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OSCILLATION OF THIRD-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to study oscillatory and asymptotic properties of the third-order nonlinear delay differential equation

$$(E) \qquad \left[a(t)\left[x''(t)\right]^{\gamma}\right]' + q(t)f(x\left[\tau(t)\right]) = 0.$$

Applying suitable comparison theorems we present new criteria for oscillation or certain asymptotic behavior of nonoscillatory solutions of (E). Obtained results essentially improve and complement earlier ones. Various examples are considered to illustrate the main results.

1. Introduction

We are concerned with oscillatory behavior of the third-order functional differential equations of the form

$$(E) \qquad \left[a(t)\left[x''(t)\right]^{\gamma}\right]' + q(t)f(x\left[\tau(t)\right]) = 0$$

In the sequel we will assume that the following conditions are always satisfied throughout this paper:

(H1) $a(t), q(t) \in C([t_0,\infty))$, $a(t), q(t)$ are positive, $\tau(t) \in C([t_0,\infty))$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$,
(H2) $\int_{t_0}^{\infty} a^{-1/\gamma}(s)\,ds < \infty$,
(H3) γ is a quotient of odd positive integers,
(H4) $f(x) \in C(-\infty,\infty)$, $xf(x) > 0$, $f'(x) \ge 0$ for $x \ne 0$ and $-f(-xy) \ge f(xy) \ge f(x)f(y)$ for $xy > 0$.

By a solution of Eq.(E) we mean a function $x(t) \in C^2[T_x,\infty)$, $T_x \ge t_0$, which has the property $a(t)(x''(t))^{\gamma} \in C^1[T_x,\infty)$ and satisfies Eq. (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise it is called to be nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Following Tanaka [23] we say that a nontrivial solution $x(t)$ of (E) is strongly decreasing if it satisfies

$$(1.1) \qquad x(t)x'(t) < 0$$

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for all sufficiently large t and it said to be strongly increasing if

$$x(t)x'(t) > 0. \tag{1.2}$$

Recently differential equations of the form (E) and its special cases have been the subject of intensive research (see enclosed references). Grace et al. in [9] have established a useful comparison principle for studying properties of (E). They have compared Eq.(E) with a couple of the first order delay differential equations in the sense that we deduce oscillation of Eq.(E) from the oscillation of this couple of equations.

Dzurina and Baculikova in [5] improve their results for the case when

$$\int_{t_0}^{\infty} a^{-1/\gamma}(s)\, ds = \infty.$$

Zhong et al. in [24] adapted Grace et al.'s method and extended some of their results to neutral differential equation

$$\left[a(t)\left\{[x(t) + p(t)x(\sigma(t))]''\right\}^{\gamma}\right]' + q(t)f(x\,[\tau(t)]) = 0. \tag{E_1}$$

So that again from oscillation of a suitable first order delay equation we deduce oscillation of (E_1)

On the other hand, Saker and Dzurina in [21] studied a particular case of Eq.(E), namely the differential equation

$$\left[a(t)\left[x''(t)\right]^{\gamma}\right]' + q(t)x^{\gamma}\,[\tau(t)] = 0. \tag{E_2}$$

They presented conditions under which every nonoscillatory solution of (E_2) tends to zero as $t \to \infty$. Those results are applicable even if the criteria presented in [9] fail.

It is useful to notice that for a very special case of (E), that is, for

$$x'''(t) + q(t)x(t) = 0, \tag{E_3}$$

Hartman and Wintner in [11] have derived that (E_3) always has a strongly decreasing solution. Thus, the effort for obtaining criteria for all nonoscillatory solutions to be strongly decreasing appeared.

Therefore, from all above mentioned results, we conclude that if the gap between t and $\tau(t)$ is small, then there exists a nonoscillatory solution of (E) and the Theorems from [9] are not applicable to deduce oscillation of (E). In this case, our goal is to prove that every nonoscillatory solution of (E) tends to zero as $t \to \infty$. While if the difference $t - \tau(t)$ is large enough, then we can study the oscillatory character of (E).

