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On the asymptotic of solution to the Dirichlet problem for hyperbolic equations in cylinders with edges

Vu Trong Luong[✉ 1] and **Nguyen Thi Hue**[2]

[1]Taybac University, Quyet Tam, Son La, Vietnam
[2]Saodo University, Chi Linh, Hai Duong, Vietnam

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Abstract. In this paper, we consider the Dirichlet problem for second-order hyperbolic equations whose coefficients depend on both time and spatial variables in a cylinder with edges. The asymptotic behaviour of the solution near the edge is studied.

Keywords: asymptotic behaviour of solutions, regularity of solutions, domains with edges, hyperbolic equations.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with the boundary $\partial\Omega$ consisting of two surfaces Γ_1, Γ_2 which intersect along a manifold l_0. Assume that in a neighbourhood of each point of l_0 the set $\overline{\Omega}$ is diffeomorphic to a dihedral angle. Assume that in a neighbourhood of each point of l_0 the set $\overline{\Omega}$ is diffeomorphic to a dihedral angle. For any $P \in l_0$, two half-spaces $T_1(P)$, and $T_2(P)$ tangent to Ω, and a two-dimensional plane $\pi(P)$ normal to l_0 are defined. We denote by $\nu(P)$ the angle in the plane $\pi(P)$ (on the side of Ω) bounded by the rays $R_1 = T_1(P) \cap \pi(P)$, $R_2 = T_2(P) \cap \pi(P)$ and by $\beta(P)$ the aperture of this angle.

In the cylinder $Q = \Omega \times \mathbb{R}$, we study a class of second-order hyperbolic equations. The Dirichlet boundary condition is given on the boundary $\partial Q = \partial\Omega \times \mathbb{R}$ of the cylinder. Our goal is to describe the behaviour of the solutions near the edges. There are some approaches to this issue. For systems or equations dealt with in [13, 6, 12] whose coefficients are independent of the time variable, B. A. Plamenevssky used Fourier transform to reduce the problem to an elliptic one with a parameter. In contrast to [13] and [12], in this paper, we consider equations with coefficients depending on both of time and spatial variables. We develop the approach suggested in [2] to demonstrate the asymptotic representation of the solution of the problem mentioned above near the edges. Furthermore, we investigate the unique solvability of the problem and the regularity of solutions in weighted Sobolev spaces.

✉ Corresponding author. Email: luongvt@utb.edu.vn

Let
$$L(x,t,\partial)u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u,$$
be a second-order partial differential operator, where $a_{ij}(x,t)$, $b_i(x,t)$ and $c(x,t)$ are real-valued functions on Q belonging to $C^\infty(Q)$. Moreover, suppose that $a_{ij} = a_{ji}$, $i,j = 1,\dots,n$, are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in \mathbb{R}$ and
$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \mu_0|\xi|^2 \tag{1.1}$$
for all $\xi \in \mathbb{R}^n\setminus\{0\}$ and $(x,t) \in Q$, $\mu_0 = \text{const} > 0$. In the present work, we consider the Dirichlet problem
$$u_{tt} + L(x,t,\partial)u = f \text{ in } Q, \tag{1.2}$$
$$u|_{\partial Q} = 0. \tag{1.3}$$
Let us introduce some functional spaces used in this paper. We denote by $H^l(\Omega)$ and $\mathring{H}^l(\Omega)$ the usual Sobolev spaces as in [1]. Let $\alpha \in \mathbb{R}$, we introduce the space $H^l_\alpha(\Omega)$ as the weighted Sobolev space of all functions u defined on Ω with the norm
$$\|u\|^2_{H^l_\alpha(\Omega)} = \sum_{0\leq|p|\leq l} \int_\Omega r^{2(\alpha+|p|-l)}|D^p u|^2 + |u|^2\,dx,$$
in which $r^2 = x_1^2 + x_2^2$, and $D^p u = \partial^{|p|}u/\partial x_1^{p_1}\dots\partial x_n^{p_n}$, $p = (p_1,\dots,p_n)$.
By $H^{l,k}(Q,\gamma)$, $H^{l,k}_\alpha(Q,\gamma)$ $(\gamma \in \mathbb{R})$ we denote the weighted Sobolev spaces of functions u defined on Q with the norms
$$\|u\|^2_{H^{l,k}(Q,\gamma)} = \int_Q \left(\sum_{0\leq|p|\leq l}|D^p u|^2 + \sum_{j=1}^{k}|u_{t^j}|^2\right)e^{-\gamma t}\,dx\,dt < +\infty$$
and
$$\|u\|^2_{H^{l,k}_\alpha(Q,\gamma)} = \int_Q \left(\sum_{0\leq|p|\leq l} r^{2(\alpha+|p|-l)}|D^p u|^2 + \sum_{j=0}^{k}|u_{t^j}|^2\right)e^{-\gamma t}\,dx\,dt < +\infty,$$
where $u_{t^k} = \frac{\partial^k u}{\partial t^k}$. The space $\mathring{H}^{l,k}(Q,\gamma)$ is the closure of $C_0^\infty(Q)$ in $H^{l,k}(Q,\gamma)$.
Finally, denote by $H^l_\alpha(Q,\gamma)$ the space of functions $u(x,t)$ defined on Q with the norm
$$\|u\|^2_{H^l_\alpha(Q,\gamma)} = \sum_{0\leq|p|+k\leq l}\int_Q \left(r^{2(\alpha+|p|+k-l)}|D^p u_{t^k}|^2 + |u|^2\right)e^{-\gamma t}\,dx\,dt.$$

Let us denote
$$B(u,v;t) = \sum_{i,j=1}^{n}\int_\Omega a_{ij}(\cdot,t)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx + \sum_{i=1}^{n}\int_\Omega b_i(\cdot,t)\frac{\partial u}{\partial x_i}v\,dx + \int_\Omega c(\cdot,t)uv\,dx,$$
the time-dependent bilinear form. Applying condition (1.1) and similar arguments as the proof of the Gårding inequality, it follows that
$$B(u,u;t) \geq \mu_0\|u\|^2_{H^1(\Omega)} - \lambda_0\|u\|^2_{L_2(\Omega)}, \text{ for a.e. } t \in \mathbb{R} \tag{1.4}$$

for all $u(x,t) \in \mathring{H}^{1,1}(Q,\gamma)$, where $\mu_0 = \text{const} > 0$, $\lambda_0 = \text{const} \geq 0$.

