*Electronic Journal of Qualitative Theory of Differential Equations*
2009, No. **34**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

# Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients

Başak KARPUZ*

**2000 AMS Subject Classification**: Primary: 34K11, Secondary: 34K40.
**Keywords and Phrases**: Delay dynamic equations, forced term, higher-order, oscillation, neutral, nonlinear, time scales.

**Abstract**

In this paper, we present a criterion on the oscillation of unbounded solutions for higher-order dynamic equations of the following form:

$$\big[x(t)+A(t)x(\alpha(t))\big]^{\Delta^n}+B(t)F(x(\beta(t)))=\varphi(t) \qquad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{$\star$}$$

where $n\in[2,\infty)_{\mathbb{Z}}$, $t_0\in\mathbb{T}$, $\sup\{\mathbb{T}\}=\infty$, $A\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is allowed to alternate in sign infinitely many times, $B\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, $F\in\mathrm{C}_{\mathrm{rd}}(\mathbb{R},\mathbb{R})$ is nondecreasing, and $\alpha,\beta\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ are unbounded increasing functions satisfying $\alpha(t),\beta(t)\leq t$ for all sufficiently large $t$. We give change of order formula for double(iterated) integrals to prove our main result. Some simple examples are given to illustrate the applicability of our results too. In the literature, almost all of the results for ($\star$) with $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$ hold for bounded solutions. Our results are new and not stated in the literature even for the particular cases $\mathbb{T}=\mathbb{R}$ and/or $\mathbb{T}=\mathbb{Z}$.

***Address**: Afyon Kocatepe University, Department of Mathematics, Faculty of Science and Arts, ANS Campus, 03200 Afyonkarahisar, Turkey.
**Email**: `bkarpuz@gmail.com`
**Web**: `http://www2.aku.edu.tr/~bkarpuz`

# 1 Introduction

This paper is concerned with the oscillatory nature of all unbounded solutions of the following higher-order delay dynamic equation:

$$\big[x(t)+A(t)x(\alpha(t))\big]^{\Delta^n}+B(t)F(x(\beta(t)))=\varphi(t)\qquad\text{for } t\in[t_0,\infty)_{\mathbb{T}},\quad(1)$$

where $n\in[2,\infty)_{\mathbb{Z}}$, $t_0\in\mathbb{T}$, $\sup\{\mathbb{T}\}=\infty$, $A\in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is allowed to oscillate in a strip of width less than 1, $B\in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, $F\in C(\mathbb{R},\mathbb{R})$ is nondecreasing, and $\alpha,\beta\in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ are increasing functions satisfying $\lim_{t\to\infty}\alpha(t)=\lim_{t\to\infty}\beta(t)=\infty$ and $\alpha(t),\beta(t)\leq t$ for all sufficiently large $t$.

During the last few decades, there has been extensive improvement in the oscillation theory of neutral difference/differential/dynamic equations, which are defined as equations in which the highest order differential operator is applied both to the unknown function and to its composition with a delay function. In simple terms, a function is said to be a delay function if it tends to infinity and takes values that are less than its variable. Neutral delay equations appear in many fields of real word mathematical modelings, and since the delay terms (as well as the coefficients), play a major role on the behavior of the solutions, studies on the solutions of such equations are significantly interesting.

In the literature, there are very few number of papers studying delay difference/differential equations with an oscillating coefficient inside the neutral part because of the technical difficulties arising in the computations. Also, all these results except [18, Theorem 2.4] restrict their conclusions on bounded solutions to succeed in revealing the asymptotic behaviour (see [10, 16, 17, 20, 21]). In [18, Theorem 2.4], the authors study asymptotic behaviour of all solutions of higher-order differential equations without restricting their attention on bounded solutions, but the other assumptions of this work are very strong; for instance, $A$ is assumed to be periodic and the delay functions are lines of slope 1. Also, we would like to mention that our method/technique can be easily modified for equations involving several coefficients, for simplicity in the proofs, we shall consider equations involving only one coefficients inside and outside the neutral part.

Our motivation for this study comes from the papers [12, 18]. In [12], the authors study oscillation and asymptotic behaviour of higher-order difference equations for various ranges of the coefficient associated to the neutral part (but not allowed to oscillate). As we cursory talk about the work accomplished in [18], it is important to mention that the method employed therein is completely different from those in[10, 16, 17, 20, 21], where all the authors could only deal with bounded solutions of (1). In this work, we combine and extend some of the results in [12, 18] by the means of the time scale theory. The readers are referred to [2, 3, 11] for fundamental studies on the oscillation theory of difference/differential equations.

