Boundary value problems for systems of second-order functional differential equations

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Abstract. Systems of second-order functional differential equations $(x'(t)+L(x)(t))' = F(x)(t)$ together with nonlinear functional boundary conditions are considered. Here $L : C^1([0,T];\mathbb{R}^n) \to C^0([0,T];\mathbb{R}^n)$ and $F : C^1([0,T];\mathbb{R}^n) \to L_1([0,T];\mathbb{R}^n)$ are continuous operators. Existence results are proved by the Leray-Schauder degree and the Borsuk antipodal theorem for α-condensing operators. Examples demonstrate the optimality of conditions.

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1 Introduction, notation

Let $J = [0,T]$ be a compact interval, $n \in \mathbb{N}$. For $a \in \mathbb{R}^n$, $a = (a_1,\dots,a_n)$, we set $|a| = \max\{|a_1|,\dots,|a_n|\}$. For any $x : J \to \mathbb{R}^n$ $(n \ge 2)$ we write $x(t) = (x_1(t),\dots,x_n(t))$ and $\int_a^b x(t)\,dt = (\int_a^b x_1(t)\,dt,\dots,\int_a^b x_n(t)\,dt)$ for $0 \le a < b \le T$.

From now on, $C^0(J;\mathbb{R})$, $C^0(J;\mathbb{R}^n)$, $C^1(J;\mathbb{R}^n)$, $C^0(J;\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n$, $L_1(J;\mathbb{R})$ and $L_1(J;\mathbb{R}^n)$ denote the Banach spaces with the norms $\|x\|_0 = \max\{|x(t)| : t \in J\}$, $\|x\| = \max\{\|x_1\|_0,\dots,\|x_n\|_0\}$, $\|x\|_1 = \max\{\|x\|,\|x'\|\}$, $\|(x,a,b)\|_* = \|x\| + |a| + |b|$, $\|x\|_{L_1}^0 = \int_0^T |x(t)|\,dt$ and $\|x\|_{L_1} = \max\{\|x_1\|_{L_1}^0,\dots,\|x_n\|_{L_1}^0\}$, respectively. $\mathcal{K}(J\times[0,\infty);[0,\infty))$ denotes the set of all functions $\omega : J\times[0,\infty) \to [0,\infty)$ which are integrable on J in the first variable, nondecreasing on $[0,\infty)$ in the second variable and $\lim\limits_{\varrho\to\infty} \dfrac{1}{\varrho}\displaystyle\int_0^T \omega(t,\varrho)\,dt = 0$.

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Denote by $\mathcal{A}_0$ the set of all functionals $\alpha : C^0(J;\mathbb{R}) \to \mathbb{R}$ which are

a) continuous, Im(α)=$\mathbb{R}$, and

b) increasing (i.e. $x, y \in C^0(J;\mathbb{R})$, $x(t) < y(t)$ for $t \in J \Rightarrow \alpha(x) < \alpha(y)$).

Here Im(α) stands for the range of α. If k is an increasing homeomorphism on $\mathbb{R}$ and $0 \le a < b \le T$, then the following functionals

$$\max\{k(x(t)) : \, a \le t \le b\}, \quad \min\{k(x(t)) : \, a \le t \le b\}, \quad \int_a^b k(x(t))\, dt$$

belong to the set $\mathcal{A}_0$. Next examples of functionals belonging to the set $\mathcal{A}_0$ can be found for example in [2], [3].

Let $\mathcal{A} = \underbrace{\mathcal{A}_0 \times \ldots \times \mathcal{A}_0}_{n}$. For each $x \in C^0(J;\mathbb{R}^n)$, $x(t) = (x_1(t), \ldots, x_n(t))$ and $\varphi \in \mathcal{A}$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, we define $\varphi(x)$ by

$$\varphi(x) = (\varphi_1(x_1), \ldots, \varphi_n(x_n)). \tag{1}$$

Let $L : C^1(J;\mathbb{R}^n) \to C^0(J;\mathbb{R}^n)$, $F : C^1(J;\mathbb{R}^n) \to L_1(J;\mathbb{R}^n)$ be continuous operators, $L = (L_1, \ldots, L_n)$, $F = (F_1, \ldots, F_n)$. Consider the functional boundary value problem (BVP for short)

$$(x'(t) + L(x)(t))' = F(x)(t), \tag{2}$$

$$\varphi(x) = A, \quad \psi(x') = B. \tag{3}$$

Here $\varphi, \psi \in \mathcal{A}$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, $\psi = (\psi_1, \ldots, \psi_n)$ and $A, B \in \mathbb{R}^n$, $A = (A_1, \ldots, A_n)$, $B = (B_1, \ldots, B_n)$.

