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Product-type system of difference equations of second-order solvable in closed form

Stevo Stević[✉ 1,2]

[1]Mathematical Institute of the Serbian Academy of Sciences,
Knez Mihailova 36/III, 11000 Beograd, Serbia
[2]Operator Theory and Applications Research Group, Department of Mathematics,
King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

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Abstract. This paper presents solutions to the following product-type second-order system of difference equations

$$x_{n+1} = \frac{y_n^a}{z_{n-1}^b}, \qquad y_{n+1} = \frac{z_n^c}{x_{n-1}^d}, \qquad z_{n+1} = \frac{x_n^f}{y_{n-1}^g}, \qquad n \in \mathbb{N}_0,$$

where $a, b, c, d, f, g \in \mathbb{Z}$, and $x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}$, $i \in \{0, 1\}$, in closed form.

Keywords: solvable system of difference equations, second-order system, product-type system, complex initial values.

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1 Introduction

Recently there has been some renewed interest in solving difference equations and systems of difference equations and their applications (see, e.g., [1–4,7,8,14,18,20,23–38,41–45]), especially after the publication of note [18] in which a method for solving a nonlinear difference equation of second-order was presented. Since the end of the 1990s there has been also some interest in concrete systems of difference equations (see, e.g., [8–13, 15–17, 23, 24, 26–28, 30–40, 42, 43, 45]). In the line of our investigations [5, 6, 19, 21, 22] (see also the references therein) we studied the long-term behavior of several classes of difference equations related to the product-type ones. Somewhat later we studied some systems which are extensions of these equations [39, 40].

Long-term behavior of positive solutions to the following system

$$x_{n+1} = \max\left\{c, \frac{y_n^p}{z_{n-1}^p}\right\}, \qquad y_{n+1} = \max\left\{c, \frac{z_n^p}{x_{n-1}^p}\right\}, \qquad z_{n+1} = \max\left\{c, \frac{x_n^p}{y_{n-1}^p}\right\}, \tag{1.1}$$

✉ Email: sstevic@ptt.rs

$n \in \mathbb{N}_0$, with positive parameters c and p, was investigated in [40]. Note that system (1.1) is obtained by the action of the max-type operator $m_c(s) = \max\{c, s\}$ on the right-hand side of the following product-type system of difference equations

$$x_{n+1} = \frac{y_n^p}{z_{n-1}^p}, \qquad y_{n+1} = \frac{z_n^p}{x_{n-1}^p}, \qquad z_{n+1} = \frac{x_n^p}{y_{n-1}^p}, \qquad n \in \mathbb{N}_0. \tag{1.2}$$

An interesting feature of system (1.2) is that it can be solved in closed form in the case of positive initial values. Namely, a simple inductive argument shows that in this case

$$\min\{x_n, y_n, z_n\} > 0, \quad \text{for every } n \geq -1.$$

Hence, it is legitimate to take the logarithm of all the three equations in (1.2) and by using the change of variables

$$u_n = \ln x_n, \qquad v_n = \ln y_n, \qquad w_n = \ln z_n, \qquad n \geq -1, \tag{1.3}$$

the system is transformed into the following linear one

$$\begin{aligned} u_{n+1} &= pv_n - pw_{n-1} \\ v_{n+1} &= pw_n - pu_{n-1} \\ w_{n+1} &= pu_n - pv_{n-1}, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{1.4}$$

Using the third equation of (1.4) in the first and second ones we obtain the following system of difference equations

$$u_{n+1} = pv_n - p^2 u_{n-2} + p^2 v_{n-3} \tag{1.5}$$
$$v_{n+1} = (p^2 - p)u_{n-1} - p^2 v_{n-2}, \qquad n \in \mathbb{N}_0. \tag{1.6}$$

Using (1.6) in (1.5) we get

$$v_{n+3} - (p^3 - 3p^2)v_n + p^3 v_{n-3} = 0, \qquad n \in \mathbb{N}_0. \tag{1.7}$$

This is a linear difference equation which can be solved in closed form, from which along with (1.6) and the third equation in (1.4) closed form formulas for u_n, v_n and w_n are obtained, and consequently by using (1.3) we obtain formulas for x_n, y_n and z_n. We leave the details to the reader as a simple exercise.

A natural question is whether system (1.2) can be solved in closed form if initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{0, 1\}$, are complex numbers and for which values of parameter p.

Motivated by all above mentioned and by our recent paper [37], here we will study the solvability of the following system of difference equations

$$x_{n+1} = \frac{y_n^a}{z_{n-1}^b}, \qquad y_{n+1} = \frac{z_n^c}{x_{n-1}^d}, \qquad z_{n+1} = \frac{x_n^f}{y_{n-1}^g}, \qquad n \in \mathbb{N}_0, \tag{1.8}$$

where $a, b, c, d, f, g \in \mathbb{Z}$, and when initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{0, 1\}$, are complex numbers different from zero (it is easy to see that a solution to the system is well-defined if and only if all initial values are different from zero). We present a constructive method for solving the system. Condition $a, b, c, d, f, g \in \mathbb{Z}$ is naturally posed, in order not to deal with multi-valued sequences.

2 Main result

Here we present our main result in this paper.

Theorem 2.1. *Consider system (1.8) with $a,b,c,d,f,g \in \mathbb{Z}$. If $x_{-i}, y_{-i}, z_{-i} \in \mathbb{C} \setminus \{0\}$, $i \in \{0,1\}$, then the system is solvable in closed form.*

Proof. Let

$$a_1 = a, \qquad b_1 = b, \qquad c_1 = c, \qquad d_1 = d, \qquad f_1 = f, \qquad g_1 = g. \tag{2.1}$$

By using the equations in (1.8) we obtain

$$x_{n+1} = \frac{y_n^{a_1}}{z_{n-1}^{b_1}} = \frac{z_{n-1}^{ca_1-b_1}}{x_{n-2}^{da_1}} = \frac{z_{n-1}^{a_2}}{x_{n-2}^{b_2}}, \tag{2.2}$$

$$y_{n+1} = \frac{z_n^{c_1}}{x_{n-1}^{d_1}} = \frac{x_{n-1}^{fc_1-d_1}}{y_{n-2}^{gc_1}} = \frac{x_{n-1}^{c_2}}{y_{n-2}^{d_2}}, \tag{2.3}$$

$$z_{n+1} = \frac{x_n^{f_1}}{y_{n-1}^{g_1}} = \frac{y_{n-1}^{af_1-g_1}}{z_{n-2}^{bf_1}} = \frac{y_{n-1}^{f_2}}{z_{n-2}^{g_2}}, \tag{2.4}$$

where we define a_2, b_2, c_2, d_2, f_2 and g_2 as follows

$$a_2 := ca_1 - b_1, \quad b_2 := da_1, \quad c_2 := fc_1 - d_1, \quad d_2 := gc_1, \quad f_2 := af_1 - g_1, \quad g_2 := bf_1.$$

