x^{2q} r^{-1}(x)\,\mathrm{d}x = \infty, \quad \textit{then } \{1, \dots, x^m\} \subseteq \mathcal{V}^-.$$

(iii)

$$\textit{If} \int_0^\infty x^{2q-1} r^{-1}(x)\,\mathrm{d}x < \infty, \quad \textit{then } \left\{x^{m+1}, \dots, x^{n-1}\right\} \cap \mathcal{V}^+ = \emptyset.$$

(iv)

$$\textit{If} \int\limits_{-\infty}^{0} x^{2q-1} r^{-1}(x)\,\mathrm{d}x < \infty, \quad \textit{then } \left\{x^{m+1}, \dots, x^{n-1}\right\} \cap \mathcal{V}^- = \emptyset.$$

Proof. The statements of the theorem follow from [10, Theorems 3.1–3.4]. $\square$

Moreover, the analysis done in [10] shows that only polynomial solutions can be simultaneously contained in the principal systems of solutions of Eq. (1.1) at ∞ and $-\infty$; i.e., only polynomial solutions contribute to criticality of Eq. (1.1). Thus, the following property of $\mathcal{V}^+ \cap \mathcal{V}^-$ is known.

Corollary 1. *Let $m \in \{0, 1, \dots, n-1\}$, $q := n - m - 1$. Suppose that*

$$\int\limits_{0}^{\infty} x^{2q} r^{-1}(x)\,\mathrm{d}x = \infty = \int\limits_{-\infty}^{0} x^{2q} r^{-1}(x)\,\mathrm{d}x.$$

Then $\{1, \dots, x^m\} \subset \mathcal{V}^+ \cap \mathcal{V}^-$, i.e., Eq. (1.1) *is at least $(m+1)$-critical. If, in addition, it is valid that*

$$\int\limits_{0}^{\infty} x^{2q-1} r^{-1}(x)\,\mathrm{d}x < \infty \quad \textit{or} \quad \int\limits_{-\infty}^{0} x^{2q-1} r^{-1}(x)\,\mathrm{d}x < \infty,$$

then $\operatorname{Lin}\{1, \dots, x^m\} = \mathcal{V}^+ \cap \mathcal{V}^-$, *i.e., Eq.* (1.1) *is $(m+1)$-critical.*

Now we prove the criterion of subcriticality of Eq. (1.1).

Theorem 2. *If*

$$\int\limits_{0}^{\infty} x^{2n-2} r^{-1}(x)\,\mathrm{d}x < \infty \tag{3.4}$$

or

$$\int\limits_{-\infty}^{0} x^{2n-2} r^{-1}(x)\,\mathrm{d}x < \infty, \tag{3.5}$$

then Eq. (1.1) *is subcritical, i.e., $\mathcal{V}^+ \cap \mathcal{V}^- = \{0\}$.*

Proof. Assume that (3.4) holds. In the second case (when (3.5) is true), one can proceed analogously. Henceforth in this proof, let $x > 0$. Considering (3.1) and

(3.2), we obtain

$$\hat{y}_i(x) = \int_0^\infty (x-t)^{n-1} t^{i-1} r^{-1}(t)\,\mathrm{d}t - y_i^{[n]}(x)$$
$$= \int_x^\infty (x-t)^{n-1} t^{i-1} r^{-1}(t)\,\mathrm{d}t, \qquad i \in \{1,\dots,n\},$$

as non-polynomial solutions of Eq. (1.1). We add that

$$\int_0^\infty (x-t)^{n-1} t^{i-1} r^{-1}(t)\,\mathrm{d}t, \qquad i \in \{1,\dots,n\},$$

are well defined linear combinations of polynomials $y_l^{[p]}$, because (see (3.4))

$$\int_0^\infty \left| \binom{n-1}{l-1} x^{l-1} \, (-t)^{n-1-l+1} \, t^{i-1} r^{-1}(t) \right| \mathrm{d}t$$
$$\le \binom{n-1}{l-1} x^{l-1} \int_0^\infty t^{n-l+i-1} r^{-1}(t)\,\mathrm{d}t < \infty, \qquad i, l \in \{1,\dots,n\}.$$

It is seen that

$$\hat{y}_i \prec \hat{y}_{i+1} \quad \text{as} \quad x \to \infty, \qquad i \in \{1,\dots,n-1\}.$$

We have

$$|\,\hat{y}_n(x)\,| = \int_x^\infty |\,x-t\,|^{n-1}\, t^{n-1} r^{-1}(t)\,\mathrm{d}t \le \int_x^\infty t^{2(n-1)} r^{-1}(t)\,\mathrm{d}t,$$

which implies (see (3.4)) that $\lim_{x\to\infty} \hat{y}_n(x) = 0$, i.e., $\hat{y}_n \prec 1$ as $x \to \infty$. Thus, we get the ordered system of the solutions

$$\hat{y}_1 \prec \cdots \prec \hat{y}_n \prec 1 \prec \cdots \prec x^{n-1} \quad \text{as} \quad x \to \infty.$$

