

SOME RESULTS OF NONTRIVIAL SOLUTIONS FOR A NONLINEAR PDE IN SOBOLEV SPACE

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ABSTRACT. In this study, we investigate the question of nonexistence of nontrivial solutions of the Robin problem

$$(P) \quad \left\{ \begin{array}{l} -\frac{\partial^2 u}{\partial x^2} - \sum_{s=1}^n \frac{\partial}{\partial y_s} a_s(y, \frac{\partial u}{\partial y_s}) + f(y, u) = 0 \text{ in } \Omega = \mathbb{R} \times D, \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial D. \end{array} \right.$$

where $a_s : D \times \mathbb{R} \rightarrow \mathbb{R}$ are H^1 - functions with constant sign such that

$$(H_1) \quad 2 \int_0^{\xi_s} a_s(y, t_s) dt_s - \xi_s a_s(y, \xi_s) \leq 0, s = 1, \dots, n$$

and $f : D \times \mathbb{R} \rightarrow \mathbb{R}$ is a real continuous locally Lipschitz function such that

$$(H_2) \quad 2F(y, u) - uf(y, u) \leq 0,$$

We show that the function

$$E(x) = \int_D |u(x, y)|^2 dy$$

is convex on \mathbb{R} . Our proof is based on energy (integral) identities.

$$\left(D = \prod_{k=1}^n]\alpha_k, \beta_k[, \varepsilon > 0 \text{ and } F(y, u) = \int_0^u f(y, \tau) d\tau \right).$$

1. INTRODUCTION

The problem of existence and nonexistence of nontrivial solutions of problems of the form

$$\left\{ \begin{array}{l} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{array} \right.$$

has been investigated by many authors under various situations. Previous works have been reported by Berestycky, Gallouet & Kavian [1], M. J. Esteban & P. L. Lions [2], Pucci & J. Serrin [9] and Pohozaev [10]. To illustrate some

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of the typical known results, let us consider Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta u + f(u) = 0, u \in C^2(\overline{\Omega}), \\ u = 0 \text{ on } \partial\Omega, \end{array} \right.$$

Under hypothesis

$$\left\{ \begin{array}{l} \nabla u \in L^2(\Omega), \\ f(0) = 0, \\ F(u) = \int_0^u f(s) ds \in L^1(\Omega), \end{array} \right.$$

where Ω is a connected unbounded domain of \mathbb{R}^N such as

$$\exists \Lambda \in \mathbb{R}^N, \|\Lambda\| = 1, \langle n(x), \Lambda \rangle \geq 0 \text{ on } \partial\Omega, \langle n(x), \Lambda \rangle \neq 0,$$

($n(x)$ is the outward normal to $\partial\Omega$ at the point x) Esteban & Lions [2] established that the Dirichlet problem does not have nontrivial solutions.

Berestycky, Gallouet & Kavian [1] established that the problem

$$\left\{ \begin{array}{l} -\Delta u - u^3 + u = 0, \\ u \in H^2(\mathbb{R}^2) \end{array} \right.$$

admits a radial solution

This same solution satisfies

$$\left\{ \begin{array}{l} -\Delta u - u^3 + u = 0, \\ u \in H^2(]0, +\infty[\times \mathbb{R}) \\ \frac{\partial u}{\partial n} = 0 \text{ on } \{0\} \times \mathbb{R}, \end{array} \right.$$

this shows that analogous Esteban-Lions result for Neumann problems is not valid.

The Pohozaev identity published in 1965 for solutions of the Dirichlet problem proved absence of nontrivial solutions for some elliptic equations when Ω is a star shaped bounded domain in \mathbb{R}^n and f a continuous function on \mathbb{R} satisfying:

$$(n - 2)F(u) - 2nuf(u) > 0,$$

where, $n = \dim \mathbb{R}^n$.

