ASYMPTOTIC STABILITY IN DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY

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Section 1. Historical Introduction.

In this paper we consider a functional differential equation of the form

$$x' = F(t, x, \int_0^t C(at-s)x(s)ds) \tag{1.1}$$

where a is a constant satisfying $0 < a < \infty$. Thus, the integral represents the memory of past positions of the solution x. We make the assumption that $\int_0^\infty |C(t)|dt < \infty$ so that this is a fading memory problem and we are interested in studying the effects of that memory over all those values of a. Very different properties of solutions emerge as we vary a and we are interested in developing an approach which handles them in a unified way.

Our study is based on a Liapunov functional $V(t, x_t)$ and wedges W_i satisfying

$$W_1(|x(t)|) \leq V(t, x_t) \leq W_2(|x(t)|) + \int_0^t D(t,s)W_3(|x(s)|)ds \tag{1.2}$$

and

$$V'(t, x_t) \leq -b(t)W_4(|x(t)|). \tag{1.3}$$

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The goal is to formulate a set of conditions on these relations which will imply asymptotic stability of the zero solution of (1.1). In this context there are four main challenges that a theorem needs to meet:

(i) x' may be unbounded for x bounded.

(ii) The necessarily fading memory must be utilized.

(iii) W_4 may be unrelated to W_3.

(iv) $b(t)$ may be near zero some of the time.

Volumes have been written about (i) through (iv) and we will give only a brief summary, together with references, so that the interested reader may trace them down.

It was recognized early in the theory of Liapunov's direct method that if the zero solution of an ordinary differential equation was stable, but not uniformly stable, then the existence of a positive definite Liapunov function with a negative definite derivative was insufficient to conclude that bounded solutions converge to zero. The difficulty is that there can be an annular ring around the point $x = 0$ through which a solution can pass infinitely often, moving so quickly that integration of V' does not send V to minus infinity effecting a desired contradiction. The mechanics are discussed in Burton [1, p. 161], for example, while a recent paper of Hatvani [7] details many of the advances. But the first significant contribution was that of Marachkov [9] who showed that it was sufficient to ask that x' be bounded for x bounded to ensure that solutions will tend to zero. That assumption became the basic one both for ordinary and functional differential equations. Much work has been done to relax that condition and in two recent papers Burton and Makay [4,5] have shown that x' could be unbounded of order $t \ln t$ and we could still conclude asymptotic stability.

It is easy to understand that the boundedness requirement on x' is highly objectionable because it is clear that in many systems the unboundedness of x' actually drives the solution to zero faster than would be possible with a bounded x'. Our work here on asymptotic stability completely avoids mention of boundedness of x'.

G. Seifert [11] seems to have been the first to offer an example which shows clearly the necessity of a fading memory for asymptotic stability. And most of our work here will focus on the fading memory. The reader may also consult a recent paper by Huang and Zhang [8] for further results on asymptotic stability and fading memory.

To show the ideas most clearly we will focus on the scalar equation

$$x' = -h(t)x - b(t)x^3 + \int_0^t C(at-s)x(s)ds \tag{1.4}$$

with a Liapunov functional

$$V(t,x_t) = x^2(t) + (1/a)\int_0^t \int_{at-s}^{\infty} |C(u)|du x^2(s)ds \tag{1.5}$$

under the assumption that

$$2h(t) \geq [1+(1/a)]\int_{(a-1)t}^{\infty} |C(v)|dv \tag{1.6}$$

so that there results the relation

$$V' \leq -2b(t)x^4. \tag{1.7}$$

Here, h and b can be unbounded.

When $C \in L^1[0,\infty)$, a is a positive constant whose magnitude will dictate the manner in which the memory fades. The integral in (1.5) can be thought of as a weighted average of x. The future position of x depends on this memory of past x.

For $a = 1$, we have the well-studied *convolution case*, and if $|C|$ decreases monotonically to zero as $t \to \infty$, then:

a) at $s = t$ we have $C(0)x(t)$ so that $x(t)$ is weighted maximally. This seems prudent since the future position should be most influenced by the present position.

b) at $s = 0$ we have $C(t)x(0)$ so that $x(0)$ is weighted mimimally; again, this seems right since things that happened long ago should have the least effect.

c) It is crucial to notice that, in contrast to the case of $a > 1$, the memory, that is the integral, $\int_0^t C(at-s)ds$, never fades away entirely; it tends to a constant as $t \to \infty$ for $C \in L^1[0,\infty)$. This means that, on one hand, we could have asymptotic stability even when h and b are zero, if the stability comes from the integal. On the other hand, if the stability does not come from the integral, then the h must be as large as a positive constant for uniform asymptotic stability.