By using (2.2), (2.3), (2.4) and the equations in (1.8), it follows that

$$x_{n+1} = \frac{z_{n-1}^{a_2}}{x_{n-2}^{b_2}} = \frac{x_{n-2}^{fa_2-b_2}}{y_{n-3}^{ga_2}} = \frac{x_{n-2}^{a_3}}{y_{n-3}^{b_3}}, \tag{2.5}$$

$$y_{n+1} = \frac{x_{n-1}^{c_2}}{y_{n-2}^{d_2}} = \frac{y_{n-2}^{ac_2-d_2}}{z_{n-3}^{bc_2}} = \frac{y_{n-2}^{c_3}}{z_{n-3}^{d_3}}, \tag{2.6}$$

$$z_{n+1} = \frac{y_{n-1}^{f_2}}{z_{n-2}^{g_2}} = \frac{z_{n-2}^{cf_2-g_2}}{x_{n-3}^{df_2}} = \frac{z_{n-2}^{f_3}}{x_{n-3}^{g_3}}, \tag{2.7}$$

where we define a_3, b_3, c_3, d_3, f_3 and g_3 as follows

$$a_3 := fa_2 - b_2, \quad b_3 := ga_2, \quad c_3 := ac_2 - d_2, \quad d_3 := bc_2, \quad f_3 := cf_2 - g_2, \quad g_3 := df_2.$$

By using (2.5), (2.6), (2.7) and the equations in (1.8), we further get

$$x_{n+1} = \frac{x_{n-2}^{a_3}}{y_{n-3}^{b_3}} = \frac{y_{n-3}^{aa_3-b_3}}{z_{n-4}^{ba_3}} = \frac{y_{n-3}^{a_4}}{z_{n-4}^{b_4}}, \tag{2.8}$$

$$y_{n+1} = \frac{y_{n-2}^{c_3}}{z_{n-3}^{d_3}} = \frac{z_{n-3}^{cc_3-d_3}}{x_{n-4}^{dc_3}} = \frac{z_{n-3}^{c_4}}{x_{n-4}^{d_4}}, \tag{2.9}$$

$$z_{n+1} = \frac{z_{n-2}^{f_3}}{x_{n-3}^{g_3}} = \frac{x_{n-3}^{ff_3-g_3}}{y_{n-4}^{gf_3}} = \frac{x_{n-3}^{f_4}}{y_{n-4}^{g_4}}, \tag{2.10}$$

where we define a_4, b_4, c_4, d_4, f_4 and g_4 as follows

$$a_4 := aa_3 - b_3, \quad b_4 := ba_3, \quad c_4 := cc_3 - d_3, \quad d_4 := dc_3, \quad f_4 := ff_3 - g_3, \quad g_4 := gf_3.$$

Let

$$x_{n+1} = \frac{z_{n-3k+2}^{a_{3k-1}}}{x_{n-3k+1}^{b_{3k-1}}}, \qquad y_{n+1} = \frac{x_{n-3k+2}^{c_{3k-1}}}{y_{n-3k+1}^{d_{3k-1}}}, \qquad z_{n+1} = \frac{y_{n-3k+2}^{f_{3k-1}}}{z_{n-3k+1}^{g_{3k-1}}},$$

where

$$\begin{aligned} &a_{3k-1} := ca_{3k-2} - b_{3k-2}, && b_{3k-1} := da_{3k-2}, && c_{3k-1} := fc_{3k-2} - d_{3k-2},\\ &d_{3k-1} := gc_{3k-2}, && f_{3k-1} := af_{3k-2} - g_{3k-2}, && g_{3k-1} := bf_{3k-2}, \end{aligned}$$

$$x_{n+1} = \frac{x_{n-3k+1}^{a_{3k}}}{y_{n-3k}^{b_{3k}}}, \qquad y_{n+1} = \frac{y_{n-3k+1}^{c_{3k}}}{z_{n-3k}^{d_{3k}}}, \qquad z_{n+1} = \frac{z_{n-3k+1}^{f_{3k}}}{x_{n-3k}^{g_{3k}}},$$

where

$$\begin{aligned} &a_{3k} := fa_{3k-1} - b_{3k-1}, && b_{3k} := ga_{3k-1}, && c_{3k} := ac_{3k-1} - d_{3k-1},\\ &d_{3k} := bc_{3k-1}, && f_{3k} := cf_{3k-1} - g_{3k-1}, && g_{3k} := df_{3k-1}, \end{aligned}$$

$$x_{n+1} = \frac{y_{n-3k}^{a_{3k+1}}}{z_{n-3k-1}^{b_{3k+1}}}, \qquad y_{n+1} = \frac{z_{n-3k}^{c_{3k+1}}}{x_{n-3k-1}^{d_{3k+1}}}, \qquad z_{n+1} = \frac{x_{n-3k}^{f_{3k+1}}}{y_{n-3k-1}^{g_{3k+1}}} \tag{2.11}$$

where

$$\begin{aligned} &a_{3k+1} := aa_{3k} - b_{3k}, && b_{3k+1} := ba_{3k}, && c_{3k+1} := cc_{3k} - d_{3k},\\ &d_{3k+1} := dc_{3k}, && f_{3k+1} := ff_{3k} - g_{3k}, && g_{3k+1} := gf_{3k}, \end{aligned}$$

for some $k \in \mathbb{N}$ such that $n \geq 3k$.