So our aim of this article is to provide a general classification of oscillatory and asymptotic behavior of the studied equation. We present criteria for (E) to be oscillatory or for every its nonoscillatory solution to be either strictly decreasing or tend to zero as $t \to \infty$.

At first we turn our attention to Theorem 2.2 from [9], which is the main result of the paper. Formulation of Theorem 2.2 in [9] does not match its proof and for all that we provide a corrected version of the theorem.

Theorem A. *[Theorem 2.2 in* [9]*] Let (H1) hold and assume that there exist two functions* $\xi(t)$ *and* $\eta(t) \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\xi'(t) > 0, \quad \eta'(t) > 0 \quad \text{and} \quad \tau(t) < \xi(t) < \eta(t) < t \quad \text{for} \quad t \geq t_0.$$

If

$$\int_{t_0}^{\infty}\left(\frac{1}{a(u)}\int_{t_0}^{u} q(s)f(\tau(s))f\left(\int_{\tau(s)}^{\infty} a^{-1/\gamma}(v)dv\right)ds\right)^{1/\gamma} du = \infty, \tag{1.3}$$

and both the first order delay equations

$$y'(t) + cq(t)f\left(\int_{T}^{\tau(t)} sa^{-1/\gamma}(s)ds\right) f\left(y^{1/\gamma}[\tau(t)]\right) = 0, \tag{E_I}$$

for any constant c*,* $0 < c < 1$*, and* $T \geq t_0$*, and*

$$z'(t) + q(t)f\left(\xi(t) - \tau(t)\right) f\left(\int_{\xi(t)}^{\eta(t)} a^{-1/\gamma}(s)ds\right) f\left(z^{1/\gamma}[\eta(t)]\right) = 0 \tag{E_{II}}$$

are oscillatory, then every solution of Eq.(E) is oscillatory.

Remark 1. *In the formulation of Theorem 2.2 in* [9]*, there is an excess term* $f(g(t))$ *included in Eq.* (E_I)*. Now, the reader can easily reconstruct the results from* [9] *pertaining to Theorem 2.2.*

2. Main results

We start our main results with the classification of the possible nonoscillatory solutions of (E).

Lemma 1. *Let* $x(t)$ *be a positive solution of* (E)*. Then either*

(i) $x''(t) > 0$*, eventually and* $x(t)$ *is either strongly increasing or strongly decreasing, or*

(ii) $x''(t) < 0$*, eventually and* $x(t)$ *is strongly increasing.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(E). We may assume that $x(t) > 0$, eventually (if it is an eventually negative, the proof is similar). Then $[a(t)[x''(t)]^{\gamma}]' < 0$, eventually. Thus, $a(t)[x''(t)]^{\gamma}$ is decreasing and of one sign and it follows from hypothesis $(H1)$ and $(H2)$ that there exists a $t_1 \geq t_0$ such that $x''(t)$ is of fixed sign for $t \geq t_1$. If we have $x''(t) > 0$, then $x'(t)$ is increasing and then either (1.1) or (1.2) hold, eventually.

On the other hand, if $x''(t) < 0$ then $x'(t)$ is decreasing, hence $x'(t)$ is of fixed sign. If we have $x'(t) < 0$, then $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the positivity of $x(t)$. Whereupon $x'(t) > 0$. The proof is complete. ∎

The following criterion eliminates *case (ii)* of Lemma 1.