Very different behavior emerges if $a > 1$. In this case, we have strongly fading memory; the integral $\int_0^t C(at-s)ds = \int_{(a-1)t}^t C(v)dv$ tends to zero as $t \to \infty$ for $C \in L^1[0, \infty)$. This will allow us to prove asymptotic stability when $h(t)$ and $b(t)$ are unbounded, and at the same time, when they are close to zero for extended periods of time.

The fact that W_4 may be unrelated to W_3, Item (iii) above, has played a major role in the study of stability and boundedness. Most motivating examples by investigators deal with problems similar to (1.4), but with $b(t) = 0$. In this case, asymptotic stability comes from the term $h(t)$ when (1.6) is strengthened by adding a positive constant α to the right hand side, resulting in $V'(t, x_t) \le -\alpha x^2$. This makes the derivative of V closely related to the memory in V and has resulted in simple proofs of asymptotic stability without any mention of boundedness of x'. Details of this type of work and references may be found in Burton [2, p. 311] for systems with unbounded delay, or Burton and Hatvani [3], Busenberg and Cooke [6], and Hatvani [7] for systems with bounded delay. But when (1.6) is not strengthened and we have (1.7) as stated, then different ideas are needed. We use the fading memory, together with integral inequalities, to avoid asking that close relationship between V' and the memory in order to conclude asymptotic stability without reference to boundedness of x'.

In the next section we present a sample of representative theorems for the particular example (1.4). Much of what is done here can also be obtained when $b(t)x^3$ is replaced by $b(t)x^n$ where n is the quotient of odd positive integers. In that case the analysis of asymptotic stability and uniform asymptotic stability would be modified in the inequalities using $(1/p) + (1/q) = 1$, while with our choice we always have $p = q = 2$.

The case of $0 < a < 1$ is also very interesting from the point of view of fading memory. It will require that $h(t)$ grow in the relation (1.6) in order to establish stability. But we then get more than simple stability; even when V' is zero, we find that we almost have asymptotic stability.

Existence theory may be found in Burton [2, p. 191] and Sawano [10]. In particular, for each $t_0 \geq 0$ and each continuous $\phi : [0, t_0] \to R$ there is a unique solution $x(t, t_0, \phi)$ of (1.4) on an interval $[t_0, \alpha)$; if the solution remains bounded, $\alpha = \infty$. Whenever we refer to an initial function ϕ it will be assumed that ϕ is continuous.

Section 2. A Volterra integro-differential equation: $a > 1$

Let

$$x' = -h(t)x - b(t)x^3 + \int_0^t C(at - s)x(s)ds \tag{2.1}$$

where

$$C \in L^1[0, \infty),\ a > 1, h(t) \geq 0, b(t) \geq b_0 \geq 0. \tag{2.2}$$

The kernel in the integral term, $C(at - s)$, determines the kind of memory of the equation.

For example, let $a > 1$; if C is positive and decreases monotonically to zero, then:

a) at $s = 0$, $C(at)$ is the weight on $x(0)$, a small amount for large t;

b) at $s = t$, the weight on $x(t)$ is $C((a - 1)t)$ which is larger than at $s = 0$; (Compare with the convolution discussion above.)

c) but for bounded x, the memory will gradually fade to zero.

The constant a is the **index of fading memory.**

Our first result contrasts (2.1) with the convolution case in that the integral is essentially a dissipative term. Even with $h = 0$, that integral is eventually dominated by $b(t)x^n$ for $b_0 > 0$ and $n \geq 1$.

THEOREM 1. *Let (2.2) hold with $b_0 > 0$. Then every solution of (2.1) is bounded and tends to zero.*

Proof. As $C \in L^1$ we can find M with

$$\int_0^t |C(at - s)|ds < M.$$

Let $t_0 \geq 0$ be given, $\phi : [0, t_0] \to R$, and $|\phi(t)| < H$ where $-b_0H^3 + MH < 0$. Then $|x(t, t_0, \phi)| < H$ for all $t > t_0$. To prove this, BWOC, suppose that $|x(t)| < H$ for

$t_0 \leq t < t_1$ and $|x(t_1)| = H$. Then define a Liapunov function $V = |x|$ and for $t_0 \leq t < t_1$ we have

$$V' \leq -b(t)|x|^3 + \int_0^t |C(at-s)||x(s)|ds \leq -b_0|x|^3 + MH$$

and so

$$V'(t_1) \leq -b_0 H^3 + MH < 0,$$

a contradiction to $|x(t_1)| = H$. This proves the boundedness of the solution. We want to drive $x(t)$ to zero.

