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Biologically motivated stability results for a general class of impulsive functional differential equations

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Abstract. Motivated by considerations on the success of integrated pest management strategies and on the associated economically relevant thresholds, this paper is concerned with the finite time dynamics of a class of impulsive functional differential systems with delay. By using an approach based on the Lyapunov–Razumikhin method, we determine sufficient conditions for the finite-time contractive stability of the null solution, these findings being then interpreted in biological terms as predictors for the success of a pest management strategy. Numerical simulations are also carried out to illustrate the feasibility of our results.

Keywords: impulsive functional differential systems, finite time stability, Lyapunov–Razumikhin method, integrated pest management.

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1 Introduction

Nonlinear phenomena with inherent discontinuities or involving abrupt perturbations or sudden behavioral changes can be successfully modeled by using hybrid dynamical systems, characterized by the coexistence of continuous and discrete dynamics. In this regard, impulsive dynamical systems can be viewed as particular types of hybrid dynamical systems which are characterized by three elements: a differential equation or system, which governs the dynamics of the model between the occurrences of the impulsive perturbations (also called resetting events), a difference equation, which describes the change of states as a result of impulsive perturbations and a criterion to decide whether or not the states of the system are to be reset (Nersesov and Haddad [28]). If the impulsive perturbations occur whenever trajectories reach a given manifold of the state space, the so-called resetting set, the corresponding impulsive dynamical system is called state-dependent, while if the impulsive perturbations occur at prescribed times, independent of the system states, then the corresponding impulsive dynamical system is called time-dependent.

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During recent years, impulsive functional differential equations (IFDEs) and impulsive ordinary differential equations (IODEs) have found their applications in mechanics (impact mechanical systems, Galyaev *et al.* [12], Tornambè [41]), aeronautics (satellite manoeuvering, Wiesel [44], Ryo *et al.* [32]), communication security (encryption via signal masking or modulation, Khadra *et al.* [20, 21]), economics (control of financial markets, Sakellaris [33]), population dynamics (hunting, harvesting or stocking of predator-prey models, Jin *et al.* [18], Zhang *et al.* [46], Apreutesei *et al.* [5]), neural networks (amplifiers with finite switching speed or perturbation by external stimuli, Chen and Shen [8], Stamov and Stamova [35]), control theory (synchronization of chaotic systems, Li *et al.* [25], Tao and Chua [40]), agriculture (integrated pest management, Tang and Cheke [39], Georgescu and Zhang [14], Shi and Chen [34]), medicine (vaccination strategies, Stone *et al.* [37], Gao *et al.* [13], immunotherapy, Bunimovich-Mendrazitsky *et al.* [7]), to mention only a few fields. See also Aihara and Suzuki [1] and Haddad *et al.* [16] for general overviews of the theory of hybrid and impulsive dynamical systems. It should also be noted that processes described by IFDEs or IODEs often exhibit dependence on their past history (see Gopalsamy [15], Kuang [22] and the references therein).

The Lyapunov stability of IFDEs and IODEs has received a lot of attention in recent years, often as a quantifier for the success of integrated pest management (IPM) strategies or of vaccination campaigns (Rafikov *et al.* [30], Georgescu and Zhang [14], Shi and Chen [34], Stone *et al.* [37], Gao *et al.* [13], Pei *et al.* [29]). However, while it is natural to gauge the success of an IPM strategy in terms of the asymptotic properties of the pest-eradication solution (or of the susceptible pest-eradication solution, if the control strategy relies on the release of infective pests, with the purpose of spreading a disease into the pest population), this characterization does not account neither for the concrete values of the action thresholds involved (the economic injury level (EIL), defined as the amount of pest injury which justifies the cost of using controls or the lowest pest density which causes economic damage, or the economic threshold (ET) (lower than the EIL), defined as the density at which control measures should be used to prevent an increasing pest population from reaching EIL) nor for the fact that pest outbreaks should usually be contained within a prescribed time frame.

For this situation, and for other similar contexts, an adaptation of Lyapunov stability concepts in order to deal with stabilization under maximal bounds within prescribed time intervals is more meaningful. To this purpose, the concept of finite-time stability was first introduced in the control literature in the early fifties (Kamenkov [19], Lebedev [24]). See also Dorato [10], Weiss and Infante [43]. Here, it is to be noted that Lyapunov stability (LAS) and finite-time stability (FTS) are essentially independent concepts, the former dealing with the long-term behavior of a system (after enough time has passed, that is) and the latter with the behavior of a system within a specified (possibly short) time frame. Particularly, LAS does not guarantee FTS due to possible artifacts and odd short-term behavior arising from particular initial conditions, while a Lyapunov unstable system can be FTS with respect to suitable boundedness thresholds when considered on a suitably small time interval. Also, from a qualitative viewpoint, LAS is an absolute concept, while FTS is not, being tied to the concrete values of the upper bounds involved and being more of a boundedness concept with strict specifications.

A more recent notion of FTS, which requires the convergence of the trajectories to the null solution in finite time has been employed in Bhat and Bernstein [6] and Nersesov and Haddad [28]. Without specifying bounding regions or time frames, but requiring convergence and being useful in certain control problems, this notion is related to LAS, but unrelated to the classical viewpoint on FTS adopted in this paper.

Sufficient conditions for the finite-time stability and finite time stabilization of impulsive and hybrid systems have recently been obtained by a number of authors (Xu and Sun [45], Zhao *et al.* [48]). However, only a few papers have considered approaches based on the use of Lyapunov functions to discuss the finite time stability of impulsive systems (Amato *et al.* [2], Ambrosino *et al.* [4], Chen *et al.* [9], Moulay and Perruquetti [26]). In this regard, Lyapunov–Razumikhin method (see Myshkis [27] for an overview), based on the use of Lyapunov guiding functions in combination with conditions which ensure the impossibility of a first breakdown (a crossing of the boundary of a vicinity, for instance) has already proven its usefulness for investigating the LAS of solutions of systems with delays.

The remaining part of this paper is structured as follows. The next section introduces certain preliminary notions and notations pertaining to a class of finite-time impulsive dynamical systems, together with several auxiliary results which are employed throughout this paper. These conditions are then used in Section 3 to obtain finite-time stability results for the model presented in Section 2, which represent the main contribution of this paper. Finally, a few concluding remarks are given in Section 4 together with numerical simulations which illustrate the feasibility of our results.

2 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}_+$ denote the set of positive real numbers and let $\mathbb{R}^n$ denote the real n-dimensional space endowed with the usual Euclidean norm $\|\cdot\|$. For any interval $J \subseteq \mathbb{R}_+$ and set $\mathbf{S} \subseteq \mathbb{R}^{\mathbf{n}}$, we shall denote by $\mathbf{C}(J, \mathbf{S})$ the set of functions $\psi\colon J \to \mathbf{S}$ which are continuous on J and by $\mathbf{PC}(J, \mathbf{S})$ the set of functions $\psi\colon J \to \mathbf{S}$ which are piecewise continuous on J and have a finite number points of discontinuity where they are continuous from the left. For $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ and $r > 0$, let us denote by $\mathbf{B}_r = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| < r\}$ the open ball of center $\mathbf{0}$ and radius r.

Let us consider the delayed system of time-dependent impulsive differential equations

$$\begin{cases} x'(t) = f(t, x_t), & t \in [T_0, T_0 + T)\backslash\mathcal{T}, \\ \Delta x(t)|_{t=t_k} = I_k(t_k, x(t_k), x_{t_k}), & 1 \le k \le N, \\ x(T_0 + \theta) = \phi(T_0 + \theta), & \theta \in [-\tau, 0]. \end{cases} \tag{2.1}$$

We assume that the initial time T_0 is a positive real number, the delay τ is strictly positive, the time horizon T is also strictly positive and finite and the time interval $(T_0, T_0 + T)$ includes a finite set of resetting points $\mathcal{T} = \{t_1, t_2, \ldots, t_N\}$. Also, f is assumed to be continuous on $([T_0, T_0 + T)\backslash\mathcal{T}) \times \mathbb{D}$, where $\mathbb{D}$ is an open set in $\mathbf{PC}([-\tau, 0], \mathbb{R}^n)$, and $\phi \in \mathbb{D}$. For each $t \ge T_0$, the function $x_t \in \mathbb{D}$ is defined by $x_t(s) = x(t + s)$, $-\tau \le s \le 0$. In the case when $\tau = \infty$, the interval $[T_0 - \tau, T_0]$ is understood to be replaced by $(-\infty, T_0]$.

For each $1 \le k \le N$, the instantaneous jump in the state of the system at the resetting point t_k is given by $\Delta x(t)|_{t=t_k} = x(t_k+) - x(t_k)$, while the impulsive perturbation $I_k \in \mathbf{C}([T_0, T_0 + T) \times \mathbb{R}^n \times \mathbf{PC}([-\tau, 0], \mathbb{R}^n), \mathbb{R}^n)$. For a finite delay τ, the norm of the function $\phi \in \mathbf{PC}([-\tau, 0], \mathbb{R}^n)$ is given by $\|\phi\|_\tau = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\|$, while in the case $\tau = \infty$ the norm is given by $\|\phi\|_\infty = \sup_{\theta \in (-\infty, 0]} \|\phi(\theta)\|$.