Lemma 2. *Let* $x(t)$ *be a positive solution of* (E)*. If*

$$\int_{t_0}^{\infty}\left(\frac{1}{a(u)}\int_{t_0}^{u} q(s)f(\tau(s))f\left[\int_{\tau(s)}^{\infty} a^{-1/\gamma}(v)dv\right]ds\right)^{1/\gamma} du = \infty, \tag{2.1}$$

then $x(t)$ *does not satisfy case (ii) of Lemma 1.*

Proof Let $x(t)$ be a positive solution of Eq.(E). We assume that $x(t)$ satisfies *case (ii)* of Lemma 1. That is $x''(t) < 0$ and $x'(t) > 0$, eventually. Then there exist a $t_1 \geq t_0$ and a constant k, $0 < k < 1$ such that $x(t) \geq ktx'(t)$ for $t \geq t_1$. Consequently,

$$x[\tau(t)] \geq k\tau(t)x'[\tau(t)] \tag{2.2}$$

for $t \geq t_2 \geq t_1$. Now Eq.(E), in view of (H4) and (2.2), implies

$$[a(t)[x''(t)]^\gamma]' + f(k)q(t)f[\tau(t)]f(x'[\tau(t)]) \leq 0.$$

An integration of this inequality yields

$$f(k)\int_{t_2}^{t} q(s)f[\tau(s)]f(x'[\tau(s)])ds \leq a(t_2)[x''(t_2)]^\gamma - a(t)[x''(t)]^\gamma. \tag{2.3}$$

On the other hand, since $-a^{1/\gamma}(t)[x''(t)]$ is increasing, there exist a constant $m > 0$ such that

$$-a^{1/\gamma}(t)x''(t) \geq m, \quad \text{for } t \geq t_2, \tag{2.4}$$

which implies

$$x'(\tau(t)) \geq \int_{\tau(t)}^{\infty} -a^{1/\gamma}(s)x''(s)a^{-1/\gamma}(s)ds \geq m\int_{\tau(t)}^{\infty} a^{-1/\gamma}(s)ds. \tag{2.5}$$

Combining (2.5) together with (2.3), and taking into account (H3), we get

$$c\left(\frac{1}{a(t)}\int_{t_2}^{t} q(s)f[\tau(s)]f\left(\int_{\tau(s)}^{\infty} a^{-1/\gamma}(v)dv\right)ds\right)^{1/\gamma} \leq -x''(t), \tag{2.6}$$

where $c = [f(m)f(k)]^{1/\gamma}$. Integrating (2.6) from t_3 to t, we have

$$c\int_{t_3}^{t}\left(\frac{1}{a(u)}\int_{t_2}^{u} q(s)f[\tau(s)]f\left(\int_{\tau(s)}^{\infty} a^{-1/\gamma}(v)dv\right)ds\right)^{1/\gamma} du \leq x'(t_3).$$

Letting $t \to \infty$ we get a contradiction to condition (2.1). Therefore, we have eliminated *case (ii)* of Lemma 1. ∎

Now we are prepared to provide oscillation and asymptotic criteria for solutions of Eq.(E).

Theorem 1. *Let (2.1) hold. If the first order delay equation*

$$y'(t) + q(t)f\left[\int_{t_0}^{\tau(t)} (\tau(t) - u)\, a^{-1/\gamma}(u)du\right] f\left[y^{1/\gamma}[\tau(t)]\right] = 0 \tag{E_4}$$

is oscillatory, then every solution of Eq.(E) is either oscillatory or strongly decreasing.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(E). We may assume that $x(t) > 0$ for $t \geq t_0$. From Lemma 2 we see that $x''(t) > 0$ and $x(t)$ is either strongly increasing or strongly decreasing.

Assume that $x(t)$ is strongly increasing, that is $x'(t) > 0$, eventually. Using the fact that $a(t)\left[x''(t)\right]^{\gamma}$ is decreasing, we are lead to

$$\begin{aligned} x'(t) &\geq \int_{t_1}^{t} x''(u)du = \int_{t_1}^{t} a^{-1/\gamma}(u)\left[a(u)(x''(u))^{\gamma}\right]^{1/\gamma} du \\ &\geq \left[a(t)(x''(t))^{\gamma}\right]^{1/\gamma} \int_{t_1}^{t} a^{-1/\gamma}(u)du. \end{aligned} \tag{2.7}$$