When

$$\Omega = J \times \omega,$$

where $J \subset \mathbb{R}$ is unbounded interval and $\omega \subset \mathbb{R}^n$ domain, Haraux & Khodja [3] established under the assumption

$$\left\{ \begin{array}{l} f(0) = 0, \\ 2F(u) - uf(u) \leq 0, \end{array} \right.$$

if we assume that $u \in H^2(J \times \omega) \cap L^\infty(J \times \omega)$ is a solution of the problems

$$\left\{ \begin{array}{l} -\Delta u + f(u) = 0 \text{ in } \Omega, \\ (u \text{ or } \frac{\partial u}{\partial n}) = 0 \text{ on } \partial(J \times \omega). \end{array} \right.$$

Then these two problems (Dirichlet and Neumann) do have only trivial solution.

When

$$f(u) = u(u+1)(u+2),$$

and

$$\Omega = \mathbb{R} \times]0, a[\quad (a < \pi),$$

Neumann problem

$$\left\{ \begin{array}{l} -\Delta u + u(u+1)(u+2) = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \end{array} \right.$$

is still open.

In this work, let $a_i, i = 1, \dots, n$ be a sequence in $H^1(D \times \mathbb{R})$ verifying

$$a_i(y, 0) = 0 \text{ in } D = \prod_{k=1}^n]\alpha_k, \beta_k[,$$

and $f : D \times \mathbb{R} \rightarrow \mathbb{R}$ a locally Lipschitz continuous function such that $f(y, 0) = 0$ in D , so that $u = 0$ is a solution of the equation

$$(1.1) \quad -\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) + f(y, u) = 0 \text{ in } \Omega = \mathbb{R} \times D.$$

We assume that

$$u \in H^2(\Omega) \cap L^\infty(\Omega),$$

and satisfies

$$(1.2) \quad u(x, s) = 0, (x, s) \in \mathbb{R} \times \partial D$$

or

$$(1.3) \quad \frac{\partial u}{\partial n}(x, s) = 0, (x, s) \in \mathbb{R} \times \partial D$$

or

$$(1.4) \quad \left(u + \varepsilon \frac{\partial u}{\partial n} \right) (x, s) = 0, (x, s) \in \mathbb{R} \times \partial D$$

Let us denote by:

$$\Gamma = \mathbb{R} \times \partial D = \Gamma_{\alpha_1} \cup \Gamma_{\beta_1} \cup \dots \cup \Gamma_{\alpha_n} \cup \Gamma_{\beta_n},$$

$$(\Gamma_{\mu_i} = \{(x, y_1, \dots, y_{i-1}, \mu_i, y_{i+1}, \dots, y_n), x \in \mathbb{R}, 1 \leq i \leq n\})$$

the boundary of Ω ,

$n(x, s) = (0, n_1(x, s), \dots, n_n(x, s))$, the outward normal to $\partial\Omega$ at the point (x, s)

and

$\left(\frac{\partial^2 u(x, y)}{\partial y_i^2}\right)_{i=1, \dots, n}$ the second derivative of u with respect to y_i at point (x, y) .

If $z \in \Omega$, $k = 1, 2, \dots, n$ and $\tau \in \{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_n, \beta_n\}$ one writes:

$$z := (x, y) = (x, y_1, \dots, y_n)$$

$$z_k^\tau := (y_1, \dots, y_{k-1}, \tau, y_{k+1}, \dots, y_n),$$

$$dz_k^* := dy_1 \dots dy_{k-1} dy_{k+1} \dots dy_n,$$

$$\int_{\alpha_1}^{\beta_1} \dots \int_{\alpha_{i-1}}^{\beta_{i-1}} \int_{\alpha_{i+1}}^{\beta_{i+1}} \dots \int_{\alpha_n}^{\beta_n} f(x, y) dy_1 \dots dy_{i-1} dy_{i+1} \dots dy_n := \int_{D_i^*} f(x, y) dz_i^*.$$

The objective of this paper is to extend the results of [3], [5] to problems (1.1) – (1.2), (1.1) – (1.3) and (1.1) – (1.4).

2. INTEGRAL IDENTITIES

We begin this section by giving an integral identity useful in the sequel.

