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# OSCILLATION AND SPECTRAL PROPERTIES OF SELF-ADJOINT EVEN ORDER DIFFERENTIAL OPERATORS WITH MIDDLE TERMS

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Abstract. Oscillation and spectral properties of even order self-adjoint differential operators of the form

$$L(y) := \frac{1}{w(t)} \sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)}, \quad r_n(t) > 0\,, w(t) > 0,$$

are investigated. A particular attention is devoted to the fourth order operators with a middle term, for which new (non)oscillation criteria are derived. Some open problems and perspectives of further research are discussed.

Dedicated to Professor László Hatvani on the occasion of his 60th birthday.

## 1. Introduction and preliminaries

In this contribution we deal with oscillation and spectral properties of the even order (formally) self-adjoint differential operator

$$L(y) := \frac{1}{w(t)} \sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)}, \quad t \in [T, \infty) \tag{1}$$

where $r_0, \dots, r_n, w$ are continuous functions and $r_n(t) > 0$, $w(t) > 0$ for $t \in [T, \infty)$. Note that one can investigate differential operator $L$ under weaker assumptions (the so-called minimal integrability assumptions) that the functions $r_0, \dots, r_{n-1}, \frac{1}{r_n}$ and $w$ are integrable on intervals $[T, b)$ for every $b > T$. However, the continuity assumption is sufficiently general for our purpose.

First we recall some basic concepts of the oscillation theory of the self-adjoint equation $L(y) = 0$, i.e. of the equation

$$\sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)} = 0. \tag{2}$$

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The substitution

$$x = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad u = \begin{pmatrix} \sum_{k=1}^{n} (-1)^{k-1} (r_k y^{(k)})^{(k-1)} \\ \vdots \\ -(r_n y^{(n)})' + r_{n-1} y^{(n-1)} \\ r_n y^{(n)} \end{pmatrix} \tag{3}$$

transforms equation (2) into the linear Hamiltonian system

$$x' = Ax + B(t)u, \quad u' = C(t)x - A^T u \tag{4}$$

with

$$B(t) = \operatorname{diag}\left\{0, \dots, 0, \frac{1}{r_n}\right\}, \quad C(t) = \operatorname{diag}\{r_0, \dots, r_{n-1}\}$$

and

$$A_{i,j} = \begin{cases} 1, & j = i+1, \ i = 1, \dots, n-1, \\ 0 & \text{elsewhere.} \end{cases}$$

In this case we say that the solution $(x, u)$ of (4) *is generated* by the solution $y$ of (2). Moreover, if $y_1, \dots, y_n$ are solutions of (2) and the columns of the matrix solution $(X, U)$ of (4) are generated by the solutions $y_1, \dots, y_n$, we say that the solution $(X, U)$ *is generated by the solutions* $y_1, \dots, y_n$.

Recall that two different points $t_1, t_2$ are said to be *conjugate* relative to system (4) if there exists a nontrivial solution $(x, u)$ of this system such that $x(t_1) = 0 = x(t_2)$. Consequently, by the above mentioned relationship between (2) and (4), these points are conjugate relative to (2) if there exists a nontrivial solution $y$ of this equation such that $y^{(i)}(t_1) = 0 = y^{(i)}(t_2)$, $i = 0, 1, \dots, n-1$. System (4) (and hence also equation (2)) is said to be *oscillatory* if for every $T \in \mathbb{R}$ there exists a pair of points $t_1, t_2 \in [T, \infty)$ which are conjugate relative to (4) (relative to (2)), in the opposite case (4) (or (2)) is said to be *nonoscillatory.* The equation $L(y) = y$, i.e. the equation

$$\sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)} = w(t) y \tag{5}$$

is said to be *conditionally oscillatory* if there exists $\lambda_0 > 0$ such that the equation

$$\sum_{k=0}^{n} (-1)^k \left(r_k(t) y^{(k)}\right)^{(k)} = \lambda w(t) y \tag{6}$$

is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. If (6) is oscillatory (nonoscillatory) for every $\lambda > 0$, then equation (6) or (5) is said to be *strongly oscillatory* (*strongly nonoscillatory*).

A conjoined basis $(X,U)$ of (4) (i.e. a matrix solution of this system with $n\times n$ matrices $X,U$ satisfying $X^T(t)U(t)=U^T(t)X(t)$ and $\operatorname{rank}(X^T,U^T)^T=n$) is said to be the *principal solution* of (4) if $X(t)$ is nonsingular for large $t$ and for any other conjoined basis $(\bar X,\bar U)$ such that the (constant) matrix $\bar X^T U-\bar U^T X$ is nonsingular, $\lim_{t\to\infty}\bar X^{-1}(t)X(t)=0$ holds. The last limit equals zero if and only if

$$\lim_{t\to\infty}\left(\int^t X^{-1}(s)B(s)X^{T-1}(s)\,ds\right)^{-1}=0, \tag{7}$$

see [22]. A principal solution of (4) is determined uniquely up to a right multiple by a constant nonsingular $n\times n$ matrix. If $(X,U)$ is the principal solution, any conjoined basis $(\bar X,\bar U)$ such that the matrix $X^T\bar U-U^T\bar X$ is nonsingular is said to be a *nonprincipal solution* of (4). Solutions $y_1,\dots,y_n$ of (2) are said to form the *principal (nonprincipal) system of solutions* if the solution $(X,U)$ of the associated linear Hamiltonian system generated by $y_1,\dots,y_n$ is a principal (nonprincipal) solution. Note that if (2) possesses a fundamental system of positive solutions $y_1,\dots,y_{2n}$ satisfying $y_i=o(y_{i+1})$ as $t\to\infty$, $i=1,\dots,2n-1$, (the so-called *ordered system* of solutions), then the "small" solutions $y_1,\dots,y_n$ form the principal system of solutions of (2).