Integrating (2.7) from t_1 to t, we have

$$\begin{aligned} x(t) &\geq \int_{t_1}^{t} \left[a(s)(x''(s))^{\gamma}\right]^{1/\gamma} \int_{t_1}^{s} a^{-1/\gamma}(u)duds \\ &\geq \left[a(t)(x''(t))^{\gamma}\right]^{1/\gamma} \int_{t_1}^{t} (t-u)\, a^{-1/\gamma}(u)du. \end{aligned}$$

There exists a $t_2 \geq t_1$ such that for all $t \geq t_2$, one gets

$$x[\tau(t)] \geq y^{1/\gamma}[\tau(t)] \int_{t_2}^{\tau(t)} (\tau(t)-u)\, a^{-1/\gamma}(u)du, \tag{2.8}$$

where $y(t) = a(t)(x''(t))^{\gamma}$. Combining (2.8) together with (E), we see that

$$\begin{aligned} -y'(t) &= q(t)f(x(\tau(t))) \geq q(t)f\left[y^{1/\gamma}[\tau(t)] \int_{t_2}^{\tau(t)} (\tau(t)-u)\, a^{-1/\gamma}(u)du\right] \\ &\geq q(t)f\left[\int_{t_2}^{\tau(t)} (\tau(t)-u)\, a^{-1/\gamma}(u)du\right] f\left[y^{1/\gamma}[\tau(t)]\right], \end{aligned}$$

where we have used $(H3)$. Thus, $y(t)$ is a positive and decreasing solution of the differential inequality

$$y'(t) + q(t)f\left[\int_{t_2}^{\tau(t)} (\tau(t)-u)\, a^{-1/\gamma}(u)du\right] f\left[y^{1/\gamma}[\tau(t)]\right] \leq 0.$$

Hence, by Theorem 1 in [19] we conclude that the corresponding differential equation (E_4) also has a positive solution, which contradicts to oscillation of (E_4). Therefore $x(t)$ is strongly decreasing. ∎

Adding an additional condition, we achieve stronger asymptotic behavior of nonoscillatory solutions of Eq.(E).

Lemma 3. *Assume that $x(t)$ is a strongly decreasing solution of Eq.(E). If*

$$\int_{t_0}^{\infty}\int_{v}^{\infty} \frac{1}{a^{1/\gamma}(u)}\left[\int_{u}^{\infty} q(s)ds\right]^{1/\gamma} dudv = \infty, \tag{2.9}$$

then $x(t)$ tends to zero as $t \to \infty$.

Proof. We may assume that $x(t)$ is positive. It is clear that there exists a finite $\lim_{t\to\infty} x(t) = \ell$. We shall prove that $\ell = 0$. Assume that $\ell > 0$.

Integrating Eq.(E) from t to ∞ and using $x[\tau(t)] > \ell$ and $(H3)$, we obtain

$$a(t)(x''(t))^{\gamma} \geq \int_{t}^{\infty} q(s)f(x[\tau(s)])ds \geq f(\ell)\int_{t}^{\infty} q(s)ds,$$

which implies

$$x''(t) \geq \frac{\ell_1}{a^{1/\gamma}(t)} \left[\int_t^\infty q(s)ds\right]^{1/\gamma},$$

where $\ell_1 = f^{1/\gamma}(\ell) > 0$. Integrating the last inequality from t to ∞, we get

$$-x'(t) \geq \ell_1 \int_t^\infty \frac{1}{a^{1/\gamma}(u)} \left[\int_u^\infty q(s)ds\right]^{1/\gamma} du.$$

Now integrating from t_1 to t, we arrive at

$$x(t_1) \geq \ell_1 \int_{t_1}^t \int_v^\infty \frac{1}{a^{1/\gamma}(u)} \left[\int_u^\infty q(s)ds\right]^{1/\gamma} dudv.$$

