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Center problem for a class of degenerate quartic systems

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Abstract. This paper, using pseudo-division algorithm, introduces a method for computing resonant focus numbers of a class of complex polynomial differential systems, establishes the necessary and sufficient conditions for existence of a center for a class of complex quartic systems with a degenerate resonant singular point.

Keywords: complex quartic systems, degenerate resonant singular point, integrability, resonant focus number.

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1 Introduction

In the qualitative theory of real planar differential systems, focal values and saddle values are two important detection quantities. In [2], the authors introduce a new efficient computational method, which combines the computation of focal values and saddle values into a unified calculation of singular point quantities for a class of complex planar differential systems. Using pseudo-divisions, Wang [21] gives an improved formal power series method for computing focal values of a class of polynomial differential systems. Using a perturbation technique based on multiple time scales, Yu [25] presents an efficient method for computing focal values of some classes of differential systems.

Żołądek [27] generalizes the notion of center to the case of a $p:-q$ resonant singular point of the following complex polynomial vector fields

$$\begin{cases} \dfrac{dx}{dt} = p\,x + X_m(x,y), \\ \dfrac{dy}{dt} = -q\,y + Y_m(x,y), \end{cases} \tag{1.1}$$

in $\mathbb{C}^2$, where $p,q \in \mathbb{N}, p \le q, (p,q)=1$, and

$$X_m(x,y) = \sum_{k=2}^{m}\sum_{j=0}^{k} a_{k,j}x^{k-j}y^j, \qquad Y_m(x,y) = \sum_{k=2}^{m}\sum_{j=0}^{k} b_{k,j}x^{k-j}y^j.$$

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Definition 1.1. System (1.1) is said to have a $p : -q$ resonant center at the origin if it admits a local first integral of the form

$$F(x,y) = x^q y^p + \sum_{k=p+q+1}^{\infty} \sum_{j=0}^{k} B_{k,j} x^{k-j} y^j. \tag{1.2}$$

Although, for system (1.1) with $p : -q = 1 : -1, 1 : -2, 1 : -3, 2 : -3, 1 : -q$, the resonant center problems have received intensive attentions, see [3–5, 8, 10–12, 18, 19], very few results are known for systems having high order nonlinearities.

For system (1.1), we can derive a formal power series of the form (1.2) with $B_{s(p+q),sp} = 0$, $s = 2, 3, \dots$, such that

$$\left.\frac{dF}{dt}\right|_{(1.1)} = \frac{\partial F}{\partial x}(px + X_m) + \frac{\partial F}{\partial y}(-qy + Y_m) = \sum_{n=1}^{\infty} W_n (x^q y^p)^{n+1}, \tag{1.3}$$

where W_n are called the n-th order $p : -q$ resonant focus numbers. For some computational methods of such quantities, see [14, 19]. For large n, the computation of W_n is very complicated, which is the main reason of slow progress in the center problems.

The only way to get the necessary conditions for a center is to compute the $p : -q$ resonant focus numbers. Before presenting a new algorithm, we start with a precise definition of pseudo-remainder of polynomials. For more details, see [6, 22–24].

Let $K[x_1, x_2, \dots, x_n]$ denote the ring of polynomials in indeterminates $x_1, x_2, \dots, x_n$ with coefficients in a field K of characteristic 0. Consider a fixed ordering on the set of indeterminates: $x_1 \prec x_2 \prec \cdots \prec x_n$. A polynomial $f \in K[x_1, x_2, \dots, x_n]$ is said to be of class i if i is the maximum index such that f has a positive degree in x_i. The class of elements of K is zero. If f is of class i the coefficient of the x_i of the maximum degree is said to be the initial of polynomial f and is denoted by $\mathrm{In}(f)$.

If f and g are two polynomials of class respectively i and j, with $i < j$, or such that $i = j$ and the degree in x_i of f is less than the degree of g, then it is possible, using the Euclidean algorithm over $K(x_1, x_2, \dots, x_{i-1})[x_i]$ to find polynomials q and r with $\deg_{x_i}(r) < \deg_{x_i}(f)$ such that

$$\mathrm{In}(f)^{\alpha} g = qf + r,$$

with α bounded by $\deg_{x_i}(f) - \deg_{x_i}(g) + 1$. The polynomial r is called the pseudo-remainder of g with respect to f, and it is denoted by $\mathrm{prem}(g, f)$. This operation is called pseudo-division.

Definition 1.2 ([23]). A sequence of polynomials $AS = [f_1, f_2, \dots, f_r]$ is called a triangular set if $r = 1$ and f_1 is not identically zero, or $r > 1$ and $0 < \mathrm{class}(f_1) < \mathrm{class}(f_2) < \cdots < \mathrm{class}(f_r) \le n$.

Definition 1.3 ([22]). Consider a triangular set $AS = [f_1, f_2, \dots, f_r]$, and a polynomial $g \in K[x_1, x_2, \dots, x_r]$. Let us pseudo-divide g by $f_r, f_{r-1}, \dots, f_1$ successively as polynomials in $x_{c_r}, \dots, x_{c_1}$, $c_i = \mathrm{class}(f_i)$, and denote the final remainder by R. Then we shall get an expression of the form:

$$I_1^{s_1} \cdots I_r^{s_r} g = \sum_{i=1}^{r} Q_i f_i + R,$$

where I_i is the initial of f_i, s_i assumes the smallest possible power achievable. R is called the pseudo-remainder of g with respect to AS, denoted as $R = \mathrm{Prem}(g, AS)$.

Now we are in a position to develop the algorithm for computing W_n in (1.3). Grouping the like terms in the second expression of (1.3), we get

$$\begin{aligned}\frac{\partial F}{\partial x}(px+X_m)+\frac{\partial F}{\partial y}(-qy+Y_m) = &\sum_{l=p+q+1}^{(p+q)(n+1)-1}\sum_{j=0}^{l} f_{l,j}x^{l-j}y^j\\ &+\sum_{j=0,j\neq p(n+1)}^{(p+q)(n+1)} f_{(p+q)(n+1),j}x^{(p+q)(n+1)-j}y^j\\ &+V_n(x^qy^p)^{n+1}+\cdots,\end{aligned}$$

where $V_n, f_{l,j}, f_{(p+q)(n+1),j}$ are polynomials in $a_{k,j}, b_{k,j}, B_{k,j}$.