From Lemma 3 it follows that no polynomial solution is in the principal system of solutions of Eq. (1.1) at ∞. Since the set $\mathcal{V}^+ \cap \mathcal{V}^-$ can contain only polynomial solutions, Eq. (1.1) is subcritical. $\square$

Theorem 3. *Eq.* (1.1) *is subcritical if and only if*

$$\int_0^\infty x^{2n-2} r^{-1}(x)\,\mathrm{d}x < \infty \quad \textit{or} \quad \int_{-\infty}^0 x^{2n-2} r^{-1}(x)\,\mathrm{d}x < \infty. \tag{3.6}$$

Proof. If

$$\int_0^\infty x^{2n-2} r^{-1}(x)\,\mathrm{d}x = \infty = \int_{-\infty}^0 x^{2n-2} r^{-1}(x)\,\mathrm{d}x,$$

then $1 \in \mathcal{V}^+ \cap \mathcal{V}^-$, i.e., Eq. (1.1) is not subcritical. See the first part of Corollary 1 for $m = 0$. The opposite case (given by (3.6)) is embodied in Theorem 2. □

The next theorem gives the sufficient and necessary condition for at least p-criticality of Eq. (1.1).

Theorem 4. *Let $n \geq 2$, $m \in \{1, \ldots, n-1\}$, $q := n - m - 1$. If Eq.* (1.1) *is at least* $(m+1)$*-critical, then*

$$\int_0^\infty x^{2q} r^{-1}(x)\,\mathrm{d}x = \infty = \int_{-\infty}^0 x^{2q} r^{-1}(x)\,\mathrm{d}x. \tag{3.7}$$

Proof. From Lemma 3 and Remark 1 (see (3.1)), we know that

$$x^m \notin \mathcal{V}^+ \cap \mathcal{V}^- \qquad \Longrightarrow \qquad \left\{x^m, \ldots, x^{n-1}\right\} \cap \mathcal{V}^+ \cap \mathcal{V}^- = \emptyset.$$

Hence, it suffices to show that $x^m \notin \mathcal{V}^+ \cap \mathcal{V}^-$ provided at least one of the integrals in (3.7) is convergent. (We recall that the space $\mathcal{V}^+ \cap \mathcal{V}^-$ contains only polynomial solutions.) We will prove the implication (cf. Theorem 1, part (iii))

$$\int_0^\infty x^{2q} r^{-1}(x)\,\mathrm{d}x < \infty \qquad \Longrightarrow \qquad x^m \notin \mathcal{V}^+. \tag{3.8}$$

Analogously, it is possible to prove

$$\int_{-\infty}^0 x^{2q} r^{-1}(x)\,\mathrm{d}x < \infty \qquad \Longrightarrow \qquad x^m \notin \mathcal{V}^-.$$

Let us consider the following linearly independent non-polynomial solutions (see again (3.1), (3.2) and consider (3.8))

$$\begin{aligned}\bar{y}_l(x) = &\int_0^x (x-t)^{n-1} t^{l-1} r^{-1}(t)\,\mathrm{d}t \\ &- \sum_{i=0}^{q} \left[(-1)^i \binom{n-1}{i} x^{n-1-i} \int_0^\infty t^{i+l-1} r^{-1}(t)\,\mathrm{d}t\right], \qquad l \in \{1, \ldots, q+1\}.\end{aligned}$$

Since it is valid

$$\begin{aligned}
\bar{y}_l^{(n)}(x) &= \left(\int_0^x (x-t)^{n-1} t^{l-1} r^{-1}(t)\,\mathrm{d}t \right)^{(n)} \\
&\quad - \sum_{i=0}^{q} \left[(-1)^i \binom{n-1}{i} \left(x^{n-1-i}\right)^{(n)} \int_0^\infty t^{i+l-1} r^{-1}(t)\,\mathrm{d}t \right] \\
&= \left(\int_0^x \left[(x-t)^{n-1}\right]^{(n-1)} t^{l-1} r^{-1}(t)\,\mathrm{d}t \right)' \\
&= \left(\int_0^x (n-1)!\, t^{l-1} r^{-1}(t)\,\mathrm{d}t \right)' = (n-1)!\, x^{l-1} r^{-1}(x), \quad l \in \{1,\dots,q+1\},
\end{aligned}$$

l'Hospital's rule gives that

$$\bar{y}_l \prec \bar{y}_{l+1} \quad \text{as} \quad x \to \infty, \qquad l \in \{1,\dots,q\}. \tag{3.9}$$