Our differential operator (1) is *singular* since $t\in[T,\infty)$, i.e. $t=\infty$ is a possible singularity in the sense that the functions $r_0,\dots,r_{n-1},\frac{1}{r_n},w$ may fail to be integrable on the whole interval $[T,\infty)$. On the other hand, the left endpoint $t=T$ is supposed to be *regular*, i.e., given any $x_0,u_0\in\mathbb{R}^n$, the associated linear Hamiltonian system (4) has a unique solution given by the initial condition $x(a)=x_0$, $u(a)=u_0$. The spectral properties of the operator $L$ are investigated in the Hilbert space

$$H=\left\{y:\ \int_T^\infty w(t)|y(t)|^2<\infty\right\}.$$

The maximal differential operator $L_{\max}$ generated by the differential expression $L$ (i.e. $L_{\max}(y)=L(y)$) is the operator with the domain

$$\mathcal{D}(L_{\max})=\{y:[a,\infty)\to\mathbb{R}:\ x_k,u_k\in\mathcal{AC}[T,\infty),\ k=1,\dots,n,\ \text{ and } L(y)\in H\},$$

where $x=(x_1,\dots,x_n)^T$, $u=(u_1,\dots,u_n)^T$ are related to $y$ by (3) and $\mathcal{AC}$ denotes the class of absolutely continuous functions. The minimal operator $L_{\min}$ is defined as the adjoint operator to the maximal operator, i.e., $L_{\min}:=(L_{\max})^*$. The domain of every self-adjoint extension $\hat L$ of the minimal operator $L_{\min}$ satisfies

$$\mathcal{D}(L_{\min})\subset\mathcal{D}(\hat L)\subset\mathcal{D}(L_{\max})$$

It is known that all self-adjoint extensions of the minimal operator have the same essential spectrum, see [20, 21, 24].

In this paper we focus our attention to the following spectral property of the operator $L$.

*Definition* 1. Operator $L$ is said to have *property* $\mathbb{BD}$ if every self-adjoint extension of $L_{\min}$ has spectrum discrete and bounded below.

The link between oscillatory and spectral properties of the operator $L$ is the following statement.

**Proposition 1.** *Operator $L$ has property $\mathbb{BD}$ if and only if the equation $L(y) = \lambda y$ is strongly nonoscillatory.*

The paper is organized as follows. In the next section we recall some general statements of oscillation theory of self-adjoint equations (2). In Section 3 we present known results concerning oscillation and spectral properties of one and two-term differential operators. Section 4 contains new results – oscillation and nonoscillation criteria for fourth order differential equations with middle terms. The last section is devoted to remarks on the results of the paper and to the formulation of some open problems.

## 2. Oscillation theory of self-adjoint equations

Our oscillation results are based on the following variational principle.

**Lemma 1.** ([15]) *Equation* (2) *is nonoscillatory if and only if there exists $T \in \mathbb{R}$ such that*

$$\mathcal{F}(y; T, \infty) := \int_T^\infty \left[ \sum_{k=0}^n r_k(t)(y^{(k)}(t))^2 \right] dt > 0$$

*for any nontrivial $y \in W^{n,2}(T, \infty)$ with compact support in $(T, \infty)$.*

In nonoscillation criteria, the following Wirtninger inequality is frequently used.

**Lemma 2.** ([15]) *Let $y \in W^{1,2}(T, \infty)$ have compact support in $(T, \infty)$ and let $M$ be a positive differentiable function such that $M'(t) \neq 0$ for $t \in [T, \infty)$. Then*

$$\int_T^\infty |M'(t)| y^2 \, dt < 4 \int_T^\infty \frac{M^2(t)}{|M'(t)|} y'^2 \, dt.$$

We will need also a statement concerning factorization of disconjugate differential operators.

**Lemma 3.** ([2]) *Suppose that equation* (2) *possesses a system of positive solutions $y_1, \dots, y_{2n}$ satisfying $y_i = o(y_{i+1})$, $i = 1, \dots, 2n-1$, as $t \to \infty$. Then the operator $L$ given by* (1) *admits the factorization for large $t$*

$$L(y) = \frac{1}{a_0(t)} \left( \frac{1}{a_1(t)} \left( \cdots \frac{r_n(t)}{a_n^2(t)} \left( \frac{1}{a_{n-1}(t)} \cdots \frac{1}{a_1(t)} \left( \frac{y}{a_0(t)} \right)' \cdots \right)' \cdots \right) \right)',$$

*where*

$$a_0 = y_1,\ a_1 = \left(\frac{y_2}{y_1}\right)',\ a_i = \frac{W(y_1,\dots,y_{i+1})W(y_1,\dots,y_{i-1})}{W^2(y_1,\dots,y_i)},\ i = 1,\dots,n-1$$

*and* $a_n = (a_0 \cdots a_{n-1})^{-1}$, $W(\cdot)$ *being the Wronskian of the functions in brackets.*

Oscillation criteria presented in this paper are based on the following general statement. This statement concerns oscillation of the equation

$$L(y) = M(y), \tag{8}$$

where the operator $M$ is given by

$$M(y) = \sum_{k=0}^{m} (-1)^k \left(\tilde{r}_k(t) y^{(k)}\right)^{(k)}$$

with $m \in \{1,\dots,n-1\}$ and $\tilde{r}_j(t) \geq 0$ for large $t$. Equation (8) is viewed as a perturbation of (nonoscilatory) equation (2). The next proposition says, roughly speaking, that (8) is oscillatory if the functions $\tilde{r}_j$ are sufficiently positive, in a certain sense. The proof of the next proposition can be found in [3, 4].

**Proposition 2.** *Suppose that* (2) *is nonoscillatory and* $y_1,\dots,y_n$ *is the principal system of solutions of this equation. Equation* (8) *is oscillatory if there exists* $c = (c_1,\dots,c_n) \in \mathbb{R}^n$ *such that one of the following conditions holds:*

(i) *We have*

$$\int^\infty \left[\sum_{k=0}^{m} \tilde{r}_k(t)(h^{(k)}(t))^2\right] dt = \infty, \qquad h := c_1 y_1 + \cdots + c_n y_n$$

(ii) *The previous integral is convergent and*

$$\limsup_{t\to\infty} \frac{\int_t^\infty \left[\sum_{k=0}^{m} \tilde{r}_k(t)(h^{(k)}(s))^2\right] ds}{c^T \left(\int^t X^{-1}(s)B(s)X^{T-1}(s)\, ds\right)^{-1} c} > 1, \tag{9}$$

*where* $(X, U)$ *is the solution of the linear Hamiltonian system associated with* (2) *generated by* $y_1,\dots,y_n$.

## 3. One and two-term differential operators

A typical method of the investigation of the spectral properties of the operator $L$ consists in assuming (this assumption is satisfied in many applications) that one of the terms in $L$, say $(-1)^j \left(r_j(t)y^{(j)}\right)^{(j)}$ for some $j \in \{1,\dots,n\}$, is dominant, in a certain sense, and then the operator $L$ is viewed as a perturbation of the one-term operator

$$L_j(y) = \frac{(-1)^j}{w(t)} \left(r_j(t)y^{(j)}\right)^{(j)}.$$

This approach has been used, e.g., in the papers [11, 16]. For this reason, we turn first our attention to one-term differential operators. We consider the differential operator of the form

$$l(y) := \frac{(-1)^n}{w(t)} \left(r(t)y^{(n)}\right)^{(n)} \tag{10}$$

and the associated equation

$$(-1)^n (r(t)y^{(n)})^{(n)} = w(t)y. \tag{11}$$

Basic results concerning property $\mathbb{BD}$ of one term differential operators are based on the following statement, usually referred to as the reciprocity principle, the proof can be found e.g. in [1].