We denote by $(\cdot,\cdot)$ the inner product in $L_2(\Omega)$. Let $f \in H^0_\alpha(Q,\gamma), \gamma > 0$, a function $u(x,t)$ is called the *generalized solution* in $H^{1,1}(Q,\gamma)$ of problem (1.2)–(1.3) if and only if $u(x,t) \in \mathring{H}^{1,1}(Q,\gamma)$, and for any $T > 0$ the equality

$$-\int_{-\infty}^{T}(u_t,v_t)\,dt + \int_{-\infty}^{T} B(u,v;t)\,dt = \int_{-\infty}^{T}(f,v)\,dt, \tag{1.5}$$

holds for all $v \in \mathring{H}^{1,1}(Q,-\gamma), v(x,t) = 0, t \geq T$.

The problems for nonstationary systems or equations in nonsmooth domains also have been investigated in [2, 3, 4, 5, 10, 11], in which the authors obtain results on the regularity of solutions in weighted Sobolev spaces and asymptotic behaviour of solutions in the neighbourhood of the conical points. However, the problems are considered in domains with conical points and with initial conditions. Different from the above-mentioned papers, we consider the problem without initial conditions in domains with edges. The paper is organized as follows. In Section 2, we present the results on the unique solvability of the problem. The regularity of the generalized solution is stated in Section 3. The main result, Theorem 4.2, is given in Section 4.

2 The unique solvability

In this section, we will establish the unique solvability and the regularity in time variable of the solution for problem (1.2)–(1.3). Furthermore, some energy estimates of the solution are proven. The solvability condition of problem (1.2)–(1.3) is: the right hand side f (external force) of (1.2) belongs to $H^0_\alpha(Q,\gamma)$ where γ is a sufficiently large positive number. This condition is more applicable than $f \in L_2(Q,\gamma)$, because $L_2(Q,\gamma) \hookrightarrow H^0_\alpha(Q,\gamma), \alpha \in [0,1]$.

Theorem 2.1. *Suppose that $f, f_t \in H^0_\alpha(Q,\gamma)$, $\gamma > 0$, $\alpha \in [0,1]$, and the coefficients of the operator L satisfy*

$$\sup\{|a_{ij}|, |a_{ijt}|, |b_i|, |c| : i,j = 1,\dots,n; (x,t) \in Q\} \leq \mu, \mu = \text{const}.$$

Then for any $\gamma > \gamma_0 = \frac{2\mu+\epsilon}{\min\{1;2\mu_0-\epsilon\}}, \epsilon \in (0,2\mu_0)$, problem (1.2)–(1.3) has a generalized solution u in the space $\mathring{H}^{1,1}(Q,\gamma)$ and

$$\|u\|^2_{H^{1,1}(Q,\gamma)} \leq C\left(\|f\|^2_{H^0_\alpha(Q,\gamma)} + \|f_t\|^2_{H^0_\alpha(Q,\gamma)}\right), \tag{2.1}$$

where C is a constant independent of u and f.

To prove the theorem, we construct an approximate sequence u^h of solution u of the problem (1.2)–(1.3). It is known that there is a smooth function $\chi(t)$ which is equal to 1 on $[1,+\infty)$, is equal to 0 on $(-\infty,0]$ and assumes values from $[0,1]$ on $[0,1]$ (see [14, Thm. 5.5] for more details). Moreover, we can suppose that all derivatives of $\chi(t)$ are bounded. Let $h \in (-\infty,0]$ be an integer. Set $f^h(x,t) = \chi(t-h)f(x,t)$ then

$$f^h = \begin{cases} f & \text{if } t \geq h+1, \\ 0 & \text{if } t \leq h. \end{cases}$$

Moreover, if $f \in H^0_\alpha(Q,\gamma)$, then $f^h \in H^0_\alpha(Q_h,\gamma)$, $f^h \in H^0_\alpha(Q,\gamma)$, $Q_h = \Omega \times (h,\infty)$, and

$$\|f^h\|^2_{H^0_\alpha(Q_h,\gamma)} = \|f^h\|^2_{H^0_\alpha(Q,\gamma)} \leq \|f\|^2_{H^0_\alpha(Q,\gamma)},$$

where $H^0_\alpha(Q_h,\gamma)$ as $H^0_\alpha(Q,\gamma)$, replacing Q by Q_h. Fixed $f \in H^0_\alpha(Q,\gamma)$, we consider the following problem in the cylinder Q_h:

$$u_{tt} + L(x,t,\partial)u = f^h(x,t) \text{ in } Q_h, \tag{2.2}$$

$$u = 0 \text{ on } S_h = \partial\Omega \times (h,\infty), \tag{2.3}$$

$$u\mid_{t=h}= 0, u_t\mid_{t=h}= 0 \text{ on } \Omega. \tag{2.4}$$

This is the initial boundary value problem for hyperbolic equations in cylinders Q_h. A function $u = u(x,t)$ is called the generalized solution in the space $H^{1,1}(Q_h,\gamma)$ of problem (2.2)–(2.4) if and only if $u \in \mathring{H}^{1,1}(Q_h,\gamma)$, $u(x,h) = 0$, and for any $T > 0$ the equality

$$-\int_h^T (u_t, v_t)\,dt + \int_h^T B(u,v;t)\,dt = \int_h^T (f,v)\,dt, \tag{2.5}$$

holds for all $v \in \mathring{H}^{1,1}(Q_h,-\gamma)$, $v(x,t) = 0$, $t \geq T$.

Lemma 2.2. *Suppose that the assumption of Theorem 2.1 is satisfied. For any h fixed, there exists a solution u^h in the space $\mathring{H}^{1,1}(Q_h,\gamma)$ of the problem (2.2)–(2.4) and the following estimate holds*

$$\|u^h\|^2_{H^{1,1}(Q_h,\gamma)} \leq C\left(\|f^h\|^2_{H^0_\alpha(Q_h,\gamma)} + \|f^h_t\|^2_{H^0_\alpha(Q_h,\gamma)}\right), \tag{2.6}$$

where C is a constant independent of h.

Proof. We will prove the existence by Galerkin's approximating method. Let $\{\omega_k(x)\}_{k=1}^\infty$ be an orthogonal basis of $\mathring{H}^1(\Omega)$ which is orthonormal in $L_2(\Omega)$. Put

$$u^N(x,t) = \sum_{k=1}^N C_k^N(t)\omega_k(x)$$

where $C_k^N(t)$, $k = 1,\ldots,N$, is the solution of the following ordinary differential system:

$$(u^N_{tt},\omega_k) + B(u^N,\omega_k;t) = (f^h,\omega_k),\ k = 1,\ldots,N, \tag{2.7}$$

with the initial conditions

$$C_k^N(h) = 0,\ C_{kt}^N(h) = 0,\ k = 1,\ldots,N. \tag{2.8}$$

Let us multiply (2.7) by $C^N_{kt}(t)$, then take the sum with respect to k from 1 to N to arrive at

$$(u^N_{tt},u^N_t) + B(u^N,u^N_t;t) = (f^h,u^N_t).$$

Since $(u^N_{tt},u^N_t) = \frac{d}{dt}\left(\frac{1}{2}\|u^N_t\|^2_{L_2(\Omega)}\right)$, we get

$$\frac{d}{dt}\left(\|u^N_t\|^2_{L_2(\Omega)}\right) + 2B(u^N,u^N_t;t) = 2(f^h,u^N_t).$$