BWOC, if $x(t)$ does not tend to zero, then there is an A which is not zero (let $A > 0$) and a sequence $t_n \to \infty$ such that $x(t_n) \to A$. By making the change $at-s = v$, $-ds = dv$ we have $\int_0^t |C(at-s)|ds = \int_{at-t}^{at} |C(v)|dv \to 0$ so there is a T such that $t \geq T$ implies that

$$-b_0 A^3 + 2H \int_0^t |C(at-s)|ds < -\alpha < 0$$

for some α. Thus, for $t \geq T$ we have $V' \leq -\beta < 0$ for some β whenever $|x(t)| \geq A - \delta$ for some $\delta > 0$. We can not have $|x(t)| \geq A - \delta$ for all large t or $V \to -\infty$. Thus, there is a $t_1 > T$ with $|x(t_1)| < A - \delta$. But there is a $t_2 > t_1$ with $|x(t_2)| = A - \delta$ and $|x(t)| < A - \delta$ on $[t_1, t_2)$. This will give a contradiction just as before. Hence, A does not exist and the solution tends to zero. This proves Theorem 1.

In the convolution case, if C is taken as a general perturbation, it will require that $h(t)$ be greater than a positive constant in order for $-h(t)x$ to dominate the integral term. Here, we see that even if $x' = -h(t)x$ is only stable, that can be sufficient to make (2.1) stable.

THEOREM 2. *Let (2.2) hold and let*

$$2h(t) \geq [1 + (1/a)] \int_{(a-1)t}^{\infty} |C(v)|dv. \tag{2.3}$$

Then the zero solution of (2.1) is stable.

Proof. Define

$$V(t,x_t)=x^2(t)+(1/a)\int_0^t\int_{at-s}^{\infty}|C(u)|du x^2(s)ds. \tag{2.4}$$

Then a calculation yields

$$V'\le 2\,x(t)[-h(t)x-b(t)x^3+\int_0^t C(at-s)x(s)ds]+$$

$$+(1/a)[\int_{(a-1)t}^{\infty}|C(v)|dv]x(t)^2-a\int_0^t|C(at-s)|x^2(s)\,ds].$$

Hence,

$$V'\le -2h(t)x^2(t)-2b(t)x^4+2x(t)\int_0^t C(at-s)x(s)ds+$$

$$+(1/a)[\int_{(a-1)t}^{\infty}|C(v)|dv]x(t)^2-a\int_0^t|C(at-s)|x^2(s)\,ds].$$

But $|2x(t)x(s)|\le x^2(t)+x^2(s)$ and so

$$\begin{aligned}V'\le &-2h(t)x^2(t)-2b(t)x^4+\int_0^t|C(at-s)|x^2(s)ds+\int_0^t|C(at-s)|ds\,x^2(t)\\ &+(1/a)\int_{(a-1)t}^{\infty}|C(v)|dv]x(t)^2-\int_0^t|C(at-s)|x^2(s)\,ds.\end{aligned}$$

Thus,

$$V'\le[-2h(t)+\int_{(a-1)t}^{at}|C(u)|du+(1/a)\int_{(a-1)t}^{\infty}|C(v)|dv]x^2-2b(t)x^4.$$

Hence, by (2.3)

$$V'\le -2b(t)x^4<-2b_0x^4\le 0. \tag{2.5}$$

Let $t_0\ge 0$ and $\epsilon>0$ be given. Then for an undetermined $\delta>0$ and $\|\phi\|<\delta$ we have

$$x^2(t,t_0,\phi)\le V(t,x_t)\le V(t_0,\phi)\le \delta^2+(1/a)\delta^2\int_0^{t_0}\int_{at_0-s}^{\infty}|C(u)|duds<\epsilon^2$$

provided that

$$\delta^2 < \epsilon^2/(1 + (1/a)\int_0^{t_0}\int_{at_0-s}^{\infty}|C(u)|duds).$$

This proves stability.

The next result illustrates how the fading memory is used to show asymptotic stability and relates the derivative of V to the kernel of the Liapunov functional. It is here that the conditions on C change if the exponent in $b(t)x^n$ changes, as mentioned in the introduction.

THEOREM 3. *Let (2.2) hold with $b(t) > 0$, $\int_0^\infty b(s)ds = \infty$, and let (2.3) hold. Suppose also that there is a B such that for each $T > 0$ if $t > T$, then*

$$\int_T^t \left\{\int_{at-s}^{\infty}|C(u)|du\right\}^2 / \, b(s)\, ds < B. \tag{2.6}$$

Then the zero solution of (2.1) is asymptotically stable.