A function $x \in C^1(J;\mathbb{R}^n)$ is said to be *a solution of BVP* (2), (3) if the vector function $x'(t)+L(x)(t)$ is absolutely continuous on J, (2) is satisfied for a.e. $t \in J$ and x satisfies the boundary conditions (3).

The aim of this paper is to state sufficient conditions for the existence results of BVP (2), (3). These results are proved by the Leray-Schauder degree and the Borsuk theorem for α-condensing operators (see e.g. [1]). In our case α-condensing operators have the form $U + V$, where U is a compact operator and V is a strict contraction. We recall that this paper is a continuation of the previous paper by the author [3], where the scalar BVP

$$(x'(t) + L_1(x')(t))' = F_1(x)(t),$$

$$\varphi_1(x) = 0, \quad \psi_1(x') = 0$$

was considered. Here $L_1 : C^0(J;\mathbb{R}) \to C^0(J;\mathbb{R})$, $F_1 : C^1(J;\mathbb{R}) \to L_1(J;\mathbb{R})$ are continuous operators and $\varphi_1, \psi_1 \in \mathcal{A}_0$ satisfy $\varphi_1(0) = 0 = \psi_1(0)$.

We assume throughout the paper that the continuous operators L and F in (2) satisfy the following assumptions:

(H_1) There exists $k \in [0, \frac{1}{2\mu})$, $\mu = \max\{1, T\}$, such that

$$\|L(x) - L(y)\| \le k\|x - y\|_1 \quad \text{for } x, y \in C^1(J; \mathbb{R}^n),$$

(H_2) There exists $\omega \in \mathcal{K}(J \times [0, \infty); [0, \infty))$ such that

$$|F(x)(t)| \le \omega(t, \|x\|_1)$$

for a.e. $t \in J$ and each $x \in C^1(J; \mathbb{R}^n)$.

Remark 1. If assumption (H_1) is satisfied then

$$\|L(x)\| \le k\|x\|_1 + \|L(0)\| \quad \text{for } x \in C^1(J; \mathbb{R}^n).$$

Example 1. Let $w \in C^0(J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, $\chi, \phi \in C^0(J; J)$ and

$$|w(t, r_1, u_1, v_1, z_1) - w(t, r_2, u_2, v_2, z_2)|$$

$$\le k \max\{|r_1 - r_2|, |u_1 - u_2|, |v_1 - v_2|, |z_1 - z_2|\}$$

for $t \in J$ and $r_i, u_i, v_i, z_i \in \mathbb{R}^n$ $(i = 1, 2)$, where $k \in [0, \frac{1}{2\mu})$. Then the Nemytskii operator $L : C^1(J; \mathbb{R}^n) \to C^0(J; \mathbb{R}^n)$,

$$L(x)(t) = w(t, x(t), x(\chi(t)), x'(t), x'(\phi(t)))$$

satisfies assumption (H_1).

Example 2. Let $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the Carathéodory conditions on $J \times \mathbb{R}^n \times \mathbb{R}^n$ and

$$|f(t, u, v)| \le \omega(t, \max\{|u|, |v|\})$$

for a.e. $t \in J$ and each $u, v \in \mathbb{R}^n$, where $\omega \in \mathcal{K}(J \times [0, \infty); [0, \infty))$. Then the Nemytskii operator $F : C^1(J; \mathbb{R}^n) \to L_1(J; \mathbb{R}^n)$,

$$F(x)(t) = f(t, x(t), x'(t))$$

satisfies assumption (H_2).

