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Continuous dependence results for set-valued measure differential problems

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Abstract. We discuss existence and continuous dependence properties of the solutions set of measure differential inclusions

$$\begin{aligned} dx(t) &\in G(t, x(t))d\mu(t), \\ x(0) &= x_0. \end{aligned} \tag{1.1}$$

where $G\colon [0,1] \times \mathbb{R}^d \to \mathcal{P}_{kc}(\mathbb{R}^d)$ is a regulated or bounded variation multifunction and μ is a Borel measure.

The significance of our study is proved by the remark that a result of continuous dependence of the solution set on the measure allows one to approximate the solutions of this problem with general measures by solutions of much simpler problems, with convenient measures (for instance discrete measures, as in numerical analysis).

First, by applying a selection principle for bounded variation multifunctions provided by S. A. Belov and V. V. Chistyakov, we prove the existence of solutions with specific properties and a continuous dependence result under bounded variation assumptions on the right-hand side.

Next, we prove a selection principle for regulated multifunctions and apply it to obtain a result concerning the existence of solutions with special features, as well as the continuous dependence of the set of these solutions with respect to the measure driving the inclusion.

Keywords: measure differential inclusion, continuous dependence, Stieltjes integral, regulated function, bounded variation, selection.

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1 Introduction

We focus on the problem (1.1), where $G\colon [0,1] \times \mathbb{R}^d \to \mathcal{P}(\mathbb{R}^d)$ is a compact convex-valued multifunction and μ is a positive regular Borel measure.

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The motivation for studying such a kind of problems comes from the fact that it contains, as special cases, differential and difference inclusions (when the involved measure is absolutely continuous, respectively discrete) and hybrid problems (when the measure is the sum of continuous and purely atomic measures) and it allows to describe systems with state discontinuities (such as the mechanical systems studied in a series of papers of Leine and van de Wouw [22]). Besides, one can thus overcome the difficulties arisen when trying to study by direct methods the behavior of hybrid systems in a general setting (for instance in the case where the perturbation moments have accumulation points, see [1,8,17,19]).

For practical reasons, we are interested in studying measure differential problems with bounded variation or, even more generally, regulated functions on the right-hand side, especially from the point of view of the possibility to obtain the solutions by the solutions of similar problem driven by approximating measures (shortly: continuous dependence results).

Due to the huge importance of this matter (since, if available, it allows to approach problem (1.1) with general measures via much simpler problems, with convenient measures), in the single-valued case it was intensively studied; in the nonlinear framework we remind of [14,15] and in the linear case of [18,25].

We are concerned here with the set-valued setting. For the most natural notion of solution (given by the integral of a selection) an existence result was provided in [10] and a continuous dependence result was given in [11] when the sequence of measures was supposed to converge in a sense strictly related to the set-valued framework.

In the present work, we want to obtain the existence of solutions with special properties for the case when the multifunction on the right-hand side is regulated, respectively of bounded variation and, for the family of these solutions, to prove continuous dependence results via usual convergence notions for measures.

Let us recall that for a more complicated notion of solution, namely that of robust solution (see [13]) the problem was investigated in [31] and continuous dependence results were obtained.

Obviously, in order to study continuous dependence of the family of solutions, the first matter is to ensure the existence of solutions. For this purpose, the main difficulty is to get selections with satisfactory properties for multifunctions.

In the case of multifunctions of bounded variation, the selection principles proved by V. V. Chistyakov and his collaborators (see [4,12]) are sufficient for our purpose. Using such principles, we prove the existence of solutions with good properties (coming from the properties of selections described above) and we prove that the family of such "good" solutions is continuously dependent on the measure driving the inclusion.

For regulated multifunctions, as far as we know, a selection principle is not available. Thus, we prove the existence of regulated selections for regulated multifunctions (in fact, we prove that we are able to find selections that are equi-regulated in the sense of [16]) and apply this result to give existence results for our problem and, again, a continuous dependence result with respect to the measure.

To make the paper self-contained we collect some known facts about convergence of measures and about regulated functions and bounded variation functions in Stieltjes integration.

2 Notions and preliminary facts.

Let μ be a positive Borel measure on $[0,1]$. The classical approach of measure driven equations [11,31] is using the Riesz Representation Theorem that characterizes finite regular Borel measures on a compact metrizable space as linear continuous functionals on the space of real continuous functions. Besides, it was shown (see [6, p. 126]) that any finite Borel measure on a Polish space, in particular on the unit interval of the real line, is regular.

On the space $\mathcal{M}$ of all positive Borel measures over $[0,1]$ there are several topologies, but we recall here only those concepts that will be used in the sequel.

Definition 2.1 ([5]).

i) A sequence $(\mu_n)_n$ of measures is said to be strongly convergent to μ if $\mu_n(A) \to \mu(A)$ for every measurable set A;

ii) We say that $(\mu_n)_n$ is weakly*-convergent to μ if $\mu_n(A) \to \mu(A)$ for every continuity set of μ.

Here by continuity set of the measure μ we mean a measurable set A such that $\mu(\partial A) = 0$.

The classical Portmanteau theorem [5] states that:

1) the sequence $(\mu_n)_n$ strongly converges to μ if and only if $\int_{[0,1]} f d\mu_n \to \int_{[0,1]} f d\mu$ for every bounded measurable function $f\colon [0,1] \to \mathbb{R}$;

2) $(\mu_n)_n$ weakly* converges to μ if and only if $\int_{[0,1]} f d\mu_n \to \int_{[0,1]} f d\mu$ for every continuous function $f\colon [0,1] \to \mathbb{R}$.

On the other side, every finite Borel measure on the real line agrees with some Lebesgue–Stieltjes measure (with respect to a bounded variation function) restricted to the class of Borel sets, see [6, Theorem 3.21]. This is the motivation for using, when necessary, instead of the preceding writing (1.1), of the form

$$dx(t) \in G(t, x(t))du(t),$$
$$x(0) = x_0$$

and seeing that the inclusion is a Stieltjes inclusion (in Lebesgue–Stieltjes or Kurzweil–Stieltjes approach).

