



Infinitely many solutions for a class of quasilinear two-point boundary value systems

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Abstract. The existence of infinitely many solutions for a class of Dirichlet quasilinear elliptic systems is established. The approach is based on variational methods.

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1 Introduction

The aim of this paper is to investigate the existence of infinitely many weak solutions for the following doubly eigenvalue quasilinear two-point boundary value system

$$\begin{cases} -(p_i - 1)|u_i'(x)|^{p_i-2}u_i''(x) = (\lambda F_{u_i}(x, u_1, \dots, u_n) + \mu G_{u_i}(x, u_1, \dots, u_n))h_i(x, u_i') & \text{in } (a, b) \\ u_i(a) = u_i(b) = 0, \quad 1 \leq i \leq n, \end{cases} \quad (D_{\lambda, \mu})$$

where $p_i > 1$ for $1 \leq i \leq n$, $\lambda > 0$, $\mu \geq 0$ are real numbers, $a, b \in \mathbb{R}$ with $a < b$, $F: [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that $F \in C^1([a, b] \times \mathbb{R}^n)$ and $F(x, 0, \dots, 0) = 0$ for all $x \in [a, b]$, $G: [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function such that $G \in C^1([a, b] \times \mathbb{R}^n)$ and $G(x, 0, \dots, 0) = 0$ for all $x \in [a, b]$ and $h_i: [a, b] \times \mathbb{R} \rightarrow]0, +\infty[$ is a bounded and continuous function with $m_i := \inf_{(x,t) \in [a,b] \times \mathbb{R}} h_i(x, t) > 0$. Here, F_{u_i} and G_{u_i} denote respectively the partial derivatives of F and G with respect to u_i for $1 \leq i \leq n$.

On the existence of multiple solutions for two-point boundary value problems of the type $(D_{\lambda, \mu})$, several results are known when $n = 1$, see for example [2, 3, 18, 23] and the references cited therein. Existence results for nonlinear elliptic systems with Dirichlet boundary conditions have also received a great deal of interest in recent years; see, for instance, the papers [11, 13, 19, 20, 22].

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For a discussion about the existence of infinitely many solutions for boundary value problems, using Ricceri's variational principle [26] and its variants ([5, Theorem 2.1] and [24, Theorem 1.1]) we refer the reader to the papers [1, 4, 6–10, 12, 14–17, 21, 27]. We also refer the reader, for instance, to the papers [25, 28] where the existence of infinitely many solutions for boundary value problems has been studied by using different approach.

In the present paper, employing a smooth version of [5, Theorem 2.1], under some hypotheses on the behavior of the nonlinear terms at infinity, under conditions on the potential of h_i for $1 \leq i \leq n$, we determine the exact collections of the parameters λ and μ in which the system $(D_{\lambda, \mu})$ admits infinitely many weak solutions (Theorem 3.1). We also list some consequences of Theorem 3.1 and one example. Here, due to the facts, no symmetric assumptions are requested on the nonlinearities, the infinitely many solutions are local minima of the energy functionals associated to the problem, and the nonlinearities depend on the term $h_i(x, u_i')$ being h_i a continuous bounded function and u_i' is the weak derivative of the component u_i of the weak solution $u = (u_1, u_2, \dots, u_n)$ of the system $(D_{\lambda, \mu})$, the application of variational methods to investigate the system $(D_{\lambda, \mu})$ is not standard.

A special case of our main result is the following theorem.

Theorem 1.1. *Let $f_1, f_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ be two positive $C^0(\mathbb{R}^2)$ -functions such that the differential 1-form $w := f_1(\sigma, \nu) d\sigma + f_2(\sigma, \nu) d\nu$ is integrable and let F be a primitive of w such that $F(0, 0) = 0$. Fix two integers $p, q > 2$, with $p \leq q$, and assume that*

$$\liminf_{\xi \rightarrow +\infty} \frac{F(\xi, \xi)}{\xi^p} = 0 \quad \text{and} \quad \limsup_{\xi \rightarrow +\infty} \frac{F(\xi, \xi)}{\xi^q} = +\infty.$$

Then, for every nonnegative arbitrary $C^1(\mathbb{R}^2)$ function $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying the condition

$$G_\infty^* := \limsup_{\xi \rightarrow +\infty} \frac{G(\xi, \xi)}{\xi^p} < +\infty,$$

and for every $\mu \in [0, \mu_G[$ where

$$\mu_G := \frac{1}{\left[\frac{p}{2^p} + \left(\frac{q}{2^p} \right)^{\frac{p}{q}} \right] G_\infty^*},$$

the system

$$\begin{cases} -(p-1)|u_1'(x)|^{p-2}u_1''(x) = f_1(u_1, u_2) + \mu G_{u_1}(u_1, u_2) & \text{in } (0, 1), \\ -(q-1)|u_2'(x)|^{q-2}u_2''(x) = f_2(u_1, u_2) + \mu G_{u_2}(u_1, u_2) & \text{in } (0, 1), \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0, \end{cases}$$

admits a sequence of pairwise distinct positive weak solutions.