When computing the n-th order resonant focus number W_n, the coefficients $f_{l,j}, f_{(p+q)(n+1),j}$ have to be zero. Thus in order to eliminate indeterminates $B_{k,j}$ from V_n, we use successive pseudo-divisions: first choosing a suitable variable order of $B_{k,j}$; secondly, rearranging some polynomials $f_{l,j}, f_{(p+q)(n+1),j}$ to get a triangular set TS_n; finally, performing successive pseudo-division of V_n+v by TS_n to get the pseudo-remainder R_n, then the n-th order $p:-q$ resonant focus number can be written as $W_n=\frac{R_n}{\mathrm{coeff}(R_n,v)}-v$, where $\mathrm{coeff}(R_n,v)$ is the coefficient of v in the polynomial R_n, and v is a new variable.

To illustrate the main idea of the algorithm, we compute the second order $1:-2$ resonant focus number W_2 of the family

$$\begin{cases}\dfrac{dx}{dt}=x\left(1+a_1x+a_2x^2+a_3yx+a_4y^2\right),\\[2mm] \dfrac{dy}{dt}=y\left(-2+b_1y+b_2x^2+b_3yx+b_4y^2\right).\end{cases}\tag{1.4}$$

Let

$$F(x,y)=x^2y+\sum_{k=4}^{9}\sum_{j=0}^{k}B_{k,j}x^{k-j}y^j+\cdots$$

and using the same notations as described in the algorithm, we have

$$\begin{aligned}\left.\frac{dF}{dt}\right|_{(1.4)} = &\sum_{l=4}^{8}\sum_{j=0}^{l} f_{l,j}x^{l-j}y^j\\ &+\sum_{j=0,j\neq 3}^{9} f_{9,j}x^{9-j}y^j\\ &+V_2(x^2y)^3+\cdots,\end{aligned}$$

where

$$V_2=6\,B_{7,1}a_4+B_{7,1}b_4+5\,B_{7,2}a_3+2\,B_{7,2}b_3+4\,B_{7,3}a_2+3\,B_{7,3}b_2+2\,B_{8,2}b_1+5\,B_{8,3}a_1.$$

Under the variable ordering

$$\begin{aligned}&B_{4,0}\prec B_{4,1}\prec B_{4,2}\prec B_{4,3}\prec B_{5,0}\prec B_{5,1}\prec B_{5,2}\prec B_{5,3}\prec B_{6,0}\prec B_{6,1}\\ &\quad\prec B_{6,3}\prec B_{7,1}\prec B_{7,2}\prec B_{7,3}\prec B_{8,2}\prec B_{8,3},\end{aligned}$$

the following sequence of polynomials

$$TS_2=[f_{4,0},f_{4,1},f_{4,2},f_{4,3},f_{5,0},f_{5,1},f_{5,2},f_{5,3},f_{6,0},f_{6,1},f_{6,3},f_{7,1},f_{7,2},f_{7,3},f_{8,2},f_{8,3}]$$

is a triangular set, where

$$
\begin{aligned}
f_{4,0} &= 4\,B_{4,0},\\
f_{4,1} &= 2\,a_1 + B_{4,1},\\
f_{4,2} &= -2\,B_{4,2} + b_1,\\
f_{4,3} &= -5\,B_{4,3},\\
f_{5,0} &= 4\,B_{4,0}a_1 + 5\,B_{5,0},\\
f_{5,1} &= 3\,B_{4,1}a_1 + 2\,B_{5,1} + 2\,a_2 + b_2,\\
f_{5,2} &= B_{4,1}b_1 + 2\,B_{4,2}a_1 - B_{5,2} + 2\,a_3 + b_3,\\
f_{5,3} &= 2\,B_{4,2}b_1 + B_{4,3}a_1 - 4\,B_{5,3} + 2\,a_4 + b_4,\\
f_{6,0} &= 4\,B_{4,0}a_2 + 5\,B_{5,0}a_1 + 6\,B_{6,0},\\
f_{6,1} &= 4\,B_{4,0}a_3 + 3\,B_{4,1}a_2 + B_{4,1}b_2 + 4\,B_{5,1}a_1 + 3\,B_{6,1},\\
f_{6,3} &= 3\,B_{4,1}a_4 + B_{4,1}b_4 + 2\,B_{4,2}a_3 + 2\,B_{4,2}b_3 + B_{4,3}a_2 + 3\,B_{4,3}b_2 + 2\,B_{5,2}b_1 + 2\,B_{5,3}a_1 - 3\,B_{6,3},\\
f_{7,1} &= 5\,B_{5,0}a_3 + 4\,B_{5,1}a_2 + B_{5,1}b_2 + 5\,B_{6,1}a_1 + 4\,B_{7,1},\\
f_{7,2} &= 5\,B_{5,0}a_4 + 4\,B_{5,1}a_3 + B_{5,1}b_3 + 3\,B_{5,2}a_2 + 2\,B_{5,2}b_2 + B_{6,1}b_1 + B_{7,2},\\
f_{7,3} &= 4\,B_{5,1}a_4 + B_{5,1}b_4 + 3\,B_{5,2}a_3 + 2\,B_{5,2}b_3 + 2\,B_{5,3}a_2 + 3\,B_{5,3}b_2 + 3\,B_{6,3}a_1 - 2\,B_{7,3},\\
f_{8,2} &= 6\,B_{6,0}a_4 + 5\,B_{6,1}a_3 + B_{6,1}b_3 + B_{7,1}b_1 + 5\,B_{7,2}a_1 + 2\,B_{8,2},\\
f_{8,3} &= 5\,B_{6,1}a_4 + B_{6,1}b_4 + 3\,B_{6,3}a_2 + 3\,B_{6,3}b_2 + 2\,B_{7,2}b_1 + 4\,B_{7,3}a_1 - B_{8,3}.
\end{aligned}
$$

By computing pseudo-remainder of $V_2 + v$ by TS_2, one gets

$$
\begin{aligned}
R_2 = &-9676800\,a_1{}^3b_1b_3 - 4838400\,a_1{}^2b_1{}^2b_2 + 921600\,a_1{}^2a_2a_4 - 42854400\,a_1{}^2a_3b_3\\
&+ 921600\,a_1{}^2a_4b_2 - 2073600\,a_1{}^2b_2b_4 - 19353600\,a_1{}^2b_3{}^2 - 921600\,a_1a_2a_3b_1\\
&- 4147200\,a_1a_2b_1b_3 - 20275200\,a_1a_3b_1b_2 - 12441600\,a_1b_1b_2b_3 - 2073600\,a_2b_1{}^2b_2\\
&- 1382400\,b_1{}^2b_2{}^2 - 2764800\,a_2a_3{}^2 + 1382400\,a_2a_4b_2 - 691200\,a_2b_2b_4 + 1382400\,a_3{}^2b_2\\
&+ 1382400\,a_3b_2b_3 - 691200\,b_2{}^2b_4 - 1382400\,v.
\end{aligned}
$$