On the other hand, for first-order dynamic equations; i.e., (1) with $n = 1$, one may find results in the papers [4, 7, 9, 14, 15, 19, 22]. To the best of our knowledge, there is not yet any paper studying oscillation and asymptotic behaviour of higher-order dynamic equations, and therefore this paper is one of the first papers dealing with this untouched problem (also see [13], where the author states necessary and sufficient conditions on all bounded solutions to be oscillatory or convergent to zero asymptotically).

Set $t_{-1} := \min_{t\in[t_0,\infty)_\mathbb{T}}\{\alpha(t), \beta(t)\}$. By a *solution* of (1), we mean a function $x \in \mathrm{C}_{\mathrm{rd}}([t_{-1},\infty)_\mathbb{T},\mathbb{R})$ satisfying $x + A(t)x\circ\alpha \in \mathrm{C}^{\mathrm{n}}_{\mathrm{rd}}([t_0,\infty)_\mathbb{T},\mathbb{R})$ and (1) for all $t \in [t_0,\infty)_\mathbb{T}$. A solution of (1) is called *oscillatory* if there exists an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_0,\infty)_\mathbb{T}$ such that $x(\xi_k)x^\sigma(\xi_k) \leq 0$ holds for all $n \in \mathbb{N}$, where the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t,\infty)_\mathbb{T}$ for $t \in \mathbb{T}$ and $x^\sigma$ stands for $x \circ \sigma$.

This paper is organized as follows: in § 2, we give some preliminaries and definitions about the time scale concept; in § 3, we state and prove our main result on the oscillation of unbounded solutions to (1); in § 4, we give some illustrative examples to show applicability of our results; and finally in § 5, we make a slight discussion concerning the previous works in the literature to mention the significance of this work.

## 2 Definitions and preliminaries

In this section, we give the basic facilities for the proof of our main result.

In the sequel, we introduce the definition of the generalized polynomials on time scales (see [1, Lemma 5] and/or [6, § 1.6]) $h_k : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ as follows:

$$h_k(t,s) := \begin{cases} 1, & k = 0 \\ \displaystyle\int_s^t h_{k-1}(\eta,s)\Delta\eta, & k \in \mathbb{N} \end{cases} \tag{2}$$

for $s,t \in \mathbb{T}$. Note that, for all $s,t \in \mathbb{T}$ and $k \in \mathbb{N}_0 := \{n \in \mathbb{Z} : n \geq 0\}$, the function $h_k$ satisfies

$$\frac{\partial}{\Delta t} h_k(t,s) = \begin{cases} 0, & k=0 \\ h_{k-1}(t,s), & k \in \mathbb{N}. \end{cases} \tag{3}$$

In particular, for $\mathbb{T}=\mathbb{Z}$, we have $h_k(t,s) = (t-s)^{(k)}/k!$ for all $s,t \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, where ${}^{(\cdot)}$ is the usual factorial function, and for $\mathbb{T}=\mathbb{R}$, we have $h_k(t,s) = (t-s)^k/k!$ for all $s,t \in \mathbb{R}$ and $k \in \mathbb{N}_0$.

**Property 1.** *Using induction and the definition given by* (2)*, it is easy to see that $h_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}_0$ and $s,t \in \mathbb{T}$ with $t \geq s$ and $(-1)^k h_k(t,s) \geq 0$ holds for all $k \in \mathbb{N}$ and $s,t \in \mathbb{T}$ with $t \leq s$. In view of the fact* (3)*, for all $k \in \mathbb{N}$, $h_k(t,s)$ is increasing in $t$ provided that $t \geq s$, and $(-1)^k h_k(t,s)$ is decreasing in $t$ provided that $t \leq s$. Moreover, $h_k(t,s) \leq (t-s)^{k-l} h_l(t,s)$ holds for all $s,t \in \mathbb{T}$ with $t \geq s$ and all $k,l \in \mathbb{N}_0$ with $l \leq k$.*

We prove the following lemma on the change of order in double (iterated) integrals, which extends [5, Theorem 10] to arbitrary time scales. However, our proof is more simple and direct.

**Lemma 1 (Change of integration order).** *Assume that $s,t \in \mathbb{T}$ and $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T} \times \mathbb{T}, \mathbb{R})$. Then*

$$\int_s^t \int_\eta^t f(\eta,\xi)\Delta\xi\Delta\eta = \int_s^t \int_s^{\sigma(\xi)} f(\eta,\xi)\Delta\eta\Delta\xi. \tag{4}$$