Integrating the equality above from h to t we find that

$$\|u^N_t(t)\|^2_{L_2(\Omega)} + 2\int_h^t B(u^N,u^N_t;\tau)\,d\tau = 2\int_h^t (f^h,u^N_t)\,d\tau. \tag{2.9}$$

Let us evaluate the right-hand side of (2.9). Integrating by parts, we have

$$2\int_h^t (f^h, u_t^N)\, d\tau = 2(f^h(t), u^N(t)) - 2\int_h^t (f_t^h, u^N)\, d\tau.$$

By the Cauchy–Schwarz inequality and the Hardy inequality, for an arbitrary positive number $\alpha \in [0,1]$, it follows from the equality above that

$$\begin{aligned} 2\left|\int_h^t (f^h, u_t^N)\, d\tau\right| &\le 2\|r^\alpha f^h\|_{L_2(\Omega)}\|r^{-\alpha}u^N\|_{L_2(\Omega)} + 2\int_h^t \|r^\alpha f_t^h\|_{L_2(\Omega)}\|r^{-\alpha}u^N\|_{L_2(\Omega)}\, d\tau \\ &\le C\|f^h\|_{H_\alpha^0(\Omega)}\|u^N\|_{H^1(\Omega)} + C\int_h^t \|f_t^h\|_{H_\alpha^0(\Omega)}\|u^N\|_{H^1(\Omega)}\, d\tau \\ &\le C(\epsilon)\|f^h\|^2_{H_\alpha^0(\Omega)} + \epsilon\|u^N\|^2_{H^1(\Omega)} + \int_h^t C(\epsilon)\|f_t^h\|^2_{H_\alpha^0(\Omega)} + \epsilon\|u^N\|^2_{H^1(\Omega)}\, d\tau, \end{aligned} \tag{2.10}$$

where $\epsilon > 0$ and $C(\epsilon)$ is a constant independent of N, h.

We consider the second term in the left-hand side of (2.9), we can write

$$\begin{aligned} B(u^N, u_t^N; t) &= \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial u^N}{\partial x_j}\frac{\partial u_t^N}{\partial x_i}\, dx + \sum_{i=1}^n \int_\Omega b_i \frac{\partial u^N}{\partial x_i} u_t^N\, dx + \int_\Omega c u^N u_t^N\, dx \\ &=: B_1 + B_2. \end{aligned} \tag{2.11}$$

It is easy to see that

$$B_1 = \frac{d}{dt}\Big(\frac{1}{2}A[u^N, u^N, t]\Big) - \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ijt}\frac{\partial u^N}{\partial x_j}\frac{\partial u^N}{\partial x_i}\, dx, \tag{2.12}$$

for the symmetric bilinear form

$$A[u^N, u^N, t] = \int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial u^N}{\partial x_j}\frac{\partial u^N}{\partial x_i}\, dx.$$

The equality (2.12) implies

$$\int_h^t B_1\, d\tau \ge \mu_0\|u^N(t)\|^2_{H^1(\Omega)} - \mu/2\int_h^t \|u^N\|^2_{H^1(\Omega)}\, d\tau, \tag{2.13}$$

and we also note

$$\left|\int_h^t B_2\, d\tau\right| \le \mu/2\Big(\int_h^t \|u^N\|^2_{H^1(\Omega)} + \|u_t^N\|^2_{L_2(\Omega)}\, d\tau\Big). \tag{2.14}$$

Combining estimates (2.10), (2.13) and (2.14), we obtain

$$\begin{aligned} \|u_t^N(t)\|^2_{L_2(\Omega)} + \|u^N(t)\|^2_{H^1(\Omega)} \le{}& \frac{2\mu+\epsilon}{\min\{1; 2\mu_0 - \epsilon\}}\int_h^t \|u_t^N\|^2_{L_2(\Omega)} + \|u^N\|^2_{H^1(\Omega)}\, d\tau \\ &+ C\left(\|f^h\|^2_{H_\alpha^0(\Omega)} + \int_h^t \|f_t^h\|^2_{H_\alpha^0(\Omega)}\, d\tau\right), \end{aligned} \tag{2.15}$$

where we used (2.12) and $0 < \epsilon < 2\mu_0$.

Thus the Gronwall–Bellman inequality yields the estimate

$$\begin{aligned}
&\|u_t^N(t)\|^2_{L_2(\Omega)} + \|u^N(t)\|^2_{H^1(\Omega)} \\
&\quad\le C\left(\|f^h(t)\|^2_{H^0_\alpha(\Omega)} + \int_h^t \|f_t^h(\tau)\|^2_{H^0_\alpha(\Omega)}\,d\tau\right) \\
&\qquad + C\gamma_0\int_h^t e^{\gamma_0(t-\tau)}\left(\|f^h(\tau)\|^2_{H^0_\alpha(\Omega)} + \int_h^\tau \|f_t^h(s)\|^2_{H^0_\alpha(\Omega)}ds\right)d\tau,
\end{aligned} \tag{2.16}$$

where $\gamma_0 = \frac{2\mu+\epsilon}{\min\{1;2\mu_0-\epsilon\}}$, $\epsilon \in (0, 2\mu_0)$. Now multiplying both sides of this inequality by $e^{-\gamma t}$, $\gamma > \gamma_0$, then integrating them with respect to t from h to ∞, we get

$$\begin{aligned}
\|u^N\|^2_{H^{1,1}(Q_h,\gamma)} &\le C\left(\int_h^\infty e^{-\gamma t}\|f^h(t)\|^2_{H^0_\alpha(\Omega)}\,dt + \int_h^\infty e^{-\gamma t}\int_h^t \|f_t^h(\tau)\|^2_{H^0_\alpha(\Omega)}\,d\tau\,dt\right) \\
&\quad + C\gamma_0\int_h^\infty e^{-\gamma t}\int_h^t e^{\gamma_0(t-\tau)}\|f^h(\tau)\|^2_{H^0_\alpha(\Omega)}\,d\tau\,dt \\
&\quad + C\gamma_0\int_h^\infty e^{-\gamma t}\int_h^t e^{\gamma_0(t-\tau)}\int_h^\tau \|f_t^h(s)\|^2_{H^0_\alpha(\Omega)}\,ds\,d\tau\,dt.
\end{aligned}$$

By the Fubini theorem and $\gamma > \gamma_0$, we conclude that

$$\|u^N\|^2_{H^{1,1}(Q_h,\gamma)} \le C\left(\|f^h\|^2_{H^0_\alpha(Q_h,\gamma)} + \|f_t^h\|^2_{H^0_\alpha(Q_h,\gamma)}\right), \tag{2.17}$$

where C is a constant.