2 Preliminaries

Our main tool to investigate the existence of infinitely many weak solutions for the system $(D_{\lambda, \mu})$ is a smooth version of Theorem 2.1 of [5] that we recall here.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi: X \rightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r])} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r])} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

and

$$\gamma := \liminf_{r \rightarrow +\infty} \varphi(r), \quad \delta := \liminf_{r \rightarrow (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r])$ admits a global minimum, which is a critical point (local minimum) of I_λ in X .

(b) If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds:

either

(b₁) I_λ possesses a global minimum,

or

(b₂) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n \rightarrow +\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds:

either

(c₁) there is a global minimum of Φ which is a local minimum of I_λ ,

or

(c₂) there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ .

Let X be the Cartesian product of n Sobolev spaces $W_0^{1,p_1}([a, b]), W_0^{1,p_2}([a, b]), \dots, W_0^{1,p_n}([a, b])$, i.e., $X = \prod_{i=1}^n W_0^{1,p_i}([a, b])$, equipped with the norm

$$\|(u_1, u_2, \dots, u_n)\| = \sum_{i=1}^n \|u'_i\|_{p_i}, \quad \text{for every } (u_1, u_2, \dots, u_n) \in X,$$

where

$$\|u'_i\|_{p_i} = \left(\int_a^b |u'_i(x)|^{p_i} dx \right)^{1/p_i}, \quad i = 1, \dots, n.$$

Since $p_i > 1$ for $i = 1, \dots, n$, X is compactly embedded in $(C([a, b]))^n$.

In the sequel, let $\underline{p} = \min\{p_i; 1 \leq i \leq n\}$, $\bar{p} = \max\{p_i; 1 \leq i \leq n\}$,

$$m_i := \inf_{(x,t) \in [a,b] \times \mathbb{R}} h_i(x, t) > 0 \quad \text{for } 1 \leq i \leq n,$$

$$M_i := \sup_{(x,t) \in [a,b] \times \mathbb{R}} h_i(x, t) \quad \text{for } 1 \leq i \leq n,$$

$\bar{M} := \max\{M_i; 1 \leq i \leq n\}$ and $\underline{m} := \min\{m_i; 1 \leq i \leq n\}$. Then, $\bar{M} \geq M_i \geq m_i > \underline{m} > 0$ for each $i = 1, \dots, n$.

In order to apply Theorem 2.1 we set

$$H_i(x, t) = \int_0^t \left(\int_0^\tau \frac{(p_i - 1)|\delta|^{p_i-2}}{h_i(x, \delta)} d\delta \right) d\tau,$$

for $1 \leq i \leq n$ and for all $(x, t) \in [a, b] \times \mathbb{R}$, and consider the functionals $\Phi, \Psi: X \rightarrow \mathbb{R}$ for each $u = (u_1, \dots, u_n) \in X$, as follows

$$\Phi(u) = \sum_{i=1}^n \int_a^b H_i(x, u'_i(x)) dx,$$

and

$$\Psi(u) = \int_a^b F(x, u_1(x), \dots, u_n(x)) dx + \frac{\mu}{\lambda} \int_a^b G(x, u_1(x), \dots, u_n(x)) dx.$$

It is well known that Ψ is a Gâteaux differentiable functional and sequentially weakly lower semicontinuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx + \frac{\mu}{\lambda} \int_a^b \sum_{i=1}^n G_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx,$$

for every $v = (v_1, \dots, v_n) \in X$, and $\Psi': X \rightarrow X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u_1, \dots, u_n)(v_1, \dots, v_n) = \sum_{i=1}^n \int_a^b \left(\int_0^{u'_i(x)} \frac{(p_i - 1)|\tau|^{p_i-2}}{h_i(x, \tau)} d\tau \right) v'_i(x) dx,$$

for every $v = (v_1, \dots, v_n) \in X$. Furthermore, Φ is sequentially weakly lower semicontinuous.