We further suppose that $f(t, 0) \equiv 0$ and $I_k(t, 0, \mathbf{0}) \equiv 0$ for all $1 \le k \le N$, where by $\mathbf{0}$ we mean the null function in $\mathbf{PC}([-\tau, 0], \mathbb{R}^n)$, so that the system (2.1) admits the null solution, whose stability we shall discuss in what follows. Let us also denote $t_0 = T_0$. To introduce a theoretical framework for our considerations, we also need the following notations, definitions and functional classes.

Definition 2.1 ([23]).

(i) A function a is said to belong to class $\mathbf{K}$ if a $\in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\mathrm{a}(0) = 0$ and $\mathrm{a}(u)$ is strictly increasing in u.

(ii) A function a is said to belong to class $\mathbf{K_1}$ if a $\in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, $\mathrm{a}(0) = 0$ and $\mathrm{a}(u)$ is nondecreasing in u.

Definition 2.2 ([17]). Let $V\colon [T_0 - \tau, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$. Then V is said to belong to class $\mathbf{V_0}$ if

(i) For all $1 \le k \le n$, V is continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and the limit

$$\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x)$$

exists and is finite;

(ii) V is locally Lipschitzian in the second variable and $V(t,0) = 0$.

Definition 2.3 ([36]). Define the upper Dini derivative of V along the solution $(t, x(t))$ of (2.1) by

$$D^+V(t,x) = \limsup_{h\to 0^+} \frac{1}{h}\big[V(t+h, x(t) + hf(t, x_t)) - V(t, x(t))\big].$$

It is known that the Dini derivatives can be used to characterize the monotonicity of continuous functions. In this regard, the following result holds.

Lemma 2.4 ([31]). *Suppose that $u \in C([a,b), \mathbb{R})$ and that*

$$Du(t) \le 0 \quad \text{for } t \in [a,b)\backslash S,$$

D being a fixed Dini derivative and S being an at most countable subset of $[a,b)$. Then u is nonincreasing on $[a,b)$.

For further details, including a general comparison lemma, see [31], Appendix 1.

We are now ready to introduce the concepts of finite-time stability and finite-time contractive stability which will be employed in what follows. Essentially, finite time stability represents the capacity of the trajectories to obey a given maximal bound within a specified time interval, provided that the initial data also satisfies a prescribed boundedness estimation. Finite-time contractive stability represents, in addition to the above, the capacity of the trajectories to “shrink” under the estimation provided for the initial data from some moment onwards.

Definition 2.5. Given an initial time T_0 and an initial condition ϕ, denote by $x(t) = x(t; T_0, \phi)$ the solution of (2.1) which satisfies the initial condition $x(t; T_0, \phi) = \phi(t - T_0)$ for $T_0 - \tau \le t \le T_0$.

- The null solution of (2.1) is said to be finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$, $\alpha \le \gamma$, if for every trajectory $x(t)$, $\|\phi\|_\tau < \alpha$ implies $x(t) \in \mathbf{B}_\gamma$ for all $t \in [T_0, T_0 + T)$.

- The null solution of (2.1) is said to be finite-time quasi-contractively stable with respect to $(\alpha, \beta, T_0, T, \|\cdot\|)$, $\beta < \alpha$, if for every trajectory $x(t)$, $\|\phi\|_\tau < \alpha$ implies that there exists a $T_1 \in (T_0, T_0 + T)$ such that $x(t) \in \mathbf{B}_\beta$ for all $t \in (T_1, T_0 + T)$.

- The null solution of (2.1) is said to be finite-time contractively stable with respect to $(\alpha, \beta, \gamma, T_0, T, \|\cdot\|)$, if it is finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$, and it is quasi-contractively stable with respect to $(\alpha, \beta, T_0, T, \|\cdot\|)$.

To the best of our knowledge, the concept of finite stability has been introduced in Kamenkov [19] and Lebedev [24], while the concept of finite time contractive stability appears for the first time in Weiss and Infante [43]. It is to be noted that finite time stability concepts require, loosely speaking, bounds on the initial conditions such that, for a finite time horizon, the solutions resulting from these initial conditions do not exceed certain thresholds. Having this in view, one would perhaps be left wondering if pest management strategies would not be better addressed in the setting of asymptotic stabilization, in which the solutions do not exceed the given thresholds forever, not only within a given time horizon.

Although this setting is perhaps more mathematically established, asymptotic stabilization may not necessarily be cost effective, desirable, or even possible, since the use of control measures (in this paper, the impulsive perturbations) comes at a price, may have negative consequences upon the ecosystem (may bring resistance to chemical control measures, or wipe out beneficial species in the process) and may be useless after the time horizon passes. Actually, quantifying the success of a control strategy in terms of the asymptotic stability properties of the null solution is a better option for disease control models, in which the permanent eradication of a disease is sought after, rather than in agricultural-based or ecosystem-based pest control models, which have different concerns, some of them seasonal or not amounting to complete species extinction.

As previously mentioned, FTS concepts are well tailored to describe the concrete problems which arise when conceiving IPM strategies. In this regard, γ can be thought as being the the economic injury level EIL, β can be thought as being the economic threshold ET and α can be considered as an estimation of the initial pest population size. Under this scenario, for a finite-time contractively stable system, after the period $[T_0, T_0 + T)$ in which control measures are employed passes the pest population size is left under the ET without ever reaching EIL. This is the "safer", proactive way, in which the pest do not get to cause sustained economic damage.

The second possible choice is to think of γ as being the carrying capacity of the environment for the given pest, β as being the EIL and α as being an estimation of the initial pest population size. This is the less demanding course of action, in which for a finite-time contractively stable system, after the period $[T_0, T_0 + T)$ in which control measures are employed passes, the pest population size is left under the EIL. Consequently, the pests are not susceptible to cause serious economic losses in the short term, although since the pest population size may have surpassed the ET, it is possible they will soon surpass the EIL as well. Also, the pests never cause irreparable damage to the environment (since the carrying capacity of the environment for the given pest is never surpassed), although economic losses may be incurred, more severe than for the first course of action.

In this regard, an approach towards the finite-time stabilization of a three-dimensional predator-pest model with diseased pest has been proposed in Amato *et al.* [3]. Since only the finite-time stability is discussed therein, the contractive stability not being of concern, the purpose of the control strategy is that the size of the pest populations (susceptibles and infectives) do not surpass their respective EILs. A maximal threshold for the predator population is also defined, the boundedness constants for all populations being used to define a polytope in the space of states inside of which all trajectories are steered by using impulsive controls.

3 Main results

In this section, we shall establish theoretical results which provide sufficient conditions for the finite-time contractive stability of the null solution of the IFDE (2.1), our approach relying upon the use of Lyapunov–Razumikhin method.

Theorem 3.1. *Assume that there exist functions W_1, $W_2 \in \mathbf{K}$, $g \in \mathbf{K_1}$, $c \in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, $p \in \mathbf{PC}(\mathbb{R}_+, \mathbb{R}_+)$ and $V \in \mathbf{V_0}$, and constants $T^* > T_0$, $\rho > 0$, $\eta > 0$, $\{\beta_k\}_{k=1}^N \subset [0, \infty)$, $\{\gamma_k\}_{k=1}^N \subset [0, \infty)$ and $M > 1$ such that the following conditions hold:*

(i) For $s > 0$, $s < g(s) \le Ms$, and $\inf_{s>0} \frac{c(s)}{s} > 0$, $\inf_{s\ge 0} p(s) > 0$;

(ii) For $t \in [T_0 - \tau, T^*)$ and $\phi \in \mathbf{PC}([-\tau, 0], \mathbf{B}_\rho)$,

$$W_1(\|x(t; T_0, \phi)\|) \le V(t, x(t; T_0, \phi)) \le W_2(\|x(t; T_0, \phi)\|);$$

(iii) For $t \in [T_0, T^*)\backslash\mathcal{T}$ and $\phi \in \mathbf{PC}([-\tau, 0], \mathbf{B}_\rho)$, if

$$g(V(t, x(t; T_0, \phi)) \exp(\eta(t - T_0))) > \frac{V(t + \theta, x(t + \theta; T_0, \phi))}{M} \quad \textit{for } \theta \in [-\tau, 0],$$

then

$$D^+V(t, x(t; T_0, \phi)) \le -p(t)c(V(t, x(t; T_0, \phi)));$$

(iv) For all $(t_k, \phi) \in \mathcal{T} \times \mathbf{PC}([-\tau, 0], \mathbf{B}_\rho)$, $1 \le k \le N$ and $\theta \in [-\tau, 0]$

$$V(t_k^+, x(t_k^+; T_0, \phi)) \le (1 + \beta_k)V(t_k, x(t_k; T_0, \phi)) + \gamma_k V(t_k + \theta, x(t_k + \theta; T_0, \phi));$$

(v)

$$\min_{1\le l\le N} \int_{t_{l-1}}^{t_l} p(u)\, du > \sup_{t>0} \int_t^{\xi M t} \frac{du}{c(u)},$$

where

$$t_0 \doteq T_0, \qquad \xi \doteq \max_{1\le l\le N} \{\exp(\tilde{m}(t_l - t_{l-1}))\},$$

with

$$\tilde{m} \doteq \min\left\{\rho, \eta, \inf_{s>0} \frac{c(s)}{s} \cdot \inf_{s\ge 0} p(s)\right\};$$

(vi) There are $\gamma \in (0, \rho)$ and $H \in (0, \tilde{m})$ with the property that there exist $T \in (0, T^* - T_0)$, $T_1 \in (T_0, T_0 + T)$, $\alpha \in (0, W_2^{-1}(\frac{W_1(\gamma)}{MM^*}))$ and $\beta \in (0, \alpha)$ such that

$$T_1 > \frac{1}{H}(\ln \frac{W_2(\alpha)MM^*}{W_1(\beta)} + \eta T_0),$$

where

$$M^* \doteq \prod_{1\le j\le N} (1 + \beta_j + M\gamma_j \exp(\eta\tau)).$$

Then the null solution of (2.1) is finite-time contractively stable with respect to $(\alpha, \beta, \gamma, T_0, T, \|\cdot\|)$.