From the inequality (2.17), by standard weak convergence arguments, we can conclude that the sequence $\{u^N\}_{N=1}^\infty$ possesses a subsequence convergent to a function $u^h \in \mathring{H}^{1,1}(Q_h,\gamma)$, which is a generalized solution of problem (2.2)–(2.4). □

Proof of Theorem 2.1: Let k be a integer less than h, denote u^k a generalized solution of the problem (2.2)–(2.4) with the replacement of h by k. We define u^h in the cylinder Q_k by setting $u^h(x,t) = 0$ for $k \le t \le h$. Put $u^{hk} = u^h - u^k$, $f^{hk} = f^h - f^k$, so u^{hk} is the generalized solution of the following problem

$$\begin{gathered}
u_{tt}^{hk} + L(x,t,\partial)u^{hk} = f^{hk}(x,t) \text{ in } Q_k, \\
u^{hk} = 0 \text{ on } S_k,\; u^{hk}\mid_{t=k} = 0, u_t^{hk}\mid_{t=k} = 0 \text{ on } \Omega.
\end{gathered}$$

According to Lemma 2.2, we have

$$\|u^{hk}\|^2_{H^{1,1}(Q,\gamma)} = \|u^{hk}\|^2_{H^{1,1}(Q_k,\gamma)} \le C(\|f^h - f^k\|^2_{H^0_\alpha(Q_k,\gamma)} + \|f_t^h - f_t^k\|^2_{H^0_\alpha(Q_k,\gamma)}).$$

It is easily seen that

$$\begin{aligned}
\|f^h - f^k\|^2_{H^0_\alpha(Q,\gamma)} = \|f^h - f^k\|^2_{H^0_\alpha(Q_k,\gamma)} &= \int_k^{h+1} e^{-\gamma t}\|f^h - f^k\|^2_{H^0_\alpha(\Omega)}\,dt \\
&\le 2\int_k^{h+1} e^{-\gamma t}\|f\|^2_{H^0_\alpha(\Omega)}\,dt.
\end{aligned}$$

As $f \in H^0_\alpha(Q,\gamma)$, we have

$$\int_k^{h+1} e^{-\gamma t}\|f\|^2_{H^0_\alpha(\Omega)}\,dt \to 0,\; h,k \to -\infty.$$

Therefore,

$$\lim \|f^h - f^k\|^2_{H^0_\alpha(Q,\gamma)} = 0,\ h,k \to -\infty.$$

Repeating this argument, we get

$$\|f^h_t - f^k_t\|^2_{H^0_\alpha(Q,\gamma)} \to 0,\ h,k \to -\infty.$$

This shows that $\{u^h\}$ is a Cauchy sequence in $\mathring{H}^{1,1}(Q,\gamma)$. Hence, $\{u^h\}$ is convergent to u in $\mathring{H}^{1,1}(Q,\gamma)$. Since u^h is a generalized solution of problem (2.2)–(2.4), for any $T > 0$, we have that the equality

$$-\int_h^T (u_{ht}, v_t)\, dt + \int_h^T B(u_h, v; t)\, dt = \int_h^T (f^h, v)\, dt, \tag{2.18}$$

holds for all $v \in \mathring{H}^{1,1}(Q,-\gamma)$, $v(x,t) = 0,\ t \geq T$. Using (2.18) when $h \to -\infty$, we obtain (1.5). It means that u is a generalized solution of the problem (1.2)–(1.3). Using (2.6), we get

$$\|u^h\|^2_{H^{1,1}(Q,\gamma)} \leq C\left(\|f\|^2_{H^0_\alpha(Q,\gamma)} + \|f_t\|^2_{H^0_\alpha(Q,\gamma)}\right).$$

From this inequality, sending $h \to -\infty$, we get (2.1). The proof of the theorem is completed.

Theorem 2.3. *If $\gamma > 0$ and $|a_{ijt}|, |b_i|, |b_{ix_i}|, |c| \leq \mu_1 e^{2\gamma t}$, a.e. $(x,t) \in Q, \mu_1 = \mathrm{const}$, then problem (1.2)–(1.3) has no more than one solution in $\mathring{H}^{1,1}(Q,\gamma)$.*

Proof. It suffices to prove that the only solution of (1.2)–(1.3) with $f \equiv 0$ is $u \equiv 0$. To verify this, for any $T > 0$, fix $0 \leq s \leq T$ and set

$$v(x,t) = \begin{cases} \int_s^t u(x,\tau)\, d\tau, & -\infty < t < s \\ 0, & s \leq t < +\infty. \end{cases}$$

Then $v \in \mathring{H}^{1,1}(Q,-\gamma)$ for any $\gamma > 0$. From the definition of generalized solution, we get

$$-\int_{-\infty}^s (u_t, v_t)\, dt + \int_{-\infty}^s B(u, v; t)\, dt = 0.$$

As $v_t = -u(t \leq s)$, so

$$\int_{-\infty}^s (u_t, u)\, dt - \int_{-\infty}^s B(v_t, v; t)\, dt = 0.$$

Therefore,

$$\int_{-\infty}^s \frac{d}{dt}\left(\frac{1}{2}\|u\|^2_{L_2(\Omega)} - \frac{1}{2}A[v,v;t]\right) dt = -\int_{-\infty}^s C(u,v;t) + D(v,v;t)\, dt,$$

where

$$C(u,v;t) = \int_\Omega \sum_{i=1}^n b_i u v_{x_i} + b_{ix_i} u v\, dx - \int_\Omega cuv\, dx$$

and

$$D(v,v;t) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ijt} v_{x_i} v_{x_j}\, dx.$$

Hence,

$$\frac{1}{2}\|u(s)\|^2_{L_2(\Omega)} + \frac{1}{2}\lim_{t\to-\infty} A[v(t), v(t); t] = -\int_{-\infty}^s C(u,v;t) + D(v,v;t)\, dt.$$

Using inequality (1.1) and the Cauchy inequality, we arrive at

$$\|u(s)\|^2_{L_2(\Omega)} + \lim_{t\to-\infty} \|v(t)\|^2_{H^1(\Omega)} \leq C\int_{-\infty}^{s} e^{2\gamma t}\left(\|u(t)\|^2_{L_2(\Omega)} + \|v(t)\|^2_{H^1(\Omega)}\right) dt. \tag{2.19}$$

Let us write

$$w(t) = \int_{-\infty}^{t} u(x,\tau)\, d\tau, t \leq s,$$

then

$$\lim_{t\to-\infty} v(t) = -w(s), v(t) = w(t) - w(s).$$

It follows readily from (2.19) that

$$\|u(s)\|^2_{L_2(\Omega)} + (1 - Ce^{2\gamma s})\|w(s)\|^2_{H^1(\Omega)} \leq C\int_{-\infty}^{s} e^{2\gamma t}\left(\|u(t)\|^2_{L_2(\Omega)} + \|w(t)\|^2_{H^1(\Omega)}\right) dt.$$