Lemma 1. *Let*

$$a_i \in H^1(D \times \mathbb{R}), i = 1, \dots, n$$

satisfy

$$a_i(\cdot, \xi_i) : \overline{D} \rightarrow \mathbb{R}, > 0 \text{ or } < 0, \forall \xi_i, i \in \{1, \dots, n\},$$

and assume $f : D \times \mathbb{R} \rightarrow \mathbb{R}$ a locally Lipschitz continuous function. Then any solution $u \in H^2(\mathbb{R} \times D) \cap L^\infty(\mathbb{R} \times D)$ of (1.1) satisfying (1.4), verifies for each $x \in \mathbb{R}$ and $\varepsilon \neq 0$ the integral identity

$$(2.1) \quad \int_D \left(-\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \sum_{i=1}^n A_i(y, \frac{\partial u}{\partial y_i}) + F(y, u) \right) (x, y) dy + \varepsilon \sum_{i=1}^n \int_{D_i^*} (A_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) + A_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i}))) dz_i^* = 0$$

Proof. Let

$$H : \mathbb{R} \rightarrow \mathbb{R}$$

the function defined by

$$H(x) = \int_D \left(-\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \sum_{i=1}^n A_i(y, \frac{\partial u}{\partial y_i}) + F(y, u) \right) (x, y) dy.$$

The hypotheses on u , a_i , $i = 1, \dots, n$ and f imply that H is absolutely continuous and thus differentiable almost everywhere on \mathbb{R} , we have

$$(2.2) \quad \begin{aligned} \frac{d}{dx}H(x) &= \int_D \left(-\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + \sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial^2 u}{\partial y_i \partial x} + f(y, u) \frac{\partial u}{\partial x} \right) (x, y) dy \\ &= \int_D \left(-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) + f(y, u) \right) \left(\frac{\partial u}{\partial x} \right) (x, y) dy \\ &+ \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\beta_i})) \frac{\partial u}{\partial x}(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\alpha_i})) \frac{\partial u}{\partial x}(x, z_i^{\alpha_i}) \right) dz_i^*. \end{aligned}$$

Indeed a simple use of Fubini's theorem and an integration by parts yields

$$\begin{aligned} \int_D a_i(y, \frac{\partial u}{\partial y_i}) \left(\frac{\partial^2 u}{\partial y_i \partial x} \right) (x, y) dy &= \int_{D_i^*} \left(\int_{\alpha_i}^{\beta_i} a_i(y, \frac{\partial u}{\partial y_i}) \left(\frac{\partial^2 u}{\partial y_i \partial x} \right) dy \right) dz_i^* \\ &= \int_{D_i^*} \left(\int_{\alpha_i}^{\beta_i} -\frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) \frac{\partial u}{\partial x}(x, y) dy \right) dz_i^* \\ &+ \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i}) \left(\frac{\partial u}{\partial x} \right) (x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u}{\partial y_i}) \left(\frac{\partial u}{\partial x} \right) (x, z_i^{\alpha_i}) \right) dz_i^* \\ &= \int_D -\frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) \left(\frac{\partial u}{\partial x} \right) (x, y) dy \\ &+ \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\beta_i})) \left(\frac{\partial u}{\partial x} \right) (x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\alpha_i})) \left(\frac{\partial u}{\partial x} \right) (x, z_i^{\alpha_i}) \right) dz_i^*. \end{aligned}$$

By summing up these formulas with respect to i and substituting them in (2.2), one obtains

$$\begin{aligned} \frac{d}{dx}H(x) &= \int_D \left(-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) + f(y, u) \right) \left(\frac{\partial u}{\partial x} \right) (x, y) dy \\ &+ \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\beta_i})) \frac{\partial u}{\partial x}(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\alpha_i})) \frac{\partial u}{\partial x}(x, z_i^{\alpha_i}) \right) dz_i^*. \end{aligned}$$

As u satisfies equation (1.1), the above expression reduces to

$$(2.3) \quad \frac{d}{dx}H(x) = \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\beta_i})) \frac{\partial u}{\partial x}(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\alpha_i})) \frac{\partial u}{\partial x}(x, z_i^{\alpha_i}) \right) dz_i^*.$$