Letting $t \to \infty$ we have a contradiction with (2.9) and so we have verified that $\lim_{t\to\infty} x(t) = 0$. ■

Combining Theorem 1 and Lemma 3 we get:

Theorem 2. *Assume that (2.1) and (2.9) holds. If the equation (E_4) is oscillatory then every solution of Eq.(E) is oscillatory or tends to zero as $t \to \infty$.*

For a special case of Eq.(E), we have:

Corollary 1. *Assume that (2.9) holds and*

$$\int_{t_0}^\infty \left(\frac{1}{a(u)} \int_{t_0}^u q(s)\tau^\beta(s) \left(\int_{\tau(s)}^\infty a^{-1/\gamma}(v)dv\right)^\beta ds\right)^{1/\gamma} du = \infty. \tag{2.10}$$

Assume that β is a quotient of odd positive integers. If the delay equation

$$y'(t) + q(t) \left[\int_{t_0}^{\tau(t)} (\tau(t) - s)a^{-1/\gamma}(s)ds\right]^\beta y^{\beta/\gamma}[\tau(t)] = 0 \tag{E_5}$$

is oscillatory then every solution of the equation

$$\left[a(t)\left[x''(t)\right]^\gamma\right]' + q(t)x^\beta\left[\tau(t)\right] = 0 \tag{E_6}$$

is oscillatory or tends to zero as $t \to \infty$.

In Theorems 1 and 2 and Corollary 1 we have established new comparison principles that enable to deduce properties of the third order nonlinear differential equation (E) from oscillation of the first order nonlinear delay equation (E_4). Consequently, taking into account oscillation criteria for (E_4), we immediately obtain results for (E).

Corollary 2. *Assume (2.9) and*

$$\int_{t_0}^\infty \left(\frac{1}{a(u)} \int_{t_0}^u q(s)\tau^\gamma(s) \left(\int_{\tau(s)}^\infty a^{-1/\gamma}(v)dv\right)^\gamma ds\right)^{1/\gamma} du = \infty \tag{2.11}$$

hold. If

$$\liminf_{t\to\infty} \int_{\tau(t)}^t q(u) \left[\int_{t_0}^{\tau(u)} (\tau(u) - s)a^{-1/\gamma}(s)ds\right]^\gamma du > \frac{1}{\mathrm{e}}, \tag{2.12}$$

then every solution of the equation

$$(E_7) \qquad \left[a(t)\left[x''(t)\right]^{\gamma}\right]' + q(t)x^{\gamma}\left[\tau(t)\right] = 0$$

is oscillatory or tends to zero as $t \to \infty$.

Proof. Condition (2.12) (see Theorem 2.1.1 in [16]) guarantees oscillation of (E_6) with $\beta = \gamma$. ∎

Now we eliminate the strongly decreasing solutions of (E) to get an oscillation result. We relax condition (2.9) and employ another one. Our method is new and complements the one presented in [9].

Theorem 3. *Let (2.1) hold and* $\tau'(t) > 0$. *Assume that there exist a function* $\xi(t) \in C^1([t_0,\infty))$ *such that*

$$(2.13) \qquad \xi'(t) \geq 0, \quad \xi(t) > t, \quad \text{and} \quad \eta(t) = \tau(\xi(\xi(t))) < t.$$

If both first order delay equations (E_4) *and*

$$(E_8) \qquad z'(t) + \left[\int_t^{\xi(t)} \frac{1}{a^{1/\gamma}(s_2)}\left(\int_{s_2}^{\xi(s_2)} q(s_1)ds_1\right)^{1/\gamma} ds_2\right] f^{1/\gamma}\left(z[\eta(t)]\right) = 0$$

are oscillatory, then Eq. (E) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq.(E). We may assume that $x(t) > 0$. From Theorem 1, we see that $x(t)$ is strongly decreasing (i.e., $x'(t) < 0$). Integration of (E) from t to $\xi(t)$ yields

$$a(t)\left(x''(t)\right)^{\gamma} \geq \int_t^{\xi(t)} q(s_1)f(x(\tau(s_1)))ds_1 \geq f(x[\tau(\xi(t))])\int_t^{\xi(t)} q(s_1)ds_1.$$