*Proof.* We set

$$g(t) := \int_s^t \int_\eta^t f(\eta,\xi)\Delta\xi\Delta\eta - \int_s^t \int_s^{\sigma(\xi)} f(\eta,\xi)\Delta\eta\Delta\xi \tag{5}$$

for $t \in \mathbb{T}$. Then, applying [6, Theorem 1.117] to (5), we have

$$\begin{aligned} g^{\Delta}(t) =& \left\{ \int_s^t \frac{\partial}{\Delta t}\left( \int_\eta^t f(\eta,\xi)\Delta\xi \right)\Delta\eta + \int_t^{\sigma(t)} f(t,\xi)\Delta\xi \right\} - \int_s^{\sigma(t)} f(\eta,t)\Delta\eta \\ =& \left\{ \int_s^t f(\eta,t)\Delta\eta + \int_t^{\sigma(t)} f(t,\xi)\Delta\xi \right\} - \int_s^{\sigma(t)} f(\eta,t)\Delta\eta \\ =& \int_t^{\sigma(t)} f(t,\xi)\Delta\xi - \int_t^{\sigma(t)} f(\eta,t)\Delta\eta \\ =& \mu(t)f(t,t) - \mu(t)f(t,t) \equiv 0 \end{aligned}$$

for all $t \in \mathbb{T}$, where [6, Theorem 1.75] is applied in the last step. Hence, $g$ is a constant function. On the other hand, we see that $g(s) = 0$ holds; i.e., $g = 0$ on $\mathbb{T}$, and this shows that (4) is true. □

We would like to point out that [8, Lemma 3] is a particular case of Lemma 1. As an immediate consequence, we can give the following generalization of Lemma 1 for $n$-fold integrals.

**Corollary 1.** *Assume that $n \in \mathbb{N}$, $s, t \in \mathbb{T}$ and $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. Then*

$$\int_s^t \int_{\eta_n}^t \cdots \int_{\eta_2}^t f(\eta_1)\Delta\eta_1\Delta\eta_2 \cdots \Delta\eta_{n+1} = (-1)^n \int_s^t h_n(s, \sigma(\eta))f(\eta)\Delta\eta. \quad (6)$$

*Proof.* The proof of the corollary makes the use of Lemma 1 and the induction principle. From Lemma 1, it is clear that (6) holds for $n = 2$. Suppose now that (6) holds for some $n \in \mathbb{N}$. Then integrating (6) over $[s, t)_{\mathbb{T}}$, and using Lemma 1, we get

$$\begin{aligned}(-1)^n \int_s^t \int_\eta^t h_n(\eta, \sigma(\xi))f(\xi)\Delta\xi\Delta\eta =& (-1)^n \int_s^t \int_s^{\sigma(\xi)} h_n(\eta, \sigma(\xi))f(\xi)\Delta\eta\Delta\xi \\ =& (-1)^{n+1} \int_s^t \int_{\sigma(\xi)}^s h_n(\eta, \sigma(\xi))f(\xi)\Delta\eta\Delta\xi \\ =& (-1)^{n+1} \int_s^t h_{n+1}(s, \sigma(\xi))f(\xi)\Delta\xi,\end{aligned}$$

which proves that (6) holds for $(n + 1)$. This completes the proof. □

The following lemma is interesting on its own.

**Lemma 2.** *Let $n \in \mathbb{N}_0$, $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$ and $\sup\{\mathbb{T}\} = \infty$, and that $s, t$ be any two points in $\mathbb{T}$. Then*

$$\int_s^\infty h_n(s, \eta)f(\eta)\Delta\eta \qquad \textit{and} \qquad \int_t^\infty h_n(t, \eta)f(\eta)\Delta\eta$$

*diverge or converge together.*

*Proof.* To complete the proof, we shall employ the induction principle. The proof is trivial for $n = 0$. Suppose that the claim holds for some $n \in \mathbb{N}$, we shall show that it is also true for $(n+1)$. Without loss of generality, we may suppose $s \geq t$. From (2) and Lemma 1, we have

$$\begin{aligned}\int_t^\infty h_{n+1}(t,\sigma(\eta))f(\eta)\Delta\eta &= \int_t^\infty \int_{\sigma(\eta)}^t h_n(\xi,\sigma(\eta))f(\eta)\Delta\xi\Delta\eta \\ &= -\int_t^\infty \int_\xi^\infty h_n(\xi,\sigma(\eta))f(\eta)\Delta\eta\Delta\xi \\ &= -\bigg[\int_s^\infty \int_\xi^\infty h_n(\xi,\sigma(\eta))f(\eta)\Delta\eta\Delta\xi \\ &\quad + \int_t^s \int_\xi^\infty h_n(\xi,\sigma(\eta))f(\eta)\Delta\eta\Delta\xi\bigg].\end{aligned} \tag{7}$$