Proof. Let us observe that since $\alpha < W_2^{-1}\left(\frac{W_1(\gamma)}{MM^*}\right)$ and $MM^* > 1$, it follows that

$$W_2(\alpha) < \frac{W_1(\gamma)}{MM^*} < W_1(\gamma) \leq W_2(\gamma),$$

which implies that $\alpha < \gamma$.

We start by proving that the null solution is finite-time stable w.r.t. $(\alpha, \gamma, T_0, T, \|\cdot\|)$. To this purpose, let us fix ϕ such that $\|\phi\|_\tau < \alpha$ and prove that $x(t; T_0, \phi) \in \mathbf{B}_\gamma$ for all $t \in [T_0, T_0 + T)$.

Choose $\epsilon \in (H, \tilde{m})$, denote

$$x(t) = x(t; T_0, \phi); \qquad V(t) = V(t, x(t))$$

and define

$$\Phi(t) = \begin{cases} V(t)\exp(\epsilon(t - T_0)), & t \in [T_0, T_0 + T) \\ V(t), & t \in [T_0 - \tau, T_0). \end{cases} \tag{3.1}$$

Define also $t_{N+1} = T_0 + T$, $\beta_0 = 0$, $\gamma_0 = 0$ and

$$M_k^* = \prod_{0 \leq j \leq k} (1 + \beta_j + M\gamma_j \exp(\eta\tau)), \qquad 0 \leq k \leq N.$$

We shall prove that

$$\Phi(t) \leq MM_k^* W_2(\alpha), \qquad t \in (t_k, t_{k+1}], \quad 0 \leq k \leq N, \tag{3.2}$$

the interval (t_N, t_{N+1}) replacing $(t_N, t_{N+1}]$ for $k = N$, inequality which will be of crucial importance in what follows. Note that all inequalities in terms of Φ are easily translatable as inequalities in terms of V by means of (3.1).

For $k = 0$, we need to prove that

$$\Phi(t) \leq MW_2(\alpha), \qquad t \in (t_0, t_1]. \tag{3.3}$$

First, it is seen by (ii) that

$$\Phi(t_0) = V(t_0) \leq W_2(\|x(t_0)\|) \leq W_2(\alpha) < MW_2(\alpha).$$

Suppose to the contrary that (3.3) does not hold, and consequently there exists $t \in (t_0, t_1]$ such that $\Phi(t) > MW_2(\alpha)$. Let us note that Φ is continuous on $(t_0, t_1]$ and define

$$t^* = \inf\{t \in (t_0, t_1] \mid \Phi(t) \geq MW_2(\alpha)\}.$$

Since $\Phi(t_0) < MW_2(\alpha)$, it is seen that $t^* \in (t_0, t_1]$. Also, due to the definition of t^*, it follows that

$$\Phi(t) < MW_2(\alpha) \quad \text{for } t \in [t_0, t^*), \qquad \Phi(t^*) = MW_2(\alpha), \tag{3.4}$$

which in turn yields using (i) that

$$g(\Phi(t^*)) > \Phi(t^*) = MW_2(\alpha). \tag{3.5}$$

Note also that

$$g(\Phi(t_0)) < M\Phi(t_0) \leq MW_2(\alpha).$$

Let us then define

$$t^{**} = \sup\{t \in (t_0, t^*] \mid g(\Phi(t)) \leq MW_2(\alpha)\},$$

and note that $t^{**} \in (t_0, t^*)$, while

$$g(\Phi(t)) > MW_2(\alpha) \quad \text{for } t \in (t^{**}, t^*], \qquad g(\Phi(t^{**})) = MW_2(\alpha). \tag{3.6}$$

By (3.1) and (3.6), one sees that

$$g(V(t)\exp(\eta(t-T_0))) \geq g(\Phi(t)) > MW_2(\alpha) \quad \text{for } t \in (t^{**}, t^*]. \tag{3.7}$$

We shall now prove the opposite inequality for $V(s)$, which would enable us to use condition (iii). For $s \in [t_0, t^*]$, one has using (3.1) and (3.4) that

$$V(s) \leq \Phi(s) \leq MW_2(\alpha).$$

while for $s \in [t_0 - \tau, t_0]$ one obtains from (ii) that

$$V(s) = V(s, x(s)) = V(s, \phi(s)) \leq W_2(\|\phi(s)\|) \leq W_2(\alpha).$$

Consequently,

$$V(s) \leq MW_2(\alpha) \quad \text{for } s \in [t^{**} - \tau, t^*]. \tag{3.8}$$

From (3.7) and (3.8), one obtains that

$$g(V(t)\exp(\eta(t-T_0))) > V(t+\theta) \quad \text{for } t \in (t^{**}, t^*] \text{ and } \theta \in [-\tau, 0].$$

We are now ready to establish the monotonicity of Φ on $[t^{**}, t^*]$ with the help of condition (iii). Using this condition, it follows that, for $t \in [t^{**}, t^*]$,

$$\begin{aligned}
D^+\Phi(t) &= D^+V(t)\exp(\epsilon(t-T_0)) + \epsilon V(t)\exp(\epsilon(t-T_0)) \\
&\leq -p(t)c(V(t))\exp(\epsilon(t-T_0)) + \epsilon V(t)\exp(\epsilon(t-T_0)) \\
&= -V(t)\exp(\epsilon(t-T_0))\big(p(t)\frac{c(V(t))}{V(t)} - \epsilon\big) \\
&\leq 0.
\end{aligned} \tag{3.9}$$

Consequently, Φ is nonincreasing on $[t^{**}, t^*]$ and therefore $\Phi(t^{**}) \geq \Phi(t^*)$. However, since

$$\Phi(t^{**}) < g(\Phi(t^{**})) = MW_2(\alpha) = \Phi(t^*),$$

this is a contradiction. It now follows that (3.2) holds for $k = 0$.

Suppose now that (3.2) holds for $0 \leq k \leq l-1$ and prove that it is also valid for $k = l$. To this purpose, we first prove that

$$\Phi(t_l) \leq M^*_{l-1}W_2(\alpha). \tag{3.10}$$

Suppose to the contrary that $\Phi(t_l) > M^*_{l-1}W_2(\alpha)$. Then either $\Phi(t) > M^*_{l-1}W_2(\alpha)$ for all $t \in (t_{l-1}, t_l]$, or there exists $t \in (t_{l-1}, t_l]$ such that $\Phi(t) \leq M^*_{l-1}W_2(\alpha)$.

In the first case, since (3.2) holds for $k = l-1$, it is seen that

$$\Phi(t) > M^*_{l-1}W_2(\alpha) \geq \frac{\Phi(t+\theta)}{M}, \qquad t \in (t_{l-1}, t_l],\ \theta \in [-\tau, 0]. \tag{3.11}$$

Note that, due to (ii) and to the fact that, by its definition, $\{M^*_k\}_{k=0}^N$ is an increasing sequence, (3.11) holds even if $\tau > t_l - t_{l-1}$. In fact, if $t + \theta$ belongs to a previous interval $(t_{k-1}, t_k]$ rather than to the working interval $(t_{l-1}, t_l]$, then the better estimation $\Phi(t+\theta) \leq MM^*_{k-1}W_2(\alpha)$ is available, rather than the required one, $\Phi(t+\theta) \le MM^*_{l-1}W_2(\alpha)$, while if $t+\theta \in [T_0-\tau, T_0]$, then $\Phi(t+\theta) \le W_2(\alpha)$. This implies that

$$\Phi(t_l) = \Phi(t_l^-) \ge \frac{\Phi(t_{l-1}^+)}{M}.$$

Then, by the definition of Φ,

$$MV(t_l^-)\exp(\epsilon(t_l - T_0)) \ge V(t_{l-1}^+)\exp(\epsilon(t_{l-1} - T_0)),$$

that is,

$$MV(t_l^-)\exp(\epsilon(t_l - t_{l-1})) > V(t_{l-1}^+).$$

Let us now define

$$\xi_1 = \max_{1\le l\le N}\{\exp(\epsilon(t_l - t_{l-1}))\}$$

and observe that $\xi_1 < \xi$. Then

$$\xi MV(t_l^-) > \xi_1 MV(t_l^-) > V(t_{l-1}^+). \tag{3.12}$$

By (i), (3.1) and (3.11), it is seen that, for $t \in (t_{l-1}, t_l]$ and $\theta \in [-\tau, 0]$,

$$g(V(t)\exp(\eta(t-T_0))) \ge g(\Phi(t)) > \Phi(t) \ge \frac{\Phi(t+\theta)}{M} \ge \frac{V(t+\theta)}{M}.$$

Using condition (iii), it follows that the inequality $D^+V(t) \le -p(t)c(V(t))$ holds for all $t \in (t_{l-1}, t_l]$. Also, from (3.12) we obtain that

$$\int_{V(t_l^-)}^{V(t_{l-1}^+)} \frac{du}{c(u)} < \int_{V(t_l^-)}^{\xi MV(t_l^-)} \frac{du}{c(u)} \le \sup_{s>0}\int_s^{\xi Ms}\frac{du}{c(u)}. \tag{3.13}$$

However, by means of condition (v), one notes that

$$\int_{V(t_l^-)}^{V(t_{l-1}^+)} \frac{du}{c(u)} \ge \int_{t_{l-1}}^{t_l} p(u)\,du \ge \inf_{1\le l\le N}\int_{t_{l-1}}^{t_l} p(u)\,du, \tag{3.14}$$

which leads to a contradiction.