By a classical solution of the system $(D_{\lambda, \mu})$, we mean a function $u = (u_1, \dots, u_n)$ such that, for $i = 1, \dots, n$, $u_i \in C^1[a, b]$, $u'_i \in AC[a, b]$, and u satisfies $(D_{\lambda, \mu})$. We say that a function $u = (u_1, \dots, u_n) \in X$ is a weak solution of the system $(D_{\lambda, \mu})$ if

$$\begin{aligned} \sum_{i=1}^n \int_a^b \left(\int_0^{u'_i(x)} \frac{(p_i - 1)|\tau|^{p_i-2}}{h_i(x, \tau)} d\tau \right) v'_i(x) dx - \lambda \int_a^b \sum_{i=1}^n F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx \\ - \mu \int_a^b \sum_{i=1}^n G_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx = 0, \end{aligned}$$

for every $v = (v_1, \dots, v_n) \in X$.

3 Main results

In this section, we present our main results. To be precise, we establish an existence result of infinitely many solutions to problem $(D_{\lambda, \mu})$.

For all $\xi > 0$ we denote by $K(\xi)$ the set

$$\left\{ (t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n |t_i| \leq \xi \right\}. \quad (3.1)$$

Let

$$p^* = \begin{cases} \bar{p} & \text{if } b - a \geq 1, \\ \underline{p} & \text{if } 0 < b - a < 1. \end{cases}$$

Put

$$A := \liminf_{\xi \rightarrow +\infty} \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\xi)} F(x, t_1, \dots, t_n) dx}{\xi^p},$$

$$B := \limsup_{(t_1, \dots, t_n) \rightarrow \infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, t_1, \dots, t_n) dx}{\sum_{i=1}^n D_i(t_i)},$$

where

$$D_i(t_i) := \int_a^{a+\frac{b-a}{4}} H_i \left(x, \frac{t_i(p_i - 1)(x - a)^{p_i - 2}}{\left(\frac{b-a}{4}\right)^{p_i - 1}} \right) dx + \int_{b-\frac{b-a}{4}}^b H_i \left(x, -\frac{t_i(p_i - 1)(b - x)^{p_i - 2}}{\left(\frac{b-a}{4}\right)^{p_i - 1}} \right) dx,$$

for each $t_i \in \mathbb{R}$, for all $i = 1, \dots, n$,

$$\lambda_1 := \frac{1}{B} \quad \text{and} \quad \lambda_2 := \frac{\left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^* - 1} \bar{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p}{A}. \quad (3.2)$$

Theorem 3.1. Assume that

(A1) $F(x, t_1, \dots, t_n) \geq 0$ for each

$$x \in \left(\left[a, a + \frac{b-a}{4} \right] \cup \left[b - \frac{b-a}{4}, b \right] \right), \quad t_i \in \mathbb{R}, \quad \forall i = 1, \dots, n,$$

(A2)

$$\liminf_{\xi \rightarrow +\infty} \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\xi)} F(x, t_1, \dots, t_n) dx}{\xi^p}$$

$$< \left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^* - 1} \bar{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p \limsup_{(t_1, \dots, t_n) \rightarrow \infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, t_1, \dots, t_n) dx}{\sum_{i=1}^n D_i(t_i)}.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ and for every nonnegative arbitrary function $G: [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$ which is measurable in $[a, b]$ and of class $C^1(\mathbb{R}^n)$ satisfying the condition

$$G_\infty := \limsup_{\xi \rightarrow +\infty} \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\xi)} G(x, t_1, \dots, t_n) dx}{\xi^p} < +\infty, \quad (3.3)$$

and for every $\mu \in [0, \mu_{G, \lambda}[$ where

$$\mu_{G,\lambda} := \frac{1 - \lambda \left(\sum_{i=1}^n \left(\frac{p_i(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p A}{\left(\sum_{i=1}^n \left(\frac{p_i(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p G_\infty},$$

the system $(D_{\lambda,\mu})$ has an unbounded sequence of weak solutions in X .

Proof. Our aim is to apply Theorem 2.1. To this end, fix λ, μ and G satisfying our assumptions. Let X be the Sobolev space $\prod_{i=1}^n W_0^{1,p_i}([a,b])$. For any $u \in X$, set

$$\Phi(u) = \sum_{i=1}^n \int_a^b H_i(x, u'_i(x)) dx, \quad (3.4)$$

and

$$\Psi(u) = \int_a^b F(x, u_1(x), \dots, u_n(x)) dx + \frac{\mu}{\lambda} \int_a^b G(x, u_1(x), \dots, u_n(x)) dx. \quad (3.5)$$

It is well known that they satisfy all regularity assumptions requested in Theorem 2.1 and that the critical points in X of the functional $I_\lambda = \Phi - \lambda\Psi$ are precisely the weak solutions of problem $(D_{\lambda,\mu})$.