From the relations in (2.11) and by using the equations in (1.8) we obtain

$$x_{n+1} = \frac{y_{n-3k}^{a_{3k+1}}}{z_{n-3k-1}^{b_{3k+1}}} = \frac{z_{n-3k-1}^{ca_{3k+1}-b_{3k+1}}}{x_{n-3k-2}^{da_{3k+1}}} = \frac{z_{n-3k-1}^{a_{3k+2}}}{x_{n-3k-2}^{b_{3k+2}}}, \tag{2.12}$$

$$y_{n+1} = \frac{z_{n-3k}^{c_{3k+1}}}{x_{n-3k-1}^{d_{3k+1}}} = \frac{x_{n-3k-1}^{fc_{3k+1}-d_{3k+1}}}{y_{n-3k-2}^{gc_{3k+1}}} = \frac{x_{n-3k-1}^{c_{3k+2}}}{y_{n-3k-2}^{d_{3k+2}}}, \tag{2.13}$$

$$z_{n+1} = \frac{x_{n-3k}^{f_{3k+1}}}{y_{n-3k-1}^{g_{3k+1}}} = \frac{y_{n-3k-1}^{af_{3k+1}-g_{3k+1}}}{z_{n-3k-2}^{bf_{3k+1}}} = \frac{y_{n-3k-1}^{f_{3k+2}}}{z_{n-3k-2}^{g_{3k+2}}}, \tag{2.14}$$

where we define a_{3k+2}, b_{3k+2}, c_{3k+2}, d_{3k+2}, f_{3k+2} and g_{3k+2} as follows

$$\begin{aligned} &a_{3k+2} := ca_{3k+1} - b_{3k+1}, && b_{3k+2} := da_{3k+1}, && c_{3k+2} := fc_{3k+1} - d_{3k+1},\\ &d_{3k+2} := gc_{3k+1}, && f_{3k+2} := af_{3k+1} - g_{3k+1}, && g_{3k+2} := bf_{3k+1}. \end{aligned}$$

By using (2.12), (2.13), (2.14) and the equations in (1.8), it follows that

$$x_{n+1}=\frac{z_{n-3k-1}^{a_{3k+2}}}{x_{n-3k-2}^{b_{3k+2}}}=\frac{x_{n-3k-2}^{fa_{3k+2}-b_{3k+2}}}{y_{n-3k-3}^{ga_{3k+2}}}=\frac{x_{n-3k-2}^{a_{3k+3}}}{y_{n-3k-3}^{b_{3k+3}}},\tag{2.15}$$

$$y_{n+1}=\frac{x_{n-3k-1}^{c_{3k+2}}}{y_{n-3k-2}^{d_{3k+2}}}=\frac{y_{n-3k-2}^{ac_{3k+2}-d_{3k+2}}}{z_{n-3k-3}^{bc_{3k+2}}}=\frac{y_{n-3k-2}^{c_{3k+3}}}{z_{n-3k-3}^{d_{3k+3}}},\tag{2.16}$$

$$z_{n+1}=\frac{y_{n-3k-1}^{f_{3k+2}}}{z_{n-3k-2}^{g_{3k+2}}}=\frac{z_{n-3k-2}^{cf_{3k+2}-g_{3k+2}}}{x_{n-3k-3}^{df_{3k+2}}}=\frac{z_{n-3k-2}^{f_{3k+3}}}{x_{n-3k-3}^{g_{3k+3}}},\tag{2.17}$$

where we define a_{3k+3}, b_{3k+3}, c_{3k+3}, d_{3k+3}, f_{3k+3} and g_{3k+3} as follows

$$\begin{aligned}
&a_{3k+3}:=fa_{3k+2}-b_{3k+2}, && b_{3k+3}:=ga_{3k+2}, && c_{3k+3}:=ac_{3k+2}-d_{3k+2},\\
&d_{3k+3}:=bc_{3k+2}, && f_{3k+3}:=cf_{3k+2}-g_{3k+2}, && g_{3k+3}:=df_{3k+2}.
\end{aligned}$$

By using (2.15), (2.16), (2.17) and the equations in (1.8) we further get

$$x_{n+1}=\frac{x_{n-3k-2}^{a_{3k+3}}}{y_{n-3k-3}^{b_{3k+3}}}=\frac{y_{n-3k-3}^{aa_{3k+3}-b_{3k+3}}}{z_{n-3k-4}^{ba_{3k+3}}}=\frac{y_{n-3k-3}^{a_{3k+4}}}{z_{n-3k-4}^{b_{3k+4}}},\tag{2.18}$$

$$y_{n+1}=\frac{y_{n-3k-2}^{c_{3k+3}}}{z_{n-3k-3}^{d_{3k+3}}}=\frac{z_{n-3k-3}^{cc_{3k+3}-d_{3k+3}}}{x_{n-3k-4}^{dc_{3k+3}}}=\frac{z_{n-3k-3}^{c_{3k+4}}}{x_{n-3k-4}^{d_{3k+4}}},\tag{2.19}$$

$$z_{n+1}=\frac{z_{n-3k-2}^{f_{3k+3}}}{x_{n-3k-3}^{g_{3k+3}}}=\frac{x_{n-3k-3}^{ff_{3k+3}-g_{3k+3}}}{y_{n-3k-4}^{gf_{3k+3}}}=\frac{x_{n-3k-3}^{f_{3k+4}}}{y_{n-3k-4}^{g_{3k+4}}},\tag{2.20}$$

where we define a_{3k+4}, b_{3k+4}, c_{3k+4}, d_{3k+4}, f_{3k+4} and g_{3k+4} as follows

$$\begin{aligned}
&a_{3k+4}:=aa_{3k+3}-b_{3k+3}, && b_{3k+4}:=ba_{3k+3}, && c_{3k+4}:=cc_{3k+3}-d_{3k+3},\\
&d_{3k+4}:=dc_{3k+3}, && f_{3k+4}:=ff_{3k+3}-g_{3k+3}, && g_{3k+4}:=gf_{3k+3}.
\end{aligned}$$

Hence, this inductive argument shows that sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$, satisfy the following recurrent relations

$$a_{3k+2}=ca_{3k+1}-b_{3k+1},\quad a_{3k+3}=fa_{3k+2}-b_{3k+2},\quad a_{3k+4}=aa_{3k+3}-b_{3k+3},\tag{2.21}$$

$$b_{3k+2}=da_{3k+1},\quad b_{3k+3}=ga_{3k+2},\quad b_{3k+4}=ba_{3k+3},\tag{2.22}$$

$$c_{3k+2}=fc_{3k+1}-d_{3k+1},\quad c_{3k+3}=ac_{3k+2}-d_{3k+2},\quad c_{3k+4}=cc_{3k+3}-d_{3k+3},\tag{2.23}$$

$$d_{3k+2}=gc_{3k+1},\quad d_{3k+3}=bc_{3k+2},\quad d_{3k+4}=dc_{3k+3},\tag{2.24}$$

$$f_{3k+2}=af_{3k+1}-g_{3k+1},\quad f_{3k+3}=cf_{3k+2}-g_{3k+2},\quad f_{3k+4}=ff_{3k+3}-g_{3k+3},\tag{2.25}$$

$$g_{3k+2}=bf_{3k+1},\quad g_{3k+3}=df_{3k+2},\quad g_{3k+4}=gf_{3k+3},\tag{2.26}$$

for $k\in\mathbb{N}_0$.