Next, we consider the second case. Let us define

$$t' = \sup\{t \in (t_{l-1}, t_l] \mid \Phi(t) \le M^*_{l-1}W_2(\alpha)\}.$$

Then $t' \in (t_{l-1}, t_l]$ and

$$\Phi(t) > M^*_{l-1}W_2(\alpha) \quad \text{for } t \in (t', t_l], \qquad \Phi(t') = M^*_{l-1}W_2(\alpha),$$

which implies that, for $t \in [t', t_l]$ and $\theta \in [-\tau, 0]$,

$$g(V(t)\exp(\eta(t-T_0))) \ge g(\Phi(t)) > \Phi(t) > M^*_{l-1}W_2(\alpha) \ge \frac{\Phi(t+\theta)}{M} \ge \frac{V(t+\theta)}{M}.$$

By applying the same argument as in the proof of (3.9), we then obtain that Φ is nonincreasing on $[t', t_l]$. In particular,

$$\Phi(t') \ge \Phi(t_l) = \Phi(t_l^-).$$

However, this contradicts the fact that

$$\Phi(t_l^-) > M_{l-1}^* W_2(\alpha) = \Phi(t').$$

As a result, we can claim that (3.10) holds. In the following we shall prove that (3.2) holds for $k = l$, that is,

$$\Phi(t) \le M M_l^* W_2(\alpha), \qquad t \in (t_l, t_{l+1}].$$

Suppose that this assertion is not true. There then exists $t \in (t_l, t_{l+1}]$ such that

$$\Phi(t) > M M_l^* W_2(\alpha).$$

Let us define

$$t'' = \inf\{t \in (t_l, t_{l+1}] \mid \Phi(t) \ge M M_l^* W_2(\alpha)\}.$$

Then $t'' \in (t_l, t_{l+1})$ and

$$\Phi(t) < M M_l^* W_2(\alpha) \ \text{ for } t \in (t_l, t''), \qquad \Phi(t'') = M M_l^* W_2(\alpha).$$

It is easy to see that

$$g(\Phi(t'')) = g(M M_l^* W_2(\alpha)) > M M_l^* W_2(\alpha)$$

and, by (i), (iv) and (3.10),

$$\begin{aligned} g(\Phi(t_l^+)) &\le M\Phi(t_l^+) \\ &= M V(t_l^+) \exp(\epsilon(t_l - T_0)) \\ &\le M\big((1+\beta_l)V(t_l) + \gamma_l V(t_l+\theta)\big) \exp(\epsilon(t_l - T_0)) \\ &< M\big((1+\beta_l)\Phi(t_l) + \gamma_l \exp(\eta\tau)\Phi(t_l+\theta)\big) \\ &< M(1+\beta_l + M\gamma_l \exp(\eta\tau)) M_{l-1}^* W_2(\alpha) \\ &= M M_l^* W_2(\alpha). \end{aligned}$$

Consequently, we may define

$$\widetilde{t} = \sup\{t \in (t_l, t''] \mid g(\Phi(t)) \le M M_l^* W_2(\alpha)\}.$$

Then $\widetilde{t} \in (t_l, t'']$ and

$$g(\Phi(t)) > M M_l^* W_2(\alpha) \quad \text{for } t \in (\widetilde{t}, t''], \qquad g(\Phi(\widetilde{t})) = M M_l^* W_2(\alpha).$$

Thus, we also have

$$\begin{aligned} g(V(t)\exp(\eta(t - T_0))) &\ge g(\Phi(t)) > M M_l^* W_2(\alpha) \ge \Phi(t+\theta) \\ &\ge \frac{V(t+\theta)}{M} \qquad \text{for } t \in (\widetilde{t}, t''] \text{ and } \theta \in [-\tau, 0]. \end{aligned}$$

As done above for the proof of (3.9), we can obtain that Φ is nonincreasing on $[\widetilde{t}, t'']$. Thus, one notes that

$$\Phi(\widetilde{t}) \ge \Phi(t''),$$

which contradicts the fact that

$$\Phi(t'') = M M_l^* W_2(\alpha) = g(\Phi(\widetilde{t})) > \Phi(\widetilde{t}).$$

By the above, it is seen that (3.2) holds globally. Consequently,

$$\Phi(t) \leq MM^* W_2(\alpha) \quad \text{for } t \in [T_0, T_0 + T)$$

and, by (ii),

$$W_1(\|x(t)\|) \leq V(t) \leq MM^* W_2(\alpha) \exp(-\epsilon(t - T_0)) \quad \text{for } t \in [T_0, T_0 + T).$$

Since $\alpha < W_2^{-1}(\frac{W_1(\gamma)}{MM^*})$, this implies

$$W_1(\|x(t)\|) < W_1(\gamma) \quad \text{for } t \in [T_0, T_0 + T)$$

and consequently

$$x(t) \in \mathbf{B}_\gamma \quad \text{for } t \in [T_0, T_0 + T).$$

It now follows that the null solution of the system (2.1) is finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$.
Similarly, using again (ii) and (3.2),

$$W_1(\|x(t)\|) \leq MM^* W_2(\alpha) \exp(-\epsilon(T_1 - T_0)) \quad \text{for } t \in (T_1, T_0 + T),$$

which implies, by the choices of ϵ and H,

$$W_1(\|x(t)\|) \leq W_1(\beta) \exp(T_1(H - \epsilon) + T_0(\epsilon - \eta)) < W_1(\beta) \quad \text{for } t \in (T_1, T_0 + T),$$

and consequently

$$x(t) \in \mathbf{B}_\beta \quad \text{for } t \in (T_1, T_0 + T).$$

It now follows that the null solution of the system (2.1) is finite-time quasi-contractively stable with respect to $(\alpha, \beta, T_0, T, \|\cdot\|)$, which finishes the proof. □

At this point, it should be noted that Theorem 3.1 (our main result, actually), is related in its purpose and approach towards proof to Theorem 3.1 of Fu and Li [11], Theorem 3.1 of Wang and Zhu [42] and Theorem 3.1 of Sun and Li [38], although these results are stated as classical exponential stability results and the impulsive perturbations are applied in a slightly different manner. While our results have a different scope than those of [11], [38] and [42] (estimations on a finite time horizon as opposed to global exponential estimations), in technical terms the approaches are directly related, and in this regard one notes that our condition (iv) is weaker than the corresponding condition (iii) in Theorem 3.1 of Fu and Li [11], while condition (i) in Theorem 3.1 of Wang and Zhu [42] and condition (i) in Theorem 3.1 of Sun and Li [38] are particular cases of our condition (i). Also, Theorem 3.1 is an improvement of our previous related result, Theorem 1 in Zhang and Georgescu [47], which does not account for the influence of delay and features a significantly stronger form of condition (iii).

Let us also elaborate upon the significance of the conditions employed in the statement of Theorem 3.1. In this regard, hypothesis (iii) of Theorem 3.1 states that if a function of $V(t)$ is larger than another function of all previous values of $V(s)$ for s in the "history" interval $[t - \tau, t]$, then $D^+V(t)$ should satisfy an estimation which in particular ensures its negative sign. That is, if V grows too large in an interval of length equal with the value of the delay, then it should decrease with at least a certain speed in order to ensure that the solution x still obeys the finite stability estimation. The use of a function g (which in concrete situations is usually a multiple of identity), subject to condition (i), leads to a more general and perhaps flexible growth condition. Common examples of functions W_1 and W_2 are multiples of the square of the norm (the norm measuring the size of the pest population), at least in the case in which V is a (possibly perturbed) multiple of the square of the norm as well. See also the examples in Section 4 for further insight.

Typical examples of impulsive perturbations I_k which appear in integrated pest management are $I_k(t_k, x(t_k)) = -p_k x(t_k)$ (proportional reduction of the pest population size due to pesticide spraying), $I_k(t_k, x(t_k)) = p_k x(t_k)$ (proportional increase of the pest population size due to birth pulses) and $I_k(t_k, x(t_k)) = \mu$ (impulsive release of a constant amount of individuals, of use especially in models with disease in the pest). In this regard, β_k's may be thought as accounting for the effects of birth pulses, although both β_k's and γ_k's can also be thought as "safety" parameters, allowing for possible errors in the estimation of V (that is, for possible errors in the estimation of the size of the pest population).