2.1 Regulated or bounded variation functions and Stieltjes integrals.

In this subsection, we focus on the properties of measures in terms of their distribution functions, treating the subject of Stieltjes integrals.

Definition 2.2. A function $f\colon [0,1] \to \mathbb{R}^d$ is said to be Kurzweil–Stieltjes integrable with respect to $u : [0,1] \to \mathbb{R}$ on $[0,1]$ (shortly, KS-integrable) if there exists $(KS) \int_0^1 f(s)du(s) \in \mathbb{R}^d$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε (a positive function) on $[0,1]$ with

$$\left\| \sum_{i=1}^{p} f(\xi_i)(u(t_i) - u(t_{i-1})) - (\mathrm{KS}) \int_0^1 f(s)du(s) \right\| < \varepsilon$$

for every δ_ε-fine partition $\{([t_{i-1}, t_i], \xi_i) : i = 1, \ldots, p\}$ of $[0,1]$.

The partition is δ-fine if $[t_{i-1}, t_i] \subset]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[$, $\forall i$.

The KS-integrability is preserved on all sub-intervals of $[0,1]$. The function

$$t \mapsto (KS) \int_0^t f(s)du(s)$$

is called the KS-primitive of f w.r.t. u on $[0,1]$.

In the whole paper, we deal with the Kurzweil–Stieltjes integral. Note that in the framework of a left-continuous function u of bounded variation, as a consequence of [26, Theorem VI.8.1], the Lebesgue–Stieltjes integrability implies the KS integrability, but the converse is not true. Moreover, the KS integral $\int_0^t f(s)du(s)$ coincides with the Lebesgue–Stieltjes integral $\int_{[0,t)} f(s)d\mu(s)$, μ being the Stieltjes measure associated to u.

Let us recall here that for a function $u\colon [0,1] \to X$ with values in a Banach space, the total variation will be denoted by $\mathrm{var}(u)$ and if it is finite then u will be said to have bounded variation (or to be a BV function). For a real-valued BV-function u, by du we denote the corresponding Stieltjes measure. It is defined for half-open sub-intervals of $[0,1]$ by

$$du([a,b)) = u(b) - u(a)$$

and it is then extended to all Borel subsets of the unit interval in the standard way.

We shall consider only positive Borel measures, therefore Stieltjes measures with left-continuous non-decreasing distribution function u.

As it can be seen in the literature concerning Kurzweil–Stieltjes integrals (we refer the reader to [21,27,28,32]), the theory of KS integration is closely related to that of regulated and bounded variation functions. In particular, regulated functions are KS-integrable with respect to bounded variation functions.

For a general Banach space X, a function $u\colon [0,1] \to X$ is said to be regulated if there exist the limits $u(t^+)$ and $u(s^-)$ for every point $t \in [0,1)$ and $s \in (0,1]$. It is well-known [20] that the set of discontinuities of a regulated function is at most countable, that any bounded variation function is regulated, regulated functions are bounded and the space $G([0,1],X)$ of regulated functions is a Banach space when endowed with the norm $\|u\|_C = \sup_{t\in[0,1]}\|u(t)\|$.

The following property of the indefinite Kurzweil–Stieltjes integral implies that the solutions that will be obtained are regulated functions.

Proposition 2.3 ([32, Proposition 2.3.16])**.** *Let $u\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}^d$ be such that the Kurzweil–Stieltjes $\int_0^1 g(s)du(s)$ exists. If u is regulated, then so is the primitive $h\colon [0,1] \to \mathbb{R}^d$, $h(t) = \int_0^t g(s)du(s)$ and for every $t \in [0,1]$,*

$$h(t^+) - h(t) = g(t)\left[u(t^+) - u(t)\right] \quad \text{and} \quad h(t) - h(t^-) = g(t)\left[u(t) - u(t^-)\right].$$

It follows that h is left-continuous, respectively right-continuous at the points where u has the same property.

Moreover, when u is of bounded variation and g is bounded, h is also of bounded variation.

The following notion is very important when looking for compactness properties.

Definition 2.4 ([16])**.** A set $\mathcal{A} \subset G([0,1],X)$ is said to be equi-regulated if for every $\varepsilon > 0$ and every $t_0 \in [0,1]$ there exists $\delta > 0$ such that, for all $x \in \mathcal{A}$:

i) for any $t_0 - \delta < t' < t_0$: $\|x(t') - x(t_0^-)\| < \varepsilon$;

ii) for any $t_0 < t'' < t_0 + \delta$: $\|x(t'') - x(t_0^+)\| < \varepsilon$.

Lemma 2.5 ([16])**.** *A pointwise convergent sequence of functions which is equi-regulated converges uniformly to its limit.*

Let us recall a very useful characterization of equiregulatedness proved in [16].

Theorem 2.6. *For a set $\mathcal{A} \subset G([0,1], \mathbb{R}^d)$ the following assertions are equivalent:*

(i) $\mathcal{A} \subset G([0,1], \mathbb{R}^d)$ *is relatively compact;*

(ii) $\mathcal{A}$ *is equi-regulated and, for every* $t \in [0,1]$, $\mathcal{A}(t) = \{x(t), x \in \mathcal{A}\}$ *is relatively compact in* $\mathbb{R}^d$*;*

(iii) The set $\mathcal{A}(0) = \{x(0), x \in \mathcal{A}\}$ *is bounded and there is an increasing continuous function* $\eta\colon [0,\infty) \to [0,\infty)$, $\eta(0) = 0$ *and an increasing function* $v\colon [0,1] \to [0,1]$, $v(0) = 0$, $v(1) = 1$ *such that*

$$\|x(t_2) - x(t_1)\| \leq \eta(v(t_2) - v(t_1)), \tag{2.1}$$

for every $x \in \mathcal{A}$ *and every* $0 \leq t_1 < t_2 \leq 1$.