Hence the second order $1:-2$ resonant focus number can be written as

$$
\begin{aligned}
W_2 &= \frac{R_2}{\mathrm{coeff}(R_2, v)} - v\\
&= 7\,a_1{}^3b_1b_3 + \frac{7}{2}\,a_1{}^2b_1{}^2b_2 - \frac{2}{3}\,a_1{}^2a_2a_4 + 31\,a_1{}^2a_3b_3 - \frac{2}{3}\,a_1{}^2a_4b_2 + \frac{3}{2}\,a_1{}^2b_2b_4\\
&\quad + 14\,a_1{}^2b_3{}^2 + \frac{2}{3}\,a_1a_2a_3b_1 + 3\,a_1a_2b_1b_3 + \frac{44\,a_1a_3b_1b_2}{3} + 9\,a_1b_1b_2b_3 + \frac{3}{2}\,a_2b_1{}^2b_2\\
&\quad + b_1{}^2b_2{}^2 + 2\,a_2a_3{}^2 - a_2a_4b_2 + \frac{1}{2}\,a_2b_2b_4 - a_3{}^2b_2 - a_3b_2b_3 + \frac{1}{2}\,b_2{}^2b_4.
\end{aligned}
$$

A general purposed Maple package *Myvalue* based on our algorithm is developed in Maple V.18 on Intel Core 2 Quad CPU Q8400, 4G RAM, and such Maple package is available for non-commercial purpose via email to: sangbo_76@163.com. Another Maple package *Liuc* based on the method [14] is also developed by us using the same computing platform. For technical comparison of these two packages, let us consider a class of cubic differential systems

$$
\begin{cases}
\dfrac{dx}{dt} = x + X_3(x, y),\\[2mm]
\dfrac{dy}{dt} = -y + Y_3(x, y),
\end{cases}
$$

in $\mathbb{C}^2$, and

$$X_3(x,y) = \sum_{k=2}^{3}\sum_{j=0}^{k} a_{k,j}x^{k-j}y^j, \qquad Y_3(x,y) = \sum_{k=2}^{3}\sum_{j=0}^{k} b_{k,j}x^{k-j}y^j.$$

Computing the first eight $1:-1$ resonant focus numbers W_j, $1 \le j \le 8$ by *Myvalue* and *Liuc* respectively, we find that the outputs (in expanded form) are the same for these two methods, and get the following experimental results on efficiency, see Table 1.1.

Method	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
Myvalue	0.015	0.015	0.046	0.203	1.281	5.640	45.968	160.968
Liuc	0.0	0.0	0.0	0.062	0.578	3.546	40.078	592.750

Table 1.1: Computing times (in CPU seconds) for the first eight resonant focus numbers

For computing W_n with n large, it is worth noting that the expansion of long polynomials in the last stage of package *Liuc* is pretty time-consuming, whereas the package *Myvalue* does not need any expansions before generating its outputs.

Consider the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = P_n(x,y), \\ \dfrac{dy}{dt} = Q_n(x,y), \end{cases} \tag{1.5}$$

where $(x,y) \in \mathbb{C}^2$, P_n, Q_n are polynomials of degree n with $(P_n, Q_n) = 1$.

Definition 1.4. The polynomial $f(x,y) \in \mathbb{C}[x,y]$ is called an algebraic partial integral of the system (1.5) if there exists a polynomial $h \in \mathbb{C}[x,y]$ such that

$$\left.\frac{df}{dt}\right|_{(1.5)} = h(x,y)\,f(x,y).$$

The polynomial h is called a cofactor. If $h \equiv 0$ then $f(x,y) = \text{const}$ is a first integral of system (1.5).

Lemma 1.5. *Suppose that system (1.5) admits m independent algebraic partial integrals $f_1, f_2, \ldots, f_m$ satisfying*

$$\left.\frac{df_k}{dt}\right|_{(1.5)} = h_k(x,y)\,f_k(x,y), \quad k = 1,2,\ldots,m.$$

If there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_m h_m = -\left(\frac{\partial P_n}{\partial x} + \frac{\partial Q_n}{\partial y}\right),$$

then the function $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}$ is an integrating factor of system (1.5).

Mattei and Moussu [16] proved the next result for all isolated singularities.

Lemma 1.6. *Assume that system (1.1) with an isolated singularity at the origin has a formal first integral $F(x,y) \in \mathbb{R}[[x,y]]$ around it. Then, there exists an analytic first integral around the singularity.*

Another mechanism to prove the integrability of system (1.5) is time-reversibility. From [20], we have the following result.

Lemma 1.7. *System* (1.5) *is time-reversible with respect to a transformation*

$$R\colon x \mapsto \gamma y, \qquad y \mapsto \gamma^{-1} x,$$

where γ is a nonzero scalar, if and only if,

$$\gamma Q_n(\gamma y, \gamma^{-1} x) = -P_n(x, y), \qquad \gamma Q_n(x, y) = -P_n(\gamma y, \gamma^{-1} x).$$

2 Main result

In the qualitative theory of planar differential systems, there are few works about degenerate singular point. Most of the work focuses on the center problem of the system

$$\begin{cases} \dfrac{dx}{dt} = y + P(x, y), \\ \dfrac{dy}{dt} = Q(x, y), \end{cases}$$

where P, Q are polynomials in x and y with degree no less than two, see [1,7,9,17].

Let us consider the real analytic system

$$\begin{cases} \dfrac{du}{dt_1} = -v(u^2+v^2)^n + \displaystyle\sum_{k=2n+2}^{\infty} U_k(u, v) = U(u, v), \\ \dfrac{dv}{dt_1} = \quad u(u^2+v^2)^n + \displaystyle\sum_{k=2n+2}^{\infty} V_k(u, v) = V(u, v), \end{cases} \tag{2.1}$$

where $U(u, v)$, $V(u, v)$ are analytic in a sufficiently small neighborhood of the origin, $U_k(u, v)$, $V_k(u, v)$ are homogeneous polynomials of degree k, and $n \geq 0$. Because the singularity $(u, v) = (0, 0)$ of system (2.1) has no characteristic directions, it is a center or a focus.

Under the transformation $u = r\cos(\theta)$, $v = r\sin(\theta)$, system (2.1) becomes

$$\begin{cases} \dfrac{dr}{dt_1} = r^{2n+1} \displaystyle\sum_{k=0}^{\infty} \phi_{2n+2+k}(\theta) r^k, \\ \dfrac{d\theta}{dt_1} = r^{2n} \displaystyle\sum_{k=0}^{\infty} \psi_{2n+2+k}(\theta) r^k. \end{cases} \tag{2.2}$$

It can be written as

$$\frac{dr}{d\theta} = r \sum_{k=0}^{\infty} R_k(\theta) r^k, \tag{2.3}$$

where the function on the right side of (2.3) is convergent in the range $\theta \in [-4\pi, 4\pi], r < r_0$, and

$$R_k(\theta + \pi) = (-1)^k R_k(\theta), \qquad k = 0, 1, 2, \ldots. \tag{2.4}$$

For sufficient small h, let

$$\Delta(h) = r(2\pi, h) - h, \qquad r = r(\theta, h) = \sum_{m=1}^{\infty} v_m(\theta) h^m \tag{2.5}$$

be the Poincaré successor function and the solution of (2.3) satisfying the initial value condition $r|_{\theta=0} = h$.