One way of interpreting Theorem 3.1 is as a controllability result. In this regard, it is seen that even large values of β_k's are allowed, on condition that they are balanced by a corresponding decrease of V between pulses, which yields the optimistic conclusion that even pests with strong reproductive potential (or perhaps with successive immigrational waves) can successfully be controlled provided that appropriate control measure are taken.

Remark 3.2. Note that the conclusions of Theorem 3.1 (boundedness estimations, in their essence) hold with the same proof if (iv) is replaced by the following condition

(iv') For all $(t_k, \phi) \in \mathcal{T} \times \mathbf{PC}([-\tau, 0], \mathbf{B}_\rho)$, $1 \le k \le N$ and $\theta \in [-\tau, 0]$

$$V(t_k^+, x(t_k^+; T_0, \phi)) \le (1+\beta_k)(1-\delta_k)V(t_k, x(t_k; T_0, \phi)) + \gamma_k V(t_k + \theta, x(t_k + \theta; T_0, \phi)),$$

where $\{\delta_k\}_{k=1}^N \subset [0, 1)$. However, M_k^*, $0 \le k \le N$, and M^* should be replaced by $\tilde{M}_k$, $0 \le k \le N$, and $\tilde{M}$, respectively, where

$$M_l^* = \prod_{0 \le j \le l} \big((1+\beta_j)(1-\delta_j) + M\gamma_j \exp(\eta\tau)\big), \quad 0 \le k \le N,$$

$$\tilde{M}_k = \max_{0 \le l \le k} M_l^*, \quad \tilde{M} = \max_{0 \le l \le N} M_l^*, \quad \delta_0 = 0.$$

Of course, one would ask which is the motivation of using an estimation of type (iv'), which showcases two distinct tendencies: one (involving β_k's) possibly increasing the pest population size and the other (involving δ_k's) causing a decrease of the population size. Actually, as mentioned before, the system (2.1) may be subject to both impulsive pest control measures (decreasing the pest population size) and pulse birth phenomena (increasing the pest population size). Even if these types of perturbations do not actually occur simultaneously, they may be thought as formally acting in this manner by choosing the appropriate β_k or δ_k's as being zero. Further remarks in this direction will be made in the next section.

From the above Theorem 3.1, by choosing $g(t) = Mt$, $p(t) \equiv p > 0$ and $c(u) \equiv u > 0$, one obtains the following practical consequence.

Corollary 3.3. *Assume that there exist functions W_1, $W_2 \in \mathbf{K}$, $c \in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, and $V \in \mathbf{V_0}$, and constants $T^* > T_0$, $\rho > 0$, $\eta > 0$, $p > 0$, $\{\beta_k\}_{k=1}^N \subset [0, \infty)$, $\{\gamma_k\}_{k=1}^N \subset [0, \infty)$ and $M > 1$ such that the following conditions hold:*

(i) For $t \in [T_0, T^*)$ and $\phi \in \mathbf{PC}([-\tau, 0], \mathbf{B}_\rho)$,

$$W_1(\|x(t; T_0, \phi)\|) \le V(t, x(t; T_0, \phi)) \le W_2(\|x(t; T_0, \phi)\|);$$

(ii) *For* $t \in [T_0, T^*) \backslash \mathcal{T}$ *and* $\phi \in \mathbf{PC}([-\tau, 0], \mathbf{B}_\rho)$, *if*

$$M^2 V(t, x(t; T_0, \phi)) > V(t+\theta, x(t+\theta; T_0, \phi)) \exp(-\eta(t - T_0)) \quad \text{for } \theta \in [-\tau, 0],$$

then

$$D^+ V(t, x(t; T_0, \phi)) \le -pc(V(t, x(t; T_0, \phi)));$$

(iii) *For all* $(t_k, \phi) \in \mathcal{T} \times \mathbf{PC}([-\tau, 0], \mathbf{B}_\rho)$, $1 \le k \le N$ *and* $\theta \in [-\tau, 0]$

$$V(t_k^+, x(t_k^+; T_0, \phi)) \le (1+\beta_k) V(t_k, x(t_k; T_0, \phi)) + \gamma_k V(t_k + \theta, x(t_k + \theta; T_0, \phi));$$

(iv)

$$\min_{1 \le k \le N} \{t_k - t_{k-1}\} > \frac{1}{p} \left(\tilde{m} \max_{1 \le k \le N} \{t_k - t_{k-1}\} + \ln M \right),$$

where

$$\tilde{m} \doteq \min\{\rho, \eta, p\};$$

(v) *There are* $\gamma \in (0, \rho)$ *and* $H \in (0, \tilde{m})$ *with the property that there exist* $T \in (0, T^* - T_0)$, $T_1 \in (T_0, T_0 + T)$, $\alpha \in (0, W_2^{-1}(\frac{W_1(\gamma)}{MM^*}))$ *and* $\beta \in (0, \alpha)$ *such that*

$$T_1 > \frac{1}{H} \left(\ln \frac{W_2(\alpha) M M^*}{W_1(\beta)} + \eta T_0 \right),$$

where

$$M^* \doteq \prod_{1 \le j \le N} (1 + \beta_j + M \gamma_j \exp(\eta \tau)).$$

Then the null solution of (2.1) *is finite-time contractively stable with respect to* $(\alpha, \beta, \gamma, T_0, T, \|\cdot\|)$.

Remark 3.4. Again, note that the conclusions of Corollary 3.3 still hold if condition (iv) is replaced by condition (iv'), provided that M_k^*, $0 \le k \le N$, and M^* are replaced by $\tilde{M}_k$, $0 \le k \le N$, and $\tilde{M}$, respectively, as done above.

To prove the LAS of a certain solution via the Lyapunov–Razumikhin method in the classical situation (i.e. no impulses), the functional V needs to be decreasing in certain circumstances. The same viewpoint has been proven true for Theorem 3.1. However, we shall observe that it is possible to obtain FTS results even if the functional V is not decreasing at all. The reason is actually twofold. Apart from the fact that for the FTS of the null solution only the fulfilment of a certain boundedness estimation for a certain interval of time is necessary, rather than the classical null convergence, the burden of stabilizing the null solution can actually be shifted to the impulsive perturbation. Actually, the hypotheses can be further weakened by giving up one of the two inequalities satisfied by g in (i) of Theorem 3.1. Of course, if V is not assumed to be decreasing anymore, stronger hypotheses should be imposed on the impulsive perturbations. Considering impulsive perturbations of this form, one then obtains the following result.

Theorem 3.5. *Assume that there exist functions* W_1, $W_2 \in \mathbf{K}$, $g \in \mathbf{K_1}$, $c \in \mathbf{C}(\mathbb{R}_+, \mathbb{R}_+)$, $p \in \mathbf{PC}(\mathbb{R}_+, \mathbb{R}_+)$ *and* $V \in \mathbf{V_0}$, *and constants* $T^* > T_0$, $m > 1$, $\eta > 0$, $\widetilde{c} > 0$, $\{\beta_k\}_{k=1}^N \subset [0, \infty)$, $\{\delta_k\}_{k=1}^N \subset [0, 1)$, $\{\gamma_k\}_{k=1}^N \subset [0, 1)$ *and*

$$m > \exp((\eta + \overline{p}\widetilde{c})\underline{\varrho}),$$

$$\underline{\varrho} \doteq \max_{1\le k\le N}\{t_k - t_{k-1}\} < \frac{\ln\left(\frac{1}{\max_{1\le k\le N}\{(1+\beta_k)(1-\delta_k)+\gamma_k\exp(\eta\tau)\}}\right)}{\eta+\overline{p}\widetilde{c}} \doteq \overline{\varrho} \tag{3.15}$$

and

$$\overline{p} \doteq \max_{1\le k\le N}\left\{\frac{1}{t_k - t_{k-1}}\int_{t_{k-1}}^{t_k} p(s)ds\right\};$$

such that the following conditions hold:

(i) *For* $s>0$, $g(s)\ge ms$, $c(s)\le \widetilde{c}s$;

(ii) *For* $(t,x(t))\in[T_0-\tau,T^*)\times \mathbf{B}_\rho$,

$$W_1(\|x(t)\|)\le V(t,x(t;T_0,\phi))\le W_2(\|x(t)\|);$$

(iii) *For any* $\phi\in \mathbf{PC}([-\tau,0],\mathbf{B}_\rho)$, *if*

$$g(V(t,x(t;T_0,\phi))\exp(\eta(t-T_0)))>\frac{V(t+\theta,\phi(\theta))}{m\exp(\eta\tau)} \quad \text{for } \theta\in[-\tau,0],\ t\notin\mathcal{T},$$

then

$$D^+V(t,x(t;T_0,\phi))\le p(t)c(V(t,x(t;T_0,\phi)));$$

(iv) *For all* $(t_k,\phi)\in\mathcal{T}\times\mathbf{PC}([-\tau,0],\mathbf{B}_\rho)$, $1\le k\le N$ *and* $\theta\in[-\tau,0]$,

$$V(t_k^+,x(t_k^+;T_0,\phi))\le(1+\beta_k)(1-\delta_k)V(t_k,x(t_k;T_0,\phi))+\gamma_kV(t_k+\theta,x(t_k+\theta;T_0,\phi));$$

(v) *There are* $\gamma\in(0,\rho)$ *and* $H\in(0,\tilde{m})$ *with the property that there exist* $T\in(0,T^*-T_0)$, $T_1\in(T_0,T_0+T)$, $\alpha\in(0,W_2^{-1}(\frac{W_1(\gamma)}{m}))$ *and* $\beta\in(0,\alpha)$ *such that*

$$T_1>\frac{1}{H}\left(\ln\frac{mW_2(\alpha)}{W_1(\beta)}+\eta T_0\right),$$

in which

$$\tilde{m}\doteq\min\{\underline{\varrho},\eta\}.$$

Then the null solution of (2.1) *is finite-time contractively stable with respect to* $(\alpha,\beta,\gamma,T_0,T,\|\cdot\|)$.