The existence results for BVP (2), (3) are given in Sec. 3. Here the optimality of our assumptions (H_1) and (H_2) is studied as well. We shall show that $k \in [0, \frac{1}{2})$ can not be replaced be the weaker assumption $k \in [0, \frac{1}{2}]$ in (H_1) provided $T \le 1$ (see Example 4), and if $k > \frac{1}{2\mu}$ in (H_2) then there exists unsolvable BVP of the type (2), (3) provided $T > 1$ (see Example 5). Example 6 shows that the condition $\lim\limits_{\varrho \to \infty} \dfrac{1}{\varrho} \displaystyle\int_0^T \omega(t, \varrho)\, dt = 0$ which appears for ω in (H_2) can not be replaced by $\limsup\limits_{\varrho \to \infty} \dfrac{1}{\varrho} \displaystyle\int_0^T \omega(t, \varrho)\, dt < \infty$.

2 Auxiliary results

For each $\alpha \in \mathcal{A}_0$, we define the function $p_\alpha \in C^0(\mathbb{R};\mathbb{R})$ by

$$p_\alpha(c) = \alpha(c).\ ^{1}$$

Then p_α is increasing on $\mathbb{R}$ and maps $\mathbb{R}$ onto itself. Hence there exists the inverse function $p_\alpha^{-1} : \mathbb{R} \to \mathbb{R}$ to p_α.

From now on, $m_{\gamma C} \in \mathbb{R}$ is defined for each $\gamma \in \mathcal{A}$, $\gamma = (\gamma_1, \dots, \gamma_n)$ and $C \in \mathbb{R}^n$, $C = (C_1, \dots, C_n)$, by

$$m_{\gamma C} = \max\{|p_{\gamma_i}^{-1}(C_i)| : i = 1, \dots, n\}. \tag{4}$$

Lemma 1. *Let $\gamma \in \mathcal{A}$, $A \in \mathbb{R}^n$ and let $\gamma(x) = A$ for some $x \in C^0(J;\mathbb{R}^n)$. Then there exists $\xi \in \mathbb{R}^n$ such that*

$$(x_1(\xi_1), \dots, x_n(\xi_n)) = (p_{\gamma_1}^{-1}(A_1), \dots, p_{\gamma_n}^{-1}(A_n)).$$

Proof. Fix $j \in \{1, \dots, n\}$. If $x_j(t) > p_{\gamma_j}^{-1}(A_j)$ (resp. $x_j(t) < p_{\gamma_j}^{-1}(A_j)$) on J, then $\gamma_j(x_j) > \gamma_j(p_{\gamma_j}^{-1}(A_j)) = A_j$ (resp. $\gamma_j(x_j) < \gamma_j(p_{\gamma_j}^{-1}(A_j)) = A_j$), contrary to $\gamma_j(A_j) = A_j$. Hence there exists $\xi_j \in \mathbb{R}$ such that $x_j(\xi_j) = p_{\gamma_j}^{-1}(A_j)$. □

Define the operators

$$\Pi : C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \to C^1(J;\mathbb{R}^n), \quad P : C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \to C^0(J;\mathbb{R}^n),$$

$$Q : C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \to L_1(J;\mathbb{R}^n)$$

by the formulas

$$\Pi(x,a)(t) = \int_0^t x(s)\, ds + a, \tag{5}$$

$$P(x,a)(t) = L(\Pi(x,a))(t) \tag{6}$$

and

$$Q(x,a)(t) = F(\Pi(x,a))(t). \tag{7}$$

Here L and F are the operators in (2).

Consider BVP

$$x(t) = a + \lambda\Big(- P(x,b)(t) + \int_0^t Q(x,b)(s)\, ds\Big), \qquad (8)_{(\lambda,a,b)}$$

$$\varphi\Big(\int_0^t x(s)\, ds + b\Big) = A, \qquad (9)_b$$

$$\psi(x) = B \tag{10}$$

[1]We identify the set of all constant scalar functions on J with R.

depending on the parameters λ, a, b, $(\lambda, a, b) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$. Here $\varphi, \psi \in \mathcal{A}$ and $A, B \in \mathbb{R}^n$.

We say that $x \in C^0(J; \mathbb{R}^n)$ is *a solution of BVP* $(8)_{(\lambda,a,b)}$, $(9)_b$, (10) for some $(\lambda, a, b) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ if $(8)_{(\lambda,a,b)}$ is satisfied for $t \in J$ and $x(t)$ satisfies the boundary conditions $(9)_b$, (10).