From (2.12)–(2.20) we easily obtain

$$x_{3n+1}=\frac{y_0^{a_{3n+1}}}{z_{-1}^{b_{3n+1}}},\qquad x_{3n+2}=\frac{z_0^{a_{3n+2}}}{x_{-1}^{b_{3n+2}}},\qquad x_{3n+3}=\frac{x_0^{a_{3n+3}}}{y_{-1}^{b_{3n+3}}},\tag{2.27}$$

$$y_{3n+1}=\frac{z_0^{c_{3n+1}}}{x_{-1}^{d_{3n+1}}},\qquad y_{3n+2}=\frac{x_0^{c_{3n+2}}}{y_{-1}^{d_{3n+2}}},\qquad y_{3n+3}=\frac{y_0^{c_{3n+3}}}{z_{-1}^{d_{3n+3}}},\tag{2.28}$$

$$z_{3n+1}=\frac{x_0^{f_{3n+1}}}{y_{-1}^{g_{3n+1}}},\qquad z_{3n+2}=\frac{y_0^{f_{3n+2}}}{z_{-1}^{g_{3n+2}}},\qquad z_{3n+3}=\frac{z_0^{f_{3n+3}}}{x_{-1}^{g_{3n+3}}},\qquad n\in\mathbb{N}_0.\tag{2.29}$$

From (2.21) and (2.22) we have that

$$a_{3k+2} = ca_{3k+1} - ba_{3k}, \qquad a_{3k+3} = fa_{3k+2} - da_{3k+1}, \qquad a_{3k+4} = aa_{3k+3} - ga_{3k+2},$$

$k \in \mathbb{N}_0$, (here we regard that $a_0 = 1$ if $b \neq 0$, due to the relation $b_{3k+4} = ba_{3k+3}$ with $k = -1$). Using the first equation in the second and third ones it follows that

$$a_{3k+3} - (cf - d)a_{3k+1} + bfa_{3k} = 0, \qquad k \in \mathbb{N}_0, \tag{2.30}$$

and

$$a_{3k+4} - aa_{3k+3} + cga_{3k+1} - bga_{3k} = 0, \qquad k \in \mathbb{N}_0. \tag{2.31}$$

Using (2.30) into (2.31) it follows that

$$a_{3k+6} + (ad + bf + cg - acf)a_{3k+3} + bdga_{3k} = 0, \qquad k \in \mathbb{N}_0. \tag{2.32}$$

Relation (2.32) means that the sequence $(a_{3k})_{k\in\mathbb{N}_0}$ annihilate the linear operator

$$\mathcal{L}(x_n) = x_{n+2} + (ad + bf + cg - acf)x_{n+1} + bdgx_n.$$

From this along with (2.30) it follows that sequence $(a_{3k+1})_{k\in\mathbb{N}_0}$ annihilate the operator too. Using these facts and the relation $a_{3k+2} = ca_{3k+1} - ba_{3k}$, it follows that that sequence $(a_{3k+2})_{k\in\mathbb{N}_0}$ also annihilate the operator. This along with the relations in (2.22) implies that sequences $(b_{3k+i})_{k\in\mathbb{N}_0}$, $i = 0, 1, 2$, annihilate the operator too. Since sequences $(c_{3k+2+i})_{k\in\mathbb{N}_0}$ and $(d_{3k+2+i})_{k\in\mathbb{N}_0}$, that is, $(f_{3k+1+i})_{k\in\mathbb{N}_0}$ and $(g_{3k+1+i})_{k\in\mathbb{N}_0}$, $i = 0, 1, 2$, satisfy the relations in (2.21) and (2.22) it follows that they also annihilate the operator.

Case $b = 0$. In this case we have that sequences $(a_{3k+i})_{k\in\mathbb{N}_0}$, $(c_{3k+i})_{k\in\mathbb{N}_0}$, $(f_{3k+i})_{k\in\mathbb{N}_0}$, satisfy the recurrent relation

$$x_{n+1} = (acf - ad - cg)x_n, \qquad n \in \mathbb{N}.$$

From this and since $a_1 = a$, $c_1 = c$, $f_1 = f$, $a_2 = ac - b$, $c_2 = cf - d$, $f_2 = af - g$, $a_3 = acf - bf - ad$, $c_3 = acf - ad - cg$, $f_3 = acf - bf - cg$, and by using the condition $b = 0$, it follows that

$$a_{3k+1} = a(acf - ad - cg)^k, \tag{2.33}$$
$$a_{3k+2} = ac(acf - ad - cg)^k, \tag{2.34}$$
$$a_{3k+3} = a(cf - d)(acf - ad - cg)^k, \tag{2.35}$$
$$c_{3k+1} = c(acf - ad - cg)^k, \tag{2.36}$$
$$c_{3k+2} = (cf - d)(acf - ad - cg)^k, \tag{2.37}$$
$$c_{3k+3} = (acf - ad - cg)^{k+1}, \tag{2.38}$$
$$f_{3k+1} = f(acf - ad - cg)^k, \tag{2.39}$$
$$f_{3k+2} = (af - g)(acf - ad - cg)^k, \tag{2.40}$$
$$f_{3k+3} = c(af - g)(acf - ad - cg)^k, \qquad k \in \mathbb{N}_0. \tag{2.41}$$

Using (2.33)–(2.41) into (2.22), (2.24) and (2.26), as well as the condition $b = 0$, it follows that

$$\begin{aligned}
b_{3k+1} &= 0, &(2.42)\\
b_{3k+2} &= ad(acf - ad - cg)^k, &(2.43)\\
b_{3k+3} &= acg(acf - ad - cg)^k, &(2.44)\\
d_{3k+1} &= d(acf - ad - cg)^k, &(2.45)\\
d_{3k+2} &= cg(acf - ad - cg)^k, &(2.46)\\
d_{3k+3} &= 0, &(2.47)\\
g_{3k+1} &= cg(af - g)(acf - ad - cg)^{k-1}, &(2.48)\\
g_{3k+2} &= 0, &(2.49)\\
g_{3k+3} &= d(af - g)(acf - ad - cg)^k, \qquad k \in \mathbb{N}_0. &(2.50)
\end{aligned}$$