We have

$$\begin{aligned}
\bar{y}_{q+1}^{(m)}(x) &= \frac{(n-1)!}{(n-m-1)!} \int_0^x (x-t)^{n-m-1} t^q r^{-1}(t)\,\mathrm{d}t \\
&\quad - \sum_{i=0}^{q} \left[(-1)^i \binom{n-1}{i} \frac{(n-1-i)!}{(n-m-1-i)!} x^{n-m-1-i} \int_0^\infty t^{q+i} r^{-1}(t)\,\mathrm{d}t \right] \\
&= \frac{(n-1)!}{q!} \int_0^x (x-t)^q t^q r^{-1}(t)\,\mathrm{d}t \\
&\quad - \sum_{i=0}^{q} \left[(-1)^i \frac{(n-1)!(n-1-i)!}{(n-1-i)!i!(q-i)!} x^{q-i} \int_0^\infty t^{q+i} r^{-1}(t)\,\mathrm{d}t \right] \\
&= \frac{(n-1)!}{q!} \int_0^x (x-t)^q t^q r^{-1}(t)\,\mathrm{d}t \\
&\quad - \frac{(n-1)!}{q!} \sum_{i=0}^{q} \left[(-1)^i \binom{q}{i} x^{q-i} \int_0^\infty t^{q+i} r^{-1}(t)\,\mathrm{d}t \right]
\end{aligned}$$

$$= \frac{(n-1)!}{q!} \int_0^x (x-t)^q t^q r^{-1}(t)\,\mathrm{d}t$$
$$- \frac{(n-1)!}{q!} \int_0^\infty \sum_{i=0}^{q} \left[(-1)^i \binom{q}{i} x^{q-i} t^i \right] t^q r^{-1}(t)\,\mathrm{d}t$$
$$= \frac{(n-1)!}{q!} \left(\int_0^x (x-t)^q t^q r^{-1}(t)\,\mathrm{d}t - \int_0^\infty (x-t)^q t^q r^{-1}(t)\,\mathrm{d}t \right)$$
$$= -\frac{(n-1)!}{q!} \int_x^\infty (x-t)^q t^q r^{-1}(t)\,\mathrm{d}t = (-1)^{q+1} \frac{(n-1)!}{q!} \int_x^\infty (t-x)^q t^q r^{-1}(t)\,\mathrm{d}t$$

and

$$\int_x^\infty (t-x)^q t^q r^{-1}(t)\,\mathrm{d}t \leq \int_x^\infty t^{2q} r^{-1}(t)\,\mathrm{d}t, \qquad x \geq 0.$$

It means that

$$\int_0^\infty x^{2q} r^{-1}(x)\,\mathrm{d}x < \infty \qquad \Longrightarrow \qquad \lim_{x\to\infty} \bar{y}_{q+1}^{(m)}(x) = 0.$$

Let the above integral be convergent (consider (3.8)). Further, by l'Hospital's rule, we get

$$\lim_{x\to\infty} \frac{\bar{y}_{q+1}(x)}{x^m} = \lim_{x\to\infty} \frac{\bar{y}_{q+1}^{(m)}(x)}{m!} = 0,$$

i.e., $\bar{y}_{q+1} \prec x^m$ as $x \to \infty$. Finally (see also (3.9)), we have

$$1 \prec \cdots \prec x^{m-1} \prec x^m, \quad \bar{y}_1 \prec \cdots \prec \bar{y}_{q+1} \prec x^m \quad \text{as} \quad x \to \infty.$$

From Lemma 3 it follows that $x^m \notin \mathcal{V}^+$, because there exist at least $m+q+1 = n$ linearly independent solutions y of Eq. (1.1) for which $y \prec x^m$ as $x \to \infty$. The implication (3.8) is proved. □

Combining the first part of Corollary 1, Theorem 2, and Theorem 4, we obtain:

Corollary 2. *Let $m \in \{0, 1, \dots, n-1\}$, $q := n - m - 1$. Eq.* (1.1) *is at least $(m+1)$-critical if and only if* (3.7) *holds.*

Corollary 3. *Let $m \in \{-1, 0, \dots, n-2\}$, $q := n - m - 1$. Eq.* (1.1) *is at most $(m+1)$-critical (where 0-critical stands for subcritical) if and only if it is valid*

$$\int_0^\infty x^{2q-2} r^{-1}(x)\,\mathrm{d}x < \infty \quad \textit{or} \quad \int_{-\infty}^0 x^{2q-2} r^{-1}(x)\,\mathrm{d}x < \infty.$$

At the end, note that the question of an equivalent criterion based on the second part of Corollary 1 remains open.

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