Lemma 2. (A priori bounds). *Let assumptions* (H_1) *and* (H_2) *be satisfied. Let* $x(t)$ *be a solution of BVP* $(8)_{(\lambda,a,b)}$, $(9)_b$, (10) *for some* $(\lambda, a, b) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$. *Then*

$$\|x\| < S, \quad |a| < (1 - k\mu)S, \quad |b| < m_{\varphi A} + ST,$$

where S *is a positive constant such that*

$$\frac{m_{\psi B} + 2k m_{\varphi A} + 2\|L(0)\|}{u} + \frac{1}{u}\int_0^T \omega(t, m_{\varphi A} + \mu u)\, dt < 1 - 2k\mu \tag{11}$$

for $u \in [S, \infty)$ *and* $m_{\varphi A}, m_{\psi B}$ *are given by* (4).

Proof. By Lemma 1 (cf. $(9)_b$ and (10)), there exist $\xi, \nu \in \mathbb{R}^n$ such that

$$\int_0^{\xi_i} x_i(s)\, ds + b_i = p_{\varphi_i}^{-1}(A_i), \quad x_i(\nu_i) = p_{\psi_i}^{-1}(B_i), \quad i = 1, \ldots, n. \tag{12}$$

Then (cf. $(8)_{(\lambda,a,b)}$)

$$p_{\psi_i}^{-1}(B_i) = a_i + \lambda\Big(-P_i(x,b)(\nu_i) + \int_0^{\nu_i} Q_i(x,b)(s)\, ds\Big), \tag{13}$$

and consequently (for $i = 1, \ldots, n$)

$$x_i(t) = p_{\psi_i}^{-1}(B_i) + \lambda\Big(P_i(x,b)(\nu_i) - P_i(x,b)(t) + \int_{\nu_i}^t Q_i(x,b)(s)\, ds\Big).$$

Hence (cf. (4), (H_1), (H_2) and Remark 1)

$$|x_i(t)| \le m_{\psi B} + 2k\|\Pi(x,b)\|_1 + 2\|L(0)\| + \int_0^T \omega(t, \|\Pi(x,b)\|_1)\, dt \tag{14}$$

for $t \in J$ and $i = 1, \ldots, n$. Since (cf. (5) and (12))

$$\begin{aligned} \|\Pi(x,b)\| &= \Big\|\Big(\int_0^t x_1(s)\, ds + b_1, \ldots, \int_0^t x_n(s)\, ds + b_n\Big)\Big\| \\ &= \Big\|\Big(\int_{\xi_1}^t x_1(s)\, ds + p_{\varphi_1}^{-1}(A_1), \ldots, \int_{\xi_n}^t x_n(s)\, ds + p_{\varphi_n}^{-1}(A_n)\Big)\Big\| \qquad (15) \\ &= \max\Big\{\Big\|\int_{\xi_i}^t x_i(s)\, ds + p_{\varphi_i}^{-1}(A_i)\Big\|_0 : i = 1, \ldots, n\Big\} \le m_{\varphi A} + T\|x\|, \end{aligned}$$

we have

$$\|\Pi(x,b)\|_1 \le \max\{m_{\varphi A} + T\|x\|,\, \|x\|\} \le m_{\varphi A} + \mu\|x\|. \tag{16}$$

Then (cf. (14)-(16))

$$\|x\| \le m_{\psi B} + 2k(m_{\varphi A} + \mu\|x\|) + 2\|L(0)\| + \int_0^T \omega(t, m_{\varphi A} + \mu\|x\|)\, dt. \tag{17}$$

Set

$$q(u) = \frac{m_{\psi B} + 2k m_{\varphi A} + 2\|L(0)\|}{u} + \frac{1}{u}\int_0^T \omega(t, m_{\varphi A} + \mu u)\, dt$$

for $u \in (0,\infty)$. Then $\lim_{u\to\infty} q(u) = 0$. Whence there exists $S > 0$ such that $q(u) < 1 - 2k\mu$ for $u \ge S$, and so (cf. (17))

$$\|x\| < S.$$

Therefore (cf. (12), (13) and (15))

$$|b_i| = \Big|p_{\varphi_i}^{-1}(A_i) - \int_0^{\xi_i} x_i(s)\, ds\Big| < m_{\varphi A} + ST,$$