Employing (2.33)–(2.50) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions to system (1.8) in this case are given by the following formulas

$$\begin{aligned}
x_{3n+1} &= y_0^{a(acf-ad-cg)^n}, &(2.51)\\
x_{3n+2} &= z_0^{ac(acf-ad-cg)^n} x_{-1}^{-ad(acf-ad-cg)^n}, &(2.52)\\
x_{3n+3} &= x_0^{a(cf-d)(acf-ad-cg)^n} y_{-1}^{-acg(acf-ad-cg)^n}, &(2.53)\\
y_{3n+1} &= z_0^{c(acf-ad-cg)^n} x_{-1}^{-d(acf-ad-cg)^n}, &(2.54)\\
y_{3n+2} &= x_0^{(cf-d)(acf-ad-cg)^n} y_{-1}^{-cg(acf-ad-cg)^n}, &(2.55)\\
y_{3n+3} &= y_0^{(acf-ad-cg)^{n+1}}, &(2.56)\\
z_{3n+1} &= x_0^{f(acf-ad-cg)^n} y_{-1}^{-cg(af-g)(acf-ad-cg)^{n-1}}, &(2.57)\\
z_{3n+2} &= y_0^{(af-g)(acf-ad-cg)^n}, &(2.58)\\
z_{3n+3} &= z_0^{c(af-g)(acf-ad-cg)^n} x_{-1}^{-d(af-g)(acf-ad-cg)^n}, \qquad n \in \mathbb{N}_0. &(2.59)
\end{aligned}$$

Case $d = 0$. In this case we have that sequences $(a_{3k+i})_{k\in\mathbb{N}_0}$, $(c_{3k+i})_{k\in\mathbb{N}_0}$, $(f_{3k+i})_{k\in\mathbb{N}_0}$, satisfy the recurrent relation

$$x_{n+1} = (acf - bf - cg)x_n, \qquad n \in \mathbb{N}.$$

From this and since $a_1 = a$, $c_1 = c$, $f_1 = f$, $a_2 = ac - b$, $c_2 = cf - d$, $f_2 = af - g$, $a_3 = acf - bf - ad$, $c_3 = acf - ad - cg$, $f_3 = acf - bf - cg$, and by using the condition $d = 0$, it follows that

$$\begin{aligned}
a_{3k+1} &= a(acf - bf - cg)^k, &(2.60)\\
a_{3k+2} &= (ac - b)(acf - bf - cg)^k, &(2.61)\\
a_{3k+3} &= f(ac - b)(acf - bf - cg)^k, &(2.62)\\
c_{3k+1} &= c(acf - bf - cg)^k, &(2.63)\\
c_{3k+2} &= cf(acf - bf - cg)^k, &(2.64)\\
c_{3k+3} &= c(af - g)(acf - bf - cg)^k, &(2.65)\\
f_{3k+1} &= f(acf - bf - cg)^k, &(2.66)
\end{aligned}$$

$$f_{3k+2} = (af-g)(acf-bf-cg)^k, \tag{2.67}$$
$$f_{3k+3} = (acf-bf-cg)^{k+1}, \qquad k \in \mathbb{N}_0. \tag{2.68}$$

Using (2.60)–(2.68) into (2.22), (2.24) and (2.26), as well as the condition $d = 0$, it follows that

$$b_{3k+1} = bf(ac-b)(acf-bf-cg)^{k-1}, \tag{2.69}$$
$$b_{3k+2} = 0, \tag{2.70}$$
$$b_{3k+3} = g(ac-b)(acf-bf-cg)^k, \tag{2.71}$$
$$d_{3k+1} = 0, \tag{2.72}$$
$$d_{3k+2} = cg(acf-bf-cg)^k, \tag{2.73}$$
$$d_{3k+3} = bcf(acf-bf-cg)^k, \tag{2.74}$$
$$g_{3k+1} = g(acf-bf-cg)^k, \tag{2.75}$$
$$g_{3k+2} = bf(acf-bf-cg)^k, \tag{2.76}$$
$$g_{3k+3} = 0, \qquad k \in \mathbb{N}_0. \tag{2.77}$$

Employing (2.60)–(2.77) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

$$x_{3n+1} = y_0^{a(acf-bf-cg)^n} z_{-1}^{-bf(ac-b)(acf-bf-cg)^{n-1}}, \tag{2.78}$$
$$x_{3n+2} = z_0^{(ac-b)(acf-bf-cg)^n}, \tag{2.79}$$
$$x_{3n+3} = x_0^{f(ac-b)(acf-bf-cg)^n} y_{-1}^{-g(ac-b)(acf-bf-cg)^n}, \tag{2.80}$$
$$y_{3n+1} = z_0^{c(acf-bf-cg)^n}, \tag{2.81}$$
$$y_{3n+2} = x_0^{cf(acf-bf-cg)^n} y_{-1}^{-cg(acf-bf-cg)^n}, \tag{2.82}$$
$$y_{3n+3} = y_0^{c(af-g)(acf-bf-cg)^n} z_{-1}^{-bcf(acf-bf-cg)^n}, \tag{2.83}$$
$$z_{3n+1} = x_0^{f(acf-bf-cg)^n} y_{-1}^{-g(acf-bf-cg)^n}, \tag{2.84}$$
$$z_{3n+2} = y_0^{(af-g)(acf-bf-cg)^n} z_{-1}^{-bf(acf-bf-cg)^n}, \tag{2.85}$$
$$z_{3n+3} = z_0^{(acf-bf-cg)^{n+1}}, \quad n \in \mathbb{N}_0. \tag{2.86}$$

Case $g = 0$. In this case we have that sequences $(a_{3k+i})_{k\in\mathbb{N}_0}$, $(c_{3k+i})_{k\in\mathbb{N}_0}$, $(f_{3k+i})_{k\in\mathbb{N}_0}$, satisfy the recurrent relation

$$x_{n+1} = (acf-ad-bf)x_n, \quad n \in \mathbb{N}.$$

From this and since $a_1 = a$, $c_1 = c$, $f_1 = f$, $a_2 = ac-b$, $c_2 = cf-d$, $f_2 = af-g$, $a_3 = acf-bf-ad$, $c_3 = acf-ad-cg$, $f_3 = acf-bf-cg$, and by using the condition $g = 0$, it follows that

$$a_{3k+1} = a(acf-ad-bf)^k, \tag{2.87}$$
$$a_{3k+2} = (ac-b)(acf-bf-ad)^k, \tag{2.88}$$
$$a_{3k+3} = (acf-bf-ad)^{k+1}, \tag{2.89}$$
$$c_{3k+1} = c(acf-bf-ad)^k, \tag{2.90}$$
$$c_{3k+2} = (cf-d)(acf-bf-ad)^k, \tag{2.91}$$

$$c_{3k+3} = a(cf-d)(acf-bf-ad)^k, \tag{2.92}$$
$$f_{3k+1} = f(acf-bf-ad)^k, \tag{2.93}$$
$$f_{3k+2} = af(acf-bf-ad)^k, \tag{2.94}$$
$$f_{3k+3} = f(ac-b)(acf-bf-ad)^k, \quad k \in \mathbb{N}_0. \tag{2.95}$$