Proof. We have already shown that the solution is stable and we have defined a Liapunov functional in (2.4) with negative derivative in (2.5).

Let $x(t)$ be a solution with $|x(t)| < 1$. It is clear that $b(t)\,x^4(t) \in L^1[0,\infty)$.

Now we want to prove that $V(t) \to 0$. If it does not, then as above there is a $\mu > 0$ such that

$$2\mu < V(t) \le x^2(t) + (1/a)\int_0^t\int_{at-s}^{\infty}|C(u)|du x^2(s)ds$$

If we can find a t_f such that for $t > t_f$ we have

$$(1/a)\int_0^t\int_{at-s}^{\infty}C(u)du\ x^2(s)ds < \mu \tag{2.7}$$

so that for $t > t_f$ we have

$$2\mu < V(t) \le x^2(t) + \mu$$

and $-x^4(t) < -\mu^2$, $-b(t)x^4(t) < -b(t)\mu^2$, a contradiction to $b(t)x^4(t) \in L^1[0,\infty)$.

Let us find a t_f such that for $t > t_f$ condition (2.7) is satisfied.

Without loss of generality, let $\mu < 1$. Let B be that of condition (2.6). Since $b(t)x^4(t) \in L^1$, there would be a t_1 such that for any $T > t_1$ then

$$\int_T^\infty b(s)\, x^4(s)\, ds < \mu^2/4B. \tag{2.8}$$

Fix a $T > t_1$; for all $t > T$ we can write $\int_0^t \int_{at-s}^\infty |C(u)|du\; x^2(s)ds$ as the sum of two integrals

$$\int_0^T \int_{at-s}^\infty |C(u)|du\; x^2(s)ds+ \tag{2.9}$$

$$+\int_T^t \int_{at-s}^\infty |C(u)|du\; x^2(s)ds. \tag{2.10}$$

Since $|x(t)| < 1$, the first integral,(2.9), satisfies

$$\begin{aligned}
&\int_0^T \int_{at-s}^\infty |C(u)|du\; x^2(s)ds \\
&\quad \le \int_0^T \int_{at-T}^\infty |C(u)|du\; ds = T\int_{at-T}^\infty |C(u)|du.
\end{aligned}$$

Also, since $C(t) \in L^1[0,\infty)$, we can pick a t_f (which depends on the fixed T) such that for $t > t_f$, $\int_{at-T}^\infty |C(v)|dv < \mu/2T$.

Thus, (2.9) satisfies

$$\int_0^T \int_{at-s}^\infty |C(u)|du\; x^2(s)ds \le T\int_{at-T}^\infty |C(u)|du < \mu/2.$$

In the second integral, (2.10), we have for $t > t_f$

$$\int_T^t \int_{at-s}^\infty |C(u)|du\; x^2(s)ds = \int_T^t \frac{\int_{at-s}^\infty |C(u)|du}{\sqrt{b(s)}}\;\sqrt{b(s)}x^2(s)ds$$

and so

$$\int_T^t \int_{at-s}^\infty |C(u)|du\; x^2(s)ds \le \left\{\int_T^t \frac{\left\{\int_{at-s}^\infty |C(u)|du\right\}^2}{b(s)}ds\right\}^{\frac{1}{2}} \left\{\int_T^t b(s)x^{2*2}(s)ds\right\}^{\frac{1}{2}}.$$

Thus, by (2.6) and (2.8)

$$\int_T^t \int_{at-s}^\infty |C(u)|du\; x^2(s)ds < \sqrt{B}\left\{\int_T^t b(u)x^4(u)du\right\}^{\frac{1}{2}} < \sqrt{B}\sqrt{\mu^2/4B} = \mu/2.$$

Putting together the two integrals we see that for $t > t_f$

$$\int_0^t \int_{at-s}^\infty |C(u)|duds < \mu/2 + \mu/2 = \mu.$$

As $a > 1$ the proof is complete.

QED.

Remark. Note that when $b(t) \geq b_0 > 0$ and when (2.3) holds, then the growth condition on $C(t)$, (2.6), is all we need for asymptotic stability.

THEOREM 4. *Let (2.2) and (2.3) hold, $b_0 \geq 0$, and suppose there is a $B > 0$ so that*

$$\int_{(a-1)t}^{at} \int_v^\infty |C(u)|du < B \tag{2.11}$$

for all $t > 0$. Then $x = 0$ is uniformly stable.

Proof. For any t_0 we have

$$\int_0^{t_0} \int_{at_0-s}^\infty |C(u)|duds = \int_{at_0-t_0}^{at_0} \int_v^\infty |C(u)|duds < B.$$

Let $\epsilon > 0$ be given and take a $\delta < \epsilon$ such that $\delta^2 + (1/a)\delta^2 B < \epsilon^2$.