Corollary 2.7. *In particular, when the set $\mathcal{A}$ is a singleton, the preceding result states that a function $x\colon [0,1] \to \mathbb{R}^d$ is regulated if and only if there is an increasing continuous function $\eta\colon [0,\infty) \to [0,\infty)$, $\eta(0) = 0$ and an increasing function $v\colon [0,1] \to [0,1]$, $v(0) = 0$, $v(1) = 1$ such that $\|x(t_2) - x(t_1)\| \leq \eta(v(t_2) - v(t_1))$, for every $0 \leq t_1 < t_2 \leq 1$.*

In fact, the proof of [16, Theorem 2.14] can be repeated in the case of a Banach space and so, this characterization is also available for a Banach-space valued regulated functions.

Remark 2.8. As pointed out in [16, Remark 2.15], if the regulated functions in Theorem 2.6 are left-continuous, then v can be chosen left-continuous as well. It can be easily seen that the reciprocal is available as well: if v is left-continuous, then by passing to the limit in inequality (2.1) and taking into account that $\eta(0) = 0$, the regulated functions are also left-continuous.

We refer the reader to [2, 9] for notions of set-valued analysis. The space $\mathcal{P}_{kc}(\mathbb{R}^d)$ of all nonempty compact convex subsets of $\mathbb{R}^d$ will be considered endowed with the Hausdorff–Pompeiu distance, D; it is well-known that it becomes a complete metric space. For $A \in \mathcal{P}_{kc}(\mathbb{R}^d)$, denote by $|A| = D(A, \{0\})$. A multifunction $\Gamma\colon \mathbb{R}^d \to \mathcal{P}_{kc}(\mathbb{R}^d)$ is upper semi-continuous at a point x_0 if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that the excess of $\Gamma(x)$ over $\Gamma(x_0)$ (in the sense of Hausdorff) is less than ε whenever $\|x - x_0\| < \delta_\varepsilon$: $\Gamma(x) \subset \Gamma(x_0) + \varepsilon B^d$, where B^d is the unit ball of $\mathbb{R}^d$.

3 Main results

Let us first remind of several definitions that were considered in literature for the notion of solution of a measure driven inclusion.

Definition 3.1 ([10]). A solution of the problem (1.1) is a function $x\colon [0,1] \to \mathbb{R}^d$ for which there exists a μ-integrable function $g\colon [0,1] \to \mathbb{R}^d$ such that $g(t) \in G(t, x(t^-))$ μ-a.e. and

$$x(t) = x_0 + \int_0^t g(s)\, d\mu(s), \quad \forall\, t \in [0,1].$$

Note that when μ is a Stieltjes measure associated to a left-continuous function, by Proposition 2.3, x is also left-continuous and so, in the preceding definition, we may write $g(t) \in G(t, x(t))$ μ-a.e.

For this notion of solution, an existence result was proved.

Theorem 3.2 ([10, Theorem 11]). *Let $G\colon [0,1] \times \mathbb{R}^d \to \mathcal{P}_{kc}(\mathbb{R}^d)$ satisfy the following hypotheses:*

1) $G(\cdot,\cdot)$ *is product Borel measurable,*

2) $G(t,\cdot)$ *is upper semi-continuous for every* $t \in [0,1]$,

3) *there exists a positive function* $M \in L^1([0,1],\mu)$ *and a constant* $N > 0$ *such that* $G(t,y) \subset [M(t) + N\|y\|]B^d$ *for all* $t \in [0,1]$ *and* $y \in R^d$.

Then there exists at least one solution for the measure differential problem (1.1).

In a series of papers of Silva and his collaborators (e.g. [31]) another definition for solution was considered (similar to [13]) based on the idea to use a reparametrization method for μ and in this way to transform the measure driven differential inclusions into usual differential inclusions.

Definition 3.3. A function $x\colon [0,1] \to \mathbb{R}^d$ is called a robust solution if

$$x(t) = x_0 + \int_0^t g(s)\, d\mu(s), \quad \forall t \in [0,1]$$

for some μ-integrable function g such that $g(t) \in \widetilde{G}(t, x(t^-); \mu(\{t\}))$ μ-a.e., where the multifunction $\widetilde{G}$ is defined on $[0,1] \times \mathbb{R}^d \times [0,\infty)$ as follows:

- if $\alpha > 0$, then $\widetilde{G}(t,v,\alpha) = \left\{ \frac{y(\alpha)-v}{\alpha} : y \in AC^1([0,\alpha]),\ \dot{y}(\sigma) \in G(t,y(\sigma))\ a.e.,\ y(0) = v \right\}$
- and if $\alpha = 0$, then $\widetilde{G}(t,v,\alpha) = G(t,v)$.

[31, Theorem 4.1] reduces the matter of the existence of robust solutions to the existence problem for a usual differential inclusion and in [31, Corollary 4.2] the existence of robust solutions is provided under Lipschitz continuity assumptions together with linear growth assumptions on the multifunction on the right-hand side.

Finally, note that there is another type of solution, called approximable solution, which was considered in the single-valued case by many authors [23,24,30].

Concerning the continuous dependence property, when $G(\cdot,\cdot)$ has closed graph and the values of G are contained in some ball, [31, Theorem 5.1] states that the set of robust solutions is continuous with respect to data, in the sense that when a sequence of measures $(\mu_i)_i$ tends to μ in the weak* topology, for any sequence $(x_i)_i$ of robust solutions corresponding to μ_i there exists a robust solution x corresponding to μ with the property that on a subsequence

$$x_i \to x \text{ (weakly}^*\text{) and } x_i(t) \to x(t) \text{ except on the atoms of } \mu.$$

For our concept of solution (given by Definition 3.1), a continuous dependence result was given in [11] when the sequence of measures was supposed to converge in some sense strictly related to the set-valued framework.

In the present paper we shall prove that a subset of "good" solutions (in a sense that will be clearly described) satisfies a similar continuous dependence result via classical convergence assumptions on the measures: Theorem 3.8 concerns the case of BV multifunctions, respectively Theorem 3.14 that of regulated multifunctions.

We will consider the following notions of convergence for measures, related to the weak* convergence.