By using the homeomorphic transformation

$$z = u + \mathrm{i}v, \qquad w = u - \mathrm{i}v, \qquad T = \mathrm{i}t_1, \tag{2.6}$$

system (2.1) is transformed into

$$\begin{cases} \dfrac{dz}{dT} = z^{n+1}w^n + \displaystyle\sum_{k=2n+2}^{\infty} Z_k(z,w), \\ \dfrac{dw}{dT} = -w^{n+1}z^n - \displaystyle\sum_{k=2n+2}^{\infty} W_k(z,w), \end{cases} \tag{2.7}$$

where $Z_k(z,w) = \sum_{\alpha+\beta=k} a_{\alpha,\beta} z^\alpha w^\beta$, $W_k(z,w) = \sum_{\alpha+\beta=k} b_{\alpha,\beta} w^\alpha z^\beta$ are homogeneous polynomials of degree k, and $a_{\alpha,\beta} = \overline{b_{\alpha,\beta}}$.

Let

$$x = z(zw)^{-\frac{(n+1)}{(2n+3)}}, \qquad y = w(zw)^{-\frac{(n+1)}{(2n+3)}}, \qquad dt = (zw)^n dT, \tag{2.8}$$

system (2.7) is transformed into

$$\begin{cases} \dfrac{dx}{dt} = x + x\displaystyle\sum_{k=1}^{\infty} \Phi_{k(2n+3)}(x,y), \\ \dfrac{dy}{dt} = -y - y\displaystyle\sum_{k=1}^{\infty} \Psi_{k(2n+3)}(x,y), \end{cases} \tag{2.9}$$

where $\Phi_{k(2n+3)}, \Psi_{k(2n+3)}$ are homogeneous polynomials of degree $k(2n+3)$. Because $z = \overline{w}$, (2.8) is a homeomorphic transformation in some open neighborhood of the origin $(z,w) = (0,0)$.

Lemma 2.1 ([14])**.** *For system (2.9), we can derive successively the terms of the following formal series*

$$F(x,y) = xy\left[1 + \sum_{m=1}^{\infty} f_{m(2n+3)}(x,y)\right], \tag{2.10}$$

such that

$$\left.\frac{dF}{dt}\right|_{(2.9)} = \sum_{m=1}^{\infty} \mu_m (xy)^{m(2n+3)+1}, \tag{2.11}$$

where $f_{m(2n+3)}$ are homogeneous polynomials of degree $m(2n+3)$.

Lemma 2.2 ([14])**.** *System (2.9) has a complex center at the origin if and only if there exists a non-zero real number s and a first integral of the form*

$$\tilde{F}(x,y) = (xy)^s\left[1 + \sum_{m=1}^{\infty} \tilde{f}_{m(2n+3)}(x,y)\right], \tag{2.12}$$

where $\tilde{f}_{m(2n+3)}$ are homogeneous polynomials of degree $m(2n+3)$. The power series in (2.12) has a non-zero convergence radius.

Definition 2.3 ([13–15])**.** For any positive integer m, the number μ_m is called the m-th singular point value of system (2.7) at the origin. And $v_{2m+1}(2\pi) \sim \mathrm{i}\pi\mu_m$ is called the m-th focal value of system (2.1) at the origin.

Definition 2.4 ([13–15]). If for all m, $\mu_m = 0$, then the origin of system (2.7) is called a complex center. If for all m, $v_{2m+1}(2\pi) = 0$, then the origin of real system (2.1) is a center.

Lemma 2.5. *The origin of system (2.7) is a complex center if and only if the origin of system (2.9) is a complex center.*

Proof. Necessity. Suppose that system (2.7) has a complex center at the origin, then for all m, $\mu_m = 0$. Hence system (2.9) has a formal first integral $F(x,y)$ of the form (2.10), so by Lemma 1.6, it has a complex center at the origin.

Sufficiency. Suppose that system (2.9) has a complex center at the origin , then by Lemma 2.2 it has an analytic first integral $\tilde{F}(x,y)$ of the form (2.12). Thus it also has an analytic first integral of the form $F(x,y) = [\tilde{F}(x,y)]^{\frac{1}{s}}$, which implies $\mu_m = 0$ for all m by Lemma 2.1, and so that the origin of system (2.7) is a complex center. □

Because system (2.9) is integrable at the origin if and only if the origin of it is a complex center, we have the following theorem.

Theorem 2.6. *The origin of system (2.7) is a complex center if and only if system (2.9) is integrable at the origin.*

As a consequence of Theorem 2.6, we have the following corollary.

Corollary 2.7. *The origin of the real system (2.1) is a center if and only if system (2.9) is integrable at the origin.*

The authors of [26] obtain the center conditions of the following system:

$$\begin{cases} \dfrac{dz}{dt} = z^2 w(1 + a\,z + b\,w), \\ \dfrac{dw}{dt} = -zw^2(2 + c\,z + d\,w). \end{cases}$$

In this paper, we consider the center problem of a class of complex quartic systems

$$\begin{cases} \dfrac{dz}{dT} = z^2 w + a_1 z^4 + a_2 z^2 w^2 + a_3 w^4, \\ \dfrac{dw}{dT} = -zw^2 + b_1 z^4 + b_2 z^2 w^2 + b_3 w^4, \end{cases} \tag{2.13}$$

where

$$a_1 = -\overline{b_3}, \qquad a_2 = -\overline{b_2}, \qquad a_3 = -\overline{b_1}. \tag{2.14}$$

Using the non-linear change (2.8) for $n = 1$, system (2.13) becomes

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{2}{5} b_1 x^6 + \dfrac{3}{5} a_1 y x^5 - \dfrac{2}{5} b_2 y^2 x^4 + \dfrac{3}{5} a_2 y^3 x^3 - \dfrac{2}{5} b_3 y^4 x^2 + \dfrac{3}{5} a_3 y^5 x, \\ \dfrac{dy}{dt} = -y + \dfrac{3}{5} b_1 y x^5 - \dfrac{2}{5} a_1 y^2 x^4 + \dfrac{3}{5} b_2 y^3 x^3 - \dfrac{2}{5} a_2 y^4 x^2 + \dfrac{3}{5} b_3 y^5 x - \dfrac{2}{5} a_3 y^6. \end{cases} \tag{2.15}$$