Proof. Since $m>1$, one easily observes using (ii) that $\alpha<\gamma$. Let us fix $\phi\in\mathbf{PC}([-\tau,0],\mathbb{R}^n)$ such that $\|\phi\|_\tau<\alpha$ and then prove that $x(t;T_0,\phi)\in\mathbf{B}_\gamma$ for all $t\in[T_0,T_0+T)$. To this purpose, let us define t_0, t_{N+1}, β_0 and δ_0 as in the proof of Theorem 3.1, choose $\epsilon\in(H,\tilde{m})$, $V(t)=V(t,x(t))$ and again define $\Phi(t)$ as in the proof of Theorem 3.1.

We shall prove that

$$\Phi(t)\le mV(t_0), \qquad t\in(t_k,t_{k+1}], \quad 0\le k\le N, \tag{3.16}$$

the interval (t_N,t_{N+1}) replacing $(t_N,t_{N+1}]$ for $k=N$, inequality which plays a role similar to the one played by (3.2) in the proof of Theorem 3.1. First, it is seen by (ii) that

$$\Phi(t_0)=V(t_0)<mV(t_0).$$

We then show that

$$\Phi(t)\le mV(t_0), \qquad t\in(t_0,t_1). \tag{3.17}$$

Suppose that (3.17) does not hold, and consequently there exists $t \in (t_0, t_1)$ such that $\Phi(t) > mV(t_0)$. Let us note that Φ is continuous on (t_0, t_1) and define

$$t^* = \inf\{t \in (t_0, t_1) \mid \Phi(t) \geq mV(t_0)\}.$$

Since $\Phi(t_0) < mV(t_0)$, it is seen that $t^* \in (t_0, t_1)$. Also, due to the definition of t^*, it follows that

$$\Phi(t) < mV(t_0) \quad \text{for } t \in [t_0, t^*), \quad \Phi(t^*) = mV(t_0).$$

Using (i), one sees that $g(\Phi(t^*)) \geq m\Phi(t^*) > mV(t_0)$.

Let us then define $t^{**} = \sup\{t \in [t_0 - \tau, t^*) \mid g(\Phi(t)) \leq mV(t_0)\}$, and note that $t^{**} \in [t_0 - \tau, t^*)$, together with

$$g(\Phi(t)) > mV(t_0) \quad \text{for } t \in (t^{**}, t^*], \qquad g(\Phi(t^{**})) = mV(t_0). \tag{3.18}$$

Also, by the definition of Φ and (3.18)

$$g(V(t)\exp(\eta(t - t_0))) \geq g(\Phi(t)) > mV(t_0) \quad \text{for } t \in (t^{**}, t^*],$$

and consequently, since $\Phi(t) \leq mV(t_0)$ for $t \in [t_0 - \tau, t^*]$,

$$g(V(t)\exp(\eta(t - t_0))) > \frac{V(t+s)}{m\exp(\eta\tau)} \quad \text{for } t \in (t^{**}, t^*] \text{ and } s \in [-\tau, 0].$$

Using condition (iii), it follows that, for $t \in [t^{**}, t^*]$,

$$\begin{aligned} V(t^{**}) &\geq V(t^*)\exp(-\int_{t^{**}}^{t^*} p(s)\frac{c(V(s))}{V(s)}\, ds) \\ &\geq V(t^*)\exp(-\widetilde{c}\int_{t_0}^{t_1} p(s)\, ds) \\ &\geq V(t^*)\exp(-\overline{p}\widetilde{c}\underline{\varrho}). \end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we then get that

$$\begin{aligned} g(\Phi(t^{**})) &= mV(t_0) = \Phi(t^*) = V(t^*)\exp(\epsilon(t^* - t_0)) \\ &\leq \exp(\overline{p}\widetilde{c}\underline{\varrho})\exp(\epsilon(t^* - T_0))V(t^{**}) \\ &= \exp(\overline{p}\widetilde{c}\underline{\varrho})\exp(\epsilon(t^* - t^{**}))\Phi(t^{**}) \\ &= \exp(\overline{p}\widetilde{c}\underline{\varrho})\Phi(t^{**})\exp(\epsilon\underline{\varrho}). \end{aligned}$$

As a result, we get that

$$g(\Phi(t^{**})) < \exp((\eta + \overline{p}\widetilde{c})\underline{\varrho})\Phi(t^{**}),$$

which contradicts (i). Consequently, the inequality (3.17) holds true. Let us now suppose that (3.16) holds for $0 \leq k \leq l-1$ and then show that it is also valid for $k = l$, that is,

$$\Phi(t) \leq mV(t_0), \qquad t \in (t_l, t_{l+1}].$$

In this regard, since

$$\max_{1\leq k\leq N}\{t_k - t_{k-1}\} < \frac{\ln\left(\frac{1}{\max_{1\leq k\leq N}\{(1+\beta_k)(1-\delta_k)+\gamma_k\exp(\eta\tau)\}}\right)}{\eta + \overline{p}\widetilde{c}},$$

note first that, for any $1 \le l \le N$,

$$\frac{1}{\max_{1\le k\le N}\{(1+\beta_k)(1-\delta_k)+\gamma_k\exp(\eta\tau)\}} \ge \exp(\underline{\varrho}\overline{p}\widetilde{c}+\eta(t_{l+1}-t_l)).$$

Consequently,

$$\begin{aligned} V(t_l^+) &\le (1+\beta_l)(1-\delta_l)V(t_l)+\gamma_l V(t_l+s) \\ &= (1+\beta_l)(1-\delta_l)\Phi(t_l)\exp(-\epsilon(t_l-t_0))+\gamma_l\Phi(t_l+s)\exp(-\epsilon(t_l+s-t_0)) \\ &\le \big[(1+\beta_l)(1-\delta_l)+\gamma_l\exp(\epsilon\tau)\big]mV(t_0)\exp(-\epsilon(t_l-t_0)) \\ &< \exp(-\underline{\varrho}\overline{p}\widetilde{c})\exp(-\epsilon(t_{l+1}-t_l))mV(t_0)\exp(-\epsilon(t_l-t_0)) \\ &= \exp(-\underline{\varrho}\overline{p}\widetilde{c})mV(t_0)\exp(-\epsilon(t_{l+1}-t_0)) \\ &< mV(t_0)\exp(-\epsilon(t_{l+1}-t_0)), \end{aligned}$$

which implies that $\Phi(t_l^+) < mV(t_0)$. In the following we shall prove that (3.16) holds for $k = l$. Suppose that this assertion is not true. There then exists $t \in (t_l, t_{l+1}]$ such that

$$\Phi(t) > mV(t_0).$$

Let us define

$$t'' = \inf\{t \in (t_l, t_{l+1}] \mid \Phi(t) \ge mV(t_0)\}.$$

Then $t'' \in (t_l, t_{l+1})$ and

$$\Phi(t'') = mV(t_0), \qquad \Phi(t) < mV(t_0) \quad \text{for } t \in (t_l, t'').$$

It is easy to see that

$$\begin{aligned} V(t_l^+) &< \exp(-\underline{\varrho}\overline{p}\widetilde{c})mV(t_0)\exp(-\epsilon(t_{l+1}-t_0)) \\ &< mV(t_0)\exp(-\epsilon(t''-t_0)) = V(t''). \end{aligned}$$

This implies that there exists $\widehat{t} \in (t_l, t'')$ such that

$$V(\widehat{t}) = \exp(-\underline{\varrho}\overline{p}\widetilde{c})mV(t_0)\exp(-\epsilon(t_{l+1}-t_0)).$$

and

$$V(\widehat{t}) < V(t) < V(t'') \quad \text{for } t \in (\widehat{t}, t'').$$

Then, for $t \in (\widehat{t}, t'')$ and $\theta \in [-\tau, 0]$, we have that either $t+\theta \in (t_0-\tau, t_l)$ or $t+\theta \in (t_l, t'')$. In both cases, one notes that

$$g(V(t)\exp(\eta(t-T_0))) > \frac{V(t+\theta)}{\exp(\epsilon\tau)} > \frac{V(t+\theta)}{m\exp(\eta\tau)} \quad \text{for } t \in (\widehat{t}, t'') \text{ and } \theta \in [-\tau, 0].$$