Let $\{\zeta_k\}$ be a real sequence of positive numbers such that $\lim_{k \rightarrow +\infty} \zeta_k = +\infty$, and

$$A = \lim_{k \rightarrow +\infty} \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\zeta_k)} F(x, t_1, \dots, t_n) dx}{\zeta_k^p}.$$

Put

$$S = \left(\sum_{i=1}^n \left(\frac{p_i(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p,$$

and

$$r_k = \frac{\zeta_k^p}{S},$$

for all $k \in \mathbb{N}$. Since $0 < h_i(x, t) \leq \overline{M}$ for each $(x, t) \in [a, b] \times \mathbb{R}$ for $i = 1, \dots, n$, from (3.4) we see that

$$\frac{1}{\overline{M}} \sum_{i=1}^n \frac{\|u'_i\|_{p_i}^{p_i}}{p_i} \leq \Phi(u_1, \dots, u_n) \leq \frac{1}{\underline{m}} \sum_{i=1}^n \frac{\|u'_i\|_{p_i}^{p_i}}{p_i} \quad \text{for all } u = (u_1, \dots, u_n) \in X. \quad (3.6)$$

Taking into account that

$$\max_{x \in [a,b]} |u_i(x)| \leq \frac{(b-a)^{\frac{p_i-1}{p_i}}}{2} \|u'_i\|_{p_i},$$

for each $u_i \in W_0^{1,p_i}([a,b])$ (see [28]), we have

$$\max_{x \in [a,b]} \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq \frac{(b-a)^{p^*-1}}{2^p} \sum_{i=1}^n \frac{\|u'_i\|_{p_i}^{p_i}}{p_i}, \quad (3.7)$$

for each $u = (u_1, u_2, \dots, u_n) \in X$. This, for each $r > 0$, along with (3.6), ensures that

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{ u \in X; \max_{i=1}^n \sum_{i=1}^n \frac{|u_i(x)|^{p_i}}{p_i} \leq \frac{(b-a)^{p^*-1} \overline{M} r}{2^p} \text{ for each } x \in [a, b] \right\}.$$

Therefore, one has

$$\begin{aligned}
 \varphi(r_k) &\leq \frac{\sup_{v \in \Phi^{-1}(-\infty, r_k]} \Psi(v)}{r_k} \\
 &\leq \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\xi_k)} F(x, t_1, \dots, t_n) dx + \frac{\mu}{\lambda} \int_a^b \max_{(t_1, \dots, t_n) \in K(\xi_k)} G(x, t_1, \dots, t_n) dx}{\xi_k^p} \\
 &\leq S \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\xi_k)} F(x, t_1, \dots, t_n) dx}{\xi_k^p} + S \frac{\mu}{\lambda} \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\xi_k)} G(x, t_1, \dots, t_n) dx}{\xi_k^p},
 \end{aligned} \tag{3.8}$$

for all $k \in \mathbb{N}$. Therefore, from assumption (A2) and the condition (3.3) one has

$$\gamma \leq \liminf_{k \rightarrow +\infty} \varphi(r_k) \leq SA + S \frac{\mu}{\lambda} G_\infty < +\infty. \tag{3.9}$$

Now, let $\{(\eta_{i,k})\} \subseteq \mathbb{R}^n$ be positive real sequences such that $\eta_{i,k} > 0$ for all $i = 1, \dots, n$ and for all $k \in \mathbb{N}$, and

$$\lim_{k \rightarrow +\infty} \left(\sum_{i=1}^n \eta_{i,k}^2 \right)^{\frac{1}{2}} = +\infty.$$

Put

$$B := \lim_{k \rightarrow +\infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx}{\sum_{i=1}^n D_i(\eta_{i,k})}. \tag{3.10}$$

Let $\{w_k = (w_{1,k}(x), \dots, w_{n,k}(x))\}$ be a sequence in X defined by

$$w_{i,k}(x) = \begin{cases} \left(\frac{4}{b-a}\right)^{p_i-1} \eta_{i,k} (x-a)^{p_i-1} & \text{if } a \leq x < a + \frac{b-a}{4}, \\ \eta_{i,k} & \text{if } a + \frac{b-a}{4} \leq x \leq b - \frac{b-a}{4}, \\ \left(\frac{4}{b-a}\right)^{p_i-1} \eta_{i,k} (b-x)^{p_i-1} & \text{if } b - \frac{b-a}{4} < x \leq b, \end{cases} \tag{3.11}$$

for each $i = 1, \dots, n$. Clearly $w_k(x) \in \prod_{i=1}^n W_0^{1,p_i}([a, b])$ for each $k \in \mathbb{N}$.