First, consider the case that $(-1)^n \int_r^\infty h_n(r,\sigma(\eta))f(\eta)\Delta\eta = \infty$ holds for all $r \in \mathbb{T}$ (see Property 1). Clearly, this implies by (7) that $(-1)^{n+1} \int_s^\infty h_{n+1}(s,\sigma(\eta))f(\eta)\Delta\eta = \infty$, and thus $(-1)^{n+1} \int_t^\infty h_{n+1}(s,\sigma(\eta))f(\eta)\Delta\eta = \infty$ since $s \geq t$. Next, consider the case that $(-1)^n \int_r^\infty h_n(r,\sigma(\eta))f(\eta)\Delta\eta < \infty$ for all $r \in \mathbb{T}$. In view of (2), (7) and Lemma 1, we get

$$\begin{aligned}\int_s^\infty h_{n+1}(s,\sigma(\eta))f(\eta)\Delta\eta &= \int_t^\infty h_{n+1}(t,\sigma(\eta))f(\eta)\Delta\eta \\ &\quad + \int_t^s \int_\xi^\infty h_n(\xi,\sigma(\eta))f(\eta)\Delta\eta\Delta\xi.\end{aligned} \tag{8}$$

Using the fact that the last term on the right-hand side of (8) is finite, we infer that $\int_s^\infty h_{n+1}(s,\sigma(\eta))f(\eta)\Delta\eta$ and $\int_t^\infty h_{n+1}(t,\sigma(\eta))f(\eta)\Delta\eta$ diverge or converge together. This proves that the claim holds for $(n+1)$, and the proof is therefore completed. $\square$

The following result is the generalization of the well-known Kneser's theorem, which is one of the most powerful tools in the oscillation theory of higher-order difference/differential equations in the discrete and the continuous cases.

**Kneser's theorem ([1, Theorem 5]).** *Let $n \in \mathbb{N}$, $f \in \mathrm{C}^{\mathrm{n}}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ and $\sup\{\mathbb{T}\} = \infty$. Suppose that $f$ is either positive or negative and $f^{\Delta^n} \not\equiv 0$ is either nonnegative or nonpositive on $[t_0,\infty)_\mathbb{T}$ for some $t_0 \in \mathbb{T}$. Then there exist $t_1 \in [t_0,\infty)_\mathbb{T}$ and $m \in [0,n)_\mathbb{Z}$ such that $(-1)^{n-m} f(t) f^{\Delta^n}(t) \geq 0$ holds for all $t \in [t_0,\infty)_\mathbb{T}$ with* *(i)* $f(t)f^{\Delta^j}(t) > 0$ *holds for all* $t \in [t_1, \infty)_{\mathbb{T}}$ *and all* $j \in [0, m)_{\mathbb{Z}}$,

*(ii)* $(-1)^{m+j} f(t)f^{\Delta^j}(t) > 0$ *holds for all* $t \in [t_1, \infty)_{\mathbb{T}}$ *and all* $j \in [m, n)_{\mathbb{Z}}$.

# 3 Main result

In this section, we give our main result on (1) under the following primary assumptions:

(H1) There exist two constants $a_1, a_2 \geq 0$ with $a_1 + a_2 < 1$ such that $-a_1 \leq A(t) \leq a_2$ for all sufficiently large $t$.

(H2) $F \in \mathrm{C_{rd}}(\mathbb{R}, \mathbb{R})$ is nondecreasing with $F(u)/u > 0$ for all $u \in \mathbb{R}\backslash\{0\}$ and $\liminf_{u\to\infty} \big[F(u)/u\big] > 0$.

(H3) $(-1)^{n-2} \int_{t_0}^{\infty} h_{n-2}(t_0, \sigma(\eta))B(\eta)\Delta\eta = \infty$.

(H4) $\liminf_{t\to\infty} \big[(-1)^{m-1} h_{n-m-1}(s, \sigma(t))h_{m-1}(\beta(t), s)/h_{n-2}(s, \sigma(t))\big] > 0$ for every fixed sufficiently large $s$ and every fixed $m \in [1, n-2]_{\mathbb{Z}}$.

(H5) There exists a bounded function $\Phi \in \mathrm{C^n_{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ satisfying $\Phi^{\Delta^n} = \varphi$ on $[t_0, \infty)_{\mathbb{T}}$.

We are now ready to state our main result.

**Theorem 1.** *Assume that (H1)–(H5) hold, then every unbounded solution of* (1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

*Proof.* On the contrary, let $x$ be an unbounded nonoscillatory solution of (1). Then, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that either $x(t), x(\alpha(t)), x(\beta(t)) > 0$ or $x(t), x(\alpha(t)), x(\beta(t)) < 0$ holds for all $t \in [t_1, \infty)_{\mathbb{T}}$.