Applying our method to compute the first thirty $1 : -1$ resonant focus numbers, we get $W_1, W_2, \ldots, W_{30}$, where the quantity W_k is reduced w.r.t. the Gröbner basis of $\{W_j : j < k\}$.

$$W_k = 0, \qquad k \in \{1, 2, \cdots, 25\} \setminus \{10, 15, 20, 25\},$$

$$W_{10} = -\frac{16}{45} a_3 {a_1}^2 a_2 - \frac{4}{15} a_1 {a_2}^3 - \frac{4}{25} a_3 a_1 {b_2}^2 + \frac{4}{25} {a_2}^2 b_1 b_3 + \frac{16}{45} b_1 b_2 {b_3}^2 + \frac{4}{15} {b_2}^3 b_3,$$

$$
\begin{aligned}
W_{15} = & \frac{2857}{6750} a_1{}^3 a_3{}^2 b_2 + \frac{427}{2700} a_1{}^2 a_2{}^3 b_3 + \frac{97}{4500} a_1{}^2 a_3 b_2{}^2 b_3 - \frac{169}{200} a_1 a_2{}^4 b_2 + \frac{3527}{3000} a_1 a_2{}^3 a_3 b_1 \\
& - \frac{97}{4500} a_1 a_2{}^2 b_1 b_3{}^2 - \frac{253}{9000} a_1 a_2 a_3 b_2{}^3 + \frac{2711}{45000} a_1 a_3{}^2 b_1 b_2{}^2 - \frac{427}{2700} a_1 b_2{}^3 b_3{}^2 + \frac{27}{100} a_2{}^5 b_1 \\
& + \frac{253}{9000} a_2{}^3 b_1 b_2 b_3 - \frac{2711}{45000} a_2{}^2 a_3 b_1{}^2 b_3 - \frac{2857}{6750} a_2 b_1{}^2 b_3{}^3 + \frac{169}{200} a_2 b_2{}^4 b_3 - \frac{3527}{3000} a_3 b_1 b_2{}^3 b_3 \\
& - \frac{27}{100} a_3 b_2{}^5, \\
W_{20} = & \frac{1235077}{103994800} a_1{}^3 a_3 b_2{}^2 b_3{}^2 - \frac{955578854125547}{4202028745200000} a_1{}^2 a_2{}^3 a_3 b_2{}^2 - \frac{5542584347}{17819109000} a_1 a_2{}^3 a_3{}^2 b_1{}^2 \\
& - \frac{564506438409643}{336162299616000} a_1 a_2{}^2 a_3 b_2{}^4 + \frac{24252192229}{386080695000} a_1 a_3{}^3 b_1{}^2 b_2{}^2 - \frac{4890698789}{8236388160} a_1 a_3 b_2{}^5 b_3 \\
& - \frac{24252192229}{386080695000} a_2{}^2 a_3{}^2 b_1{}^3 b_3 + \frac{5542584347}{17819109000} a_3{}^2 b_1{}^2 b_2{}^3 b_3 + \frac{481204102950071}{882426036492000} a_1{}^2 a_2{}^4 b_2 b_3 \\
& - \frac{1235077}{103994800} a_1{}^2 a_2{}^2 b_1 b_3{}^3 + \frac{2683884447677231}{3268244579600000} a_1 a_2{}^5 b_1 b_3 \\
& + \frac{955578854125547}{2521217247120000} a_1 a_2{}^3 b_2{}^3 b_3 - \frac{481204102950071}{882426036492000} a_1 a_2 b_2{}^4 b_3{}^2 \\
& + \frac{564506438409643}{336162299616000} a_2{}^4 b_1 b_2{}^2 b_3 - \frac{616607103869689}{756365174136000} a_2 a_3 b_1 b_2{}^4 b_3 \\
& + \frac{955578854125547}{1890912935340000} a_1{}^2 a_2 a_3 b_2{}^3 b_3 - \frac{1396275623989}{2316484170000} a_1{}^2 a_3{}^2 b_1 b_2{}^2 b_3 \\
& + \frac{616607103869689}{756365174136000} a_1 a_2{}^4 a_3 b_1 b_2 + \frac{1396275623989}{2316484170000} a_1 a_2{}^2 a_3 b_1{}^2 b_3{}^2 \\
& + \frac{9548870928976103}{8824260364920000} a_1 a_2 a_3{}^2 b_1 b_2{}^3 - \frac{9548870928976103}{8824260364920000} a_2{}^3 a_3 b_1{}^2 b_2 b_3 \\
& - \frac{955578854125547}{2521217247120000} a_1{}^2 a_2{}^6 - \frac{184074380363917}{156875739820800} a_1 a_2{}^5 b_2{}^2 + \frac{136361}{14856400} a_1{}^3 a_2{}^3 b_3{}^2 \\
& - \frac{136361}{14856400} a_1{}^2 b_2{}^3 b_3{}^3 + \frac{1125970354387}{3088645560000} a_2{}^4 b_1{}^2 b_3{}^2 + \frac{184074380363917}{156875739820800} a_2{}^2 b_2{}^5 b_3 \\
& - \frac{955578854125547}{1890912935340000} a_1{}^3 a_2{}^4 a_3 - \frac{1125970354387}{3088645560000} a_1{}^2 a_3{}^2 b_2{}^4 - \frac{2707696721}{2969851500} a_2{}^5 a_3 b_1{}^2 \\
& - \frac{637748287543}{8404057490400} a_2 a_3 b_2{}^6 + \frac{2707696721}{2969851500} a_3{}^2 b_1 b_2{}^5 + \frac{637748287543}{8404057490400} a_2{}^6 b_1 b_2,
\end{aligned}
$$

and W_{25}, W_{30} are very complicated so we do not present these polynomials here, but the interested reader can easily compute them using any computer algebra system.

If condition (2.14) holds, by applying suitable non-degenerate similarity transformation and time scaling, system (2.13) becomes one of the two forms:

$$
\begin{cases}
\dfrac{dz}{dT} = z^2 w + a_1 z^4 + z^2 w^2 + a_3 w^4, \\
\dfrac{dw}{dT} = -z w^2 + b_1 z^4 - z^2 w^2 + b_3 w^4.
\end{cases}
\tag{2.16}
$$

$$
\begin{cases}
\dfrac{dz}{dT} = z^2 w + a_1 z^4 + a_3 w^4, \\
\dfrac{dw}{dT} = -z w^2 + b_1 z^4 + b_3 w^4,
\end{cases}
\tag{2.17}
$$

where

$$
a_1 = -\overline{b_3}, \qquad a_3 = -\overline{b_1}. \tag{2.18}
$$

Theorem 2.8. *If condition (2.18) holds, system (2.16) has a complex center at the origin if and only if* $a_1 = -b_3,\ a_3 = -b_1$.