Now observe that $\left(u + \varepsilon \frac{\partial u}{\partial n}\right)(x, s) = 0$ on $\partial\Omega$, is equivalent to

$$(2.4) \quad \begin{cases} \left(u - \varepsilon \frac{\partial u}{\partial y_i}\right)(x, y_1, \dots, y_{i-1}, \alpha_i, y_{i+1}, \dots, y_n) = 0 \\ \left(u + \varepsilon \frac{\partial u}{\partial y_i}\right)(x, y_1, \dots, y_{i-1}, \beta_i, y_{i+1}, \dots, y_n) = 0 \end{cases} \quad x \in \mathbb{R}, \alpha_i < y_i < \beta_i.$$

This allows to write formula (2.3) in the following form

$$\begin{aligned} \frac{d}{dx}H(x) &= \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, (-\varepsilon^{-1})u(x, z_i^{\beta_i})) \frac{\partial u}{\partial x}(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \varepsilon^{-1}u(x, z_i^{\alpha_i})) \frac{\partial u}{\partial x}(x, z_i^{\alpha_i}) \right) dz_i^* \\ &= -\varepsilon \sum_{i=1}^n \int_{D_i^*} \frac{\partial}{\partial x} \left(A_i(z_i^{\beta_i}, -\varepsilon^{-1}u(x, z_i^{\beta_i})) + A_i(z_i^{\alpha_i}, \varepsilon^{-1}u(x, z_i^{\alpha_i})) \right) dz_i^*, \end{aligned}$$

i.e

$$\frac{d}{dx} \left(H(x) + \varepsilon \sum_{i=1}^n \int_{D_i} \left(A_i(z_i^{\alpha_i}, \varepsilon^{-1}u(x, z_i^{\alpha_i})) + A_i(z_i^{\beta_i}, (-\varepsilon^{-1})u(x, z_i^{\beta_i})) \right) dz_i^* \right) = 0.$$

Integrating this expression, with respect to x one obtains

$$H(x) + \varepsilon \sum_{i=1}^n \int_{D_i^*} \left(A_i(z_i^{\alpha_i}, \varepsilon^{-1}u(x, z_i^{\alpha_i})) + A_i(z_i^{\beta_i}, (-\varepsilon^{-1})u(x, z_i^{\beta_i})) \right) dz_i^* = \text{const.}$$

Since

$$u(x, y) \in H^2(\mathbb{R} \times D),$$

one must get

$$\int_{-\infty}^{+\infty} \left(H(x) + \varepsilon \sum_{i=1}^n \int_{D_i^*} \left(A_i(z_i^{\alpha_i}, \varepsilon^{-1}u(x, z_i^{\alpha_i})) + A_i(z_i^{\beta_i}, (-\varepsilon^{-1})u(x, z_i^{\beta_i})) \right) dz_i^* \right) dx < \infty.$$

We conclude that the constant is null which is the desired result.

Lemma 2. *Let u be chosen as in Lemma 1.1. If one assumes u to be solution of problems (1.1) – (1.3) or (1.1) – (1.4), then for each $x \in \mathbb{R}$, the solution u verifies*

$$(2.5) \quad \int_D \left(-\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \sum_{i=1}^n A_i(y, \frac{\partial u}{\partial y_i}) + F(y, u) \right) (x, y) dy = 0.$$

Proof. To prove (2.5) it suffices to show that the second term of (2.1) vanishes if u verifies (1.2) or (1.3), i.e

$$\int_{D_i^*} \left(A_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) + A_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i})) \right) dz_i^* = 0.$$

If one supposes that $u(x, s) = 0$ for $(x, s) \in \mathbb{R} \times \partial D$, it is immediate, that

$$A_i(z_i^{\alpha_i}, 0) = A_i(z_i^{\beta_i}, 0), \forall i = 1, \dots, n.$$

Now if the boundary condition is $\frac{\partial u}{\partial n}(x, s) = 0$ for $(x, s) \in \mathbb{R} \times \partial D$, then

$$\frac{\partial u}{\partial n}(x, s) = \langle \nabla u, n \rangle(x, s) = 0 \quad (x, s) \in \mathbb{R} \times \partial D,$$