First, let $x(t), x(\alpha(t)), x(\beta(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. For convenience in the notation, we set

$$y(t) := x(t) + A(t)x(\alpha(t)) \qquad \text{and} \qquad z(t) := y(t) - \Phi(t) \tag{9}$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. Therefore, from (1), we have

$$z^{\Delta^n}(t) = -B(t)F(x(\beta(t))) \leq 0 (\not\equiv 0) \tag{10}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$, which indicates that $z^{\Delta^j}$ is eventually monotonic and of single sign for all $j \in [0, n)_{\mathbb{Z}}$. Then, $\ell_0 \in [-\infty, \infty]_{\mathbb{R}}$ holds, where $\ell_j := \lim_{t\to\infty} z^{\Delta^j}(t)$ for $j \in [0, n)_{\mathbb{Z}}$. Now, we prove that $\ell_0 = \infty$ is true. Since $x$ is unbounded, there exists an increasing divergent sequence $\{\zeta_k\}_{k\in\mathbb{N}} \subset [t_1, \infty)_{\mathbb{T}}$ such that $x(\zeta_k) \geq \sup\{x(t) : t \in [t_1, \zeta_k)_{\mathbb{T}}\}$ holds for all $k \in \mathbb{N}$ and $\{x(\zeta_k)\}_{k\in\mathbb{N}}$ is divergent. Then, from (H1) and (9), we have $z(\zeta_k) \geq \big(1-a_1\big)x(\zeta_k) - \Phi(\zeta_k)$ for all $k \in \mathbb{N}$, which proves $\ell_0 = \infty$ by considering (H4) and letting $k \to \infty$. Hence, by Kneser's theorem, we have $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and $m \in [1, n)_{\mathbb{Z}}$ such that $n - m$ is odd, $z^{\Delta^j} > 0$ for all $j \in [0, m)_{\mathbb{Z}}$ and $(-1)^{m+j} z^{\Delta^j} > 0$ for all $j \in [m, n)_{\mathbb{Z}}$ on $[t_2, \infty)_{\mathbb{T}}$. By Taylor's formula (see [6, Theorem 1.109, Theorem 1.112, Theorem 1.113]), we have

$$\begin{aligned} z(t) &= \sum_{j=0}^{m-1} z^{\Delta^j}(t_2) h_j(t, t_2) + \int_{t_2}^{t} h_{m-1}(t, \sigma(\eta)) z^{\Delta^m}(\eta)\Delta\eta \\ &\geq z^{\Delta^{m-1}}(t_2) h_{m-1}(t, t_2) \end{aligned}$$

for all $t \in [t_2\infty)_{\mathbb{T}}$. It follows from Property 1 (recall that $\eta$ in the integral above varies in $[t_2, t)_{\mathbb{T}}$ and thus $\sigma(\eta) \in [t_2, t]_{\mathbb{T}}$) that

$$\liminf_{t\to\infty} \left( \frac{z(t)}{h_{m-1}(t, t_2)} \right) \geq z^{\Delta^{m-1}}(t_2) > 0. \tag{11}$$

On the other hand, by the decreasing nature of $z^{\Delta^m}$, we observe that $\ell_m$ is finite and thus $\ell_j = 0$ for all $j \in (m, n)_{\mathbb{Z}}$ (see [1, Lemma 7]). Repeatedly integrating (10) over $[t, \infty)_{\mathbb{T}} \subset [t_2, \infty)_{\mathbb{T}}$ for $(n - m)$-times, and applying Corollary 1 (recall that $(n - m - 1)$ is even), we deduce that

$$z^{\Delta^m}(t_2) - \ell_m = \int_{t_2}^{\infty} h_{n-m-1}(t_2, \sigma(\eta)) B(\eta) F(x(\beta(\eta)))\Delta\eta.$$

Therefore, we have

$$\int_{t_2}^{\infty} h_{n-m-1}(t_2, \sigma(\eta)) B(\eta) F(x(\beta(\eta)))\Delta\eta < \infty,$$

which, together with (H3), implies and Lemma 2 that

$$\liminf_{t\to\infty} \left( \frac{(-1)^{m-1} h_{n-m-1}(t_2, \sigma(t)) F(x(\beta(t)))}{h_{n-2}(t_2, \sigma(t))} \right) = 0. \tag{12}$$