Choose T_1 so small that $1 - Ce^{2\gamma T_1} \geq 1/2$, then we have

$$\|u(s)\|^2_{L_2(\Omega)} + \|w(s)\|^2_{H^1(\Omega)} \leq 2C\int_{-\infty}^{s} e^{2\gamma t}\left(\|u(t)\|^2_{L_2(\Omega)} + \|w(t)\|^2_{H^1(\Omega)}\right) dt$$

for all $s \leq T_1$. Consequently the Gronwall inequality implies $u \equiv 0$ on $(-\infty, T_1]$. In view of the uniqueness of the solution of the problem with initial condition (2.2)–(2.4), $u \equiv 0$ holds on $\mathbb{R}$. □

By the same arguments as in the proof of Theorem 2.1 together with inductive arguments (cf. [2]), we obtain the following theorem:

Theorem 2.4. *Let $h \in \mathbb{N}^*$, assume that*

(i) $\sup\{|a_{ijt^{k+1}}|, |b_{it^k}|, |c_{t^k}| : i,j = 1,\dots,n; (x,t) \in \overline{Q}, k \leq h\} \leq \mu,$

(ii) $f_{t^k} \in H^0_\alpha(Q,\gamma_0),\ k \leq h+1.$

Then for an arbitrary real number γ satisfying $\gamma > \gamma_0$, the generalized solution $u \in \mathring{H}^{1,1}(Q,\gamma)$ of problem (1.2)–(1.3) has derivatives with respect to t up to order h belonging to $\mathring{H}^{1,1}(Q,\gamma)$, and

$$\|u_{t^h}\|^2_{H^{1,1}(Q,\gamma)} \leq C\sum_{j=0}^{h+1} \|f_{t^j}\|^2_{H^0_\alpha(Q,\gamma)} \tag{2.20}$$

where C is a constant independent of u and f.

3 Regularity of the generalized solution

We reduce the operator with coefficients at $P \in l_0,\ t \in \mathbb{R}$

$$L_0^{(2)} := -\sum_{i,j=1}^{2} a_{ij}(P,t)\frac{\partial^2}{\partial x_i \partial x_j},$$

to its canonical form. After a linear transformation of coordinates, it can be realized that via this reduction T_1 and T_2 go over into hyperplanes T_1' and T_2', respectively. The angle β at (P,t) is transformed to

$$\omega(P,t) = \arctan\frac{[a_{11}(P,t)a_{22}(P,t) - a^2_{12}(P,t)]^{1/2}}{a_{22}(P,t)\cot\beta - a_{12}(P,t)}.$$

Clearly, the value $\omega(P,t)$ does not depend on the method by which $L_0^{(2)}$ is reduced to its canonical form. The function $\omega(P,t)$ is infinitely differentiable and $\omega(P,t) > 0$. Then, we have the following theorem.

Theorem 3.1. *Let the assumptions of Theorem 2.4 be satisfied for a given positive interger $h+1$. Furthermore, assume $\alpha \in [0,1]$ and $1-\alpha < \frac{\pi}{\omega}$. Then the generalized solution $u \in \mathring{H}^{1,1}(Q,\gamma)$ of problem (1.2)–(1.3) has derivatives with respect to t up to order h, $u_{t^h} \in H_\alpha^{2,0}(Q,\gamma)$ and*

$$\|u_{t^h}\|^2_{H_\alpha^{2,0}(Q,\gamma)} \le C\sum_{k=0}^{h+2} \|f_{t^k}\|^2_{H_\alpha^0(Q,\gamma)},$$

where C is a constant independent of u, f.

Proof. We will prove the assertion of the theorem by induction on h. Firstly, we consider the case $h=0$. It is easy to see that $u(\cdot,t_0)$, $t_0 \in \mathbb{R}$, is the generalized solution of the following problem:

$$L(x,t_0,\partial)u = F(x,t_0) \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0,$$

where $F(x,t_0) = f(x,t_0) - u_{tt}(x,t_0) \in H_\alpha^0(\Omega)$. From [8, Thm. 2], we get $u(x,t_0) \in H_\alpha^2(\Omega)$ and

$$\begin{aligned}\|u(\cdot,t_0)\|^2_{H_\alpha^2(\Omega)} &\le C\left[\|F(\cdot,t_0)\|^2_{H_\alpha^0(\Omega)} + \|u\|^2_{L_2(\Omega)}\right]\\ &\le C\left[\|f\|_{H_\alpha^0(\Omega)} + \|u_{tt}\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)}\right].\end{aligned} \tag{3.1}$$

Multiplying both sides of the above inequality by $e^{-t_0\gamma}$, then integrating with respect to t_0 on $\mathbb{R}$ and using estimates from Theorem 2.4, we obtain

$$\|u\|^2_{H_\alpha^{2,0}(Q,\gamma)} \le C\sum_{k=0}^{2} \|f_{t^k}\|^2_{H_\alpha^0(Q,\gamma)}.$$

Thus, the assertion of the theorem is valid for $h=0$.

Next, suppose that the assertion of the theorem is true for $h-1$, we will prove that it also holds for $k=h$. Differentiating h times both sides of (1.2) with respect to t, we find

$$L_{u_{t^h}} = f_{t^h} - u_{t^{h+2}} - \sum_{k=0}^{h-1}\binom{k}{h} L_{t^{h-k}}u_{t^k} := F. \tag{3.2}$$

By using the assumptions of the theorem and the inductive assumption, we obtain $f_{t^h} \in H_\alpha^0(\Omega)$, $u_{t^{h+2}} \in L_2(\Omega) \subset H_\alpha^0(\Omega)$, $\alpha \in [0,1]$, and $u_{t^k} \in H_\alpha^0(\Omega)$, $k \le h-1$. Therefore, $F(\cdot,t_0) \in H_\alpha^0(\Omega)$, a.e. $t_0 \in \mathbb{R}$. By using [8, Thm. 2] again, we get $u_{t^h} \in H_\alpha^2(\Omega)$ for a.e. $t_0 \in \mathbb{R}$ and

$$\|u_{t^h}\|^2_{H_\alpha^2(\Omega)} \le C\|F\|^2_{H_\alpha^0(\Omega)} \le C\left[\|f_{t^h}\|^2_{H_\alpha^0(\Omega)} + \|u_{t^{h+2}}\|^2_{L_2(\Omega)} + \sum_{k=0}^{h-1}\|u_{t^k}\|^2_{L_2(\Omega)}\right]. \tag{3.3}$$

Multiplying both sides of (3.3) by $e^{-t_0\gamma}$, then integrating with respect to t_0 on $\mathbb{R}$ and using estimates from Theorem 2.4 again, we obtain

$$\|u_{t^h}\|^2_{H_\alpha^{2,0}(Q,\gamma)} \le C\sum_{k=0}^{h+2} \|f_{t^k}\|^2_{H_\alpha^0(Q,\gamma)}.$$

It means that the assertion of the theorem is valid for $k=h$. The proof is completed. □

From now on, let the assumption of Theorem 2.4 be satisfied for a given positive integer $h+1$.