Then

$$x''(t) \geq \frac{f^{1/\gamma}(x[\tau(\xi(t))])}{a^{1/\gamma}(t)}\left(\int_t^{\xi(t)} q(s_1)ds_1\right)^{1/\gamma}.$$

Integrating from t to $\xi(t)$ once more, we get

$$\begin{aligned} -x'(t) &\geq \int_t^{\xi(t)} \frac{f^{1/\gamma}(x[\tau(\xi(s_2))])}{a^{1/\gamma}(s_2)}\left(\int_{s_2}^{\xi(s_2)} q(s_1)ds_1\right)^{1/\gamma} ds_2 \\ &\geq f^{1/\gamma}(x[\eta(t)])\int_t^{\xi(t)} \frac{1}{a^{1/\gamma}(s_2)}\left(\int_{s_2}^{\xi(s_2)} q(s_1)ds_1\right)^{1/\gamma} ds_2. \end{aligned}$$

Finally, integrating from t to ∞, one gets

$$(2.14) \quad x(t) \geq \int_t^{\infty} f^{1/\gamma}(x[\eta(s_3)])\int_{s_3}^{\xi(s_3)} \frac{1}{a^{1/\gamma}(s_2)}\left(\int_{s_2}^{\xi(s_2)} q(s_1)ds_1\right)^{1/\gamma} ds_2 ds_3.$$

Let us denote the right hand side of (2.14) by $z(t)$. Then $z(t) > 0$ and one can easily verify that $z(t)$ is a solution of the differential inequality

$$z'(t) + \left[\int_t^{\xi(t)} \frac{1}{a^{1/\gamma}(s_2)}\left(\int_{s_2}^{\xi(s_2)} q(s_1)ds_1\right)^{1/\gamma} ds_2\right] f^{1/\gamma}\left(z[\eta(t)]\right) \leq 0$$

Then Theorem 1 in [19] shows that the corresponding differential equation (E_8) has also a positive solution. This contradiction finishes the proof. ∎

For the special case of Eq. (E) with $f(u) = u^\beta$, we immediately have:

Corollary 3. *Let (2.10) hold and $\tau'(t) > 0$. Let β be a quotient of odd positive integers. Assume that there exist a function $\xi(t) \in C^1([t_0, \infty))$ such that (2.13) holds. If both Eq.(E_5) and*

$$(E_9) \qquad z'(t) + \left[\int_t^{\xi(t)} \frac{1}{a^{1/\gamma}(s_2)} \left(\int_{s_2}^{\xi(s_2)} q(s_1) ds_1 \right)^{1/\gamma} ds_2 \right] z^{\beta/\gamma}[\eta(t)] = 0$$

are oscillatory, then equation (E_6) is oscillatory.

When choosing $\xi(t)$ we are very particular about two conditions $\xi(t) > t$ and $\tau(\xi(\xi(t))) < t$ hold. Unfortunately there is no general rule how to choose function $\xi(t)$ to obtain the best result for oscillation of (E_8). We suggest for function $\xi(t)$ *"to be close to"* the inverse function of $\tau(t)$. In the next example the reader can see the details.