From (H4) and (12), we see that

$$\liminf_{t\to\infty}\left(\frac{F(x(\beta(t)))}{h_{m-1}(\beta(t),t_2)}\right)=0, \tag{13}$$

and by (H2) and (13), we get

$$\liminf_{t\to\infty}\left(\frac{x(t)}{h_{m-1}(t,t_2)}\right)=0. \tag{14}$$

In view of (11) and (14), we learn that

$$\liminf_{t\to\infty}\widetilde{x}(t)=0, \tag{15}$$

where $\widetilde{x}(t):=x(t)/z(t)$ for $t\in[t_2,\infty)_{\mathbb{T}}$. Now, set

$$\widetilde{y}(t):=\widetilde{x}(t)+\widetilde{A}(t)\widetilde{x}(\alpha(t))\qquad\text{and}\qquad\widetilde{A}(t):=\frac{A(t)z(\alpha(t))}{z(t)} \tag{16}$$

for $t\in[t_2,\infty)_{\mathbb{T}}$. By (H1) and the increasing nature of $z$, we have $-a_1\leq\widetilde{A}(t)\leq a_2$ for all $t\in[t_2,\infty)_{\mathbb{T}}$. Thus, from (H4), (9) and $\ell_0=\infty$, we see that $\lim_{t\to\infty}z(t)/y(t)=1$ holds, which indicates

$$\lim_{t\to\infty}\widetilde{y}(t)=1 \tag{17}$$

since we have $\widetilde{y}(t)=y(t)/z(t)$ for all $t\in[t_2,\infty)_{\mathbb{T}}$.

Now, we prove that $\widetilde{x}$ is bounded; that is, $\widetilde{\ell}$ is a finite constant, where $\widetilde{\ell}:=\limsup_{t\to\infty}\widetilde{x}(t)$. Otherwise, $\widetilde{\ell}=\infty$ holds, and there exists an increasing divergent sequence $\{\zeta_k\}_{k\in\mathbb{N}}\subset[t_2,\infty)_{\mathbb{T}}$ such that $\widetilde{x}(\zeta_k)\geq\sup\{\widetilde{x}(t):t\in[t_2,\zeta_k)_{\mathbb{T}}\}$ holds for all $k\in\mathbb{N}$ and $\{\widetilde{x}(\zeta_k)\}_{k\in\mathbb{N}}$ is divergent. Hence, for all $k\in\mathbb{N}$, we get

$$\begin{aligned}\widetilde{y}(\zeta_k)\geq&\widetilde{x}(\zeta_k)-a_1\widetilde{x}(\alpha(\zeta_k))\\ \geq&(1-a_1)\widetilde{x}(\zeta_k),\end{aligned}$$

which implies $\widetilde{y}(\zeta_k)\to\infty$ by letting $k\to\infty$. This contradicts (17). Thus, $\widetilde{\ell}$ is finite.

Finally, we prove $\widetilde{\ell}=0$; i.e.,

$$\lim_{t\to\infty}\widetilde{x}(t)=0. \tag{18}$$

Let $\{\xi_k\}_{n\in\mathbb{N}}, \{\varsigma_k\}_{k\in\mathbb{N}} \subset [t_2,\infty)_{\mathbb{T}}$ be two increasing divergent subsequences satisfying $\lim_{k\to\infty}\widetilde{x}(\xi_k) = 0$ (see (15)) and $\lim_{k\to\infty}\widetilde{x}(\varsigma_k) = \widetilde{\ell}$. Since $\widetilde{x}$ is bounded, we may assume existence of the limits $\lim_{k\to\infty}\widetilde{x}(\alpha(\xi_k))$ and $\lim_{k\to\infty}\widetilde{x}(\alpha(\varsigma_k))$ (due to Bolzano-Weierstrass theorem, there always exist such subsequences of $\{\widetilde{x}(\alpha(\xi_k))\}_{k\in\mathbb{N}}$ and $\{\widetilde{x}(\alpha(\varsigma_k))\}_{k\in\mathbb{N}}$), and note that both of these limits cannot exceed $\widetilde{\ell}$. Then, for all $k\in\mathbb{N}$, we get

$$\widetilde{y}(\varsigma_k) - \widetilde{y}(\xi_k) \geq \widetilde{x}(\varsigma_k) - a_1\widetilde{x}(\alpha(\varsigma_k)) - \widetilde{x}(\xi_k) - a_2\widetilde{x}(\alpha(\xi_k)). \tag{19}$$

In view of (17), by letting $k\to\infty$ in (19), we get $0 \geq (1-a_1-a_2)\widetilde{\ell}$, which proves $\widetilde{\ell}=0$ by (H1); that is, (18) is true. Hence, by letting $t\to\infty$ in (16), we obtain $\lim_{t\to\infty}\widetilde{y}(t) = 0$ because of (18) and boundedness of $\widetilde{A}$. However, this contradicts (17), and thus (1) cannot admit eventually positive unbounded solutions.