Theorem 3.2. *Assume that $f_{t^k} \in H^h_\alpha(Q,\gamma_0)$, $k \le 2$, and*

$$h+1-\alpha < \frac{\pi}{\omega},\ \alpha \in [0,1].$$

Then the generalized solution u of problem (1.2)–(1.3) belongs to $H^{2+h}_\alpha(Q,\gamma)$. In addition,

$$\|u\|^2_{H^{2+h}_\alpha(Q,\gamma)} \le C\sum_{k=0}^{2} \|f_{t^k}\|^2_{H^h_\alpha(Q,\gamma)}, \tag{3.4}$$

where C is a constant independent of u, f.

Proof. We have

$$\begin{aligned}
\|u\|^2_{H^2_\alpha(Q,\gamma)} &= \sum_{|p|+k\le 2}\int_Q \left(r^{2(\alpha+|p|+k-2)}|D^p u_{t^k}|^2 + |u|^2\right)e^{-\gamma t}\,dx\,dt\\
&= \sum_{|p|\le 2}\int_Q \left(r^{2(\alpha+|p|-2)}|D^p u|^2 + |u|^2\right)e^{-\gamma t}\,dx\,dt\\
&\quad + \sum_{|p|\le 1}\int_Q \left(r^{2(\alpha+|p|-1)}|D^p u_t|^2\right)e^{-\gamma t}\,dx\,dt + \int_Q r^{2\alpha}|u_{tt}|^2 e^{-\gamma t}\,dx\,dt\\
&= \|u\|^2_{H^{2,0}_\alpha(Q,\gamma)} + \|u_t\|^2_{H^{1,0}_\alpha(Q,\gamma)} + \|u_{tt}\|^2_{H^0_\alpha(Q,\gamma)}\\
&= \sum_{k=0}^{2}\|u_{t^k}\|^2_{H^{2-k,0}_\alpha(Q,\gamma)}.
\end{aligned}$$

Therefore, $u \in H^2_\alpha(Q,\gamma)$ by Theorem 3.1 and Theorem 2.4. Moreover, we have

$$\|u\|^2_{H^2_\alpha(Q,\gamma)} = \sum_{k=0}^{2}\|u_{t^k}\|^2_{H^{2-k,0}_\alpha(Q,\gamma)} \le C\sum_{k=0}^{2}\|f_{t^k}\|^2_{H^0_\alpha(Q,\gamma)}.$$

Thus, the theorem is valid for $h=0$. Suppose that the assertion of the theorem is true for $h-1$, we will prove that it also holds for $k=h$. It is easy to see that

$$\|u\|^2_{H^{2+h}_\alpha(Q,\gamma)} = \sum_{k=0}^{h+2}\|u_{t^k}\|^2_{H^{h+2-k,0}_\alpha(Q,\gamma)}. \tag{3.5}$$

Hence, we will prove that

$$u_{t^k} \in H^{h+2-k,0}_\alpha(Q,\gamma), k=0,\dots,h \tag{3.6}$$

and

$$\|u_{t^k}\|^2_{H^{h+2-k,0}_\alpha(Q,\gamma)} \le C\sum_{s=0}^{k+2}\|f_{t^s}\|^2_{H^{h-k,0}_\alpha(Q,\gamma)},\ k\le h. \tag{3.7}$$

By using Theorem 3.1, this holds for $k=h$. Suppose that it holds for $k=h,h-1,\dots,j+1$, we will prove that it holds for $k=j$. Returning one more time to (3.2) $(h=j)$, we get

$$Lu_{t_j} = f_{t_j} - u_{t^{j+2}} - \sum_{k=0}^{j-1}\binom{j}{k}Lu_{t^{j-k}}u_{t^k} := F_1.$$

Notice that $f_{t^j} \in H^h_\alpha(\Omega) \subset H^{h-j}_\alpha(\Omega)$ for a.e. $t \in \mathbb{R}$ (by the assumptions of theorem), $u_{t^{j+2}} \in H^{h-j}_\alpha(\Omega)$ for a.e. $t \in \mathbb{R}$ (by (3.6) which holds for $k = j+2$), $u_{t^k} \in H^{h+1-k}_\alpha(\Omega) \subset H^{h-j}_\alpha(\Omega), k = 0, \dots, j-1$, (by the valid inductive assumption for $k = h-1$).

It implies that $F_1(\cdot, t) \in H^{h-j}_\alpha(\Omega)$, a.e. $t \in \mathbb{R}$. From [8, Thm. 2], we obtain

$$u_{t^j} \in H^{h+2-j}_\alpha(\Omega), \text{ a.e. } t \in \mathbb{R}$$

and

$$\begin{aligned} \|u_{t^j}\|^2_{H^{h+2-j}_\alpha(\Omega)} &\le C\|F_1\|_{H^{h-j}_\alpha(\Omega)} \\ &\le C\Big[\|f_{t^j}\|^2_{H^{h-j}_\alpha(\Omega)} + \|u_{t^{j+2}}\|^2_{H^{h-j}_\alpha(\Omega)} + \sum_{k=0}^{j-1} \|u_{t^k}\|^2_{H^{h-j}_\alpha(\Omega)}\Big]. \end{aligned} \tag{3.8}$$

Multiplying both sides of (3.8) by $e^{-\gamma t}$, then integrating on $\mathbb{R}$, we arrive at

$$\|u_{t^j}\|^2_{H^{h+2-j,0}_\alpha(Q,\gamma)} \le C\sum_{k=0}^{j+2} \|f_{t^k}\|^2_{H^{h-j,0}_\alpha(Q,\gamma)}.$$

It means that (3.6) and (3.7) are true for $k = j$. Thus, (3.6) and (3.7) hold for all $k = 0, 1, \dots, h$. From (3.5), we get

$$\|u\|^2_{H^{h+2}_\alpha(Q,\gamma)} \le C\sum_{k=0}^{2} \|f_{t^k}\|^2_{H^h_\alpha(Q,\gamma)}.$$

The proof is completed. □

4 Asymptotics of the solution in a neighbourhood of the edge

According to the previous section, if $k+1-\alpha < \frac{\pi}{\omega}, \alpha \in [0,1]$ and $f, f_t, f_{tt} \in H^k_\alpha(Q,\gamma)$, then the solution $u \in H^{2+k}_\alpha(Q,\gamma)$. Now we study the solution in the case $\frac{\pi}{\omega} < k+1-\alpha$. In this case, we can find an asymptotic representation of u in the neighbourhood of $l_0 : x_1 = x_2 = 0$. In this section, we use the notations $y_1 = x_1$, $y_2 = x_2$, $y = (y_1, y_2)$, $z_i = x_{i+2}$, $z = (z_1, \dots, z_{n-2})$, $r = x_1^2 + x_2^2$ and (r, φ) for the polar coordinates of the point $y = (y_1, y_2) \in \Omega_z = \Omega \cap \{z = \text{const}\}$. Set $Q_z = \Omega_z \times \mathbb{R}$. To begin with, we present the following lemma.