Theorem 2.9. *If condition (2.18) holds, system (2.17) has a complex center at the origin if and only if one of the following conditions holds:*

(i) $b_1^3 b_3^5 - a_1^5 a_3^3 = 0,\ a_1 b_3 \neq 0$;

(ii) $a_1 = b_1 = 0$;

(iii) $a_1 = b_3 = 0,\ b_1 \neq 0$.

3 Proof of the Theorems 2.8 and 2.9

Using the transformation (2.8) for $n = 1$, system (2.16) becomes

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{2}{5} b_1 x^6 + \dfrac{3}{5} a_1 y x^5 + \dfrac{2}{5} y^2 x^4 + \dfrac{3}{5} y^3 x^3 - \dfrac{2}{5} b_3 y^4 x^2 + \dfrac{3}{5} a_3 y^5 x, \\ \dfrac{dy}{dt} = -y + \dfrac{3}{5} b_1 y x^5 - \dfrac{2}{5} a_1 y^2 x^4 - \dfrac{3}{5} y^3 x^3 - \dfrac{2}{5} y^4 x^2 + \dfrac{3}{5} b_3 y^5 x - \dfrac{2}{5} a_3 y^6. \end{cases} \tag{3.1}$$

Lemma 3.1. *System (3.1) is integrable at the origin if and only if* $a_1 = -b_3,\ a_3 = -b_1$.

Proof. Necessity. Let $W_1, W_2, \ldots, W_{30}$ be the first thirty $1 : -1$ resonant focus numbers of system (2.15). By substituting $a_2 = 1,\ b_2 = -1$ into these numbers respectively, one gets the first thirty $1 : -1$ resonant focus numbers $W_1', W_2', \ldots, W_{30}'$ of system (3.1).

Computing a Gröbner basis of the ideal $\langle W_1', W_2', \ldots, W_{30}' \rangle$ with respect to the graded reverse lexicographical order with $b_1 \succ a_3 \succ b_3 \succ a_1$, we obtain a list of polynomials $G = \{b_3 + a_1, b_1 + a_3\}$. The vanishing of G leads to $a_1 = -b_3, a_3 = -b_1$.

Sufficiency. In the case $a_1 = -b_3, a_3 = -b_1$, system (3.1) takes the form

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{2}{5} b_1 x^6 - \dfrac{3}{5} b_3 x^5 y + \dfrac{2}{5} x^4 y^2 + \dfrac{3}{5} x^3 y^3 - \dfrac{2}{5} b_3 x^2 y^4 - \dfrac{3}{5} b_1 x y^5, \\ \dfrac{dy}{dt} = -y + \dfrac{3}{5} b_1 x^5 y + \dfrac{2}{5} b_3 x^4 y^2 - \dfrac{3}{5} x^3 y^3 - \dfrac{2}{5} x^2 y^4 + \dfrac{3}{5} b_3 x y^5 + \dfrac{2}{5} b_1 y^6. \end{cases} \tag{3.2}$$

From Lemma 1.7, we know that system (3.2) is time-reversible w.r.t. the transformation $x \mapsto y$, $y \mapsto x$. So by the symmetry principle, the origin of such system is a resonant center, and hence system (3.2) is integrable at the origin. □

Lemma 3.2. *If condition (2.18) holds, system (3.1) is integrable at the origin if and only if* $a_1 = -b_3$, $a_3 = -b_1$.

Proof. In view of the consistence of condition (2.18) and condition $a_1 = -b_3,\ a_3 = -b_1$, the result follows by Lemma 3.1. □

Using the transformation (2.8) for $n = 1$, system (2.17) becomes

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{2}{5} b_1 x^6 + \dfrac{3}{5} a_1 y x^5 - \dfrac{2}{5} b_3 y^4 x^2 + \dfrac{3}{5} a_3 y^5 x, \\ \dfrac{dy}{dt} = -y + \dfrac{3}{5} b_1 y x^5 - \dfrac{2}{5} a_1 y^2 x^4 + \dfrac{3}{5} b_3 y^5 x - \dfrac{2}{5} a_3 y^6. \end{cases} \tag{3.3}$$

Lemma 3.3. *System (3.3) is integrable at the origin if and only if one of the following conditions holds:*

(1) $b_1^3 b_3^5 - a_1^5 a_3^3 = 0,\ a_1 b_3 \neq 0;$

(2) $a_1 = a_3 = 0,\ b_1 b_3 \neq 0;$

(3) $a_1 = b_1 = 0;$

(4) $a_1 = b_3 = 0,\ b_1 \neq 0;$

(5) $a_3 = b_3 = 0,\ a_1 \neq 0;$

(6) $b_1 = b_3 = 0,\ a_1 a_3 \neq 0.$

Proof. Necessity. Let $W_1, W_2, \ldots, W_{30}$ be the first thirty $1 : -1$ resonant focus numbers of system (2.15). By substituting $a_2 = b_2 = 0$ into these numbers respectively, one gets the first thirty $1 : -1$ resonant focus numbers $W_1', W_2', \ldots, W_{30}'$ of system (3.3).

Computing a Gröbner basis of the ideal $\langle W_1', W_2', \ldots, W_{30}' \rangle$ with respect to the graded reverse lexicographical order with $b_1 \succ a_3 \succ b_3 \succ a_1$ and we get a list of polynomials

$$
\begin{aligned}
G = \Big\{ & 16504950\, {a_1}^6 {a_3}^3 b_3 - 6113731\, {a_1}^5 {a_3}^4 b_1 - 16504950\, a_1 {b_1}^3 {b_3}^6 + 6113731\, a_3 {b_1}^4 {b_3}^5, \\
& - {a_1}^7 {a_3}^3 {b_3}^2 + {a_1}^2 {b_1}^3 {b_3}^7 \Big\}.
\end{aligned}
$$

The vanishing of G gives rise to six cases in the Lemma.