Using (2.87)–(2.95) into (2.22), (2.24) and (2.26), as well as the condition $g = 0$, it follows that

$$b_{3k+1} = b(acf-bf-ad)^k, \tag{2.96}$$
$$b_{3k+2} = ad(acf-ad-bf)^k, \tag{2.97}$$
$$b_{3k+3} = 0, \tag{2.98}$$
$$d_{3k+1} = ad(cf-d)(acf-bf-ad)^{k-1}, \tag{2.99}$$
$$d_{3k+2} = 0, \tag{2.100}$$
$$d_{3k+3} = b(cf-d)(acf-bf-ad)^k, \tag{2.101}$$
$$g_{3k+1} = 0, \tag{2.102}$$
$$g_{3k+2} = bf(acf-bf-ad)^k, \tag{2.103}$$
$$g_{3k+3} = adf(acf-bf-ad)^k, \quad k \in \mathbb{N}_0. \tag{2.104}$$

Employing (2.87)–(2.104) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

$$x_{3n+1} = y_0^{a(acf-ad-bf)^n} z_{-1}^{-b(acf-bf-ad)^n}, \tag{2.105}$$
$$x_{3n+2} = z_0^{(ac-b)(acf-bf-ad)^n} x_{-1}^{-ad(acf-ad-bf)^n}, \tag{2.106}$$
$$x_{3n+3} = x_0^{(acf-bf-ad)^{n+1}}, \tag{2.107}$$
$$y_{3n+1} = z_0^{c(acf-bf-ad)^n} x_{-1}^{-ad(cf-d)(acf-bf-ad)^{n-1}}, \tag{2.108}$$
$$y_{3n+2} = x_0^{(cf-d)(acf-bf-ad)^n}, \tag{2.109}$$
$$y_{3n+3} = y_0^{a(cf-d)(acf-bf-ad)^n} z_{-1}^{-b(cf-d)(acf-bf-ad)^n}, \tag{2.110}$$
$$z_{3n+1} = x_0^{f(acf-bf-ad)^n}, \tag{2.111}$$
$$z_{3n+2} = y_0^{af(acf-bf-ad)^n} z_{-1}^{-bf(acf-bf-ad)^n}, \tag{2.112}$$
$$z_{3n+3} = z_0^{f(ac-b)(acf-bf-ad)^n} x_{-1}^{-adf(acf-bf-ad)^n}, \qquad n \in \mathbb{N}_0. \tag{2.113}$$

Case $bdg \neq 0$. Let $\lambda_{1,2}$ be the roots of the characteristic polynomial

$$P(\lambda) = \lambda^2 - (acf-ad-bf-cg)\lambda + bdg,$$

of difference equation

$$u_{n+2} - (acf-ad-bf-cg)u_{n+1} + bdgu_n = 0, \qquad n \in \mathbb{N}. \tag{2.114}$$

It is known that the general solution of equation (2.114) has the following form

$$u_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n, \qquad n \in \mathbb{N},$$

if $(acf-ad-bf-cg)^2 \neq 4bdg$, where α_1 and α_2 are arbitrary constants, while in the case $(acf-ad-bf-cg)^2 = 4bdg$, the general solution has the following form

$$u_n = (\beta_1 n + \beta_2)\lambda_1^n, \qquad n \in \mathbb{N},$$

where β_1 and β_2 are arbitrary constants.

By some calculation and using the values for a_i, b_i, c_i, d_i, f_i, g_i for $i \in \{1,2,3,4\}$, if $(acf-ad-bf-cg)^2 \neq 4bdg$, we have that

$$a_{3k+1} = \frac{(a\lambda_1+bg)\lambda_1^k - (bg+a\lambda_2)\lambda_2^k}{\lambda_1-\lambda_2} \tag{2.115}$$

$$a_{3k+2} = (ac-b)\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{\lambda_1-\lambda_2} \tag{2.116}$$

$$a_{3k+3} = \frac{(acf-bf-ad-\lambda_2)\lambda_1^{k+1} + (\lambda_1-acf+bf+ad)\lambda_2^{k+1}}{\lambda_1-\lambda_2} \tag{2.117}$$

$$b_{3k+1} = b\frac{(acf-bf-ad-\lambda_2)\lambda_1^{k} + (\lambda_1-acf+bf+ad)\lambda_2^{k}}{\lambda_1-\lambda_2} \tag{2.118}$$

$$b_{3k+2} = d\frac{(a\lambda_1+bg)\lambda_1^k - (bg+a\lambda_2)\lambda_2^k}{\lambda_1-\lambda_2}, \tag{2.119}$$

$$b_{3k+3} = g(ac-b)\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{\lambda_1-\lambda_2}, \tag{2.120}$$

$$c_{3k+1} = \frac{(c\lambda_1+bd)\lambda_1^k - (bd+c\lambda_2)\lambda_2^k}{\lambda_1-\lambda_2} \tag{2.121}$$

$$c_{3k+2} = (cf-d)\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{\lambda_1-\lambda_2} \tag{2.122}$$

$$c_{3k+3} = \frac{(acf-ad-cg-\lambda_2)\lambda_1^{k+1} + (\lambda_1-acf+ad+cg)\lambda_2^{k+1}}{\lambda_1-\lambda_2} \tag{2.123}$$

$$d_{3k+1} = d\frac{(acf-ad-cg-\lambda_2)\lambda_1^{k} + (\lambda_1-acf+ad+cg)\lambda_2^{k}}{\lambda_1-\lambda_2} \tag{2.124}$$

$$d_{3k+2} = g\frac{(c\lambda_1+bd)\lambda_1^k - (bd+c\lambda_2)\lambda_2^k}{\lambda_1-\lambda_2}, \tag{2.125}$$

$$d_{3k+3} = b(cf-d)\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{\lambda_1-\lambda_2}, \tag{2.126}$$