Next, let $x(t), x(\alpha(t)), x(\beta(t)) < 0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$. Set $\widehat{x} := -x$, $\widehat{\varphi} := -\varphi$ on $[t_0,\infty)_{\mathbb{T}}$, and $\widehat{F}(u) := -F(-u)$ for $u\in\mathbb{R}$. Then, from (1), eventually positive $\widehat{x}$ satisfies the following equation

$$\left[\widehat{x}(t) + A(t)\widehat{x}(\alpha(t))\right]^{\Delta^n} + B(t)\widehat{F}(\widehat{x}(\beta(t))) = \widehat{\varphi}(t) \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}} \tag{20}$$

for which the conditions (H1)–(H5) hold. Applying the first part of the proof for (20), we get the desired contradiction that (20) cannot have eventually positive unbounded solutions; i.e., (1) cannot have eventually negative unbounded solutions. Thus, all unbounded solutions of (1) are oscillatory, and the proof is hence completed. □

# 4 Some applications

We have the following application of Theorem 1 on the well-known time scale $\mathbb{T}=\mathbb{Z}$.

**Example 1.** *Let $\mathbb{T}=\mathbb{Z}$, and consider the following second-order neutral delay difference equation:*

$$\left[x(t) + \left(-\frac{1}{3}\right)^t x(t-2)\right]^{\Delta^2} + 48x(t-1) = 0 \quad \text{for } t\in[1,\infty)_{\mathbb{Z}}. \tag{21}$$

*Here, we have $n = 2$, $A(t) = (-1/3)^t$, $\alpha(t) = t-2$, $B(t) \equiv 48$, $F(u) = u$, $\beta = t-1$ and $\varphi(t) \equiv 0$ for $t \in [1,\infty)_{\mathbb{Z}}$. In this case, we may pick $a_1 = a_2 = 1/3$ to satisfy (H1), and we have $h_0(t,s) \equiv 1$ for $s,t \in [1,\infty)_{\mathbb{Z}}$, which indicates that (H3) and (H4) hold. It is trivial that (H2) and (H5) hold. Since all the conditions of Theorem 1 are satisfied, all unbounded solutions of* (21) *are oscillatory. Such an unboundedly oscillating solution is $x(t) = (-3)^t$ for $t \in [1,\infty)_{\mathbb{Z}}$.*

Next, we give another example for $\mathbb{T} = \mathbb{R}$.

**Example 2.** *Let $\mathbb{T} = \mathbb{R}$, and consider the following fourth-order neutral delay differential equation:*

$$\left[x(t) + 2\mathrm{e}^{-t/2}\cos(t/2)x(t/2)\right]^{\Delta^4} + 4\mathrm{e}^{2\pi}x(t-2\pi) = \sin(t) \quad \text{for } t \in [1,\infty)_{\mathbb{R}}. \tag{22}$$

*Here, we have $n = 4$, $A(t) = 2\mathrm{e}^{-t/2}\cos(t/2)$, $\alpha(t) = t/2$, $B(t) \equiv 4\mathrm{e}^{2\pi}$, $F(u) = u$, $\beta = t-2\pi$ and $\varphi(t) = \sin(t)$ for $t \in [1,\infty)_{\mathbb{R}}$. Since $\lim_{t\to\infty} A(t) = 0$ holds, we may pick $a_1 = a_2 = 1/4$ for (H1) to hold, and we have $h_1(t,s) = t-s$ and $h_2(t,s) = (t-s)^2/2$ for $s,t \in [1,\infty)_{\mathbb{R}}$, which indicates that (H3), (H4) holds. Obviously, (H2) and (H5) hold with $\Phi(t) = \sin(t)$ for $t \in [1,\infty)_{\mathbb{R}}$. All the conditions of Theorem 1 are satisfied, hence all unbounded solutions of* (22) *are oscillatory. It can be easily verified that $x(t) = \mathrm{e}^t\sin(t)$ for $t \in [1,\infty)_{\mathbb{R}}$ is an unboundedly oscillating solution of* (22).

# 5 Final comments

Theorem 1 extends and improves [18, Theorem 2.4] (for unbounded solutions). It is pointed out in [2, § 6.4] (for differential equations) that it would be a significant interest when $|A| > 1$ holds, but unfortunately, the results [3, Lemma 6.4.2(ii), Theorem 6.4.4, Theorem 6.4.8] for the case $|A| < 1$ are wrong (the sequences picked in the proof of [3, Lemma 6.4.2] may not always exist, and thus, [3, Theorem 6.4.4, Theorem 6.4.8] are wrong because they depend on [3, Lemma 6.4.2]). Therefore, Theorem 1 (for $\mathbb{T} = \mathbb{R}$) not only corrects some of the results of [2, § 6.4] (for unbounded solutions and oscillating $A$ in a strip of width less than 1) but also improves by replacing $\int_{t_0}^{\infty} B(\eta)\mathrm{d}\eta = \infty$ with the weaker one $\int_{t_0}^{\infty} (\eta - t_0)^{n-2}B(\eta)\mathrm{d}\eta = \infty$, indeed it is trivial that $\int_{t_0}^{\infty} B(\eta)\mathrm{d}\eta = \infty$ implies $\int_{t_0}^{\infty} (\eta - t_0)^{n-2}B(\eta)\mathrm{d}\eta = \infty$, when

$n \in [2, \infty)_{\mathbb{Z}}$. However, as is mentioned in [2, § 6.4], it is indeed further more difficult when $A$ oscillates in a strip of which width exceeds 1.