Definition 3.4.

i) We say that a sequence of measures $(\mu_n)_n$ reg-weakly* converges to μ if for every regulated function $f\colon [0,1] \to \mathbb{R}_+$, $\int_0^t f(s)d(\mu_n - \mu)(s) \to 0$ for every $t \in [0,1]$.

ii) The sequence $(\mu_n)_n$ is called càglàd-weakly* convergent to μ if for every left-continuous regulated function (known as càglàd function in probability theory) $f\colon [0,1] \to \mathbb{R}_+$, $\int_0^t f(s)d(\mu_n - \mu)(s) \to 0$ for every $t \in [0,1]$.

Notice that this definition implies that for every regulated (respectively càglàd) $\mathbb{R}^d$-valued function, $\int_0^t f(s)d(\mu_n - \mu)(s) \to 0$ for every $t \in [0,1]$.

3.1 BV multifunctions.

Let us recall that [10, Theorem 11] (see Theorem 3.2 above) gives the existence of solutions for the measure differential problem (1.1). We shall see that if G satisfies additional conditions, then we can prove the existence of solutions with additional properties.

Theorem 3.5. *Let $\mu \in \mathcal{M}$ be the Stieltjes measure associated to a left-continuous nondecreasing function and let $G\colon [0,1] \times \mathbb{R}^d \to \mathcal{P}_{kc}(\mathbb{R}^d)$ satisfy the following hypotheses.*

1) $G(t, \cdot)$ is upper semi-continuous for every $t \in [0,1]$.

2) For every BV-function $x\colon [0,1] \to \mathbb{R}^d$, the map $G(\cdot, x(\cdot))$ has bounded variation with respect to the Hausdorff–Pompeiu distance.

3) For every $R > 0$ there exists $M_R > 0$ such that for every BV-function x whose variation $\operatorname{var}(x) \leq R$:

$$\operatorname{var}(G(\cdot, x(\cdot))) \leq M_R.$$

Moreover, suppose that one can find $R_0 > 0$ satisfying the inequality

$$\mu([0,1])(|G(0, x_0)| + M_{R_0}) \leq R_0.$$

Then there exists at least one solution for the measure differential problem (1.1) such that $x(t) = x_0 + \int_0^t g(s)\, d\mu(s)$, $\forall\, t \in [0,1]$ and $g(t) \in G(t, x(t))$ is of bounded variation with $\operatorname{var}(g) \leq M_{R_0}$.

Proof. Our proof is based on an iteration procedure. More precisely, we construct a sequence of approximate solutions (which are BV-functions) which is shown to have a convergent subsequence due to some compactness properties.

So, let $x_0(t) = x_0$ for $t \in [0,1]$. Suppose that we have already constructed a BV-function x_n on $[0,1]$ with $\operatorname{var}(x_n) \leq R_0$ and choose x_{n+1} by following a scheme described in the sequel.

Using hypothesis 2), we apply [4, Theorem 2] and obtain the existence of a BV-selection $g_n(t) \in G(t, x_n(t)), \forall t \in [0,1]$ with $\operatorname{var}(g_n) \leq \operatorname{var}(G(\cdot, x_n(\cdot)))$. Define now

$$x_{n+1}(t) = x_0 + \int_0^t g_n(s)\, d\mu(s), \quad \forall t \in [0,1].$$

Since g_n is bounded, by Proposition 2.3, x_{n+1} is of bounded variation. Besides, hypothesis 3) implies that

$$\begin{aligned}
\operatorname{var}(x_{n+1}) &\leq \int_0^1 \|g_n(s)\|\, d\mu(s) \\
&\leq \int_0^1 \|g_n(0)\| + \operatorname{var}(g_n)\, d\mu(s) \\
&\leq \mu([0,1])(|G(0, x_0)| + M_{R_0}) \leq R_0
\end{aligned}$$

and so, the procedure can be continued.

Note that the sequence $(g_n)_n$ is bounded in variation by M_{R_0} and so, by Helly's selection principle, one can extract a subsequence $(g_{n_k})_k$ pointwise convergent to a BV-function g.

Next, by a convergence result, [28, Theorem I.4.24] (it can be applied since the functions g_n are bounded in variation by M_{R_0}, therefore they are uniformly bounded as well), we deduce that $\int_0^t g_{n_k}(s)d\mu(s) \to \int_0^t g(s)d\mu(s)$ and so, if we note by

$$x(t) = x_0 + \int_0^t g(s)d\mu(s),$$

it follows that $x_{n_k} \to x$ pointwisely.

We assert that x is a solution for our measure driven differential inclusion (i.e, $g(t) \in G(t, x(t))$). This comes from hypothesis 1): for each $t \in [0,1]$ and $\varepsilon > 0$, $G(t, x_{n_k}(t)) \subset G(t, x(t)) + \varepsilon B^d$, for all k greater than some $k_{\varepsilon,t}$, whence $g(t) \in G(t, x(t))$ as pointwise limit of $(g_{n_k})_k$. □

Remark 3.6. Conditions under which hypothesis 2) is verified can be found for instance in [7]. As for our hypothesis 3), it is satisfied by a large category of set-valued functions, e.g. by Lipschitz continuous multifunctions.

Indeed, let G verify the condition

$$D(G(t_1, x_1), G(t_2, x_2)) \leq K(|t_1 - t_2| + \|x_1 - x_2\|), \forall t_1, t_2 \in [0,1], x_1, x_2 \in \mathbb{R}^d$$

with $K \cdot \mu([0,1]) < 1$. Then for any BV-function x,

$$\mathrm{var}(G(\cdot, x(\cdot))) \leq K + K\,\mathrm{var}(x)$$

and so, we can take $M_R = K + KR$. Any

$$R_0 > \frac{\mu([0,1])(|G(0,x_0)| + K)}{1 - \mu([0,1])K}$$

satisfies the inequality $\mu([0,1])(|G(0,x_0)| + M_{R_0}) \leq R_0$.

We can obtain, under the assumptions of previous theorem, the continuous dependence on the measure of the set of solutions with described properties.