**Proposition 3.** *Equation* (11) *is nonoscillatory if and only if the so-called reciprocal equation* (*related to* (11) *by the substitution* $z = r(t)y^{(n)}$)

$$(-1)^n \left(\frac{z^{(n)}}{w(t)}\right)^{(n)} = \frac{1}{r(t)} z \tag{12}$$

*is nonoscillatory.*

Now we present two oscillation criteria for equation (11) in case $r(t) = t^\alpha$, i.e. we consider the equation

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} = w(t)y, \tag{13}$$

where $\alpha$ is a real constant.

**Theorem 1.** ([6, 17]) *Let* $\alpha \notin \{1, 3, \dots, 2n-1\}$, $\alpha < 2n-1$, $\int^\infty w(t)\, dt < \infty$,

$$M := \limsup_{t\to\infty} t^{2n-1-\alpha} \int_t^\infty w(s)\, ds \tag{14}$$

*and*

$$\hat{\gamma}_{n,\alpha} := \frac{\prod_{k=0}^{2n-1} (2n-1-2k-\alpha)^2}{4^n(2n-1-\alpha)}. \tag{15}$$

*If* $M < \hat{\gamma}_{n,\alpha}$ *then* (13) *is nonoscillatory and if* $M > \hat{\gamma}_{n,\alpha}$ *the this equation is oscillatory.*

**Theorem 2.** ([6, 13]) *Let* $\alpha \in \{1, 3, \dots, 2n-1\}$, $\int^\infty w(t) t^{2n-1-\alpha} dt < \infty$,

$$K := \limsup_{t\to\infty} \lg t \int_t^\infty w(s) s^{2n-1-\alpha}\, ds \tag{16}$$

*and*

$$\rho_{n,\alpha} := \frac{[n!(n-m-1)]^2}{4}, \quad m := \frac{2n-1-\alpha}{2}. \tag{17}$$

*If* $K < \rho_{n,\alpha}$ *then* (13) *is nonoscillatory and if* $K > \rho_{n,\alpha}$ *then this equation is oscillatory.*

Note that the oscillation parts of the previous theorem were proved in [13, 17] under the stronger assumption $M > (2n-1-\alpha)[(n-1)!]^2$ for $\alpha \notin \{1, 3, \dots, 2n-1\}$ and $K > 4\rho_{n,\alpha}$ if $\alpha \in \{1, 3, \dots, 2n-1\}$. In the form presented in Theorems 1, 2 (with the improved value of the oscillation constant) these criteria were proved in the recent paper [6].

Consider now the differential operator

$$\tilde{l}(y) = (-1)^n t^\alpha \left(r(t) y^{(n)}\right)^{(n)}, \tag{18}$$

The reciprocal equation of the equation $\tilde{l}(y) = y$ is the equation

$$(-1)^n \left(t^\alpha z^{(n)}\right)^{(n)} = \frac{1}{r(t)} z.$$

Applying the previous oscillation and nonoscillation criteria to this equation we get the following necessary and sufficient condition for property $\mathbb{BD}$ of $\tilde{l}$. For $\alpha = 0$ this is the classical condition of Tkachenko [15] (sufficiency) and of Lewis [18] (necessity). For $\alpha \notin \{1, 3, \dots, 2n-1\}$, $\alpha < 2n-1$, this condition is formulated in [17]. The case $\alpha \in \{1, 3, \dots, 2n-1\}$ is treated in [13].

**Theorem 3.** *Operator* $\tilde{l}$ *has property* $\mathbb{BD}$ *if and only if*

(i) $\alpha \notin \{1, 3, \dots, 2n-1\}$ *and*

$$\lim_{t\to\infty} t^{2n-1-\alpha} \int_t^\infty \frac{ds}{r(s)} = 0,$$

(ii) $\alpha \in \{1, 3, \dots, 2n-1\}$ *and*

$$\lim_{t\to\infty} \lg t \int_t^\infty \frac{s^{2n-1-\alpha}}{r(s)}\, ds = 0.$$

Observe that if $M = \gamma_{n,\alpha}$ in (14) resp. $K = \rho_{n,\alpha}$ in (16), Theorems 1 and 2 do not apply. A typical example of a function $w$ for which this happens is

$$w(t) = \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}, \quad \gamma_{n,\alpha} := (2n-1-\alpha)\hat{\gamma}_{n,\alpha}, \quad \alpha \notin \{1, 3, \dots, 2n-1\},$$

with $\hat{\gamma}_{n,\alpha}$ given by (15), resp.

$$w(t) = \frac{\rho_{n,\alpha}}{t^{2n-\alpha}\lg^2 t}, \quad \alpha \in \{1, 3, \dots, 2n-1\}.$$

This fact was a motivation for the research of the papers [5, 9, 10] where the equations

$$(-1)^n \left(t^\alpha y^{(n)}\right)^n - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = w(t)y, \ \alpha \notin \{1, 3, \dots, 2n-1\}, \tag{19}$$

and

$$(-1)^n \left(t^\alpha y^{(n)}\right)^n - \frac{\rho_{n,\alpha}}{t^{2n-\alpha}\lg^2 t} y = w(t)y, \ \alpha \in \{1, 3, \dots, 2n-1\}, \tag{20}$$

were investigated. In the next criteria equations (19) and (20) are viewed as a perturbation of the equations

$$(-1)^n \left(t^\alpha y^{(n)}\right)^n - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}} y = 0, \quad \alpha \notin \{1, 3, \dots, 2n-1\}, \tag{21}$$

and

$$(-1)^n \left(t^\alpha y^{(n)}\right)^n - \frac{\rho_{n,\alpha}}{t^{2n-\alpha}\lg^2 t} y = 0, \quad \alpha \in \{1, 3, \dots, 2n-1\}. \tag{22}$$

In the easier case $\alpha \notin \{1, 3, \dots, 2n-1\}$ we have obtained the following result.

**Theorem 4.** ([6, 8, 9]) *Let $\alpha \notin \{1, 3, \dots, 2n-1\}$,*

$$\tilde{M} := \limsup_{t\to\infty} \lg t \int_t^\infty w(s)s^{2n-1-\alpha}\, ds$$

*and*

$$\tilde{\gamma}_{n,\alpha} := \frac{\prod_{k=0}^{n-1}(2n-1-\alpha-2k)^2}{4^n} \left[\sum_{k=0}^{n-1} \frac{1}{(2n-1-\alpha-2k)^2}\right]. \tag{23}$$

*If $\tilde{M} < \tilde{\gamma}_{n,\alpha}$ then* (19) *is nonoscillatory and if $\tilde{M} > \tilde{\gamma}_{n,\alpha}$ then this equation is oscillatory.*

Note that similarly to the case treated in Theorem 1, the oscillation part of the previous theorem was proved in [5, 9] under the stronger assumption $\tilde{M} > 4\tilde{\gamma}_{n,\alpha}$ and this assumption is weakened to the form as presented in Theorem 4 in [8] using the method introduced in [6].