Assume that we have $|\phi| < \delta$ on $[0, t_0]$ and let $x(t)$ be the solution $x(t, t_0, \phi)$.

Then,

$$x^2(t) \leq V(t, x_t) \leq V(t_0, x_{t_0}) \leq x^2(t_0) + (1/a)\int_0^{t_0}\int_{at_0-s}^\infty |C(u)|du\phi^2(s)ds.$$
$$\leq \delta^2 + (1/a)\delta^2 \int_0^{t_0}\int_{at_0-s}^\infty |C(u)|duds \leq \delta^2 + (1/a)\delta^2 B < \epsilon^2.$$

Thus for any $|\phi| < \delta$, and any t_0 the solution $x(t, t_0, \phi)$ satisfies $|x(t)| < \epsilon$. QED

REMARK: We can use the same Liapunov functional for all values of a and the proof of uniform stability always proceeds the same way. Here, for $a > 1$ the function

$\int_t^\infty |C(u)|du$ is not quite required to be integrable to infinity. Our condition holds, for example, if this function is $1/(t+1)$. But for $a = 1$ integrability to infinity is the exact requirement of this method. When $0 < a < 1$ we can not find a condition at all for uniform stability. Uniform stability means that the behavior of solutions with similar initial functions, but different starting times, is much the same. And this is consistent with the rapidly fading memory with $a > 1$. But for $a < 1$ more and more of the memory is retained. In fact, upon change of variable, the limits on the memory become $(a-1)t$ to at. If C is even and if $|C(t)|$ decreases monotonically as t increases, then that interval always includes the largest part of C. Thus, there is increasingly more weight on the initial function so that different behavior must be expected as t_0 increases.

THEOREM 5. *Let (2.2) and (2.3) hold, $b_0 > 0$, and let $\int_t^\infty |C(u)|du \in L^1[0,\infty)$. Then the zero solution of (2.1) is uniformly asymptotically stable.*

Proof. The zero solution is uniformly stable by the previous theorem.

Let us prove uniform asympototic stability. For $\epsilon = 1$, find the δ of uniform stability. Let $\gamma > 0$ be given. We will find $T > 0$ such that $[t_0 \geq 0, |\phi(t)| < \delta$ on $[0, t_0], t \geq t_0 + T]$ implies that $|x(t, t_0, \phi)| < \gamma$.

In the following, $x(t)$ will denote any solution $x(t, t_0, \phi)$ described above and $V(t)$ will denote $V(t, x_t)$.

Since $V'(t) \leq 0$, if we find a t_f such that

$$V(t_f) < \gamma^2 \tag{2.12}$$

then

$$x^2(t) \leq V(t) \leq V(t_f) < \gamma^2$$

for all $t \geq t_f$. We will now find a T so that for any such solution, there will be a $t_f \in [t_0, t_0 + T]$.

Since $\int_v^\infty |C(u)|du \in L^1$, there is a T_1 such that for all $T > T_1$,

$$\int_{(a-1)T}^\infty \int_v^\infty |C(u)|du\, dv \leq a\gamma^2/3.$$

Thus for $t \geq T$ we have

$$\int_T^t \int_{at-s}^{\infty} |C(u)|du\, ds = \int_{at-t}^{at-T} \int_v^{\infty} |C(u)|du\, dv$$

$$\leq \int_{(a-1)t}^{\infty} \int_v^{\infty} |C(u)|du\, dv \leq a\gamma^2/3. \tag{2.13}$$

Fix a $T_2 > T_1$; for all $t > T_2$ we we can write $\int_0^t \int_{at-s}^{\infty} |C(u)|du\, x^2(s)ds$ as the sum of two integrals

$$\int_0^{T_2} \int_{at-s}^{\infty} |C(u)|du\, x^2(s)ds+ \tag{2.14}$$

$$+\int_{T_2}^t \int_{at-s}^{\infty} |C(u)|du\, x^2(s)ds. \tag{2.15}$$

Since $|x(t)| < 1$, the first integral, (2.14), satisfies

$$\int_0^{T_2} \int_{at-s}^{\infty} |C(u)|du\, x^2(s)ds \leq \int_0^{T_2} \int_{at-T_2}^{\infty} |C(u)|du\, ds$$

$$\leq T_2 \int_{at-T_2}^{\infty} |C(u)|du.$$

Also, since $C(t) \in L^1[0, \infty)$, we can pick a T_3 (which depends on the fixed T_2) such that for $t > T_3$ then

$$\int_{at-T_2}^{\infty} |C(v)|dv < a\gamma^2/3T_2. \tag{2.16}$$

By (2.16), the first integral, (2.14), satisfies

$$\int_0^{T_2} \int_{at-s}^{\infty} |C(u)|du\, x^2(s)ds \leq T_2 \int_{at-T_2}^{\infty} |C(u)|du < a\gamma^2/3.$$