i.e

$$\left. \begin{array}{l} \frac{\partial u}{\partial x}(x, z_i^{\alpha_i}) = \frac{\partial u}{\partial x}(x, z_i^{\beta_i}) = 0 \\ \frac{\partial u}{\partial y_1}(x, z_1^{\alpha_1}) = \frac{\partial u}{\partial y_1}(x, z_1^{\beta_1}) = 0 \\ \vdots \\ \frac{\partial u}{\partial y_i}(x, z_i^{\alpha_i}) = \frac{\partial u}{\partial y_i}(x, z_i^{\beta_i}) = 0 \\ \vdots \\ \frac{\partial u}{\partial y_n}(x, z_n^{\alpha_n}) = \frac{\partial u}{\partial y_n}(x, z_n^{\beta_n}) = 0 \end{array} \right|, x \in \mathbb{R}, \alpha_i \leq y_i \leq \beta_i,$$

consequently

$$a_i(z_i^{\alpha_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\alpha_i})) = a_i(z_i^{\beta_i}, \frac{\partial u}{\partial y_i}(x, z_i^{\beta_i})) = 0, \forall i = 1, \dots, n.$$

because

$$a_i(x, 0) = 0, \forall x \in \overline{D}, \forall i = 1, \dots, n..$$

Finally one gets

$$A_i(z_i^{\alpha_i}, 0) = A_i(z_i^{\beta_i}, 0) = 0, \forall i = 1, \dots, n.$$

3. MAIN RESULTS

The goal of this section is to establish the nonexistence of nontrivial solutions to Robin problem.

Theorem 1. Let $a_i, i = 1, \dots, n$ and f satisfying respectively

$$(3.1) \quad \begin{aligned} a_i(\cdot, \xi_i) : \bar{D} &\rightarrow \mathbb{R}, > 0 \text{ or } < 0, \\ &\forall \xi_i, \forall i \in \{1, \dots, n\} \\ 2A_i(y, \xi_i) - a_i(y, \xi_i) \xi_i &\leq 0, \end{aligned}$$

$$(3.2) \quad 2F(y, u) - uf(y, u) \leq 0,$$

and assume

$$u \in H^2(\Omega) \cap L^\infty(\Omega)$$

to be a solution of (1.1) – (1.4). Then the function

$$x \mapsto E(x) = \int_D |u(x, y)|^2 dy \text{ is convex on } \mathbb{R}.$$

Proof. To begin the proof, we see that almost everywhere in $\Omega = \mathbb{R} \times D$, we have

$$\left(u \frac{\partial^2 u}{\partial x^2}\right)(x, y) = \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} (u^2) - \left|\frac{\partial u}{\partial x}\right|^2\right)(x, y).$$

In fact by multiplying equation (1.1) by $\frac{u}{2}$ and integrating the new equation over D , we obtain

$$(3.3) \quad \begin{aligned} 0 &= \int_D \left(-\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) + f(y, u) \right) \frac{u}{2}(x, y) dy \\ &= \int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \frac{1}{2} \left|\frac{\partial u}{\partial x}\right|^2 - \frac{1}{2} \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) u + \frac{u}{2} f(y, u)(x, y) \right) dy. \end{aligned}$$

A simple use of Fubini's theorem and an integration by parts yields,

$$\begin{aligned} \int_D \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) (u)(x, y) dy &= \int_{D_i^*} \left(\int_{\alpha_i}^{\beta_i} \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) u dy_i \right) dz_i^* = \\ &= - \int_D a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} (x, y) dy + \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i}) u(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i}) u(x, z_i^{\alpha_i}) \right) dz_i^* \end{aligned}$$

Instead of (3.3), we obtain

$$\int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \frac{1}{2} \left|\frac{\partial u}{\partial x}\right|^2 + \frac{1}{2} \sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} + \frac{1}{2} uf(y, u) \right) (x, y) dy$$

$$= \frac{1}{2} \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i}) u(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i}) u(x, z_i^{\alpha_i}) \right) dz_i^*$$