Sufficiency. When condition (1) holds, system (3.3) is of the form

$$
\begin{cases}
\dfrac{dx}{dt} = x - \dfrac{2}{5} b_1 x^6 + \dfrac{3}{5} a_1 y x^5 - \dfrac{2}{5} b_3 y^4 x^2 + \dfrac{3}{5} \dfrac{{b_3}^{\frac{5}{3}} b_1}{{a_1}^{\frac{5}{3}}} y^5 x, \\
\dfrac{dy}{dt} = -y + \dfrac{3}{5} b_1 y x^5 - \dfrac{2}{5} a_1 y^2 x^4 + \dfrac{3}{5} b_3 y^5 x - \dfrac{2}{5} \dfrac{{b_3}^{\frac{5}{3}} b_1}{{a_1}^{\frac{5}{3}}} y^6.
\end{cases}
\tag{3.4}
$$

From Lemma 1.7, we know that system (3.4) is time-reversible w.r.t. the transformation

$$
x \mapsto \gamma_0 y, \qquad y \mapsto {\gamma_0}^{-1} x, \quad \text{where} \quad \gamma_0 = \left(\frac{-b_3}{a_1} \right)^{\frac{1}{3}},
$$

so by the symmetry principle, the origin of it is a resonant center, and hence system (3.4) is integrable.

If condition (2) holds, system (3.3) is reduced to

$$
\begin{cases}
\dfrac{dx}{dt} = x - \dfrac{2}{5} b_1 x^6 - \dfrac{2}{5} b_3 y^4 x^2, \\
\dfrac{dy}{dt} = -y + \dfrac{3}{5} b_1 y x^5 + \dfrac{3}{5} b_3 y^5 x.
\end{cases}
\tag{3.5}
$$

We will show that for system (3.5) there exists a formal first integral in the form $F(x,y) = \sum_{n=1}^{\infty} v_n(y) x^n$, where functions $v_n(y)$ should satisfy the first-order linear differential equation

$$
\frac{dv_n}{dy} = \frac{n}{y} v_n - \frac{2}{5}(n-1) b_3 y^3 v_{n-1} - \frac{2}{5y}(n-5) b_1 v_{n-5} + \frac{3}{5} b_3 y^4 v_{n-1}' + \frac{3}{5} b_1 v_{n-5}'. \tag{3.6}
$$

Solving this equation, we obtain

$$
\begin{aligned}
v_1(y) &= y, \quad v_2(y) = \frac{1}{15}\, b_3 y^5, \quad v_3(y) = \frac{11}{450}\, {b_3}^2 y^9, \\
v_4(y) &= \frac{77}{6750}\, {b_3}^3 y^{13}, \quad v_5(y) = \frac{2387}{405000}\, {b_3}^4 y^{17}, \\
v_6(y) &= \frac{1}{30375000} \left(97867\, {b_3}^5 y^{20} - 1215000\, b_1\right) y, \\
v_7(y) &= \frac{41}{911250000}\, b_3 \left(40579\, {b_3}^5 y^{20} - 2430000\, b_1\right) y^5, \\
v_8(y) &= \frac{1}{13668750000}\, {b_3}^2 \left(14498297\, {b_3}^5 y^{20} + 1104435000\, b_1\right) y^9, \\
v_9(y) &= \frac{11}{1640250000000}\, {b_3}^3 \left(93579917\, {b_3}^5 y^{20} + 9263160000\, b_1\right) y^{13}, \\
v_{10}(y) &= \frac{11}{2733750000000}\, {b_3}^4 \left(93579917\, {b_3}^5 y^{20} + 10979010000\, b_1\right) y^{17}.
\end{aligned}
$$

taking the integration constants for $n = 1$ and for $n > 1$ equal to 1 and 0, respectively. We will show by induction that

$$
\begin{aligned}
v_{5k+1}(y) &= y P_{k,1}(y^{20}), \\
v_{5k+2}(y) &= y^5 P_{k,2}(y^{20}), \\
v_{5k+3}(y) &= y^9 P_{k,3}(y^{20}), \\
v_{5k+4}(y) &= y^{13} P_{k,4}(y^{20}), \\
v_{5k+5}(y) &= y^{17} P_{k,5}(y^{20}),
\end{aligned}
$$

for $k \geq 1$, where $P_{k,j}$ are polynomials of degree k. Hence we assume that for $k = s$, the assertion is true. We then solve the linear differential equation (3.6) for $n = 5s + 6$ and obtain

$$
v_{5s+6}(y) = -\frac{1}{5} y^{(5s+6)} \int y^{-(5s+7)} g_{20s+21}(y)\, dy, \tag{3.7}
$$

taking the integration constant to be 0, where

$$
g_{20s+21}(y) = 2(5s+1) b_1 v_{5s+1} + 10(s+1) b_3 y^4 v_{5s+5} - 3y(b_3 y^4 v'_{5s+5} + b_1 v'_{5s+1}).
$$

According to the hypothesis for $k = s$, we can see that the integrand of (3.7) involves no terms of y^{-1} and g_{20s+21} is a polynomial of degree $20s + 21$. Consequently, $v_{5s+6}(y) = v_{5(s+1)+1}(y)$ must be of the form

$$
v_{5s+6}(y) = y P_{s+1,1}(y^{20}),
$$

where $P_{s+1,1}$ is a polynomial of degree $s + 1$. In a similar way, we can also prove that $v_{5s+7}(y)$, $v_{5s+8}(y)$, $v_{5s+9}(y)$, $v_{5s+10}(y)$ are of the forms

$$
\begin{aligned}
v_{5s+7}(y) &= y^5 P_{s+1,2}(y^{20}), \\
v_{5k+8}(y) &= y^9 P_{s+1,3}(y^{20}), \\
v_{5k+9}(y) &= y^{13} P_{s+1,4}(y^{20}), \\
v_{5k+10}(y) &= y^{17} P_{s+1,5}(y^{20}).
\end{aligned}
$$

Hence, we have proved that system (3.5) admits a formal first integral of the form $F(x, y) = \sum_{n=1}^{\infty} v_n(y) x^n$. Consequently it has an analytic first integral in some neighborhood of the origin.

If condition (3) holds, system (3.3) is reduced to

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{2}{5}\, b_3 y^4 x^2 + \dfrac{3}{5}\, a_3 y^5 x, \\ \dfrac{dy}{dt} = -y + \dfrac{3}{5}\, b_3 y^5 x - \dfrac{2}{5}\, a_3 y^6. \end{cases} \tag{3.8}$$

We will show that for system (3.8) there exists a formal first integral in the form $F(x,y) = \sum_{n=1}^{\infty} v_n(x) y^n$, where functions $v_n(x)$ should satisfy the first-order linear differential equation

$$\frac{dv_n}{dx} = \frac{n}{x} v_n - \frac{3}{5}(n-4) b_3 v_{n-4} + \frac{2(n-5)a_3}{5x} v_{n-5} + \frac{2}{5} b_3 x v'_{n-4} - \frac{3}{5} a_3 v'_{n-5}. \tag{3.9}$$