In the more difficult case $\alpha \in \{1, 3, \dots, 2n-1\}$ we have not been able to find a “unifying” limit which “separates” oscillatory and nonoscillatory equations, but we proved the following result.

**Theorem 5.** *Let $\alpha \in \{1, 3, \dots, 2n-1\}$.*

(i) *If*

$$\int^{\infty} w(t)s^{2n-1-\alpha}\lg s\,ds = \infty, \tag{24}$$

*then* (20) *is oscillatory.*

(ii) *If the second order equation*

$$(t\lg t\,u')' + \frac{w(t)t^{2n-1-\alpha}\lg t}{4\rho_{n,\alpha}}u = 0 \tag{25}$$

*is nonoscillatory, then* (20) *is also nonoscillatory. In particular,* (20) *is nonoscillatory provided*

$$\lim_{t\to\infty} \lg(\lg t)\int_t^{\infty} w(s)s^{2n-1-\alpha}\lg s\,ds < \rho_{n,\alpha}.$$

The case when we consider two-term differential operators can be regarded as a model of the situation when not only *one* term is dominant in the differential operator $L$ given by (1), but *two* terms are dominant. One term is again a term $(-1)^j\left(r_j(t)y^{(j)}\right)^{(j)}$ for some $j \in \{1, 2, \dots, n\}$ and the second one is the term $r_0(t)y$. This leads then to differential equations of the form

$$(-1)^j(r_j(t)y^{(j)})^{(j)} + r_0(t)y = w(t)y$$

and equations (19) and (20) are just of this form.

Theorems 4 and 5 give the following conditions for property $\mathbb{BD}$ of the operators

$$\tilde{L}(y) := \frac{1}{w(t)}\left[(-1)^n\left(t^{\alpha}y^{(n)}\right)^n - \frac{\gamma_{n,\alpha}}{t^{2n-\alpha}}y\right], \quad \alpha \notin \{1, 3, \dots, 2n-1\}, \tag{26}$$

and

$$\hat{L}(y) := \frac{1}{w(t)}\left[(-1)^n\left(t^{\alpha}y^{(n)}\right)^n - \frac{\rho_{n,\alpha}}{t^{2n-\alpha}\lg^2 t}y\right], \quad \alpha \in \{1, 3, \dots, 2n-1\}. \tag{27}$$

The result given in the next theorem is new, it is not given in the above mentioned references [9, 10]. However, as we will see, this statement follows easily from Theorems 4 and 5.

**Theorem 6.** *Let the differential operators $\tilde{L}$ and $\hat{L}$ be given by* (26), (27), *respectively.*

(i) *Let $\alpha \notin \{1, 3, \dots, 2n-1\}$, then $\tilde{L}$ has property $\mathbb{BD}$ if and only if*

$$\lim_{t\to\infty} \lg t\int_t^{\infty} w(s)s^{2n-1-\alpha}\,ds = 0. \tag{28}$$

(ii) *Let* $\alpha \in \{1,3,\dots,2n-1\}$. *If* $\hat{L}$ *has property* $\mathbb{BD}$ *then*

$$\int^{\infty} w(t)t^{2n-1-\alpha}\lg t\,dt < \infty. \tag{29}$$

*Conversely, if*

$$\lim_{t\to\infty} \lg(\lg t)\int_t^{\infty} w(s)s^{2n-1-\alpha}\lg s\,ds = 0. \tag{30}$$

*then* $\hat{L}$ *has property* $\mathbb{BD}$.

**Proof.** (i) Let $\alpha \notin \{1,3,\dots,2n-1\}$. Note that if $\int^{\infty} w(t)t^{2n-1-\alpha}\,dt = \infty$, then the equation $\tilde{L}(y) = y$ is (strongly) oscillatory by Proposition 2 since the function $y = t^{\frac{2n-1-\alpha}{2}}$ is contained in the principal system of solution of the equation $\tilde{L}(y) = 0$. Hence, $\tilde{L}$ cannot have property $\mathbb{BD}$ by Proposition 1.
Suppose now that $\tilde{L}$ has property $\mathbb{BD}$ and the limit (28) is not zero, i.e.

$$\limsup_{t\to\infty} \lg t\int_t^{\infty} w(s)s^{2n-1-\alpha}\,ds = \varepsilon > 0.$$

Hence, for $\lambda > \frac{\tilde{\gamma}_{n,\alpha}}{\varepsilon}$ we have

$$\limsup_{t\to\infty} \lg t\int_t^{\infty} \lambda w(s)s^{2n-1-\alpha}\,ds > \tilde{\gamma}_{n,\alpha}$$

and thus equation (19) with $\lambda w(t)$ instead of $w(t)$ is oscillatory. This means that this equation is not strongly nonoscillatory and this contradicts to Proposition 1. Conversely, suppose that (28) holds. Then for every $\lambda > 0$

$$\lim_{t\to\infty} \lg t\int_t^{\infty} \lambda w(s)s^{2n-1-\alpha}\,ds = 0 < \tilde{\gamma}_{n,\alpha}.$$

Hence, (19) is strongly nonoscillatory by Theorem 5 and $\tilde{L}$ has property $\mathbb{BD}$ by Proposition 1.
(ii) Let $\alpha \in \{1,3,\dots,2n-1\}$. If $\hat{L}$ has property $\mathbb{BD}$ and the integral in (29) is divergent, i.e.,

$$\int^{\infty} w(t)t^{2n-1-\alpha}\lg t\,dt = \infty,$$

then the equation $\hat{L}(y) = y$ is oscillatory by Theorem 5, a contradiction with Proposition 1. Conversely, if (30) holds, we have

$$\lim_{t\to\infty} \lg(\lg t)\int_t^{\infty} \lambda w(s)s^{2n-1-\alpha}\lg s\,ds = 0 < \rho_{n,\alpha}$$

which means that the second order equation

$$(t \lg t\, u')' + \frac{\lambda w(t) t^{2n-1-\alpha} \lg t}{4\rho_{n,\alpha}} u = 0$$

is nonoscillatory for every $\lambda > 0$.