From (2.4) it follows that

$$\begin{aligned} & \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i}) u(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i}) u(x, z_i^{\alpha_i}) \right) dz_i^* \\ &= \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i})) u(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) u(x, z_i^{\alpha_i}) \right) dz_i^* \\ &= \sum_{i=1}^n \int_{D_i^*} \left(\frac{1}{-\varepsilon^{-1}} a_i(z_i^{\beta_i}, -\varepsilon^{-1} u(x, z_i^{\beta_i})) (-\varepsilon^{-1}) u(x, z_i^{\beta_i}) - \frac{1}{\varepsilon^{-1}} a_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) \varepsilon^{-1} u(x, z_i^{\alpha_i}) \right) dz_i^* \end{aligned}$$

i.e

$$\begin{aligned} & \int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} + \frac{1}{2} u f(y, u) \right) (x, y) dy \\ & + \frac{\varepsilon}{2} \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i})) (-\varepsilon^{-1}) u(x, z_i^{\beta_i}) + a_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) \varepsilon^{-1} u(x, z_i^{\alpha_i}) \right) dz_i^* = 0, \end{aligned}$$

Combining this formula and (2.1) we obtain

$$\begin{aligned} & \int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} + \frac{1}{2} u f(y, u) \right) (x, y) dy \\ & + \frac{\varepsilon}{2} \sum_{i=1}^n \int_{D_i^*} \left(a_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i})) (-\varepsilon^{-1}) u(x, z_i^{\beta_i}) + a_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) \varepsilon^{-1} u(x, z_i^{\alpha_i}) \right) dz_i^* \\ & = \int_D \left(-\frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \sum_{i=1}^n A_i(y, \frac{\partial u}{\partial y_i}) + F(y, u) \right) (x, y) dy \\ & + \varepsilon \sum_{i=1}^n \int_{D_i^*} \left(A_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) + A_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i})) \right) dz_i^* \end{aligned}$$

i.e

$$\begin{aligned} & \int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \left| \frac{\partial u}{\partial x} \right|^2 \right) (x, y) dy \\ & = \int_D \left(\sum_{i=1}^n \left(A_i(y, \frac{\partial u}{\partial y_i}) - \frac{1}{2} a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} \right) + F(y, u) - \frac{1}{2} u f(y, u) \right) (x, y) dy \end{aligned}$$

$$\begin{aligned}
& +\varepsilon \sum_{i=1}^n \int_{D_i^*} \left(A_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) - \frac{1}{2} a_i(z_i^{\alpha_i}, \varepsilon^{-1} u(x, z_i^{\alpha_i})) \varepsilon^{-1} u(x, z_i^{\alpha_i}) \right) dz_i^* \\
& +\varepsilon \sum_{i=1}^n \int_{D_i^*} \left(A_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i})) - \frac{1}{2} a_i(z_i^{\beta_i}, (-\varepsilon^{-1}) u(x, z_i^{\beta_i})) (-\varepsilon^{-1}) u(x, z_i^{\beta_i}) \right) dz_i^*.
\end{aligned}$$

Hypotheses (3.1) and (3.2) imply that

$$\frac{d^2}{dx^2} \left(\int_D |u(x, y)|^2 dy \right) \geq 4 \int_D \left| \frac{\partial u}{\partial x}(x, y) \right|^2 dy \geq 0, \forall x \in \mathbb{R}.$$

This completes the proof.

Remark 1. *The convexity of the function $E(x)$ on \mathbb{R} implies the triviality of the solution $u(x, y)$ of the problem (1.1) – (1.4).*

Theorem 2. *Let the function $a_i, i = 1, \dots, n$ and f be as described as in Theorem 3.1. We assume $u \in H^2(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1) – (1.2) or (1.1) – (1.3), then the function $E(x)$ defined above is convex on \mathbb{R} .*

Proof. By similar arguments as in the proof of Theorem 3.1, we obtain

$$\begin{aligned}
& \int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} + \frac{1}{2} u f(y, u) \right) (x, y) dy \\
& = \frac{1}{2} \sum_{i=1}^n \int_{D_i} \left(a_i(z_i^{\beta_i}, \frac{\partial u(x, z_i^{\beta_i})}{\partial y_i}) u(x, z_i^{\beta_i}) - a_i(z_i^{\alpha_i}, \frac{\partial u(x, z_i^{\alpha_i})}{\partial y_i}) u(x, z_i^{\alpha_i}) \right) dz_i^*
\end{aligned}$$

Now if $u(x, s) = 0$ or $\frac{\partial u}{\partial n}(x, s) = 0$, for $(x, s) \in \mathbb{R} \times \partial D$ this formula reduces to

$$\int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) - \frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} + \frac{1}{2} u f(y, u) \right) (x, y) dy = 0$$