In the second integral, (2.15), because of (2.13), we have for $t > T_3$ that

$$\int_{T_2}^t \int_{at-s}^{\infty} |C(u)|du\, x^2(s)ds \leq \int_{T_2}^t \int_{at-s}^{\infty} |C(u)|\, duds \leq a\gamma^2/3.$$

Notice that we have

$$V(t_0) \leq 1 + (1/a) \int_0^{t_0} \int_{at_0-s}^{\infty} |C(u)|duds \leq 1 + (B/a)$$

where $B = \int_0^\infty \int_v^\infty |C(u)| du dv$. It is clear that $x^4(t) \in L^1$. Furthermore, there is a $T_4 > 0$ such that in all intervals of length T_4 there is at least one point t_i such that $|x(t_i)|^2 < \gamma^2/3$. (T_4 depends on the γ but does not depend on the particular $x(t)$.) Indeed, if for all $t > t_1 \geq t_0$, $|x(t)|^2 \geq \gamma^2/3$ then $V(t_1) \leq V(t_0)$ and

$$V(t) \leq V(t_1) - \int_{t_1}^{t} 2b_0 x^4(t) \leq V(t_0) - 2b_0 \left(\gamma^2/3\right)^2 (t - t_1) \leq 1 + (B/a) - 2b_0 \left(\gamma^2/3\right)^2 (t - t_1)$$

so that for $t - t_1 > T_4 \stackrel{def}{=} [1 + (B/a)]/2b_0 \left(\gamma^2/3\right)^2$ we would have $V(t) < 0$, a contradiction.

Let us define $T = T_3 + T_4$. Then for $t > t_0 + T_3$ the two integrals are small, and there is a $t_f \in [t_0 + T_3, t_0 + T_3 + T_4]$ such that $|x(t_f)|^2 < \gamma^2/3$. We have

$$V(t_f) \leq x^2(t_f) + (1/a) \int_0^{t_f} \int_{at_f - s}^{\infty} C(u) du \; x^2(s) ds < \gamma^2/3 + \gamma^2/3 + \gamma^2/3 = \gamma^2.$$

Therefore $x^2(t) \leq V(t) \leq V(t_f) \leq \gamma^2$ so that $|x(t)| \leq \gamma$.

QED.

Example of a function $C(u)$ that satisfies conditions (2.6) and (2.11)

Let $C(u) = 1/(1+u)^\lambda$ which is in L^1 if $\lambda > 1$. In the following we will neglect constants of integration.

We have

$$\int_v^\infty C(u) du = \frac{-1}{\lambda - 1} (1 + u)^{(1-\lambda)} |_v^\infty = \frac{1}{\lambda - 1} (1 + v)^{(1-\lambda)}$$

and

$$\begin{aligned} \int^t \left\{ \int_v^\infty |C(u)| du \right\}^2 dv &= \int^t \left\{ \frac{1}{\lambda - 1} (1 + v)^{(1-\lambda)} \right\}^2 dv \\ &= \frac{1}{(\lambda - 1)^2 (3 - 2\lambda)} (1 + t)^{3 - 2\lambda} \end{aligned}$$

which is bounded for all $t > 0$ as long as $\lambda > 3/2$, and so condition (2.6) in Theorem 3 is satisfied.

Also we have

$$\int_t^\infty \int_v^\infty |C(u)|dudv = \frac{1}{(\lambda-1)(\lambda-2)} \frac{1}{(1+t)^{(\lambda-2)}}.$$

For $\lambda > 2$, by making t large, the expression can be made as small as we want. Therefore the integral conditions in Theorems 4 and 5 are satisfied.

Let us take $b(t) = \dfrac{1}{1+t} > 0$. Then $\int \frac{1}{1+s} ds = \ln(1+t)$ and $\int_0^t b(s)\, ds$ is unbounded. The condition (2.6) is, in this case,

$$\int \frac{\left\{\int_{at-s}^\infty |C(u)|du\right\}^2}{b(s)} ds = \frac{1}{(\lambda-1)^2} \int \frac{(1+s)}{(1+at-s)^{(2\lambda-2)}} ds.$$