Hence, by Theorem 5, the equation $\hat{L}(y) = \lambda y$ has the same property and hence the operator $\hat{L}$ has property $\mathbb{BD}$ by Proposition 1. $\square$

## 4. Fourth order operators with a middle term

In this section we present new oscillation and nonoscillation criteria for the fourth order differential equation

$$y^{(\mathrm{iv})} - (q(t)y')' + p(t)y = 0 \tag{31}$$

where this equation is viewed as a perturbation of the Euler differential equation

$$L_{\nu,\gamma}(y) := y^{(\mathrm{iv})} - \nu \left(\frac{y'}{t^2}\right)' - \frac{\gamma}{t^4} y = 0. \tag{32}$$

We follow essentially the same idea as in the papers [7, 8, 14], where two-term differential operators are investigated. Our ultimate aim is to study general three-term differential operators and equations

$$L(y) := (-1)^n \left(r(t)y^{(n)}\right)^{(n)} + (-1)^m \left(q(t)y^{(m)}\right)^{(m)} + p(t)y = 0 \tag{33}$$

with $m \in \{1, \dots, n-1\}$. Similarly to [7, 14], we "test" first the situation in the most simple case of the fourth order equation in order to see where are the main problems. Then, solving these problems in this simple case, we are going to "attack" the general three-term equation (33). This idea is applied in the case of two-term operators in [8].

We start with an elementary statement concerning the fourth order Euler equation (32).

**Lemma 4.** *Let $\nu \geq -\frac{5}{2}$. Equation* (32) *is nonoscillatory if and only if*

$$\nu + \frac{1}{4} - \frac{4}{9}\gamma \geq 0. \tag{34}$$

*If $\nu = \frac{4}{9}\gamma - \frac{1}{4}$ then this equation possesses a fundamental system of solutions*

$$y_1 = t^{\frac{3}{2} - \tilde{n}u}, \quad y_2 = t^{\frac{3}{2}}, \quad y_3 = t^{\frac{3}{2}} \lg t, \quad y_4 = t^{\frac{3}{2} + \tilde{\nu}}, \tag{35}$$

*where* $\tilde{\nu} := \sqrt{\frac{5}{2}+\nu}$*. Moreover, the differential operator* $L_{\nu,\gamma}$ *admits Polya's factorization*

$$L_{\nu,\gamma}(y) = \frac{1}{t^{\frac{3}{2}}}\left\{t^{1+\tilde{\nu}}\left[t^{1-2\tilde{\nu}}\left(t^{1+\tilde{\nu}}\left(\frac{y}{t^{\frac{3}{2}}}\right)'\right)'\right]'\right\}', \tag{36}$$

*and we have*

$$\int_T^\infty \left[y''^2 + \nu\frac{y'^2}{t^2} - \gamma\frac{y^2}{t^4}\right] dt = \int_T^\infty t^{1-2\tilde{\nu}}\left\{\left[t^{1+\tilde{\nu}}\left(\frac{y}{t^{\frac{3}{2}}}\right)'\right]'\right\}^2 dt$$

*for every* $y \in W^{2,2}(T,\infty)$ *with compact support in* $(T,\infty)$*.*

**Proof.** We will prove that (32) is nonoscillatory for $\nu, \gamma$ satisfying (34); oscillation of (32) if this inequality is violated will be proved later as a consequence of a more general result.

By the Wirtinger inequality given in Lemma 2,

$$\begin{aligned}\int_T^\infty y''^2\,dt + \nu\int_T^\infty \frac{y'^2}{t^2}\,dt - \gamma\int_T^\infty \frac{y^2}{t^4}\,dt &> \left(\nu+\frac{1}{4}\right)\int_T^\infty \frac{y'^2}{t^2}\,dt - \gamma\int_T^\infty \frac{y^2}{t^4}\,dt\\ &> \left[\frac{9}{4}\left(\nu+\frac{1}{4}\right)-\gamma\right]\int_T^\infty \frac{y^2}{t^4}\,dt.\end{aligned}$$

Hence, (32) is nonoscillatory by Lemma 1 if (34) holds.

Now suppose that equality in (34) holds, i.e.,

$$\nu + \frac{1}{4} - \frac{4}{9}\gamma = 0. \tag{37}$$

We look for a solution of (32) in the form $y(t) = t^\lambda$. Substituting into (32) we obtain

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3) - \nu\lambda(\lambda-3) - \gamma = 0.$$

The substitution $\mu = \lambda - \frac{3}{2}$ converts the last equation into the equation

$$\left(\mu^2-\frac{9}{4}\right)\left(\mu^2-\frac{1}{4}\right) - \nu\left(\mu^2-\frac{9}{4}\right) - \gamma = 0.$$

If (37) holds, this equation has the roots $\mu_{1,2} = 0$ and $\mu_{3,4} = \pm\sqrt{\nu+\frac{5}{2}} = \pm\tilde{\nu}$, and this means that (35) is really a fundamental system of solutions of (32).

In order to prove the factorization formula (36), it suffices to apply Lemma 3 with $y_1 = t^{\frac{3}{2}}$ and $y_2 = t^{\frac{3}{2}-\tilde{\nu}}$. Finally, multiplying (36) by $y$, integrating the obtained equation from $T$ to $\infty$, and using integration by parts, we get the last formula stated in Lemma 4. $\square$

Next, we study oscillation and nonoscillation of the equation $L_{\mu,\gamma}(y) = p(t)y$ in the critical case (37).

**Theorem 7.** *Suppose that* (37) *holds and* $p(t) \geq 0$ *for large* $t$.

(i) *If the second order equation*

$$(tu')' + \frac{t^3 p(t)}{\tilde{\nu}^2} u = 0 \tag{38}$$

*is nonoscillatory, then the equation*

$$y^{(\mathrm{iv})} - \nu \left( \frac{y'}{t^2} \right)' - \frac{\gamma}{t^4} y = p(t)y \tag{39}$$

*is also nonoscillatory. In particular,* (39) *is nonoscillatory provided* $p(t) \geq \frac{\tilde{\nu}^2}{4t^4 \lg^2 t}$ *for large* $t$ *and*

$$\lim_{t\to\infty} \lg(\lg t) \int_t^\infty \left[ p(s) - \frac{\tilde{\nu}^2}{4s^4 \lg^2 s} \right] s^3 \lg s \, ds < \frac{\tilde{\nu}^2}{4}. \tag{40}$$

(ii) *If*

$$\int^\infty \left[ p(t) - \frac{\tilde{\nu}^2}{4t^4 \lg^2 t} \right] t^3 \lg t \, dt = \infty, \tag{41}$$