Hence, we have

$$\begin{aligned}
 \Phi(w_k) &= \sum_{i=1}^n \int_a^b H_i(x, w'_{i,k}) dx \\
 &= \sum_{i=1}^n \left[\int_a^{a+\frac{b-a}{4}} H_i\left(x, \frac{\eta_{i,k}(p_i-1)(x-a)^{p_i-2}}{\left(\frac{b-a}{4}\right)^{p_i-1}}\right) dx + \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} H_i(x, 0) dx \right. \\
 &\quad \left. + \int_{b-\frac{b-a}{4}}^b H_i\left(x, -\frac{\eta_{i,k}(p_i-1)(b-x)^{p_i-2}}{\left(\frac{b-a}{4}\right)^{p_i-1}}\right) dx \right] \\
 &= \sum_{i=1}^n D_i(\eta_{i,k}).
 \end{aligned} \tag{3.12}$$

On the other hand, since G is nonnegative and bearing assumption (A1) in mind, from (3.5) one has

$$\begin{aligned}\Psi(w_k) &= \int_a^b F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx + \frac{\mu}{\lambda} \int_a^b G(x, \eta_{1,k}, \dots, \eta_{n,k}) dx \\ &\geq \int_a^b F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx \\ &\geq \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx,\end{aligned}\tag{3.13}$$

and so

$$I_\lambda(w_k) = \Phi(w_k) - \lambda\Psi(w_k) \leq \sum_{i=1}^n D_i(\eta_{i,k}) - \lambda \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx.$$

Now, consider the following cases.

If $B < +\infty$, let $\epsilon \in]0, B - \frac{1}{\lambda}[$. From (3.10), there exists ν_ϵ such that

$$\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx > (B - \epsilon) \sum_{i=1}^n D_i(\eta_{i,k}), \quad \text{for all } k > \nu_\epsilon,$$

and so

$$I_\lambda(w_k) < \sum_{i=1}^n D_i(\eta_{i,k}) - \lambda(B - \epsilon) \sum_{i=1}^n D_i(\eta_{i,k}) = \sum_{i=1}^n D_i(\eta_{i,k}) [1 - \lambda(B - \epsilon)].$$

Since $1 - \lambda(B - \epsilon) < 0$, and taking into account (3.6) and (3.12) one has

$$\lim_{k \rightarrow +\infty} I_\lambda(w_k) = -\infty.$$

If $B = +\infty$, fix $M > \frac{1}{\lambda}$. From (3.10), there exists ν_M such that

$$\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, \eta_{1,k}, \dots, \eta_{n,k}) dx > M \sum_{i=1}^n D_i(\eta_{i,k}), \quad \text{for all } k > \nu_M,$$

and moreover,

$$I_\lambda(w_k) < \sum_{i=1}^n D_i(\eta_{i,k}) [1 - \lambda M].$$

Since $1 - \lambda M < 0$, and arguing as before, we have

$$\lim_{k \rightarrow +\infty} I_\lambda(w_k) = -\infty.$$

Taking into account that

$$\left] \frac{1}{B'}, \frac{S}{A} \left[\subseteq \left] 0, \frac{1}{\gamma} \left[,$$

and that I_λ does not possess a global minimum, from part (b) of Theorem 2.1, there exists an unbounded sequence $\{u_k\}$ of critical points which are the weak solutions of $(D_{\lambda,\mu})$. So, our conclusion is achieved. \square

Proof of Theorem 1.1. Since f_1 and f_2 are positive, then F is nonnegative in \mathbb{R}_+^2 . Moreover, one has that the functions $t_1 \rightarrow F(t_1, t_2)$, $t_2 \in \mathbb{R}$, and $t_2 \rightarrow F(t_1, t_2)$, $t_1 \in \mathbb{R}$ are increasing in \mathbb{R} and, hence, $\max_{(t_1, t_2) \in K(\xi)} F(t_1, t_2) \leq F(\xi, \xi)$ for every $\xi \in \mathbb{R}^+$. Therefore,

$$\liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \max_{(t_1, t_2) \in K(\xi)} F(t_1, t_2) dx}{\xi^p} \leq \liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 F(\xi, \xi) dx}{\xi^p} = \liminf_{\xi \rightarrow +\infty} \frac{F(\xi, \xi)}{\xi^p} = 0.$$