Solving this equation, we obtain

$$v_k(x) = x^k, \qquad k = 1,2,3,4.$$
$$v_5(x) = \frac{1}{15}\, x^2 \left(15\, x^3 + b_3\right).$$

taking the integration constants equal to 1. We will show by induction that the functions $v_n(x)$ are polynomials of degree n. Hence, we assume that for $k = 1,2,\ldots,n-1$ there exist k-th degree polynomials $v_k(x)$ satisfying (3.9). We then solve the linear differential equation (3.9) for $k = n$ and obtain

$$v_n(x) = \left(1 - \frac{1}{5}\int x^{-(n+1)} g_{n-3}(x)\, dx\right) x^n, \tag{3.10}$$

where

$$g_{n-3}(x) = -2b_3 v'_{n-4} x^2 + (3nb_3 v_{n-4} + 3a_3 v'_{n-5} - 12 b_3 v_{n-4})x - 2(n-5)a_3 v_{n-5},$$

taking the integration constant equals to 1. Using the induction hypothesis that

$$\deg(v_{n-5}(x)) = n-5, \qquad \deg(v_{n-4}(x)) = n-4,$$

we find that the degree of $g_{n-3}(x)$ is at most $n-3$. Now, we must study whether the integral can give any logarithmic terms. Therefore, we must prove that terms involving x^{-1} do not appear in the integrand of (3.10). Since the exponents that can appear in the integrand are of the form

$$-(s+4), \qquad s = 0,1,2,\ldots,n-3,$$

there can be no logarithmic terms in (3.10) and $v_n(x)$ is an n-th degree polynomial in x. Hence, we have proved that system (3.8) admits a formal first integral of the form $F(x,y) = \sum_{n=1}^{\infty} v_n(x) y^n$. Consequently it has an analytic first integral around the origin.

If condition (4) holds, system (3.3) is of the form

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{2}{5}\, b_1 x^6 + \dfrac{3}{5}\, a_3 y^5 x, \\ \dfrac{dy}{dt} = -y + \dfrac{3}{5}\, b_1 y x^5 - \dfrac{2}{5}\, a_3 y^6. \end{cases} \tag{3.11}$$

From Lemma 1.7, we know that system (3.11) is time-reversible w.r.t. the transformation

$$x \mapsto \gamma_0 y, \qquad y \mapsto {\gamma_0}^{-1} x, \quad \text{where} \quad \gamma_0 = \sqrt[5]{\frac{-a_3}{b_1}},$$

so by the symmetry principle, the origin of system is a resonant center, and hence system (3.11) is integrable.

If condition (5) holds, system (3.3) is of the form

$$\begin{cases} \dfrac{dx}{dt} = x - \dfrac{2}{5}\,b_1 x^6 + \dfrac{3}{5}\,a_1 y x^5, \\ \dfrac{dy}{dt} = -y + \dfrac{3}{5}\,b_1 y x^5 - \dfrac{2}{5}\,a_1 y^2 x^4. \end{cases} \tag{3.12}$$

We will show that for system (3.12) there exists a formal first integral in the form $F(x,y) = \sum_{n=1}^{\infty} v_n(y)x^n$, where functions $v_n(y)$ should satisfy the first-order linear differential equation

$$\frac{dv_n}{dy} = \frac{n}{y}v_n + \frac{3}{5}(n-4)a_1 v_{n-4} - \frac{2(n-5)b_1}{5y}v_{n-5} - \frac{2}{5}a_1 y v'_{n-4} + \frac{3}{5}b_1 v'_{n-5}. \tag{3.13}$$

Solving this equation, we obtain

$$\begin{aligned} v_k(y) &= y^k, \qquad k = 1,2,3,4. \\ v_5(y) &= \frac{1}{15}\,y^2\left(15\,y^3 - a_1\right), \end{aligned}$$

taking the integration constants equal to 1. We will show by induction that the functions $v_n(y)$ are polynomials of degree n. Hence, we assume that for $k = 1,2,\ldots,n-1$ there exist k-th degree polynomials $v_k(y)$ satisfying (3.13). We then solve the linear differential equation (3.13) for $k = n$ and obtain

$$v_n(y) = \left(1 - \frac{1}{5}\int y^{-(n+1)} g_{n-3}(y)\,dy\right) y^n, \tag{3.14}$$

where

$$g_{n-3}(y) = 2a_1 v'_{n-4} y^2 + (-3na_1 v_{n-4} + 12a_1 v_{n-4} - 3b_1 v'_{n-5})y + 2(n-5)b_1 v_{n-5},$$

taking the integration constant equals to 1. Using the induction hypothesis that

$$\deg(v_{n-5}(y)) = n-5, \qquad \deg(v_{n-4}(y)) = n-4,$$

we find that the degree of $g_{n-3}(y)$ is $n-3$. Now, we must study whether the integral can give any logarithmic terms. Therefore, we must prove that terms involving y^{-1} do not appear in the integrand of (3.14). Since the exponents that can appear in the integrand are of the form

$$-(s+4), \qquad s = 0,1,2,\ldots,n-3,$$

there can be no logarithmic terms in (3.14) and $v_n(y)$ is an n-th degree polynomial in y. Hence, we have proved that system (3.12) admits a formal first integral of the form $F(x,y) = \sum_{n=1}^{\infty} v_n(y)x^n$. Consequently it has an analytic first integral around the origin.

If condition (6) holds, system (3.3) is reduced to

$$\begin{cases} \dfrac{dx}{dt} = x + \dfrac{3}{5}\,a_1 y x^5 + \dfrac{3}{5}\,a_3 y^5 x, \\ \dfrac{dy}{dt} = -y - \dfrac{2}{5}\,a_1 y^2 x^4 - \dfrac{2}{5}\,a_3 y^6. \end{cases} \tag{3.15}$$

By the transformation $x \mapsto y$, $y \mapsto x$, $t \mapsto -t$, system (3.15) can be transformed into the form of (3.5), and therefore system (3.15) is integrable at the origin. □

Lemma 3.4. *If condition* (2.18) *holds, system* (3.3) *is integrable at the origin if and only if one of the following conditions holds:*

(i) $b_1^3 b_3^5 - a_1^5 a_3^3 = 0,\ a_1 b_3 \neq 0$;

(ii) $a_1 = b_1 = 0$;

(iii) $a_1 = b_3 = 0,\ b_1 \neq 0$.

Proof. Elementary computations shows that condition (2.18) is consistent with conditions (1), (3), (4) in Lemma 3.3, and is inconsistent with the other conditions (2), (5), (6). Hence by Lemma 3.3, the result follows. □

The proof for Theorems 2.8 and 2.9. According to Theorem 2.6 in Section 2, we can arrive at Theorems 2.8 and 2.9 by using Lemmas 3.2 and 3.4, respectively.

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