$$\begin{aligned}
|a_i| &= \Big|p_{\psi_i}^{-1}(B_i) + \lambda\Big(P_i(x,b)(\nu_i) - \int_0^{\nu_i} Q_i(x,b)(s)\, ds\Big)\Big| \\
&\le m_{\psi B} + k\|\Pi(x,b)\|_1 + \|L(0)\| + \int_0^T \omega(t, \|\Pi(x,b)\|_1)\, dt \\
&\le m_{\psi B} + k(m_{\varphi A} + \mu S) + \|L(0)\| + \int_0^T \omega(t, m_{\varphi A} + \mu S)\, dt \\
&< k\mu S + (1-2k\mu)S = (1-k\mu)S
\end{aligned}$$

for $i = 1,\ldots,n$, and consequently

$$|a| < (1-k\mu)S, \quad |b| < m_{\varphi A} + ST. \qquad \square$$

Lemma 3. *Let assumption* (H_2) *be satisfied,* $\varphi,\psi \in \mathcal{A}$, $A, B \in \mathbb{R}^n$ *and* $S > 0$ *be a constant such that* (11) *is satisfied for* $u \ge S$. *Set*

$$\begin{aligned}
\Omega = \Big\{(x,a,b) :\ & (x,a,b) \in C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n, \\
& \|x\| < S,\, |a| < S,\, |b| < m_{\varphi A} + ST\Big\}
\end{aligned} \tag{18}$$

and let $\Gamma : \bar\Omega \to C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ *be given by*

$$\Gamma(x,a,b) = \Big(a,\, a + \varphi\Big(\int_0^t x(s)\, ds + b\Big) - A,\, b + \psi(x) - B\Big). \tag{19}$$

Then

$$\mathrm{D}(I - \Gamma, \Omega, 0) \ne 0, \tag{20}$$

where "D" *denotes the Leray-Schauder degree and* I *is the identity operator on* $C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Proof. Let $U : [0,1] \times \bar{\Omega} \to C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} U(\lambda, x, a, b) &= \Big(a,\, a + \varphi\Big(\int_0^t x(s)\,ds + b\Big) - (1-\lambda)\varphi\Big(-\int_0^t x(s)\,ds - b\Big) - \lambda A, \\ &\qquad b + \psi(x) - (1-\lambda)\psi(-x) - \lambda B\Big). \end{aligned}$$

By the theory of homotopy and the Borsuk antipodal theorem, to prove (20) it is sufficient to show that

(j) $U(0,\cdot)$ is an odd operator,

(jj) U is a compact operator, and

(jjj) $U(\lambda, x, a, b) \neq (x, a, b)$ for $(\lambda, x, a, b) \in [0,1] \times \partial\Omega$.

Since

$$\begin{aligned} U(0, -x, -a, -b) &= \Big(-a, -a + \varphi\Big(-\int_0^t x(s)\,ds - b\Big) - \varphi\Big(\int_0^t x(s)\,ds + b\Big), \\ &\qquad -b + \psi(-x) - \psi(x)\Big) = -U(0, x, a, b) \end{aligned}$$

for $(x, a, b) \in \bar{\Omega}$, U is an odd operator.

The compactness of U follows from the properties of φ, ψ and applying the Bolzano-Weierstrass theorem.

Assume that $U(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0)$ for some $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega$, $a_0 = (a_{01}, \ldots, a_{0n})$, $b_0 = (b_{01}, \ldots, b_{0n})$. Then

$$x_0(t) = a_0, \quad t \in J, \tag{21}$$

$$\varphi(a_0 t + b_0) = (1-\lambda_0)\varphi(-a_0 t - b_0) + \lambda_0 A, \tag{22}$$

$$\psi(a_0) = (1-\lambda_0)\psi(-a_0) + \lambda_0 B, \tag{23}$$

and consequently (cf. (22) and (23))

$$\varphi_i(a_{0i} t + b_{0i}) = (1-\lambda_0)\varphi_i(-a_{0i} t - b_{0i}) + \lambda_0 A_i, \tag{24}$$

$$\psi_i(a_{0i}) = (1-\lambda_0)\psi_i(-a_{0i}) + \lambda_0 B_i \tag{25}$$

for $i = 1, \ldots, n$. Fix $i \in \{1, \ldots, n\}$. If $a_{0i} > 0$ then $\psi_i(-a_{0i}) < \psi_i(a_{0i})$, and so (cf. (25)) $\psi_i(a_{0i}) \leq (1-\lambda_0)\psi_i(a_{0i}) + \lambda_0 B_i$. Therefore