To achieve our goal, let us recall a special case of [29, Theorem 2.8] for the situation where the involved measures are Borel measures on the unit interval.

Theorem 3.7. *Suppose $f_n \to f$ in μ_n-measure, f is uniformly μ_n-integrable and*

$$\mu(A) \leq \liminf \mu_n(A), \quad \text{for every measurable } A.$$

Then $\int_{[0,1]} f_n d\mu_n \to \int_{[0,1]} f d\mu$ if and only if $(f_n)_n$ is uniformly μ_n-integrable.

The meaning of this notion is the following [29, Lemma 2.5]: a sequence $(f_n)_n$ is uniformly μ_n-integrable if $\sup_n \int_{[0,1]} f_n d\mu_n < \infty$ and for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that if $A_n, n \in \mathbb{N}$ are measurable and $\sup_n \mu_n(A_n) < \delta_\varepsilon$, then $\sup_n \int_{A_n} |f_n| d\mu_n < \varepsilon$.

We proceed now to give the main result of this section. Denote by $\mathcal{S}_n$ and $\mathcal{S}$ the set of solutions for the problem (1.1) driven by μ_n and μ respectively, via BV selections with variation bounded by M_{R_0}.

Theorem 3.8. *Let G satisfy the assumptions of Theorem 3.5 and $\mu,(\mu_n)_n \subset \mathcal{M}$ be Stieltjes measures associated to left-continuous nondecreasing functions satisfying the following conditions:*

$$\mu(A) \leq \liminf_n \mu_n(A), \quad \text{for every measurable } A,$$

and

$$\mu_n([0,1]) \leq \frac{R_0}{|G(0,x_0)| + M_{R_0}}, \quad \forall n.$$

Then for every sequence $(x_n)_n \subset \mathcal{S}_n$ there exists $x \in \mathcal{S}$ towards which a subsequence $(x_{n_k})_k$ converges pointwisely and $(dx_{n_k})_k$ converges reg-weakly to dx.*

Proof. The hypothesis of the existence theorem are verified for μ and μ_n for all $n \in \mathbb{N}$, therefore the sets $\mathcal{S}_n$ and $\mathcal{S}$ are nonempty.

Let $(x_n)_n$ be a sequence of solutions for our problem driven by the measures μ_n, respectively. Then there exist $g_n(t) \in G(t, x_n(t))$ such that $x_n(t) = x_0 + \int_0^t g_n(s)\, d\mu_n(s)$, $\forall\, t \in [0,1]$ and g_n is of bounded variation with $\operatorname{var}(g_n) \leq M_{R_0}$.

Obviously, the sequence $(g_n)_n$ is bounded in variation, whence Helly's selection principle implies that one can find a subsequence $(g_{n_k})_k$ pointwise convergent to a BV-function g. Let us show that

$$x(t) = x_0 + \int_0^t g(s) d\mu(s)$$

has the property that $(x_{n_k})_k$ converges pointwisely to x and dx_{n_k} converges reg-weakly* to dx.

To this goal, note that $(g_n)_n$ is uniformly μ_n-integrable since it is uniformly bounded by M_{R_0}, therefore we can apply Theorem 3.7 and obtain that $\int_0^t g_{n_k}(s)\, d\mu_{n_k}(s) \to \int_0^t g(s) d\mu(s)$.

As for the last part of the assertion, take an arbitrary regulated function $h : [0,1] \to \mathbb{R}$. By the substitution [32, Theorem 2.3.19], for any $t \in [0,1]$:

$$\left\| \int_0^t h(s) dx_{n_k}(s) - \int_0^t h(s) dx(s) \right\| = \left\| \int_0^t h(s) g_{n_k}(s) d\mu_{n_k}(s) - \int_0^t h(s) g(s) d\mu(s) \right\|$$

which tends to 0 as $k \to \infty$ again by Theorem 3.7.

It remains to prove that $x \in \mathcal{S}$. This is a consequence of the semi-continuity property of multifunction G since it implies that for each $t \in [0,1]$ and $\varepsilon > 0$, $G(t, x_{n_k}(t)) \subset G(t, x(t)) + \varepsilon B^d$, for all k greater than some $k_{\varepsilon,t}$. $\square$

Corollary 3.9. *If the sequence $(\mu_n)_n$ strongly converges to μ and*

$$\mu_n([0,1]) \leq \frac{R_0}{|G(0,x_0)| + M_{R_0}}, \quad \forall n,$$

then for every sequence $(x_n)_n \subset \mathcal{S}_n$ there exists $x \in \mathcal{S}$ towards which a subsequence $(x_{n_k})_k$ converges pointwisely and $(dx_{n_k})_k$ converges reg-weakly to dx.*

3.2 Regulated multifunctions.

We take now into consideration the framework of measure differential inclusions with regulated multifunctions on the right-hand side: a multifunction $F : [0,1] \to \mathcal{P}_{kc}(X)$ is said to be regulated if there exist, in the Hausdorff–Pompeiu metric, the limits $F(t^+)$ and $F(s^-)$ for every points $t \in [0,1)$ and $s \in (0,1]$.

As in this framework a selection principle is not available yet, we start by presenting such a result.

Lemma 3.10. *Let $F\colon [0,1] \to \mathcal{P}_{kc}(\mathbb{R}^d)$ be a regulated multifunction. Then it has regulated selections. Moreover, if (as in Corollary 2.7) the increasing continuous function $\eta\colon [0,\infty) \to [0,\infty)$, $\eta(0) = 0$ and the increasing function $v\colon [0,1] \to [0,1]$, $v(0) = 0$, $v(1) = 1$ satisfy the condition*

$$D(F(t_2), F(t_1)) \le \eta(v(t_2) - v(t_1))$$

for every $0 \le t_1 < t_2 \le 1$, then there exists a selection f satisfying the condition

$$\|f(t_2) - f(t_1)\| \le d\eta(v(t_2) - v(t_1)), \tag{3.1}$$

for every $0 < t_1 < t_2 \le 1$.