*then* (39) *is oscillatory.*

**Proof.** We skip details of the proof since it is similar to that of [7, Theorem 1], where the case $\nu = 0$, $\gamma = \frac{9}{16}$ is investigated. To prove the nonoscillation part (i), we use Lemma 4 and the Wirtinger inequality of Lemma 2. According to Lemma 1, it suffices to find $T \in \mathbb{R}$ such that

$$\int_T^\infty \left[ y''^2 + \nu \frac{y'^2}{t^2} - \gamma \frac{y^2}{t^4} \right] dt - \int_T^\infty p(t) y^2 \, dt > 0 \tag{42}$$

for every nontrivial $y \in W^{2,2}(T,\infty)$ with compact support in $(T,\infty)$. We have by Lemma 4

$$\begin{aligned}
&\int_T^\infty \left[y''^2 + \nu\frac{y'^2}{t^2} - \gamma\frac{y^2}{t^4}\right] dt - \int_T^\infty p(t)y^2\, dt \\
= \ &\int_T^\infty t^{1-2\tilde\nu}\left\{\left[t^{1+\tilde\nu}\left(\frac{y}{t^{\frac{3}{2}}}\right)'\right]'\right\}^2 dt - \int_T^\infty p(t)y^2\, dt \\
\geq \ &\tilde\nu^2\int_T^\infty t\left[\left(\frac{y}{t^{\frac{3}{2}}}\right)'\right]^2 dt - \int_T^\infty t^3p(t)\left(\frac{y}{t^{\frac{3}{2}}}\right)^2 dt \\
= \ &\tilde\nu^2\int_T^\infty \left[tu'^2 - \frac{t^3p(t)}{\tilde\nu^2}u^2\right] dt,
\end{aligned}$$

where $u = \frac{y}{t^{\frac{3}{2}}}$. Since the second order equation (38) is nonoscillatory, the last integral is positive for every $u \in W^{1,2}(T,\infty)$ with compact support in $(T,\infty)$ if $T$ is sufficiently large. Hence, (42) holds which means that (39) is nonoscillatory. To prove the sufficiency of (40) for the nonoscillation of (38) we rewrite this equation into the form

$$(tu')' + \frac{1}{4t\lg^2 t}u + \left[\frac{t^3p(t)}{\tilde\nu^2} - \frac{1}{4t\lg^2 t}\right]u.$$

The transformation $u = \sqrt{\lg t}\, v$ transforms the last equation into the equation (see, e.g., [23])

$$(t\lg t\ v')' + \left[p(t) - \frac{\tilde\nu^2}{4t^4\lg^2}\right]\frac{t^3\lg t}{\tilde\nu^2}v = 0. \tag{43}$$

By the classical Hille nonoscillation criterion, the equation

$$(r(t)x')' + c(t)x = 0$$

with $c(t) \geq 0$, $\int^\infty c(t)\, dt < \infty$ and $\int^\infty r^{-1}(t)\, dt = \infty$ is nonoscillatory if

$$\lim_{t\to\infty}\left(\int^t r^{-1}(s)\, ds\right)\left(\int_t^\infty c(s)\, ds\right) < \frac{1}{4}. \tag{44}$$

Now, applying Hille's nonoscillation criterion to (43) gives just (40).

In the proof of (ii), let $T \in \mathbb{R}$ be arbitrary and $t_3 > t_2 > t_1 > t_0 > T$ (these values will be specified later). Denote $h(t) = t^{\frac{3}{2}}\sqrt{\lg t}$, let $f \in C^2[t_0,t_1]$ be any function satisfying $f(t_0) = 0 = f'(t_0)$, $f(t_1) = h(t_1)$, $f'(t_1) = h'(t_1)$, and let $g$ be the solution of equation (32) satisfying

$$g(t_2) = h(t_2),\ g'(t_2) = h'(t_2),\ g(t_3) = 0 = g'(t_3). \tag{45}$$

Define the function $y \in W^{2,2}(T,\infty)$ with compact support in $(T,\infty)$ as follows:

$$y(t) = \begin{cases} 0, & t \le t_0, \\ f(t), & t_0 \le t \le t_1, \\ h(t), & t_1 \le t \le t_2, \\ g(t), & t_2 \le t \le t_3, \\ 0, & t \ge t_3. \end{cases}$$

and denote

$$K := \int_{t_0}^{t_1} \left[ f''^2 - \frac{\nu}{t^2} f'^2 - \left( \frac{\gamma}{t^4} + p(t) \right) f^2 \right] dt.$$

By a direct computation, we have

$$\begin{aligned} & \int_{t_1}^{t_2} \left( h''^2 + \nu \frac{h'^2}{t^2} - \frac{\gamma}{t^4} h^2 \right) dt \\ = \;& \frac{3}{2} (\nu + 1) \lg t_2 + \frac{5 + 2\nu}{8} \int_{t_1}^{t_2} \frac{dt}{t \lg t} + \tilde{K} + o\left(\lg^{-1} t_2\right), \end{aligned}$$

as $t_2 \to \infty$, where $\tilde{K}$ is a real constant.

Computing the integral

$$\mathcal{F}(g; t_2, t_3) := \int_{t_2}^{t_3} \left[ g''^2 + \nu t^{-2} g'^2 - \gamma t^{-4} g^2 \right] dt,$$

denote by $\binom{x}{u}$ the solution of the linear Hamiltonian system (further LHS) associated with (32) generated by the solution $g$, and let $\tilde{h} = \binom{h}{h'}$. Then,

$$\begin{aligned} \mathcal{F}(g; t_2, t_3) &= \int_{t_2}^{t_3} \left[ u^T B(t) u + x^T C(t) x \right] dt \\ &= \int_{t_2}^{t_3} \left[ u^T (x' - Ax) + x^T C(t) x \right] dt \\ &= u^T x \Big|_{t_2}^{t_3} + \int_{t_2}^{t_3} x^T \left[ -u' - A^T u + C(t) x \right] dt \\ &= -u^T(t_2) x(t_2). \end{aligned}$$

Let $(X,U)$ be the principal solution of the LHS associated with (32), i.e., this solution is generated by $y_1 = t^{\frac{3}{2}-\tilde{\nu}}$, $y_2 = t^{\frac{3}{2}}$. Then we have

$$\begin{aligned}
x(t) &= X(t)\int_t^{t_3} X^{-1}BX^{T-1}\,ds\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2),\\
u(t) &= \left[U(t)\int_t^{t_3} X^{-1}BX^{T-1}\,ds - X^{T-1}(t)\right]\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2),
\end{aligned}$$

(see e.g. [2] or [7]) and hence

$$\begin{aligned}
-u^T(t_2)x(t_2) &= \tilde{h}^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)\\
&\quad - \tilde{h}^T(t_2)U(t_2)X^{-1}(t_2)\tilde{h}(t_2).
\end{aligned}$$

Since $(X,U)$ is the principal solution, the first term on the right-hand side of the last expression tends to zero as $t_3 \to \infty$ ($t_2$ being fixed). Concerning the second term, by a direct computation similar to that in [7],

$$\tilde{h}^T(t_2)U(t_2)X^{-1}(t_2)\tilde{h}(t_2) = \frac{3}{2}(\nu+1)\lg t_2 + \hat{K} + o(1)$$

as $t_2 \to \infty$, where $\hat{K}$ is a positive real constant.