We can now employ (2.5) to transform this identity into the following form

$$\begin{aligned}
& \int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{2} \sum_{i=1}^n a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} + \frac{1}{2} u f(y, u) \right) (x, y) dy \\
& = \int_D \left(-\frac{1}{2} \left| \frac{\partial u(x, y)}{\partial x} \right|^2 + \sum_{i=1}^n A_i(y, \frac{\partial u(x, y)}{\partial y_i}) + F(y, u(x, y)) \right) dy
\end{aligned}$$

i.e

$$\int_D \left(-\frac{1}{4} \frac{\partial^2}{\partial x^2} (u^2) + \left| \frac{\partial u}{\partial x} \right|^2 \right) (x, y) dy$$

$$= \int_D \left(\sum_{i=1}^n \left(A_i(y, \frac{\partial u(x, y)}{\partial y_i}) - \frac{1}{2} a_i(y, \frac{\partial u}{\partial y_i}) \frac{\partial u}{\partial y_i} \right) + F(y, u) - \frac{1}{2} u f(y, u) \right) (x, y) dy.$$

Our assumptions on a_i , and f imply the desired result.

4. APPLICATIONS

A practical tool for characterizing the assumption (3.1) or (3.2) of Theorem 3.1 is the following Proposition.

Proposition 1. *Let*

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

a Lipschitzian real function such that

$$f(0) = 0.$$

We suppose that f is concave on $] -\infty, 0[$ and convex on $]0, +\infty[$. Then the function f satisfies the assumption (3.1) or (3.2) of Theorem 3.1.

Application 4.1: Taking

$$a_i(y, \frac{\partial u(x, y)}{\partial y_i}) = \frac{\partial u(x, y)}{\partial y_i}$$

then the equation (1.1) becomes

$$(4.1) \quad -\Delta u + f(y, u) = 0 \text{ in } \Omega$$

Application 4.2: We can put

$$a_i(y, \frac{\partial u}{\partial y_i}(x, y)) = c_i \frac{\partial u(x, y)}{\partial y_i}$$

with c_i are reals constants. In this case (1.1) can be rewritten as

$$(4.2) \quad -\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n c_i \frac{\partial^2 u}{\partial y_i^2} + f(y, u) = 0 \text{ in } \Omega$$

Application 4.3: We can also put

$$a_i(y, \frac{\partial u(x, y)}{\partial y_i}) = p_i(y) \frac{\partial u(x, y)}{\partial y_i}$$

with $p_i(y) < 0$ or > 0 in \bar{D} , it follows that the equation(1.1) is equivalent to

$$(4.3) \quad -\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(p_i(y) \frac{\partial u}{\partial y_i} \right) + f(y, u) = 0 \text{ in } \Omega$$

We observe that in this three applications, we have

$$2A_i(y, \xi_i) - a_i(y, \xi_i) \xi_i \equiv 0, \forall \xi_i, i = 1, \dots, n.$$

5. EXAMPLES

To conclude this work, let us give a few simple examples illustrating the use of Theorem 3.1.

Example 1. *The problem*

$$(5.1) \quad \left\{ \begin{array}{l} -\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) + \theta(y) |u|^{p-1} u = 0 \text{ in } \Omega = \mathbb{R} \times D \\ (u + \varepsilon \frac{\partial u}{\partial n})(x, s) = 0, (x, s) \in \mathbb{R} \times \partial D \end{array} \right.$$

where

$$\theta : \bar{D} \rightarrow \mathbb{R},$$

is a nonnegative continuous real function, $p \geq 1$ does not have nontrivial solutions.

Indeed,

$$2F(y, u) - uf(y, u) = \theta(y) \left(\frac{2}{p+1} - 1 \right) |u|^{p+1} \leq 0.$$

Theorem 3.1 give the desired result.

Example 2. *Let $\rho : \bar{D} \rightarrow \mathbb{R}$, be a continuous function . The problem*

$$(5.2) \quad \left\{ \begin{array}{l} -\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i(y, \frac{\partial u}{\partial y_i}) \right) + \rho(y) u = 0 \text{ in } \mathbb{R} \times D \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 \text{ on } \mathbb{R} \times \partial D \end{array} \right.$$

considered in $H^2(\mathbb{R} \times D) \cap L^\infty(\mathbb{R} \times D)$ does not have nontrivial solutions.