It follows that from condition (iii), for $t \in [\widehat{t}, t'']$,

$$V(t'') \le V(\widehat{t})\exp(\underline{\varrho}\overline{p}\widetilde{c}).$$

As a result, we obtain that

$$\begin{aligned} V(t'') &\le \exp(-\underline{\varrho}\overline{p}\widetilde{c})mV(t_0)\exp(-\epsilon(t_{l+1}-t_0))\exp(\underline{\varrho}\overline{p}\widetilde{c}) \\ &< mV(t_0)\exp(-\epsilon(t_{l+1}-t_0)) \\ &< mV(t_0)\exp(-\epsilon(t''-t_0)) = V(t''), \end{aligned}$$

which leads to a contradiction. By the above, it is seen that (3.16) holds. Consequently,

$$\Phi(t) \leq mV(T_0) \quad \text{for } t \in [t_0, t_0 + T)$$

and, by (ii),

$$W_1(\|x(t)\|) \leq V(t) \leq mV(t_0)\exp(-\epsilon(t - t_0)) \quad \text{for } t \in [t_0, t_0 + T).$$

Since, from (ii),

$$V(t_0) \leq W_2(\|x(t_0)\|) < W_2(\alpha),$$

one sees using (3.16) and (ii) that

$$W_1(\|x(t)\|) \leq mW_2(\alpha)\exp(-\epsilon(t - t_0)) < W_1(\gamma) \quad \text{for } t \in [t_0, t_0 + T)$$

and consequently

$$x(t) \in \mathbf{B}_\gamma \quad \text{for } t \in [T_0, T_0 + T).$$

It now follows that the null solution of the system (2.1) is finite-time stable with respect to $(\alpha, \gamma, T_0, T, \|\cdot\|)$.
Similarly, using again (ii) and (3.16),

$$W_1(\|x(t)\|) \leq mW_2(\alpha)\exp(-\epsilon(T_1 - T_0)) \quad \text{for } t \in (T_1, T_0 + T),$$

which implies, by the choices of ϵ and H,

$$\begin{aligned} W_1(\|x(t)\|) &\leq W_1(\beta)\exp(T_1(H - \epsilon) + T_0(\epsilon - \eta)) \\ &< W_1(\beta), \quad \text{for } t \in (T_1, T_0 + T), \end{aligned}$$

and consequently

$$x(t) \in \mathbf{B}_\beta \quad \text{for } t \in (T_1, T_0 + T).$$

It now follows that the null solution of the system (2.1) is finite-time quasi-contractively stable with respect to (α, β, γ, T_0, T, $\|\cdot\|$), which completes the proof. □

Note that condition (3.15) precludes the pest population size from increasing each time the impulsive perturbations occur, as the denominator of the fraction under the logarithm should be subunitary, so that the logarithm be positive. This is certainly conceivable, since if (iii) does not ensure anymore that the size of the pest population is decreasing then one should ensure that the size of the pest population is decreasing by using impulses. Actually, it might be possible to weaken (3.15), as it might not be necessary to decrease the size of the pest population each time the impulsive perturbations occur, but only in certain circumstances. This, however, is subject for further research.

From the above result, by particularizing $g(t) = mt$, $p(t) \equiv p > 0$ and $c(t) = \widetilde{c}t$, one may obtain the following practical consequence.

Corollary 3.6. *Assume that there exist functions $W_1, W_2 \in \mathbf{K}$, $V \in \mathbf{V_0}$ and $p \in \mathbf{PC}(\mathbb{R}_+, \mathbb{R}_+)$, and constants $T^* > T_0$, $\eta > 0$, $p > 0$, $\tilde{c} > 0$, $\{\beta_k\}_{k=1}^N \subset [0, \infty)$, $\{\delta_k\}_{k=1}^N \subset [0, 1)$, $\{\gamma_k\}_{k=1}^N \subset [0, 1)$*

$$m > \exp((\eta + p\widetilde{c})\underline{\varrho})$$

and

$$\underline{\varrho} \doteq \max_{1 \leq k \leq N}\{t_k - t_{k-1}\} < \frac{\ln\left(\frac{1}{\max_{1\leq k\leq N}\{(1+\beta_k)(1-\delta_k)+\gamma_k\exp(\eta\tau)\}}\right)}{\eta + p\widetilde{c}} \doteq \overline{\varrho}$$

such that the following conditions hold:

(i) for $(t,x(t)) \in [T_0,T^*) \times \mathbf{B}_\rho$,

$$W_1(\|x(t)\|) \le V(t,x(t;T_0,\phi)) \le W_2(\|x(t)\|);$$

(ii) for any $\phi \in \mathbf{PC}([-\tau,0],\mathbf{B}_\rho)$*, if*

$$m^2 V(t,x(t;T_0,\phi)) > V(t+\theta,\phi(\theta))\exp(-\eta(t-T_0+\tau)) \text{ for } \theta \in [-\tau,0],\ t \notin \mathcal{T},$$

then

$$D^+V(t,x(t;T_0,\phi)) \le p\tilde{c}V(t,x(t;T_0,\phi));$$

(iii) for all $(t_k,\phi) \in \mathcal{T} \times \mathbf{PC}([-\tau,0],\mathbf{B}_\rho)$*,* $1 \le k \le N$ *and* $\theta \in [-\tau,0]$*,*

$$V(t_k^+,x(t_k^+;T_0,\phi)) \le (1+\beta_k)(1-\delta_k)V(t_k,x(t_k;T_0,\phi)) + \gamma_k V(t_k+\theta,x(t_k+\theta;T_0,\phi));$$

(iv) there are $\gamma \in (0,\rho)$ *and* $H \in (0,\tilde{m})$ *with the property that there exist* $T \in (0,T^*-T_0)$*,* $T_1 \in (T_0,T_0+T)$*,* $\alpha \in (0,W_2^{-1}(\frac{W_1(\gamma)}{m}))$ *and* $\beta \in (0,\alpha)$ *such that*

$$T_1 > \frac{1}{H}\left(\ln\frac{mW_2(\alpha)}{W_1(\beta)} + \eta T_0\right),$$

in which

$$\tilde{m} \doteq \min\{\underline{\rho},\eta\}.$$

Then the null solution of (2.1) *is finite-time contractively stable with respect to* $(\alpha,\beta,\gamma,T_0,T,\|\cdot\|)$*.*

4 Numerical examples

We now attempt to illustrate the applicability of our abstract results. Although these examples are mostly of an academic nature, showcasing our techniques, rather than actively modelling concrete situations encountered in integrated pest management, they are still able to describe the dynamics of species whose evolution depend on their past history and are subject to impulsive perturbation.

Example 1

We first consider the time-dependent impulsive dynamical system given by the following delayed ordinary differential equation which is subject to impulsive perturbations given below

$$\begin{cases} x'(t) = \exp\left(-\frac{t}{2}\right)x(t-1) - 3x(t), & t \in [0,6)\setminus\left\{\frac{k}{2},\ k=1,2,\dots,11\right\}, \\ \triangle x(t) = (\sqrt{1000}-1)x(t), & t = \frac{3k-2}{2},\ k=1,2,3,4, \\ \triangle x(t) = (\sqrt{0.000188}-1)x(t), & t = \frac{3k-1}{2},\ k=1,2,3,4, \\ \triangle x(t) = \sqrt{2x^2(t)+x^2(t-1)} - x(t), & t = \frac{3k}{2},\ k=1,2,3, \\ \phi(t) \equiv 10^4, & t \in [-1,0). \end{cases} \tag{4.1}$$

The resetting conditions can now be expressed as

- $x\left(\frac{3k-2}{2}^+\right) = x\left(\frac{3k-2}{2}\right)\sqrt{1000}, \quad k=1,2,3,4;$

- $x\left(\frac{3k-1}{2}^+\right) = x\left(\frac{3k-1}{2}\right)\sqrt{0.000188}, \quad k = 1, 2, 3, 4;$
- $x\left(\frac{3k}{2}^+\right) = \sqrt{2x^2\left(\frac{3k}{2}\right) + x^2\left(\frac{3k}{2} - 1\right)}, \quad k = 1, 2, 3.$

Note that both the differential equation and some of the impulsive perturbations are now subject to delay. Let $T_0 = 0$, $\eta = 1$, $c(s) = s$, $g(s) = 2s$, $T = 6$, $T^* = 7$, $W_1(\|x\|) = 0.98\|x\|^2$, $W_2(\|x\|) = 1.01\|x\|^2$, $V(t, x(t)) = x^2(t)$, $\tau = 1$, $H = 0.99$, $\gamma = 10^6$, $\alpha = 750$, $\beta = 80$ and $\rho = 10^6$. Then

$$V\left(x\left(\frac{3k-2}{2}^+\right)\right) = 1000V\left(x\left(\frac{3k-2}{2}\right)\right),$$
$$V\left(x\left(\frac{3k-1}{2}^+\right)\right) = 0.000188V\left(x\left(\frac{3k-1}{2}\right)\right),$$

and consequently

$$\beta_{\frac{3k-2}{2}} = 999, \quad \delta_{\frac{3k-2}{2}} = 0, \quad \gamma_{\frac{3k-2}{2}} = 0,$$
$$\beta_{\frac{3k-1}{2}} = 0, \quad \delta_{\frac{3k-1}{2}} = 0.999812, \quad \gamma_{\frac{3k-1}{2}} = 0.$$