On the other hand, one has

$$h_1(u'_1) = 1 \quad \text{and} \quad h_1(u'_1) = 1.$$

By simple calculations, we see that

$$H_1(t_1) = \frac{|t_1|^p}{p} \quad \text{and} \quad H_2(t_2) = \frac{|t_2|^q}{q}.$$

Moreover,

$$D_1(t_1) = \frac{4^{p-1}|t_1|^p}{p} [(p-1)^{p-2} + (1-p)^{p-2}],$$

and

$$D_2(t_2) = \frac{4^{q-1}|t_2|^q}{q} [(q-1)^{q-2} + (1-q)^{q-2}].$$

Since $p \leq q$, one has

$$\begin{aligned} D_1(t_1) + D_2(t_2) &\leq \frac{4^{q-1}|t_1|^p}{p} [(q-1)^{q-2} + (1+q)^{q-2}] + \frac{4^{q-1}|t_2|^q}{q} [(q-1)^{q-2} + (1+q)^{q-2}] \\ &\leq \frac{4^{q-1}[(q-1)^{q-2} + (1+q)^{q-2}]}{p} (|t_1|^p + |t_2|^q) \\ &\leq \frac{4^{q-1}[(q-1)^{q-2} + (1+q)^{q-2}]}{p} (|t_1|^q + |t_2|^q). \end{aligned}$$

Then

$$\begin{aligned} +\infty &= \frac{p}{4^{q-1}[(q-1)^{q-2} + (1+q)^{q-2}]} \frac{1}{2} \limsup_{\xi \rightarrow +\infty} \frac{F(\xi, \xi)}{\xi^q} \\ &\leq \limsup_{\xi \rightarrow +\infty} \frac{F(\xi, \xi)}{D_1(\xi) + D_2(\xi)} \leq \limsup_{(t_1, t_2) \rightarrow \infty} \frac{F(t_1, t_2)}{D_1(t_1) + D_2(t_2)}. \end{aligned}$$

Now, arguing as before we obtain

$$G_\infty^* = \limsup_{\xi \rightarrow +\infty} \frac{\int_0^1 \max_{(t_1, t_2) \in K(\xi)} G(t_1, t_2) dx}{\xi^p} \leq \limsup_{\xi \rightarrow +\infty} \frac{G(\xi, \xi)}{\xi^p} \leq +\infty.$$

Therefore, since one has also that

$$\begin{aligned} \mu_G &= \frac{1}{\left[\left(\frac{p}{2^p} \right) + \left(\frac{q}{2^p} \right)^{\frac{p}{q}} \right] G_\infty^*}, \\ \lambda_1 &= 0, \end{aligned}$$

and

$$\lambda_2 = +\infty.$$

Theorem 3.1, taking into account the positivity of f and g , ensures the conclusion. \square

We now exhibit an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.2. Put $p_1 = p_2 = 2$, $[a, b] = [0, 1]$ and consider the increasing sequence of positive real numbers given by

$$a_1 = 2, \quad a_{k+1} = ka_k^2 + 2,$$

for every $k \in \mathbb{N}$. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function such that

$$F(t_1, t_2) = \begin{cases} (a_{k+1})^4 e^{-\frac{1}{1-[(t_1-a_{k+1})^2+(t_2-a_{k+1})^2]}+1}} & (t_1, t_2) \in \bigcup_{k \geq 1} B((a_{k+1}, a_{k+1}), 1), \\ 0 & \text{otherwise,} \end{cases}$$

where $B((a_{k+1}, a_{k+1}), 1)$ denotes the open unit ball of center (a_{k+1}, a_{k+1}) and radius 1.

Now, put

$$h_1(y) = h_2(y) = \frac{1}{2 + \cos y},$$

for each $y \in \mathbb{R}$. By simple calculations, we see that

$$H_1(y) = H_2(y) = y^2 - \cos y + 1,$$

for each $y \in \mathbb{R}$, and

$$D_1(t_1) = \frac{16t_1^2 - \cos(4t_1) + 1}{2} \quad \text{and} \quad D_2(t_2) = \frac{16t_2^2 - \cos(4t_2) + 1}{2}.$$