Summarizing the above computations

$$\begin{aligned}
\mathcal{F}(y;t_0,t_3) &\le K + \frac{3}{2}(\nu+1)\lg t_2 + \frac{5+2\nu}{8}\int_{t_1}^{t_2}\frac{dt}{t\lg t}\\
&\quad + \tilde{K} + o(1) - \int_{t_1}^{t_2} q(t)t^3\lg t\,dt\\
&\quad + \tilde{h}^T(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2)\\
&\quad - \left[\frac{3}{2}(\nu+1)\lg t_2 + \hat{K} + o(1)\right].
\end{aligned}$$

Now, let $t_2 > t_1$ be such that the following conditions are satisfied:

(a) $\int_{t_1}^{t_2}\left(q(t) - \frac{5+2\tilde{\nu}}{8t^4\lg^2 t}\right)t^3\lg t\,dt > K + \tilde{K} + 1$,

(b) the sum of all terms $o(1)$ (as $t_2 \to \infty$) is less than 1.

Let $t_3 > t_2$ be such that

$$\tilde{h}(t_2)X^{T-1}(t_2)\left(\int_{t_2}^{t_3} X^{-1}BX^{T-1}\,ds\right)^{-1} X^{-1}(t_2)\tilde{h}(t_2) < \hat{K}.$$

For $t_2, t_3$ chosen in this way we have

$$\mathcal{F}(y;T,\infty) \le K - (K + \tilde{K} + 1) + \tilde{K} - \hat{K} + \hat{K} + 1 \le 0.$$

Therefore, equation (39) is oscillatory by Lemma 1. ☐

Substituting $p(t) = \frac{\lambda}{t^4 \lg^2 t}$ in the previous statement we have the folowing statement.

**Corollary 1.** *The equation*

$$y^{(\mathrm{iv})} - \nu \left(\frac{y'}{t^2}\right)' - \frac{\gamma}{t^4} y = \frac{\lambda}{t^4 \lg^2 t} y$$

*with* $\frac{9}{4}\left(\nu + \frac{1}{4}\right) - \gamma = 0$ *is nonoscillatory if and only if* $\lambda \le \frac{\tilde{\nu}^2}{4} = \frac{2\nu+5}{8}$.

The next corollary completes the proof of Lemma 4.

**Corollary 2.** *If* $\frac{9}{4}\left(\nu + \frac{1}{4}\right) - \gamma < 0$ *then* (32) *is oscillatory.*

**Proof.** Let $\varepsilon > 0$ be such that $\frac{9}{4}\left(\nu + \frac{1}{4}\right) - (\gamma - \varepsilon) = 0$ and denote $\tilde{\gamma} = \gamma - \varepsilon$. Then (32) can be written in the form

$$L_{\nu,\tilde{\gamma}}(y) = \frac{\varepsilon}{t^4} y.$$

The function $p(t) = \varepsilon t^{-4}$ obviously satisfies (41), so (32) is oscillatory. ☐

Now we will deal with the equation

$$y^{(\mathrm{iv})} - \nu \left(\frac{y'}{t^2}\right)' - \frac{\gamma}{t^4} y = -(q(t)y')' \tag{46}$$

which will be viewed again as a perturbation of (32).

**Theorem 8.** *Suppose that* (37) *holds.*

(i) *If* $\tilde{\nu}^2 - t^2 q(t) > 0$ *for large* $t$ *and the second order equation*

$$\left[t\left(1 - \frac{t^2 q(t)}{\tilde{\nu}^2}\right) u'\right]' - \frac{3}{2\tilde{\nu}^2} t^{\frac{3}{2}} \left(q(t)\sqrt{t}\right)' u = 0 \tag{47}$$

*is nonoscillatory, then* (46) *is also nonoscillatory. In particular,* (46) *is nonoscillatory if* $\int^\infty t^{-1}(\tilde{\nu}^2 - t^2 q(t))^{-1}\, dt = \infty$, $(q(t)\sqrt{t})' \le 0$ *for large* $t$, *and*

$$\lim_{t\to\infty} \left(\int^t s^{-1}(\tilde{\nu}^2 - s^2 q(s))^{-1}\, ds\right) \left(\int_t^\infty s^{\frac{3}{2}} \left|\left(q(s)\sqrt{s}\right)'\right|\right) < \frac{\tilde{\nu}^2}{6}. \tag{48}$$

(ii) *If* $q(t) \ge 0$ *for large* $t$ *and*

$$\limsup_{t\to\infty} \lg t \int_t^\infty q(s) s\, ds > \frac{4\tilde{\nu}^2}{9} \tag{49}$$

*then* (47) *is oscillatory.*

**Proof.** (i) To prove nonoscillation of (46) we will use again Lemmas 1, 2 and the factorization of the operator $L_{\nu,\gamma}$ given in Lemma 4. We have

$$
\begin{aligned}
&\int_T^\infty \left[ y''^2 + \nu \frac{y'^2}{t^2} - \gamma \frac{y^2}{t^4} \right] dt - \int_T^\infty q(t) y'^2\, dt \\
= \;& \int_T^\infty t^{1-2\tilde\nu} \left\{ \left[ t^{1+\tilde\nu} \left( \frac{y}{t^{\frac{3}{2}}} \right)' \right]' \right\}^2 dt - \int_T^\infty q(t) y'^2\, dt \\
\geq \;& \tilde\nu^2 \int_T^\infty t \left[ \left( \frac{y}{t^{\frac{3}{2}}} \right)' \right]^2 - \int_T^\infty q(t) y'^2\, dt \\
= \;& \tilde\nu^2 \int_T^\infty \left\{ \left[ t \left( 1 - \frac{t^2 q(t)}{\tilde\nu^2} \right) \right] u'^2 + \frac{1}{\tilde\nu^2} \left[ \frac{3t^{\frac{3}{2}}}{2} \left( q(t)\sqrt{t} \right)' \right] u^2 \right\} dt, \qquad (50)
\end{aligned}
$$

where $u = t^{-\frac{3}{2}} y$ and we have used the identity

$$
\int_T^\infty q(t) y'^2\, dt = \int_T^\infty \left[ t^3 q(t) u'^2 - \frac{3}{2} t^{\frac{3}{2}} \left( q(t)\sqrt{t} \right)' u^2 \right] dt
$$

which holds for every $y \in W^{1,2}(T,\infty)$ with compact support in $(T,\infty)$. Since the equation (47) is nonoscillatory, the integral (50) is positive by Lemma 1 and this implies, again by Lemma 1, that (46) is nonoscillatory as well.