$$\lambda_0 \psi_i(a_{0i}) \leq \lambda_0 B_i. \tag{26}$$

For $\lambda_0 = 0$ we obtain (cf. (25)) $\psi_i(a_{0i}) = \psi_i(-a_{0i})$, a contradiction. Let $\lambda_0 \in (0, 1]$. Then (cf. (26)) $\psi_i(a_{0i}) \le B_i$ and

$$0 < a_{0i} \le p_{\psi_i}^{-1}(B_i) \le m_{\psi B}. \tag{27}$$

If $a_{0i} < 0$ then $\psi_i(a_{0i}) < \psi_i(-a_{0i})$ and (cf. (25)) $\psi_i(a_{0i}) \ge (1-\lambda_0)\psi_i(a_{0i}) + \lambda_0 B_i$. Hence

$$\lambda_0 \psi_i(a_{0i}) \ge \lambda_0 B_i. \tag{28}$$

For $\lambda_0 = 0$ we obtain (cf. (25)) $\psi_i(a_{0i}) = \psi_i(-a_{0i})$, which is impossible. Let $\lambda_0 \in (0, 1]$. Then (cf. (28))

$$0 > a_{0i} \ge p_{\psi_i}^{-1}(B_i) \ge -m_{\psi B}. \tag{29}$$

From (27) and (29) we deduce

$$|a_{0i}| \le m_{\psi B}. \tag{30}$$

Assume that $a_{0i}t + b_{0i} > 0$ for $t \in J$. Then $\varphi_i(-a_{0i}t - b_{0i}) < \varphi_i(a_{0i}t + b_{0i})$, and so (cf. (24)) $\lambda_0 \ne 0$ and $\varphi_i(a_{0i}t + b_{0i}) \le (1-\lambda_0)\varphi_i(a_{0i}t + b_{0i}) + \lambda_0 A_i$. Hence

$$\varphi_i(a_{0i}t + b_{0i}) \le A_i.$$

If $a_{0i}t + b_{0i} > p_{\varphi_i}^{-1}(A_i)$ for $t \in J$ then $A_i \ge \varphi_i(a_{0i}t + b_{0i}) > \varphi_i(p_{\varphi_i}^{-1}(A_i)) = A_i$, a contradiction. Thus there is $\xi_i \in J$ such that

$$0 < a_{0i}\xi_i + b_{0i} \le p_{\varphi_i}^{-1}(A_i) \le m_{\varphi A}. \tag{31}$$

Let $a_{0i}t + b_{0i} < 0$ for $t \in J$. Then $\varphi_i(a_{0i}t + b_{0i}) < \varphi_i(-a_{0i}t - b_{0i})$ and (24) implies that $\lambda_0 \ne 0$ and $\varphi_i(-a_{0i}t - b_{0i}) \le A_i$. If $-a_{0i}t - b_{0i} > p_{\varphi_i}^{-1}(A_i)$ for $t \in J$ then $A_i \ge \varphi_i(-a_{0i}t - b_{0i}) > \varphi_i(p_{\varphi_i}^{-1}(A_i)) = A_i$, a contradiction. Hence there exists $\nu_i \in J$ such that

$$0 < -a_{0i}\nu_i - b_{0i} \le p_{\varphi_i}^{-1}(A_i) \le m_{\varphi A}. \tag{32}$$

We have proved that there exists $\tau_i \in J$ such that (cf. (31) and (32))

$$|a_{0i}\tau_i + b_{0i}| \le m_{\varphi A},$$

and consequently (cf. (30))

$$|b_{0i}| \le |a_{0i}\tau_i + b_{0i}| + |a_{0i}\tau_i| \le m_{\varphi A} + T m_{\psi B}. \tag{33}$$

Since (cf. (11)) $m_{\psi B} < (1 - k\mu)S \le S$, it follows that (cf. (21), (30) and (33))

$$\|x_0\| < S, \quad |a| < S, \quad |b| < m_{\varphi A} + ST,$$

contrary to $(x_0, a_0, b_0) \in \partial\Omega$. □

3 Existence results, examples

The main result of this paper is given in the following theorem.