Proof. By Rådström's embedding theorem, F can be seen as a Banach-space valued regulated function for which we can use the characterization given by Corollary 2.7. So, there are an increasing continuous function $\eta\colon [0,\infty) \to [0,\infty)$, $\eta(0) = 0$ and an increasing function $v\colon [0,1] \to [0,1]$, $v(0) = 0$, $v(1) = 1$ such that

$$D(F(t_2), F(t_1)) \le \eta(v(t_2) - v(t_1))$$

for every $0 \le t_1 < t_2 \le 1$.

Consider now the Steiner selection $f(t)$ of $F(t)$ [2, p. 366]:

$$f(t) = \frac{1}{\mathrm{Vol}(B^d)} \int_{B^d} m(\partial\sigma(F(t), p))dp$$

where $\mathrm{Vol}(B^d)$ is the measure of the unit ball in the d-dimensional space, the subdifferential $\partial\sigma(F(t), p)$ of the support function $\sigma(F(t), \cdot)$ is given by

$$\partial\sigma(F(t), p) = \{x \in F(t); \langle p, x\rangle = \sigma(F(t), p)\},$$

and $m(\partial\sigma(F(t), p))$ is the element of $\partial\sigma(F(t), p)$ of minimal norm.

It satisfies (by [2, Theorem 9.4.1]) the following property:

$$\|f(t_2) - f(t_1)\| \le dD(F(t_2), F(t_1)), \quad \forall 0 \le t_1 < t_2 \le 1.$$

It follows that $\|f(t_2) - f(t_1)\| \le d\eta(v(t_2) - v(t_1))$, for all t_1, t_2 in $[0,1]$, whence the selection f is regulated (by Corollary 2.7) and the assertion is proved. □

We shall make use of this result to obtain the existence of solutions with special properties for measure differential inclusions.

Theorem 3.11. *Let $\mu \in \mathcal{M}$ be the Stieltjes measure associated to a left-continuous nondecreasing function and $G\colon [0,1] \times \mathbb{R}^d \to \mathcal{P}_{kc}(\mathbb{R}^d)$ satisfy:*

1) *$G(t, \cdot)$ is upper semi-continuous for each $t \in [0,1]$;*

2) *for every $x \in BV([0,1], \mathbb{R}^d)$, the map $G(\cdot, x(\cdot))$ is regulated;*

3) *for every $R > 0$, there exists an increasing continuous function $\eta_R\colon [0,\infty) \to [0,\infty)$, $\eta_R(0) = 0$ and an increasing function $v_R\colon [0,1] \to [0,1]$, $v_R(0) = 0$, $v_R(1) = 1$ such that*

$$D\left(G(t_2, x(t_2)), G(t_1, x(t_1))\right) \le \eta_R(v_R(t_2) - v_R(t_1)),$$

for every BV-function $x \in G([0,1], \mathbb{R}^d)$ with $\mathrm{var}(x) \le R$ and every $0 \le t_1 < t_2 \le 1$.

Suppose that for some $R_0 > 0$:

$$\mu([0,1])(|G(0,x_0)| + d\eta_{R_0}(1)) \leq R_0.$$

Then there exists at least one BV-solution for the measure differential problem (1.1) defined by

$$x(t) = x_0 + \int_0^t g(s)d\mu(s)$$

such that $\|g(t_2) - g(t_1)\| \leq d\eta_{R_0}(v_{R_0}(t_2) - v_{R_0}(t_1))$, for every $0 \leq t_1 < t_2 \leq 1$.

Proof. Following the method applied in Theorem 3.5, we construct a sequence of approximate solutions (which are BV-functions) and we show that it has a convergent subsequence.

Let $x_0(t) = x_0$ for $t \in [0,1]$. If we have already constructed a BV-function x_n on $[0,1]$ with $\mathrm{var}(x_n) \leq R_0$, we choose x_{n+1} in the following manner.

Using hypotheses 2) and 3), we apply Theorem 3.10 and obtain the existence of a regulated selection $g_n(t) \in G(t, x_n(t))$ with $\|g_n(t_2) - g_n(t_1)\| \leq d\eta_{R_0}(v_{R_0}(t_2) - v_{R_0}(t_1))$ for every $0 \leq t_1 < t_2 \leq 1$.

Consider

$$x_{n+1}(t) = x_0 + \int_0^t g_n(s)\, d\mu(s), \quad \forall t \in [0,1]$$

which is, by Proposition 2.3, of bounded variation and satisfies

$$\begin{aligned}
\mathrm{var}(x_{n+1}) &\leq \int_0^1 \|g_n(s)\|\, d\mu(s) \\
&\leq \int_0^1 (\|g_n(0)\| + d\eta_{R_0}(v_{R_0}(s) - v_{R_0}(0)))\, d\mu(s) \\
&\leq \mu([0,1])(|G(0,x_0)| + d\eta_{R_0}(1)) \leq R_0.
\end{aligned}$$

We assert now that the sequence $(g_n)_n$ satisfies assumption iii) in Theorem 2.6. Indeed, the inequality (2.1) is valid and $\{g_n, n \in \mathbb{N}\}(0) \subset G(0, \{x_n(0), n \in \mathbb{N}) = G(0, \{x_0\})$ is bounded.

Thus the sequence $(g_n)_n$ is relatively compact in $G([0,1], \mathbb{R}^d)$ and so, one can extract a subsequence $(g_{n_k})_k$ uniformly convergent to a regulated function g.

Next, by the convergence [28, Theorem I.4.17], $\int_0^t g_{n_k}(s)d\mu(s) \to \int_0^t g(s)d\mu(s)$ and so, denoting by $x(t) = x_0 + \int_0^t g(s)d\mu(s)$, $x_{n_k} \to x$ pointwisely.

We assert that x is a solution for our measure driven differential inclusion.