Lemma 4.1. *Suppose that the following hypotheses are satisfied:*

(i) $f_{t^s} \in H^{k,0}_\alpha(Q,\gamma)$, $s \le h$.

(ii) $k - \alpha < \frac{\pi}{\omega} < k+1-\alpha < \frac{2\pi}{\omega}$, $\alpha \in [0,1]$.

Let u be the solution of (1.2)–(1.3), $u \equiv 0$ outside some neighbourhood of l_0. Then

$$u(y,z,t) = c(z,t)r^{\frac{\pi}{\omega}}\Phi(z,\varphi,t) + u_1(y,z,t)$$

where $c_{t^s} \in L_2(Q_z,\gamma)$, $(u_1)_{t^s} \in H^{k+2,0}_\alpha(Q_z,\gamma)$, $s \le h$ *and* $\Phi \in C^\infty$.

Proof. Using (i), we get from Theorem 3.2 that $u_{t^s} \in H^{k+1,0}_\alpha(Q,\gamma)$, $s \le h$, particularly, $u_z \in H^k_\alpha(\Omega)$, $u_{ttz} \in H^k_\alpha(\Omega)$ for a.e. $t \in \mathbb{R}$. On the other hand, we have

$$Lu_z = f_z - u_{ttz} - L_z u =: f_1,$$

where $L_z = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ijz}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} b_{iz}\frac{\partial}{\partial x_i} + c_z$ and $f_1 \in H_{\alpha}^{k-1}(\Omega)$ for a.e. $t \in \mathbb{R}$. Using Theorem 2 in [8], we obtain $u_z \in H_{\alpha}^{k+1}(\Omega)$ for a.e. $t \in \mathbb{R}$. Therefore, equality (1.2) can be rewritten as follows

$$L_0^{(2)} u = F \tag{4.1}$$

where $F \in H_{\alpha}^{k}(\Omega)$ for a.e. $t \in \mathbb{R}$. Now we can apply Theorem 1' in [9] to get

$$u(y,z,t) = c(z,t) r^{\frac{\pi}{\omega}} \Phi(z,\varphi,t) + u_1(y,z,t) \tag{4.2}$$

where $\Phi \in C^{\infty}, u_1 \in H_{\alpha}^{k+2}(\Omega_z)$ and

$$|c(z,t)|^2 \le C\left(\|F\|^2_{H_{\alpha}^{k}(\Omega_z)} + \|u\|^2_{L_2(\Omega_z)}\right),$$
$$\|u_1\|^2_{H_{\alpha}^{k+2}(\Omega_z)} \le C\left(\|F\|^2_{H_{\alpha}^{k}(\Omega_z)} + \|u\|^2_{L_2(\Omega_z)}\right), \; z \in l_0, \; t \in \mathbb{R}.$$

Therefore, $c \in L_2(Q_z,\gamma),\ u_1 \in H_{\alpha}^{k+2,0}(Q_z,\gamma)$. It implies that this lemma holds for $h = 0$. Suppose the lemma is true for $h-1$, we will prove that it also holds for $k = h$. Denoting $v = u_{t^h}$ and differentiating both sides of (4.1) with respect to t, h times, we find

$$L_0^{(2)} v = F_{t^h} - \sum_{j=1}^{h} \binom{h}{j} L_{0t^j}^{(2)} u_{t^{h-j}}. \tag{4.3}$$

Set $S_0 = r^{\frac{\pi}{\omega}}\Phi$, we have

$$\sum_{j=1}^{h} \binom{h}{j} L_{0t^j}^{(2)} u_{t^{h-j}} = \sum_{j=1}^{h} \binom{h}{j} L_{0t^j}^{(2)} (cS_0)_{t^{h-j}} + \sum_{j=1}^{h} \binom{h}{j} L_{0t^j}^{(2)} (u_1)_{t^{h-j}}. \tag{4.4}$$

The first term of the right hand side of (4.4) can be rewritten in the following form:

$$\begin{aligned}
\sum_{j=1}^{h} \binom{h}{j} L_{0t^j}^{(2)} (cS_0)_{t^{h-j}} &= \sum_{j=1}^{h} \binom{h}{j} L_{0t^j}^{(2)} \left(\sum_{i=0}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} S_{0t^i}\right) \\
&= \sum_{j=1}^{h} \binom{h}{j} \sum_{i=0}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} L_{0t^j}^{(2)} S_{0t^i} \\
&= \sum_{j=1}^{h} \binom{h}{j} \sum_{i=1}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} L_{0t^j}^{(2)} S_{0t^i} + \sum_{j=1}^{h} \binom{h}{j} c_{t^{h-j}} L_{0t^j}^{(2)} S_0 \\
&= \sum_{j=0}^{h} \binom{h}{j} \sum_{i=1}^{h-j} \binom{h-j}{i} c_{t^{h-j-i}} L_{0t^j}^{(2)} S_{0t^i} \\
&\quad + \sum_{j=1}^{h} \binom{h}{j} c_{t^{h-j}} L_{0t^j}^{(2)} S_0 - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L_0^{(2)} S_{0t^i} \\
&= F_1 - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L_0^{(2)} S_{0t^i}.
\end{aligned}$$

From the assumptions of the lemma and the inductive assumptions, we find that $F_1 \in H_{\alpha}^{k}(\Omega_z)$. Hence, from equality (4.4), we obtain

$$\sum_{j=1}^{h} \binom{h}{j} L_{0t^j}^{(2)} u_{t^{h-j}} = F_2 - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L_0^{(2)} S_{0t^i},$$

where $F_2 \in H^k_\alpha(\Omega_z)$. Employing the equality above, we get from (4.3) that

$$L_0^{(2)} v = F_3 + \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} L_0^{(2)} S_{0t^i}. \tag{4.5}$$

Thus,

$$L_0^{(2)} \left(v - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} S_{0t^i} \right) = F_3,$$

where $F_3 \in H^k_\alpha(\Omega_z)$. Analogously to the case $h = 0$, we get

$$v - \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} S_{0t^i} = d(z,t) S_0 + u_2(y,z,t).$$

Therefore,

$$u_{t^h} = \sum_{i=1}^{h} \binom{h}{i} c_{t^{h-i}} S_{0t^i} + d(z,t) S_0 + u_2(y,z,t), \tag{4.6}$$

where $d \in L_2(Q_z, \gamma), u_2 \in H^{k+2,0}_\alpha(Q_z, \gamma)$. By the assumption (i), it implies that u is differentiable with respect to t. Then, it can be seen that the functions $c(z,\cdot)$, $u_1(y,z,\cdot)$ are differentiable with respect to t. Combining (4.2) and (4.6), we conclude that

$$c_{t^h} = d \in L_2(Q_z, \gamma),\ (u_1)_{t^h} = u_2 \in H^{k+2,0}_\alpha(Q_z, \gamma).$$

The proof is completed. $\square$

Now, we come to the main results.