(ii) We use Proposition 2, part (ii) with $L = L_{\nu,\gamma}$, $M(y) = -(q(t)y')'$ and $c = e_2 = (0,1)^T$. Let $y_1 = t^{\frac{3}{2}-\tilde\nu}$, $y_2 = t^{\frac{3}{2}}$ and

$$
X(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}, \quad U(t) = \begin{pmatrix} -y_1''' + \nu \frac{y_1'}{t^2} & -y_2''' + \nu \frac{y_2'}{t^2} \\ y_1'' & y_2'' \end{pmatrix}.
$$

Then $\binom{X}{U}$ is the principal solution of LHS associated with (32) and by a direct computation, we have

$$
c^T \left( \int^t X^{-1} B X^{T-1}\, ds \right)^{-1} c = \left( \int^t X^{-1} B X^{T-1}\, ds \right)^{-1}_{2,2} = \frac{\tilde\nu^2}{\lg t}(1 + o(1))
$$

as $t \to \infty$. Hence (9) becomes

$$
\limsup_{t\to\infty} \frac{\frac{9}{4}\int_t^\infty q(s)s\, ds}{\frac{\tilde\nu^2}{\lg t}} > 1,
$$

and this inequality is equivalent to (49). □

**Corollary 3.** *Let $q(t) = \frac{\beta}{t^2 \lg^2 t}$. Then (46) is oscillatory for $\beta > \frac{4\tilde\nu^2}{9}$ and it is nonoscillatory for $\beta < \frac{\tilde\nu^2}{9}$.*

## 5. Remarks and Comments

(i) The last Corollary reveals a typical phenomenon in the application of the variational principle in the oscillation theory of even order self-adjoint differential equations. Using Theorem 8, we are not able to decide the oscillatory nature of (46) with $q(t) = \frac{\beta}{t^2 \lg^2 t}$ if $\beta$ is between $\frac{\tilde{\nu}^2}{9}$ and $\frac{4\tilde{\nu}^2}{9}$. We already mentioned this phenomenon in Section 3, e.g., equation (13) with $\alpha \in \{1, 3, \dots, 2n-1\}$ was originally proved (using a variational principle) to be nonoscillatory if the upper limit $K$ given by (16) is less than $\rho_{n,\alpha}/4$ and oscillatory if $K > \rho_{n,\alpha}$, i.e., oscillation constant is four-times bigger than the nonoscillation constant. In [6] we developed a method which enables us to remove this gap and to prove that the "right" oscillation constant equals the nonoscillation one (compare Theorem 4 and the comment below this theorem). However, this method does not apply (at least, not directly) to the perturbation in a middle term and it is a subject of the present investigation how to modify this method to be applicable also to this case. We conjecture that (46) is oscillatory if the upper limit in (49) is $> \frac{\tilde{\nu}^2}{9}$, i.e., that the oscillation constant is actually four times less than stated in Theorem 8.

(ii) As we have already mentioned before, our ultimate goal is to study perturbations in middle terms of general self-adjoint, even order, differential equations and operators. Here we study fourth order equations in order to "recognize" the main difficulties. Concerning a higher order extension of our results, consider the $2n$-th order Euler-type self-adjoint equation

$$L_\nu(y) := (-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} + \sum_{k=0}^{n-1} (-1)^k \nu_k \left(\frac{y^{(k)}}{t^{2n-\alpha-2k}}\right)^{(k)} = 0 \tag{51}$$

with $\alpha \notin \{1, 3, \dots, 2n-1\}$, where $\nu_0, \dots, v_{n-1}$ are real constants. Based on the results of the of the previous section, we conjecture that there exists a "critical hyperplane"

$$\alpha_0 \nu_0 + \cdots + \alpha_{n-1} \nu_{n-1} = \beta, \tag{52}$$

where $\alpha_0, \dots, \alpha_{n-1}, \beta$ are real constants, such that (51) is nonoscillatory for $\nu = (\nu_0, \dots, n_{n-1})$ situated "above" this hyperplane (i.e. $\geq$ holds in (52) instead of equality) and oscillatory in the opposite case. Then, similarly to the fourth order case, one may investigate the equation of the form

$$L_\nu(y) = (-1)^m \left(q(t) y^{(m)}\right)^{(m)}, \quad m \in \{0, \dots, n-1\}$$

as a perturbation of (51).

(iii) It is known ([10, 13], compare also Theorem 2) that if $\alpha \in \{1, 3, \dots, 2n-1\}$, the equation

$$(-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} = \frac{\lambda}{t^{2n-\alpha}} y$$

is no longer conditionally oscillatory (in fact, it is strongly oscillatory, in contrast to the case $\alpha \notin \{1, 3, \dots, 2n-1\}$), and to get a conditionally oscillatory equation we need to consider the equation

$$(53) \qquad (-1)^n \left(t^\alpha y^{(n)}\right)^{(n)} = \frac{\lambda}{t^{2n-\alpha} \lg^2 t} y.$$

Now, let us look for an equation with middle terms which is a natural extension of (53). We conjecture (again based on the computations for fourth order equations) that such an extension is the equation

$$(54) \qquad \begin{aligned} \hat{L}_\nu(y) \;&=\; (-1)^\alpha \left(t^\alpha y^{(n)}\right)^{(n)} + \sum_{k=\mu+1}^{n-1} (-1)^k \nu_k \left(\frac{y^{(k)}}{t^{2n-\alpha-2k}}\right)^{(k)} \\ &\quad + \sum_{k=0}^{\mu} (-1)^k \nu_k \left(\frac{y^{(k)}}{t^{2n-\alpha-2k} \lg^2 t}\right)^{(k)} = 0, \end{aligned}$$

where $\mu = \frac{2n-1-\alpha}{2}$. We also conjecture that one can find a "critical hyperplane" of the form (52) for this equation as well, and that this hyperplane "separates" oscillatory and nonoscillatory equations (54). Having proved such a result, similarly to the previous remark and also to Theorem 5, we can investigate oscillation of the equation

$$\hat{L}_\nu(y) = (-1)^m \left(q(t) y^{(m)}\right)^{(m)}, \quad m \in \{0, \dots, n-1\},$$

with $\nu = (\nu_0, \dots, \nu_{n-1})$ on the critical hyperplane, viewed as a perturbation of (54).

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