Also,

$$V\left(\frac{3k}{2}^+\right) = 2x^2\left(\frac{3k}{2}\right) + x^2\left(\frac{3k}{2} - 1\right)$$

and consequently

$$\beta_{\frac{3k}{2}} = 1, \quad \delta_{\frac{3k}{2}} = 0, \quad \gamma_{\frac{3k}{2}} = 1.$$

One then sets

$\beta_{0.5} = 999$	$\delta_{0.5} = 0$	$\gamma_{0.5} = 0$
$\beta_1 = 0$	$\delta_1 = 0.999812$	$\gamma_1 = 0$
$\beta_{1.5} = 1$	$\delta_{1.5} = 0$	$\gamma_{1.5} = 1$
$\beta_2 = 999$	$\delta_2 = 0$	$\gamma_2 = 0$
$\beta_{2.5} = 0$	$\delta_{2.5} = 0.999812$	$\gamma_{2.5} = 0$
$\beta_3 = 1$	$\delta_3 = 0$	$\gamma_3 = 1$
$\beta_{3.5} = 999$	$\delta_{3.5} = 0$	$\gamma_{3.5} = 0$
$\beta_4 = 0$	$\delta_4 = 0.999812$	$\gamma_4 = 0$
$\beta_{4.5} = 1$	$\delta_{4.5} = 0$	$\gamma_{4.5} = 1$
$\beta_5 = 999$	$\delta_5 = 0$	$\gamma_5 = 0$
$\beta_{5.5} = 0$	$\delta_{5.5} = 0.999812$	$\gamma_{5.5} = 0$

and

$$M^* = \prod_{j=0.5,1,1.5,\ldots,5.5} \left[(1+\beta_j)(1-\delta_j) + M\gamma_j \exp(\eta\tau)\right] \approx 1.0275.$$

Also, if $4x^2(t) > x^2(s)\exp(-t)$, $s \in [t-1, t]$, that is,

$$2x(t)\exp\left(\frac{t}{2}\right) \geq x(s), \qquad s \in [t-1, t],$$

one sees that

$$D^+V(t) = 2x(t)x'(t) = 2x(t)\left[\exp\left(-\frac{t}{2}\right)x(t-1) - 3x(t)\right]$$
$$\leq -2x^2(t),$$

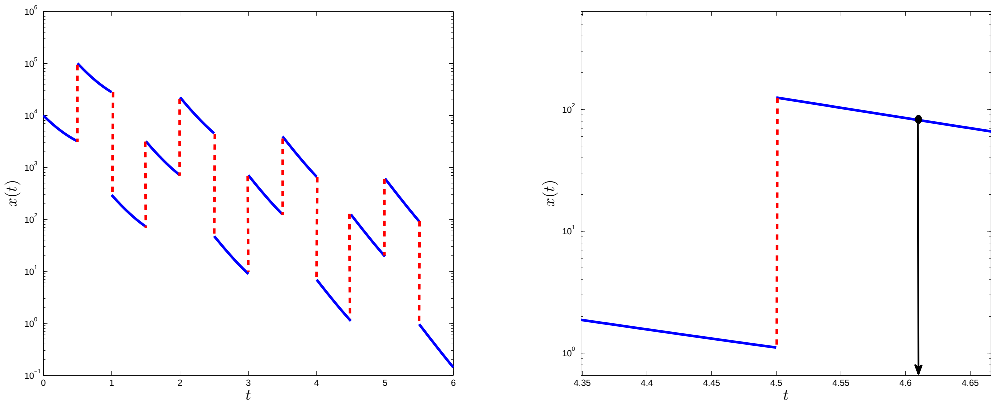

Figure 4.1: The solution of the system (4.1). Red dashed lines represent the impulsive perturbations.

which indicates that $p = 2$ and

$$T_1 > \frac{1}{0.982} \ln \frac{1.01 * (750)^2 * 1.0275}{0.98 * 6400} \approx 4.6165.$$

After analyzing the above data and using Corollary 3.3, we conclude that the system (2.1) is finite-time contractively stable with respect to $(750, 80, 10^6, 0, 7, \|\cdot\|)$ as shown in Figure 4.1.

Example 2

We now consider the second hybrid dynamical system given by

$$\begin{cases} x'(t) = \exp\left(-\frac{t}{2}\right) x\left(t - \frac{1}{2}\right) ds - (10 + \sqrt[4]{e})x(t), & t \neq 1, 2, \ldots, 7, \\ \triangle x(t) = \sqrt{0.05x^2(t) + 0.05x^2\left(t - \frac{1}{2}\right)} - x(t), & t = 1, 3, 5, 7, \\ \triangle x(t) = \sqrt{0.03x^2(t) + 0.05x^2\left(t - \frac{1}{2}\right)} - x(t), & t = 2, \\ \triangle x(t) = \sqrt{0.02x^2(t) + 0.05x^2\left(t - \frac{1}{2}\right)} - x(t), & t = 4, \\ \triangle x(t) = \sqrt{0.04x^2(t) + 0.05x^2\left(t - \frac{1}{2}\right)} - x(t), & t = 6, \\ \phi(t) \equiv 10^{10}, & t \in \left[-\frac{1}{2}, 0\right). \end{cases} \tag{4.2}$$

Let $T_0 = 0$, $\eta = 1$, $c(s) = s$, $g(s) = 9s$, $N = 7$, $T = 8$, $T^* = 9$, $W_1(\|x\|) = 0.98\|x\|^2$, $W_2(\|x\|) = 1.01\|x\|^2$, $V(t, x(t)) = x^2(t)$, $\tau = \frac{1}{2}$, $H = 0.9933$, $\gamma = 10^{10}$, $\alpha = 5000$, $\beta = 2000$ and $\rho = 10^{10}$. One then notes that

$\beta_1 = 0$	$\delta_1 = 0.95$	$\gamma_1 = 0.05$
$\beta_2 = 1$	$\delta_2 = 0.985$	$\gamma_2 = 0.05$
$\beta_3 = 0$	$\delta_3 = 0.95$	$\gamma_3 = 0.05$
$\beta_4 = 1$	$\delta_4 = 0.99$	$\gamma_4 = 0.05$
$\beta_5 = 0$	$\delta_5 = 0.95$	$\gamma_5 = 0.05$
$\beta_6 = 1$	$\delta_6 = 0.98$	$\gamma_6 = 0.05$
$\beta_7 = 0$	$\delta_7 = 0.95$	$\gamma_7 = 0.05$

and then $\overline{\varrho} = \frac{\ln \frac{1}{0.13243606}}{1+0.5444} \approx 1.31 > \underline{\varrho} = 1$. Also, if $81x^2(t) \geq x^2(s)\exp(-(\frac{1}{2}+t))$, $s \in [t-\frac{1}{2}, t]$, that is,

$$9x(t)\exp\left(\frac{1}{4}+\frac{t}{2}\right) \geq x(s), \qquad s \in \left[t-\frac{1}{2}, t\right],$$

one sees that

$$\begin{aligned} D^+V(t) &= 2x(t)x'(t) = 2x(t)\left[\exp\left(-\frac{t}{2}\right)x\left(t-\frac{1}{2}\right) - (10+\sqrt[4]{e})x(t)\right] \\ &\leq 2(8\sqrt[4]{e}-10)x^2(t), \end{aligned}$$

which indicates that $p = 16\sqrt[4]{e} - 20 \approx 0.5444$ and

$$T_1 > \frac{1}{0.9933}\ln\frac{9\cdot(5000)^2\cdot 1.01}{0.98\cdot(2000)^2} \approx 4.0873.$$

After analyzing the above data and using Corollary 3.6, we conclude that the system (4.2) is finite-time contractively stable with respect to $(5000,\ 2000,\ 10^{10},\ 0,\ 9,\ \|\cdot\|)$ as shown in Figure 4.2.

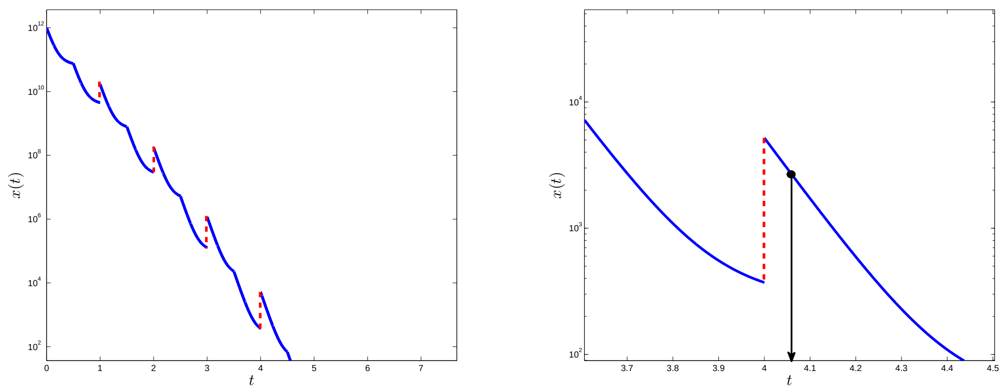

Figure 4.2: The solution of the system (4.2). Red dashed lines represent the impulsive perturbations.

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