Example 1. *Let us consider third order differential equation*

$$(E_{10}) \qquad [t^2 x''(t)]' + bx(\lambda t) = 0, \qquad b > 0, \quad \lambda \in (0,1), \quad t \geq 1.$$

It is easy to verify that (2.10) holds for (E_{10}). Now (E_5) reduces to

$$y'(t) + b(\lambda t - \ln \lambda t - 1)\, y(\lambda t) = 0$$

and the oscillation criterion (2.12) takes the form

$$\lim_{t\to\infty} b \left[t^2 \frac{\lambda - \lambda^3}{2} + t \ln t(\lambda - 1) + t \ln \lambda(2\lambda - 1) \right] > \frac{1}{\mathrm{e}},$$

which is evidently fulfilled. Choosing $\xi(t) = \alpha t$ with $1 < \alpha < \frac{1}{\sqrt{\lambda}}$ equation (E_9) takes the form

$$z'(t) + b(\alpha - 1) \ln \alpha\, z[\lambda \alpha^2 t] = 0$$

and the oscillation criterion (2.12) reduces to

$$\lim_{t\to\infty} b(\alpha - 1) \ln \alpha (1 - \lambda \alpha^2)\, t > \frac{1}{\mathrm{e}},$$

which is satisfied. All conditions of Corollary 3 are satisfied and hence Eq. (E_{10}) is oscillatory.

Remark 2. *In the proof of Lemma 2 we have recognized that condition(2.1) eliminates case (ii) of Lemma 2.1. On the other hand, Theorems 1 and 3 implies that oscillation of both equations (E_4) and (E_8) eliminates case (ii) of Lemma 2.1. Therefore if (2.1) is not satisfied, we have the following result:*

Theorem 4. *Let $\tau'(t) > 0$. Assume that there exist a function $\xi(t) \in C^1([t_0, \infty))$ such that (2.13) holds If both first order delay equations (E_4) and (E_8) are oscillatory, then every positive solution $x(t)$ of Eq. (E) satisfies case (ii) of Lemma 2.1.*

Example 2. *Let us consider third order delay differential equation*

$$(E_{11}) \qquad \left[t^5(x''(t))^3\right]' + \frac{b}{t^2}\,x\,(\lambda t) = 0, \qquad b > 0, \quad \lambda \in (0,1), \quad t \geq 1.$$

It is easy to verify that (2.1) fails for Eq.(E_{11}). On the other hand, the corresponding equation (E_4), namely,

$$y'(t) + \frac{b}{t^2}\left(\frac{3}{2}\,\lambda t - \frac{9}{2}(\lambda t)^{1/3} + 3\right)^3 z(\lambda t) = 0$$

is oscillatory for every b. Moreover, setting $\xi(t) = \alpha t$, with $1 < \alpha < \lambda^{-1/2}$, we see that the corresponding equation (E_8), namely,

$$z'(t) + \frac{b^{1/3}}{t}\left(\frac{\alpha - 1}{\alpha}\right)^{4/3} y(\alpha^2\lambda t) = 0$$

is oscillatory if

$$b^{1/3}\left(\frac{\alpha-1}{\alpha}\right)^{4/3} \ln\frac{1}{\alpha^2\lambda} > \frac{1}{\mathrm{e}},$$

which for $\alpha = (\sqrt{\lambda}+1)/(2\sqrt{\lambda})$ reduces to

$$(2.15) \qquad b^{1/3}\left(\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}\right)^{4/3} \ln\frac{4}{\lambda + 2\sqrt{\lambda}+1} > \frac{1}{\mathrm{e}}.$$

Therefore Theorem 4 ensures that every positive solution $x(t)$ of Eq.(E_{11}) satisfies case (ii) of Lemma 2.1 provided that (2.15) holds. We note that for $b = 1/128\lambda^{3/2}$ one such solution of Eq.(E_{11}) is $x(t) = t^{1/2}$.

In this paper we provide a full classification of nonoscillatory solutions of (E). Our partial results guarantee described asymptotic behavior of all nonoscillatory solutions (boundedness, convergence to zero and nonexistence). Our criteria improve and properly complement known results even for simple cases of (E). Our conclusions are precedented by illustrative examples that confirm upgrading of known oscillation criteria. If we apply known/new criteria for both nonlinear first order equations (E_4) and (E_8) to be oscillatory, we obtain more general criteria for asymptotic properties of nonlinear third order equation (E).

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