Theorem 4.2. *Suppose that hypotheses of Lemma 4.1 are satisfied. Then the following representation holds*

$$u(x,t) = c(x,t) r^{\frac{\pi}{\omega}} \Phi(z,\varphi,t) + u_1(x,t),$$

where $c_{t^s} \in H^{k+2,0}_{\alpha+\frac{\pi}{\omega}}(Q,\gamma)$, $(u_1)_{t^s} \in H^{k+2,0}_\alpha(Q,\gamma)$, $s \le h$, *and* $\Phi \in C^\infty$.

Proof. From Lemma 4.1, we have the following representation:

$$u(x,t) = c(z,t) r^{\frac{\pi}{\omega}} \Phi(z,t,\varphi) + u_1(x,t), \tag{4.7}$$

where $c_{t^s} \in L_2(Q_z,\gamma)$, $(u_1)_{t^s} \in H^{k+2,0}_\alpha(Q_z,\gamma)$, $s \le h$. Considering the differential operator

$$D_1 = \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \varphi},$$

in coordinates x_1, x_2, it has the form

$$D_1 = \Phi_1 \frac{\partial}{\partial x_1} + \Phi_2 \frac{\partial}{\partial x_2}$$

here Φ_1, Φ_2 are infinite differentiable functions. From representation (4.7), we find

$$D_1 u = \frac{\pi}{\omega} c(z,t) r^{\frac{\pi}{\omega}-1} \Phi_3(z,t,\varphi) + D_1 u_1(x,t). \tag{4.8}$$

Moreover,

$$u_1 \in H^{k+2,0}_\alpha(Q_z,\gamma),\ \int_{Q_z} \left(r^{2\alpha} \frac{\partial^{k+2} u_1}{\partial x_1^{k_1} \partial x_2^{k_2}} \right) e^{-\gamma t}\, dx_1\, dx_2\, dt < \infty. \tag{4.9}$$

By similar arguments to the proof of Lemma 4.1, we obtain

$$u_z, u \in H_\alpha^{k+1,0}(Q,\gamma).$$

Therefore,

$$(D_1 u)_z \in H_\alpha^{k,0}(Q,\gamma) \text{ and } \int_Q \left(r^{2(\alpha-k)}|(D_1u)_z|\right)e^{-\gamma t}\,dx\,dt \le \int_Q \left(r^{2\alpha}|f|^2\right)e^{-\gamma_0 t}\,dx\,dt < \infty. \quad (4.10)$$

Combining (4.9) and (4.10), we get

$$r^{-\frac{\pi}{\omega}+1}D_1u_1 \in H^{k+1,0}_{\alpha+\frac{\pi}{\omega}-1}(Q_z,\gamma)$$
$$r^{-\frac{\pi}{\omega}+1}(D_1u)_z \in H^{k+1,0}_{\alpha+\frac{\pi}{\omega}-1}(Q,\gamma).$$

On the other hand, equality (4.8) yields the equality

$$(r^{-\frac{\pi}{\omega}+1}D_1u)_y = (r^{-\frac{\pi}{\omega}+1}D_1u_1)_y.$$

Consequently,

$$r^{-\frac{\pi}{\omega}+1}D_1u \in H^{k+1,0}_{\alpha+\frac{\pi}{\omega}-1}(Q,\gamma). \quad (4.11)$$

Now write

$$c_1(x,t) = \frac{\omega}{\pi}r^{-\frac{\pi}{\omega}+1}D_1u\Phi_3,$$

then (4.11) implies $c_1 \in H^{k+1,0}_{\alpha+\frac{\pi}{\omega}-1}(Q,\gamma)$. From Lemma 2 in [9], we obtain that there is $\widetilde{c}_1 \in H^{k+2,0}_{\alpha+\frac{\pi}{\omega}}(Q,\gamma)$, $(\widetilde{c}_1)_{t^s} \in H^{k+2,0}_{\alpha+\frac{\pi}{\omega}}(Q,\gamma)$, $s \le h$ such that

$$\int_Q \left(|c_1-\widetilde{c}_1|^2 r^{2(\alpha+\frac{\pi}{\omega}-k-2)}\right)e^{-\gamma t}\,dx\,dt < \infty. \quad (4.12)$$

Utilizing (4.8) and the fact that $u_1 \in H_\alpha^{k+2,0}(Q_z,\gamma)$, we get

$$\int_Q \left(|c-c_1|^2 r^{2(\alpha+\frac{\pi}{\omega}-k-2)}\right)e^{-\gamma t}\,dx\,dt < \infty. \quad (4.13)$$

We can rewrite representation (4.7) in the following form

$$\begin{aligned} u(x,t) &= \widetilde{c}_1(x,t)r^{\frac{\pi}{\omega}}\Phi(z,\varphi,t) + [c-\widetilde{c}_1]r^{\frac{\pi}{\omega}}\Phi(z,\varphi,t) + u_1(x,t) \\ &= \widetilde{c}_1(x,t)r^{\frac{\pi}{\omega}}\Phi(z,\varphi,t) + u_2(x,t), \end{aligned} \quad (4.14)$$

where $u_2 \in H_\alpha^{k+2,0}(Q_z,\gamma)$, $z \in l_0$. Since u is differentiable with respect to z and $u_2 = u - \widetilde{c}_1 r^{\frac{\pi}{\omega}}\Phi$, we get that $Lu_2 \in H_\alpha^k(\Omega)$ and $\int_\Omega r^{2(\alpha-k-2)}|u_2|dx < \infty$. From Lemma 2 in [7], we obtain $u_2 \in H_\alpha^{k+2}(\Omega)$. Hence, $u_2 \in H_\alpha^{k+2,0}(Q,\gamma)$.

To prove $(u_2)_{t^s} \in H_\alpha^{k+2,0}(Q,\gamma)$, $s \le h$, we can use arguments analogous to the proof of Lemma 4.1. The theorem is proved. □

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