The integral is bounded for $\lambda > 2$. Indeed,

$$\begin{aligned} \frac{(1+s)}{(1+at-s)^{(2\lambda-2)}} &= -\frac{(1+at-s)}{(1+at-s)^{(2\lambda-2)}} + \frac{(2+at)}{(1+at-s)^{(2\lambda-2)}} \\ &= -\frac{1}{(1+at-s)^{(2\lambda-3)}} + \frac{(2+at)}{(1+at-s)^{(2\lambda-2)}} \end{aligned}$$

and the primitive of the first term is, for $0 \le s \le t$,

$$\int \frac{1}{(1+at-s)^{(2\lambda-3)}} ds = \frac{-1}{(4-2\lambda)} \frac{1}{(1+at-s)^{(2\lambda-4)}}.$$

The primitive of the second term is

$$\int \frac{1}{(1+at-s)^{(2\lambda-2)}} ds = \frac{-1}{(3-2\lambda)} \frac{1}{(1+at-s)^{(2\lambda-3)}}.$$

For $\lambda > 2$, both primitves are bounded functions.

For example, for $\lambda = 3$ we have, for $t > T > 1$

$$\begin{aligned} \int_T^t \frac{(1+s)}{(1+at-s)^{(2\times 3-2)}} ds &= \frac{1}{2} \frac{(at+2)}{(1+at-t)^2} - \frac{1}{(1+at-t)} \\ &\quad - \frac{1}{2} \frac{(at+2)}{(1+at-T)^2} + \frac{1}{(1+at-T)} \\ &\le \frac{1}{2} \frac{(a+2/t)}{(1/t+a-T/t)^2} + \frac{1}{(1+at-T)} \\ &< \frac{1}{2} \frac{a}{a^2} + 1 = \frac{1}{2a} + 1 = B. \end{aligned}$$

REMARK. In this example:

(i) For $\lambda > 1$ and $b_0 > 0$, we have boundedness and solutions converging to zero.

(ii) For $\lambda > 1$ and (2.3) we have stability.

(iii) For $\lambda > 3/2$, $b_0 > 0$ and (2.3), we have (2.6) and, by Theorem 3, asymptotic stability.

(iv) For $\lambda > 2$, $b_0 > 0$ and (2.3), the condition $\int_v^\infty |C(u)|du \in L^1$ obtains and Theorem 5 yields uniform asymptotic stability.

(v) For $\lambda > 2$, $b(t) = \dfrac{1}{1+t}$ and (2.3), then (2.6) obtains and Theorem 3 yields asymptotic stability.

Section 3. Consequences of Varying a

We are now coming to a point where we can begin to understand differences in the behavior of

$$x' = -h(t)x - b(t)x^3 + \int_0^t C(t-s)x(s)ds, \tag{3.1}$$

the convolution case,

$$x' = -h(t)x - b(t)x^3 + \int_0^t C(at-s)x(s)ds, a > 1, \tag{3.2}$$

and

$$x' = -h(t)x - b(t)x^3 + \int_0^t C(at-s)x(s)ds, \tag{3.3}$$

where we assume for (3.3) that

$$C \in L^1[0,\infty), 0 < a < 1, h(t) > 0, b(t) \geq b_0 \geq 0 \tag{3.4}$$

holds.

In (3.1) and (3.2) we need $C : [0,\infty) \to R$, while (3.3) requires $C : R \to R$. In all cases, the integral represents the memory of x' concerning the past position of x. It is a weighted average with $C(at-s)$ being the weight assigned to x at time s. In every stability result we discuss it is assumed that $\int_0^\infty |C(t)|dt < \infty$. To develop a picture of what is going on, let $C(0) = -1$ and $C(t)$ tend to 0 as $t \to \infty$. At $s = t$ we have $C(at - t)x(t)$ so in (3.1) x' always remembers perfectly the present value of x, while x' in (3.2) gradually fails to perceive the present value of x. But x' in (3.3) can perfectly perceive $x(t)$ if $C(t) \to -1$ as $t \to -\infty$, a condition which will be allowed for stability if $h(t)$ grows like t, but will not be allowed for AS.

In all cases, for a fixed $s < t$, as $t \to \infty$ we have $C(at - s)x(s) \to 0$ and so all past positions of x are gradually forgotten. This is to be expected for AS since an interesting example of Seifert [11] shows that asymptotic stability will generally require that the past position of x be forgotten with time.

We now look at some of the details.

THEOREM 6. *Let* (3.4) *hold with*

$$2h(t) \geq [1 + (1/a)] \int_{(a-1)t}^{\infty} |C(v)| dv. \tag{3.5}$$

Then the zero solution of (3.3) *is stable.*

Proof. We follow exactly the proof of Theorem 2 and define

$$V(t, x_t) = x^2(t) + (1/a) \int_0^t \int_{at-s}^{\infty} |C(u)| du x^2(s) ds. \tag{3.6}$$

Since the computations do not involve the range of a we have

$$V' \leq -2b(t)x^4 \leq -2b_0 x^4 \leq 0, \tag{3.7}$$

just as before. All the details are the same as in the proof of Theorem 2.