By definition, F is nonnegative and $F(0, 0) = 0$. Further it is a simple matter to verify that $F \in C^1(\mathbb{R}^2)$. We will denote by f_1 and f_2 respectively the partial derivative of F respect to t_1 and t_2 . Now, for every $k \in \mathbb{N}$, the restriction $F(t_1, t_2)|_{B((a_{k+1}, a_{k+1}), 1)}$ attains its maximum in (a_{k+1}, a_{k+1}) and one has $F(a_{k+1}, a_{k+1}) = (a_{k+1})^4$. Clearly

$$\limsup_{(t_1, t_2) \rightarrow \infty} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t_1, t_2) dx}{D_1(t_1) + D_2(t_2)} = \frac{1}{2} \limsup_{(t_1, t_2) \rightarrow \infty} \frac{F(t_1, t_2)}{D_1(t_1) + D_2(t_2)} = +\infty,$$

since

$$\lim_{k \rightarrow +\infty} \frac{F(a_{k+1}, a_{k+1})}{D_1(a_{k+1}) + D_2(a_{k+1})} = \lim_{k \rightarrow +\infty} \frac{a_{k+1}^4}{16a_{k+1}^2 - \cos(4a_{k+1}) + 1} = +\infty.$$

On the other hand, by setting $\zeta_k = a_{k+1} - 1$ for every $k \in \mathbb{N}$, one has

$$\max_{|t_1|+|t_2| \leq \zeta_k} F(t_1, t_2) = a_k^4, \quad \forall k \in \mathbb{N}.$$

Then

$$\lim_{k \rightarrow +\infty} \frac{\max_{|t_1|+|t_2| \leq \zeta_k} F(t_1, t_2)}{(a_{k+1} - 1)^2} = 0,$$

since

$$\liminf_{\xi \rightarrow +\infty} \frac{\int_0^1 \max_{|t_1|+|t_2| \leq \xi} F(t_1, t_2) dx}{\xi^2} = 0.$$

Hence, condition (A2) is provided.

Now, let $G: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function defined by

$$G(t_1, t_2) = 1 - \cos(t_1 t_2).$$

By definition $G \in C^1(\mathbb{R}^2)$ and $G(0, 0) = 0$. For any sequence $\{\rho_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow +\infty} \rho_k = +\infty$, since $|t_1| + |t_2| \leq \rho_k$, one has

$$\max_{|t_1|+|t_2| \leq \rho_k} G(t_1, t_2) \leq 2.$$

Then,

$$0 \leq G_\infty = \limsup_{\xi \rightarrow +\infty} \frac{\int_0^1 \max_{|t_1|+|t_2| \leq \xi} G(t_1, t_2) dx}{\xi^2} \leq 0.$$

All hypotheses of Theorem 3.1 are satisfied. Then for all $(\lambda, \mu) \in]0, +\infty[\times]0, +\infty[$, the system

$$\begin{cases} -u_1''(x) = \left(\lambda f_1(u_1, u_2) + \mu G_{u_1}(u_1, u_2) \right) \frac{1}{2 + \cos(u_1'(x))}, \\ -u_2''(x) = \left(\lambda f_2(u_1, u_2) + \mu G_{u_2}(u_1, u_2) \right) \frac{1}{2 + \cos(u_2'(x))}, \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0, \end{cases}$$

admits a sequence of weak solutions which is unbounded in $W_0^{1,2}([0, 1]) \times W_0^{1,2}([0, 1])$.

Remark 3.3. Under the conditions $A = 0$ and $B = +\infty$, Theorem 3.1 concludes that for every $\lambda > 0$ and for each

$$\mu \in \left[0, \frac{1}{\left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p} G_\infty \right],$$

the system $(D_{\lambda, \mu})$ admits infinitely many weak solutions in X . Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Remark 3.4. Put

$$\widehat{\lambda}_1 = \lambda_1,$$

and

$$\widehat{\lambda}_2 = \frac{1}{\lim_{k \rightarrow +\infty} \frac{\int_a^b \sup_{(t_1, \dots, t_n) \in K(b_k)} F(x, t_1, \dots, t_n) dx - \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, a_{1,k}, \dots, a_{n,k}) dx}{\frac{b_k^p}{\left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p} - \sum_{i=1}^n D_i(a_{i,k})}}.$$

We explicitly observe that assumption (A2) in Theorem 3.1 could be replaced by the following more general condition

(A3) there exist $n + 1$ sequences $\{a_{i,k}\}$ for $i = 1, \dots, n$ and $\{b_k\}$ with

$$\sum_{i=1}^n D_i(a_{i,k}) < \frac{b_k^p}{\left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p},$$

for every $k \in \mathbb{N}$ and $\lim_{k \rightarrow +\infty} b_k = +\infty$ such that

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(b_k)} F(x, t_1, \dots, t_n) dx - \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, a_{1,k}, \dots, a_{n,k}) dx}{\frac{b_k^p}{\left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p} - \sum_{i=1}^n D_i(a_{i,k})} \\ & < \left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p \limsup_{(t_1, \dots, t_n) \rightarrow \infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, t_1, \dots, t_n) dx}{\sum_{i=1}^n D_i(t_i)}, \end{aligned}$$

where $K(b_k) = \{(t_1, \dots, t_n) \mid \sum_{i=1}^n |t_i| \leq b_k\}$ (see (3.1)).