$$f_{3k+1} = \frac{(f\lambda_1+dg)\lambda_1^k - (dg+f\lambda_2)\lambda_2^k}{\lambda_1-\lambda_2} \tag{2.127}$$

$$f_{3k+2} = (af-g)\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{\lambda_1-\lambda_2} \tag{2.128}$$

$$f_{3k+3} = \frac{(acf-bf-cg-\lambda_2)\lambda_1^{k+1} + (\lambda_1-acf+bf+cg)\lambda_2^{k+1}}{\lambda_1-\lambda_2} \tag{2.129}$$

$$g_{3k+1} = g\frac{(acf-bf-cg-\lambda_2)\lambda_1^{k} + (\lambda_1-acf+bf+cg)\lambda_2^{k}}{\lambda_1-\lambda_2} \tag{2.130}$$

$$g_{3k+2} = b\frac{(f\lambda_1+dg)\lambda_1^k - (dg+f\lambda_2)\lambda_2^k}{\lambda_1-\lambda_2}, \tag{2.131}$$

$$g_{3k+3} = d(af-g)\frac{\lambda_1^{k+1}-\lambda_2^{k+1}}{\lambda_1-\lambda_2}, \tag{2.132}$$

for $k \in \mathbb{N}_0$.

By using (2.115)–(2.132) in (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

$$x_{3n+1} = y_0^{\frac{(a\lambda_1+bg)\lambda_1^n-(bg+a\lambda_2)\lambda_2^n}{\lambda_1-\lambda_2}} z_{-1}^{-b\frac{(acf-bf-ad-\lambda_2)\lambda_1^n+(\lambda_1-acf+bf+ad)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.133}$$

$$x_{3n+2} = z_0^{(ac-b)\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} x_{-1}^{-d\frac{(a\lambda_1+bg)\lambda_1^n-(bg+a\lambda_2)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.134}$$

$$x_{3n+3} = x_0^{\frac{(acf-bf-ad-\lambda_2)\lambda_1^{n+1}+(\lambda_1-acf+bf+ad)\lambda_2^{n+1}}{\lambda_1-\lambda_2}} y_{-1}^{-g(ac-b)\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}}, \tag{2.135}$$

$$y_{3n+1} = z_0^{\frac{(c\lambda_1+bd)\lambda_1^n-(bd+c\lambda_2)\lambda_2^n}{\lambda_1-\lambda_2}} x_{-1}^{-d\frac{(acf-ad-cg-\lambda_2)\lambda_1^n+(\lambda_1-acf+ad+cg)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.136}$$

$$y_{3n+2} = x_0^{(cf-d)\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} y_{-1}^{-g\frac{(c\lambda_1+bd)\lambda_1^n-(bd+c\lambda_2)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.137}$$

$$y_{3n+3} = y_0^{\frac{(acf-ad-cg-\lambda_2)\lambda_1^{n+1}+(\lambda_1-acf+ad+cg)\lambda_2^{n+1}}{\lambda_1-\lambda_2}} z_{-1}^{-b(cf-d)\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}}, \tag{2.138}$$

$$z_{3n+1} = x_0^{\frac{(f\lambda_1+dg)\lambda_1^n-(dg+f\lambda_2)\lambda_2^n}{\lambda_1-\lambda_2}} y_{-1}^{-g\frac{(acf-bf-cg-\lambda_2)\lambda_1^n+(\lambda_1-acf+bf+cg)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.139}$$

$$z_{3n+2} = y_0^{(af-g)\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}} z_{-1}^{-b\frac{(f\lambda_1+dg)\lambda_1^n-(dg+f\lambda_2)\lambda_2^n}{\lambda_1-\lambda_2}}, \tag{2.140}$$

$$z_{3n+3} = z_0^{\frac{(acf-bf-cg-\lambda_2)\lambda_1^{n+1}+(\lambda_1-acf+bf+cg)\lambda_2^{n+1}}{\lambda_1-\lambda_2}} x_{-1}^{-d(af-g)\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}}, \qquad n \in \mathbb{N}_0. \tag{2.141}$$

If $(acf - ad - bf - cg)^2 = 4bdg$, that is, if $\lambda_1 = \lambda_2 = (acf - ad - bf - cg)/2$, we have that

$$a_{3k+1} = \left(\left(a + \frac{\lambda_1}{d}\right)k + a\right)\lambda_1^k \tag{2.142}$$

$$a_{3k+2} = (ac - b)(k+1)\lambda_1^k \tag{2.143}$$

$$a_{3k+3} = \left(\frac{acf - bf - ad - \lambda_1}{\lambda_1}(k+1) + 1\right)\lambda_1^{k+1} \tag{2.144}$$

$$b_{3k+1} = b\left(\frac{acf - bf - ad - \lambda_1}{\lambda_1}k + 1\right)\lambda_1^k \tag{2.145}$$

$$b_{3k+2} = d\left(\left(a + \frac{\lambda_1}{d}\right)k + a\right)\lambda_1^k, \tag{2.146}$$

$$b_{3k+3} = g(ac - b)(k+1)\lambda_1^k, \tag{2.147}$$

$$c_{3k+1} = \left(\left(c + \frac{\lambda_1}{g}\right)k + c\right)\lambda_1^k \tag{2.148}$$

$$c_{3k+2} = (cf - d)(k+1)\lambda_1^k \tag{2.149}$$

$$c_{3k+3} = \left(\frac{acf - ad - cg - \lambda_1}{\lambda_1}(k+1) + 1\right)\lambda_1^{k+1} \tag{2.150}$$

$$d_{3k+1} = d\left(\frac{acf - ad - cg - \lambda_1}{\lambda_1}k + 1\right)\lambda_1^k \tag{2.151}$$

$$d_{3k+2} = g\left(\left(c + \frac{\lambda_1}{g}\right)k + c\right)\lambda_1^k, \tag{2.152}$$

$$d_{3k+3} = b(cf - d)(k+1)\lambda_1^k, \tag{2.153}$$

$$f_{3k+1} = \left(\left(f + \frac{\lambda_1}{b}\right)k + f\right)\lambda_1^k \tag{2.154}$$

$$f_{3k+2} = (af-g)(k+1)\lambda_1^k \tag{2.155}$$
$$f_{3k+3} = \left(\frac{acf-bf-cg-\lambda_1}{\lambda_1}(k+1)+1\right)\lambda_1^{k+1} \tag{2.156}$$
$$g_{3k+1} = g\left(\frac{acf-bf-cg-\lambda_1}{\lambda_1}k+1\right)\lambda_1^k \tag{2.157}$$
$$g_{3k+2} = b\left(\left(f+\frac{\lambda_1}{b}\right)k+f\right)\lambda_1^k, \tag{2.158}$$
$$g_{3k+3} = d(af-g)(k+1)\lambda_1^k, \tag{2.159}$$

for $k \in \mathbb{N}_0$.