To see this, by hypothesis 1): for each $t \in [0,1]$ and $\varepsilon > 0$, $G(t, x_{n_k}(t)) \subset G(t, x(t)) + \varepsilon B$, for all k greater than some $k_{\varepsilon,t}$ and so, $g(t) \in G(t, x(t))$. □

Remark 3.12. In fact, hypothesis 3) requires that any family of BV-functions which is bounded in variation is brought, by the superposition operator, into an equi-regulated family of multifunctions. It is not difficult to check, by a calculus similar to that in Remark 3.6, that Lipschitz continuous multifunctions have a more general property: bring equi-regulated families into equi-regulated families of multifunctions.

In order to provide the continuous dependence in this framework, of regulated multifunctions, let $\widetilde{\mathcal{S}}_n$ and $\widetilde{\mathcal{S}}$ be the solutions set given by Theorem 3.11 for problem (1.1) corresponding to μ_n and μ, respectively.

Theorem 3.13. *Suppose that the hypotheses on G of the preceding theorem are satisfied. If $(\mu_n)_n \in \mathcal{M}$ is a sequence of Stieltjes measures associated to left-continuous nondecreasing functions which reg-weakly *-converges to μ and*

$$\mu_n([0,1]) \leq \frac{R_0}{|G(0,x_0)| + d\eta_{R_0}(1)}, \quad \forall n,$$

then for every $x_n \in \widetilde{\mathcal{S}}_n$ one can find an element $x \in \widetilde{\mathcal{S}}$ and a subsequence pointwisely convergent to x such that $(dx_{n_k})_k$ reg-weakly-converges to dx.*

Proof. Let for every $n \in \mathbb{N}$, $x_n \in \widetilde{\mathcal{S}}_n$ and g_n regulated with $g_n(t) \in G(t, x_n(t))$ for all $t \in [0,1]$ such that (3.1) holds and $x_n(t) = x_0 + \int_0^t g_n(s) d\mu_n(s)$. In the same way as in the proof of the existence theorem it can be proved that the sequence $(g_n)_n$ is satisfying the hypothesis (iii) of Theorem 2.6, so it is relatively compact in the topology of uniform convergence. One can extract a subsequence $(g_{n_k})_k$ uniformly convergent towards a regulated function $g\colon [0,1] \to \mathbb{R}^d$.
Let $x(t) = x_0 + \int_0^t g(s) d\mu(s)$ for every $t \in [0,1]$ (it is well defined since regulated functions are KS-integrable with respect to BV-functions). Then

$$\begin{aligned}\|x_{n_k}(t) - x(t)\| &= \left\| \int_0^t g_{n_k}(s) d\mu_{n_k}(s) - \int_0^t g(s) d\mu(s) \right\| \\ &\leq \int_0^t \|(g_{n_k} - g)(s)\| \, d\mu_{n_k}(s) + \left\| \int_0^t g(s) d(\mu_{n_k} - \mu)(s) \right\|.\end{aligned}$$

The first term tends to 0 uniformly in $t \in [0,1]$ by the definition of KS-integral since $(g_{n_k})_k$ tends uniformly to g and for each $[a,b) \subset [0,1]$,

$$\mu_n([a,b)) \leq \frac{R_0}{|G(0,x_0)| + d\eta_{R_0}(1)}, \quad \forall n,$$

while the second term tends to 0 because $(\mu_n)_n$ reg-weakly *-converges to μ and g is regulated. Otherwise said, $x_{n_k}(t) \to x(t)$ pointwisely.

Besides, by hypothesis, for every $\varepsilon > 0$ and $t \in [0,1]$ there exists $k_{\varepsilon,t} \in \mathbb{N}$ such that

$$G(t, x_{n_k}(t)) \subset G(t, x(t)) + \varepsilon B,$$

for all k greater than $k_{\varepsilon,t}$. It follows that $g(t) \in G(t, x(t))$ (therefore $x(t) = x_0 + \int_0^t g(s) d\mu(s) \in \mathcal{S}$) and so, the first part of the statement is proved.

As for the second statement, by the Substitution [32, Theorem 2.3.19], for any regulated $h\colon [0,1] \to \mathbb{R}$ and for any $t \in [0,1]$:

$$\left\| \int_0^t h(s) dx_{n_k}(s) - \int_0^t h(s) dx(s) \right\| = \left\| \int_0^t h(s) g_{n_k}(s) d\mu_{n_k}(s) - \int_0^t h(s) g(s) d\mu(s) \right\|$$

whence

$$\begin{aligned}&\left\| \int_0^t h(s) dx_{n_k}(s) - \int_0^t h(s) dx(s) \right\| \\ &\quad = \left\| \int_0^t h(s) \left(g_{n_k} - g\right)(s) d\mu_{n_k}(s) + \int_0^t h(s) g(s) d(\mu_{n_k}(s) - \mu(s)) \right\| \\ &\quad \leq \left\| \int_0^t h(s) \left(g_{n_k} - g\right)(s) d\mu_{n_k}(s) \right\| + \left\| \int_0^t h(s) g(s) d(\mu_{n_k}(s) - \mu(s)) \right\| \\ &\quad \leq \int_0^t |h(s)| \, \|g_{n_k}(s) - g(s)\| \, d\mu_{n_k}(s) + \left\| \int_0^t h(s) g(s) d(\mu_{n_k}(s) - \mu(s)) \right\|\end{aligned}$$

and the sum is arbitrarily small when $k \to \infty$. Indeed: the first term is small because h is bounded, while in the second term this is a consequence of the fact that the product of two regulated functions is still regulated. □

Corollary 3.14. *Suppose that the multifunction G in Theorem 3.13 is left-continuous. If $(\mu_n)_n$ càglàd-weakly *-converges to μ, then for every $x_n \in \widetilde{\mathcal{S}}_n$ one can find an element $x \in \widetilde{\mathcal{S}}$ and a subsequence pointwisely convergent to x such that $(dx_{n_k})_k$ càglàd-weakly*-converges to dx.*

Proof. The proof follows from Remark 2.8: we are able to choose the function v to be left-continuous and so, we are able to find left-continuous regulated selections in the whole proof of Theorem 3.13. □

Remark 3.15. It is not difficult to see from the proof of Theorem 3.13 that if the reg-weak*-convergence of $(\mu_n)_n$ towards μ is uniform in $t \in [0,1]$, namely $\left| \int_0^t g(s)d(\mu_n - \mu)(s) \right| \to 0$ for every regulated function $g\colon [0,1] \to \mathbb{R}_+$, uniformly in $t \in [0,1]$, then the extracted subsequence is uniformly convergent to x.