Obviously, from (A3) we obtain (A2), by choosing $a_{i,k} = 0$ for all $k \in \mathbb{N}$. Moreover, if we assume (A3) instead of (A2) and set

$$r_k = \frac{b_k^p}{\left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p},$$

for all $k \in \mathbb{N}$, by the same arguing as inside in Theorem 3.1, we obtain

$$\begin{aligned} \varphi(r_k) &= \inf_{u \in \Phi^{-1}(]-\infty, r_k])} \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty, r_k])} \Psi(v) \right) - \Psi(u)}{r_k - \Phi(u)} \\ &\leq \frac{\sup_{v \in \Phi^{-1}(]-\infty, r_k])} \Psi(v) - \left[\int_a^b F(x, u_1(x), \dots, u_n(x)) dx + \frac{\mu}{\lambda} \int_a^b G(x, u_1(x), \dots, u_n(x)) dx \right]}{r_k - \sum_{i=1}^n \int_a^b H_i(x, u'_i(x)) dx} \\ &\leq \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(b_k)} F(x, t_1, \dots, t_n) dx - \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, a_{1,k}, \dots, a_{n,k}) dx}{\frac{b_k^p}{\left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p} - \sum_{i=1}^n D_i(a_{i,k})}. \end{aligned}$$

We have the same conclusion as in Theorem 3.1 with Λ replaced by $\Lambda' :=]\widehat{\lambda}_1, \widehat{\lambda}_2[$.

Here, we point out two simple consequences of Theorem 3.1.

Corollary 3.5. *Assume that (A1) holds and*

$$(B1) \quad \liminf_{\xi \rightarrow +\infty} \frac{\int_a^b \max_{(t_1, \dots, t_n) \in K(\xi)} F(x, t_1, \dots, t_n) dx}{\xi^p} < \left(\sum_{i=1}^n \left(p_i \frac{(b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p,$$

$$(B2) \quad \limsup_{(t_1, \dots, t_n) \rightarrow \infty} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x, t_1, \dots, t_n) dx}{\sum_{i=1}^n D_i(t_i)} > 1,$$

where $\sum_{i=1}^n D_i(t_i)$ is given as in assumption (A2).

Then, for every nonnegative function $G: [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$ which is measurable in $[a, b]$ and of class $C^1(\mathbb{R}^n)$ satisfying (3.3) and for every $\mu \in [0, \mu_G[$ where

$$\mu_G := \frac{1 - \left(\sum_{i=1}^n \left(\frac{p_i (b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p A}{\left(\sum_{i=1}^n \left(\frac{p_i (b-a)^{p^*-1} \overline{M}}{2^p} \right)^{\frac{1}{p_i}} \right)^p G_\infty},$$

the system

$$\begin{cases} -(p_i - 1)|u_i'(x)|^{p_i-2} u_i''(x) = (F_{u_i}(x, u_1, \dots, u_n) + \mu G_{u_i}(x, u_1, \dots, u_n)) h_i(x, u_i'), & x \in (a, b), \\ u_i(a) = u_i(b) = 0, \end{cases}$$

for $1 \leq i \leq n$, has an unbounded sequence of weak solutions in X .

Theorem 3.6. *Assume that the assumptions (A1) and (A2) in Theorem 3.1 hold.*

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ where λ_1 and λ_2 are given in (3.1), the system

$$\begin{cases} -(p_i - 1)|u_i'(x)|^{p_i-2} u_i''(x) = \lambda F_{u_i}(x, u_1, \dots, u_n), & x \in (a, b), \\ u_i(a) = u_i(b) = 0, \end{cases}$$

for $1 \leq i \leq n$, has an unbounded sequence of weak solutions in X .

Remark 3.7. We observe that in Theorem 3.1 we can replace $\xi \rightarrow +\infty$ with $\xi \rightarrow 0^+$, and arguing in the same way as in the proof of Theorem 3.1, but using conclusion (c) of Theorem 2.1, the system $(D_{\lambda, \mu})$ has a sequence of weak solutions, which strongly converges to 0 in X .

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