## Acknowledgements

The author is indebt to express his sincere gratitude to the anonymous referee for his/her valuable suggestions and comments shaping the paper's present form.


## References


[1] R. P. Agarwal and M. Bohner. Basic calculus on time scales and some of its applications. *Results Math.*, vol. 35, no. 1-2, pp. 3–22, (1999).

[2] R. P. Agarwal, M. Bohner and W. T. Li. *Nonoscillation and Oscillation: Theory for Functional Differential Equations.* Marcel Dekker, 2004.

[3] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan. *Discrete Oscillation Theory.* Hindawi Publishing Corporation, New York, 2005.

[4] R. P. Agarwal and M. Bohner. An oscillation criterion for first order delay dynamic equations. *Funct. Differ. Equ.*, 2009. (To appear)

[5] E. Akın. Cauchy functions for dynamic equations on a measure chain. *J. Math. Anal. Appl.*, vol. 267, no. 1, pp. 97–115, (2002).

[6] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications.* Birkhäuser, Boston, 2001.

[7] M. Bohner. Some oscillation criteria for first order delay dynamic equations. *Far East J. Appl. Math.*, vol. 18, no. 3, pp. 289–304, (2005).

[8] M. Bohner and S. Stević. Asymptotic behavior of second-order dynamic equations. *Appl. Math. Comput.*, vol. 188, pp. 1503–1512, (2007).

[9] M. Bohner, B. Karpuz and Ö. Öcalan. Iterated oscillation criteria for delay dynamic equations of first order. *Adv. Difference Equ.*, vol. 2008, aid. 458687, 1–12 p., (2008).

[10] Y. Bolat and Ö. Akın. Oscillatory behaviour of higher order neutral type nonlinear forced differential equation with oscillating coefficients. *J. Math. Anal. Appl.*, vol. 290, pp. 302–309, (2004).

[11] I. Győri and G. Ladas. *Oscillation Theory of Delay Differential Equations: With Applications*. Oxford University Press, New York, 1991.

[12] B. Karpuz, R. N. Rath and S. K. Rath. On oscillation and asymptotic behaviour of a higher order functional difference equation of neutral type. *Int. J. Difference Equ.*, 2009. (To appear)

[13] B. Karpuz. Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations. (Submitted)

[14] B. Karpuz and Ö. Öcalan. Necessary and sufficient conditions on the asymptotic behaviour of solutions of forced neutral delay dynamic equations. *Nonlinear Anal.*, (2009). (To appear, doi:10.1016/j.amc.2008.12.075).

[15] B. Karpuz, Ö. Öcalan and R. N. Rath. Necessary and sufficient conditions on the oscillatory and asymptotic behaviour of solutions to neutral delay dynamic equations. *Electron. J. Differential Equations*, (2009). (To appear)

[16] I. Kır and Y. Bolat. Oscillation criteria for higher-order sublinear neutral delay difference equations with oscillating coefficients. *Int. J. Difference Equ.*, vol. 1, no. 2, pp. 219–223, (2006).

[17] M. A. Özcan and Y. Bolat. Oscillation criteria for high-order sublinear neutral delay forced difference equations with oscillating coefficients. *Demonstratio Math.*, vol. 40, no. 4, pp. 853–857, (2007).

[18] N. Parhi and R. N. Rath. On oscillation of solutions of forced nonlinear neutral differential equations of higher order II. *Ann. Polon. Math.*, vol. 81, no. 2, pp. 101–110, (2003).

[19] B. G. Zhang and X. Deng. Oscillation of delay differential equations on time scales. *Math. Comput. Model.*, vol. 36, no. 11-13, pp. 1307–1318, (2002).

[20] X. L. Zhou. Oscillatory and asymptotic properties of higher order nonlinear neutral difference equations with oscillating coefficients. *Appl. Math. Lett.*, vol. 21, no. 11, pp. 1142–1148, (2008).

[21] X. L. Zhou and R. Yu. Oscillatory behavior of higher order nonlinear neutral forced differential equations with oscillating coefficients. *Comp. Math. Appl.*, vol. 6, no. 6, pp. 1562–1568, (2008).

[22] Z. Q. Zhu and Q. R. Wang. Existence of nonoscillatory solutions to neutral dynamic equations on time scales. *J. Math. Anal. Appl.*, vol. 335, pp. 751–762, (2007).

(Received August 30, 2008)