A simple calculation gives

$$2F(y, u) - uf(y, u) \equiv 0.$$

and

$$\begin{aligned} \frac{1}{4} \frac{d^2}{dx^2} \left(\int_D (|u(x, y)|^2 dx) \right) &= \int_D \left(\left| \frac{\partial u}{\partial x} \right|^2 + F(y, u) - \frac{1}{2} uf(y, u) \right) dx \\ &= \int_D \left| \frac{\partial u}{\partial x} \right|^2 (x, y) dx \geq 0 \end{aligned}$$

Example 3. *Let*

$$\theta_1, \theta_2 : \bar{D} \rightarrow \mathbb{R},$$

be two continuous nonnegative functions, $p, q \geq 1$ and

$$f(y, u) = mu + \theta_1(y) |u|^{p-1} u + \theta_2(y) |u|^{q-1} u, m \in \mathbb{R}.$$

The problem

$$(5.3) \quad \begin{cases} -\frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^n \frac{\partial}{\partial y_i} \left(a_i \left(y, \frac{\partial u}{\partial y_i} \right) \right) + f(y, u) = 0 & \text{in } \mathbb{R} \times D \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 & \text{on } \mathbb{R} \times \partial D \end{cases}$$

does not have nontrivial solutions.

It suffices to remark that,

$$2F(y, u) - uf(y, u) = \theta_1(y) \left(\frac{2}{p+1} - 1 \right) |u|^{p+1} + \theta_2(y) \left(\frac{2}{q+1} - 1 \right) |u|^{q+1} \leq 0$$

and then apply theorem 3.1.

REFERENCES

- [1] **H. BERESTYCKY, T. GALLOUET AND O. KAVIAN**, Equations de champs scalaires non-linéaires dans \mathbb{R}^2 . C. R. Acad. Sc. Paris, Série 1, **297**, (1983), 307-310.
- [2] **M. J. ESTEBAN AND P. L. LIONS**, Existence and nonexistence results for semi linear elliptic problems in unbounded domains. Proc. Roy. Soc. Edimburgh 93-A(1982), 1-14.
- [3] **A. HARAUX AND B. KHODJA**, Caractère trivial de la solution de certaines équations aux dérivées partielles non linéaires dans des ouverts cylindriques de \mathbb{R}^N . Portugaliae Mathematica. Vol. 42, Fasc. 2, (1982), 209-219.
- [4] **B. KHODJA**, Nonexistence of solutions for semilinear equations and systems in cylindrical domains. Comm. Appl. Nonlinear Anal. (2000), 19-30.
- [5] **B. KHODJA**, A nonexistence result for a nonlinear PDE with Robin condition, International Journal of Mathematics and Mathematical Sciences, volume 2006, Article ID 62601, 1-12
- [6] **B. KHODJA AND A. MOUSSAOUI**, Nonexistence results for semilinear systems in unbounded domains, Electron. J. Diff. Eqns., Vol. 2009 (2009), No. 02, 1-11.
- [7] **E. MITIDIERI**, Nonexistence of positive solutions of semilinear elliptic systems in \mathbb{R}^N . Diff. and Int. Equations, 9 (1996), 465-479.
- [8] **M. H. PROTTER AND H. F. WEINBERGER**, *Maximum principle in differential equations*, Prentice-Hall, Englewood Cliffs, New Jersey, (1967).
- [9] **PUCCI & J. SERRIN**, A general variational identity, Ind. Univers. Mat. Journal. Vol. 35, N3 (1986), 681-703.
- [10] **S. I. POHOZAEV**, Eigenfunctions of the equation $\Delta u + 1f(u) = 0$. Soviet. Math. Dokl. 6 (1965), 1408-1411.
- [11] **D. DE FIGUEIREDO**, Semilinear elliptic systems . Nonlinear Functional Analysis and Application to differential Equations (1998), 122-152, ICTP Trieste ITALY, 21 April-9 May 1997. World Scientific.

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