By using (2.142)–(2.159) into (2.27), (2.28) and (2.29) we obtain that the well-defined solutions of system (1.8) in this case are given by the following formulas

$$x_{3n+1} = y_0^{((a+\frac{\lambda_1}{d})n+a)\lambda_1^n} z_{-1}^{-b(\frac{acf-bf-ad-\lambda_1}{\lambda_1}n+1)\lambda_1^n}, \tag{2.160}$$
$$x_{3n+2} = z_0^{(ac-b)(n+1)\lambda_1^n} x_{-1}^{-d((a+\frac{\lambda_1}{d})n+a)\lambda_1^n}, \tag{2.161}$$
$$x_{3n+3} = x_0^{(\frac{acf-bf-ad-\lambda_1}{\lambda_1}(n+1)+1)\lambda_1^{n+1}} y_{-1}^{-g(ac-b)(n+1)\lambda_1^n}, \tag{2.162}$$
$$y_{3n+1} = z_0^{((c+\frac{\lambda_1}{g})n+c)\lambda_1^n} x_{-1}^{-d(\frac{acf-ad-cg-\lambda_1}{\lambda_1}n+1)\lambda_1^n}, \tag{2.163}$$
$$y_{3n+2} = x_0^{(cf-d)(n+1)\lambda_1^n} y_{-1}^{-g((c+\frac{\lambda_1}{g})n+c)\lambda_1^n}, \tag{2.164}$$
$$y_{3n+3} = y_0^{(\frac{acf-ad-cg-\lambda_1}{\lambda_1}(n+1)+1)\lambda_1^{n+1}} z_{-1}^{-b(cf-d)(n+1)\lambda_1^n}, \tag{2.165}$$
$$z_{3n+1} = x_0^{((f+\frac{\lambda_1}{b})n+f)\lambda_1^n} y_{-1}^{-g(\frac{acf-bf-cg-\lambda_1}{\lambda_1}n+1)\lambda_1^n}, \tag{2.166}$$
$$z_{3n+2} = y_0^{(af-g)(n+1)\lambda_1^n} z_{-1}^{-b((f+\frac{\lambda_1}{b})n+f)\lambda_1^n}, \tag{2.167}$$
$$z_{3n+3} = z_0^{(\frac{acf-bf-cg-\lambda_1}{\lambda_1}(n+1)+1)\lambda_1^{n+1}} x_{-1}^{-d(af-g)(n+1)\lambda_1^n}, \qquad n \in \mathbb{N}_0. \tag{2.168}$$

finishing the proof of the theorem. □

Remark 2.2. To get formulas (2.115)–(2.132) and (2.142)–(2.159) we need to know values of a_i, b_i, c_i, d_i, f_i, g_i, $i = \overline{1,6}$. Since the expressions of these initial values become more and more complicated when index i increases, if we want to get the formulas by hand then the process is time-consuming because of much calculations. Hence, we suggest the following procedure which facilitates getting the formulas. Namely, recurrent relations (2.21)–(2.26) can be used in a natural way to calculate also a_i, b_i, c_i, d_i, f_i, g_i, for $i \in \{-2,-1,0\}$. This is possible since relations (2.21)–(2.26) define values of a_i, b_i, c_i, d_i, f_i, g_i uniquely, for all non-positive i, when $bdg \neq 0$. For example, if in the relation $b_{3k+4} = ba_{3k+3}$, we choose $k=-1$, then we get $b_1 = ba_0$, from which it follows that $a_0 = 1$ (here the assumption $b \neq 0$ is used). Using this fact in the relation $a_{3k+4} = aa_{3k+3} - b_{3k+3}$, with $k=-1$, we obtain that $a_1 = a \cdot 1 - b_0$, from which it follows that $b_0 = 0$. From this and the relation $b_{3k+3} = ga_{3k+2}$, with $k=-1$, we get $a_{-1} = 0$ (here the assumption $g \neq 0$ is used). Continuing in this way it can be obtained that

$$\begin{array}{llllll}
a_{-2} = -\frac{1}{d}, & b_{-2} = -\frac{c}{d}, & c_{-2} = -\frac{1}{g}, & d_{-2} = -\frac{f}{g}, & f_{-2} = -\frac{1}{b}, & g_{-2} = -\frac{a}{b} \\
a_{-1} = 0, & b_{-1} = -1, & c_{-1} = 0, & d_{-1} = -1, & f_{-1} = 0, & g_{-1} = -1 \\
a_0 = 1, & b_0 = 0, & c_0 = 1, & d_0 = 0, & f_0 = 1, & g_0 = 0.
\end{array}$$

Using these "initial values" along with a_i, b_i, c_i, d_i, f_i, g_i, $i = \overline{1,3}$, all the calculations in getting formulas (2.115)–(2.132) and (2.142)–(2.159) become somewhat simpler.

From the proof of Theorem 2.1 we obtain the following corollary.

Corollary 2.3. *Consider system (1.8) with $a, b, c, d, f, g \in \mathbb{Z}$. Assume that $x_{-i}, y_{-i}, z_{-i}, \in \mathbb{C} \setminus \{0\}$, $i \in \{0, 1\}$. Then the following statement are true.*

(a) If $b = 0$, then the general solution to system (1.8) is given by (2.51)–(2.59).

(b) If $d = 0$, then the general solution to system (1.8) is given by (2.78)–(2.86).

(c) If $g = 0$, then the general solution to system (1.8) is given by (2.105)–(2.113).

(d) If $bdg \neq 0$ and $(ad + bf + cg - acf)^2 \neq 4bdg$, then the general solution to system (1.8) is given by (2.133)–(2.141).

(e) If $bdg \neq 0$ and $(ad + bf + cg - acf)^2 = 4bdg$, then the general solution to system (1.8) is given by (2.160)–(2.168).

Remark 2.4. Formulas (2.51)–(2.59), (2.78)–(2.86), (2.105)–(2.113), (2.133)–(2.141) and (2.160)–(2.168) can be used in describing the long term behavior of well-defined solutions of system (1.8). We leave the formulations and proofs of these results to the reader as some exercises.

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