We shall now see a situation when the condition in the preceding remark is satisfied (it is a consequence of [18, Lemma 2.2]).

Remark 3.16. If $(u_n)_n$ is a sequence of BV functions bounded in variation and uniformly-convergent to u (in other words, two-norm-convergent, see [3]), then for every regulated function $g\colon [0,1] \to \mathbb{R}_+$, $\left| \int_0^t g(s)d(u_n - u)(s) \right| \to 0$ uniformly in $t \in [0,1]$.

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References

[1] J.-P. Aubin, *Impulsive differential inclusions and hybrid systems: a viability approach*, Lecture Notes, Univ. Paris, 2002.

[2] J.-P. Aubin, H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990. MR1048347

[3] K. K. Aye, P. Y. Lee, The dual of the space of functions of bounded variation, *Math. Bohem.* **131**(2006), No. 1, 1–9. MR2210998

[4] S. A. Belov, V. V. Chistyakov, A selection principle for mappings of bounded variation, *J. Math. Anal. Appl.* **249**(2000), 351–366. MR1781229; url

[5] P. Billingsley, *Weak convergence of measures: Applications in probability*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 5, Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1971. MR0310933

[6] A. M. Bruckner, J. B. Bruckner, B. S. Thomson, *Real analysis*, Prentice–Hall, 1997.

[7] D. Bugajewska, On the superposition operator in the space of functions of bounded variation, revisited, *Math. Comput. Modell.* **522**(2010), 791–796. MR2661764; url

[8] M. K. Çamlibel, W. P. M. H. Heemels, A. J. van der Schaft, J. M. Schumacher, Solution concepts for hybrid dynamical systems, *IFAC*, 2002.

[9] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer, Berlin, 1977. MR0467310

[10] M. Cichoń, B. Satco, Measure differential inclusions – between continuous and discrete, *Adv. Diff. Equations* **2014**, No. 56, 18 pp. MR3348625; url

[11] M. Cichoń, B. Satco, On the properties of solutions set for measure driven differential inclusions, *AIMS Proceedings (Proceedings of the 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications)*, accepted.

[12] V. V. Chistyakov, D. Repoš, Selections of bounded variation under the excess restrictions, *J. Math. Anal. Appl.* **331**(2007), 873–885. MR2313687; url

[13] G. Dal Maso, F. Rampazzo, On systems of ordinary differential equations with measures as controls, *Differential Integral Equations* **4**(1991), 739–765. MR1108058

[14] M. Federson, J. G. Mesquita, A. Slavík, Measure functional differential equations and functional dynamic equations on time scales, *J. Differential Equations* **252**(2012), 3816–3847. MR2875603; url

[15] M. Federson, J. G. Mesquita, A. Slavík, Basic results for functional differential and dynamic equations involving impulses, *Math. Nachr.* **286**(2013), No. 2–3, 181–204. MR3021475; url

[16] D. Fraňková, Regulated functions, *Math. Bohem.* **116**(1991), No. 1, 20–59. MR1100424

[17] R. Goebel, A. R. Teel, Solutions to hybrid inclusions via set and graphical convergence with stability theory applications, *Automatica* **42**(2006), 573–587. MR2200351; url

[18] Z. Halas, M. Tvrdý, Continuous dependence of solutions of generalized linear differential equations on a parameter, *Funct. Differ. Equ.* **16**(2009), 299–313. MR2533442

[19] W. P. M. H Heemels, M. K. Çamlibel, A. J. van der Schaft, J. M. Schumacher, On the existence and uniqueness of solution trajectories to hybrid dynamical systems, in: *Nonlinear and Hybrid Control in Automotive Applications*, 2002, pp. 391–422. url

[20] C. S. Hönig, *Volterra Stieltjes-integral equations*, North-Holland, Amsterdam, 1975. MR0499969

[21] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.* **7**(1957), No. 82, 418–449. MR0111875

[22] R. I. Leine, N. van de Wouw, Uniform convergence of monotone measure differential inclusions: with application to the control of mechanical systems with unilateral constraints, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **18**(2008), No. 5, 1435–1457. MR2427134; url

[23] B. Miller, E. Y. Rubinovitch, *Impulsive control in continuous and discrete-continuous systems*, Kluwer Academic Publishers, Dordrecht, 2003. MR2024011; url

[24] B. Miller, The generalized solutions of ordinary differential equations in the impulse control problems, *J. Math. Systems Estim. Control* **4**(1994), No. 3, 1–21. MR1410411

[25] G. A. Monteiro, M. Tvrdý, Generalized linear differential equations in a Banach space: Continuous dependence on a parameter, *Discrete Contin. Dyn. Syst.* **33**(2013), 283–303. MR2972960

[26] S. Saks, *Theory of the integral*, Monografie Matematyczne, Vol. 7, New York, 1937. MR0167578

[27] Š. Schwabik, *Generalized ordinary differential equations*, World Scientific, 1992. MR1200241; url

[28] Š. Schwabik, M. Tvrdý, O. Vejvoda, *Differential and integral equations. Boundary problems and adjoints*, D. Reidel Publishing Co., Dordrecht-Boston, Mass.-London, 1979. MR542283

[29] R. Serfozo, Convergence of Lebesgue integrals with varying measures, *Sankhyā Ser. A* **44**(1982), No. 3, 380–402. MR705462

[30] A. N. Sesekin, S. T. Zavalishchin, *Dynamic impulse systems*, Dordrecht, Kluwer Academic, 1997. MR1441079; url

[31] G. N. Silva, R. B. Vinter, Measure driven differential inclusions, *J. Math. Anal. Appl.* **202**(1996), 727–746. MR1408351; url

[32] M. Tvrdý, *Differential and integral equations in the space of regulated functions*, habilitation thesis, Prague, 2001.