But now something interesting happens which we did not see in the earlier examples. The integral in the Liapunov functional becomes unbounded whenever $|x(t)|$ is bounded strictly away from zero. In this case we say that the kernel expands relative to zero, as discussed in Burton [1; p. 175]. Thus, if (3.5) holds, then $V' \leq 0$ so V is bounded along a solution. This means that the integral in V is bounded along a solution. And that means that $x(t)$ must stay near 0 most of the time EVEN WHEN $b_0 = 0$. Therefore, the integral in V plays a similar role as a negative definite derivative on V. Under these conditions, even with $b_0 = 0$ we find that $\int_{at}^{t} x^2(s) ds$ is bounded as $t \to \infty$. This is less than $x \in L^2$ and less than AS, but it is akin to both of those properties.

THEOREM 7. *Under the same conditions as in Theorem 6, if*

$$\int_0^\infty |C(s)|ds = M > 0$$

then there is a constant K with $M\int_{at}^t x^2(s)ds < K$.

Proof. We have shown the stability. Let $x(t) = x(t, t_0, \phi)$ be a fixed solution of (3.3) with $|x(t)| < 1$ so that $V'(t, x_t) \leq 0$ and, hence, $V(t, x_t) \leq V(t_0, \phi)$. Thus, both terms in V are bounded and so we have

$$\int_0^t \int_{at-s}^\infty |C(v)|dvx^2(s)ds < K$$

for some $K > 0$. Now

$$\begin{aligned} K > &\int_0^t \int_{at-s}^\infty |C(v)|dvx^2(s)ds \\ \geq &\int_{at}^t \int_{at-s}^\infty |C(v)|dvx^2(s)ds \\ \geq &\int_{at}^t \int_0^\infty |C(v)|dvx^2(s)ds \\ \geq &M\int_{at}^t x^2(s)ds. \end{aligned}$$

REMARK. In the last two results we have left C fairly unrestricted for $t < 0$. But if we want to prove AS it seems that we need to put a similar condition on C for $t < 0$ as we had for $t > 0$. If we do this, then the proof of AS which we used for $a > 1$ can be employed also for $a < 1$ by using one simple trick. One of our stated goals was to obtain asymptotic stability from our Liapunov relations even when the derivative of V is unrelated to the upper bound on V. And we can do that here just as before. However, to shorten the details we will ask that $b_0 \geq 0$, but strengthen (3.5). We will use a technique that is independent of that close relation.

THEOREM 8. *Let* (3.4) *hold and suppose that there is an $\alpha > 0$ with*

$$2h(t) \geq [1 + (1/a)]\int_{(a-1)t}^\infty |C(v)|dv + \alpha. \tag{3.8}$$

Suppose also that

$$\int_{-\infty}^{\infty} |C(u)|du = H < \infty. \tag{3.9}$$

Then the zero solution of (3.3) is asymptotically stable.

Proof. We define V in (3.6) and obtain

$$V' \leq -\alpha x^2(t). \tag{3.10}$$

Let $x(t)$ be a fixed solution. As in the proof of Theorem 3 it will suffice to prove that

$$\begin{aligned} &\int_0^t \int_{at-s}^{\infty} |C(u)|du x^2(s)ds \\ = &\int_0^t [\int_{at-s}^{t-s} |C(u)|du + \int_{t-s}^{\infty} |C(u)|du] x^2(s)ds \\ &=: I_1 + I_2 \end{aligned}$$

tends to zero as $t \to \infty$. We easily argue that I_2 tends to 0, being the convolution of an L^1 function with a function tending to zero. Let $\epsilon > 0$ be given and fix $T > 0$ so that

$$H \int_T^{\infty} x^2(s)ds < \epsilon/2.$$

Next, take t_2 so large that $t > t_2$ implies that

$$T \int_{at-T}^{t} |C(u)|du < \epsilon/2.$$

Then for $t > t_2$ we have

$$\begin{aligned} I_1 = &\int_0^T \int_{at-s}^{t-s} |C(u)|du x^2(s)ds + \int_T^t \int_{at-s}^{t-s} |C(u)|du x^2(s)ds \\ \leq &T \int_{at-T}^{t} |C(u)|du + H \int_T^t x^2(s)ds \\ \leq &(\epsilon/2) + (\epsilon/2). \end{aligned}$$

This will allow us to complete the proof.

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