Theorem 1. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then for each* $\varphi, \psi \in \mathcal{A}$ *and* $A, B \in \mathbb{R}^n$, *BVP* (2), (3) *has a solution.*

Proof. Fix $\varphi, \psi \in \mathcal{A}$ and $A, B \in \mathbb{R}^n$. Let S be a positive constant such that (11) is satisfied for $u \geq S$ and $\Omega \subset C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ be defined by (18). Let $U, V : \bar{\Omega} \to C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$U(x,a,b) = \left(a + \int_0^t Q(x,b)(s)\,ds,\ a + \varphi\Big(\int_0^t x(s)\,ds + b\Big) - A,\ b + \psi(x) - B\right),$$

$$V(x,a,b) = \big(-P(x,b)(t), 0, 0\big)$$

and let $W,\ Z : [0,1] \times \bar{\Omega} \to C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$W(\lambda,x,a,b) = \left(a + \lambda \int_0^t Q(x,b)(s)\,ds,\ a + \varphi\Big(\int_0^t x(s)\,ds + b\Big) - A,\ b + \psi(x) - B\right),$$

$$Z(\lambda,x,a,b) = \lambda V(x,a,b).$$

Then $W(0,\cdot) + Z(0,\cdot) = \Gamma(\cdot)$ and $W(1,\cdot) + Z(1,\cdot) = U(\cdot) + V(\cdot)$, where Γ is defined by (19). By Lemma 3, $\mathrm{D}(I - W(0,\cdot) - Z(0,\cdot), \Omega, 0) \neq 0$, and consequently, by the theory of homotopy (see e.g. [1]), to show that

$$\mathrm{D}(I - U - V, \Omega, 0) \neq 0 \tag{34}$$

it suffices to prove:

(i) W is a compact operator,

(ii) there exists $m \in [0,1)$ such that

$$\|Z(\lambda,x,a,b) - Z(\lambda,y,c,d)\|_* \leq m\|(x,a,b) - (y,c,d)\|_*$$

for $\lambda \in [0,1]$ and $(x,a,b), (y,c,d) \in \bar{\Omega}$,

(iii) $W(\lambda,x,a,b) + Z(\lambda,x,a,b) \neq (x,a,b)$ for $(\lambda,x,a,b) \in [0,1] \times \partial\Omega$.

The continuity of W follows from that of Q, φ and ψ. We claim that $W([0,1] \times \bar{\Omega})$ is a relatively compact subset of the Banach space $C^0(J;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Indeed, let $\{(\lambda_j, x_j, a_j, b_j)\} \subset [0,1] \times \bar{\Omega}$, $x_j = (x_{j1}, \dots, x_{jn})$, $a_j = (a_{j1}, \dots, a_{jn})$, $b_j = (b_{j1}, \dots, b_{jn})$ $(j \in \mathbb{N})$. Then (cf. (7), (H_2) and (18))

$$\Big|a_{ji} + \lambda \int_0^t Q_i(x_j, b_j)(s)\,ds\Big| \le |a_{ji}| + \int_0^T \Big|Q_i(x_j, b_j)(s)\Big|\,ds$$

$$< S + \int_0^T \omega(t, \|\Pi(x_j, b_j)\|_1)\,dt \le S + \int_0^T \omega(t, \mu\|x_j\| + |b_j|)\,dt$$

$$\le S + \int_0^T \omega(t, m_{\varphi A} + S(\mu + T))\,dt,$$

$$\Big|\int_{t_1}^{t_2} Q_i(x_j, b_j)(s)\,ds\Big| \le \Big|\int_{t_1}^{t_2} \omega(t, m_{\varphi A} + S(\mu + T))\,dt\Big|,$$

$$\Big|a_{ji} + \varphi_i\Big(\int_0^t x_{ji}(s)\,ds + b_{ji}\Big) - A_i\Big|$$
$$< S + \max\{|p_{\varphi_i}(-m_{\varphi A} - 2ST)|,\, |p_{\varphi_i}(m_{\varphi A} + 2ST)|\} + |A|$$

and

$$|b_{ji} + \psi_i(x_{ji}) - B_i| < m_{\varphi A} + ST + \max\{|p_{\psi_i}(-S)|,\, |p_{\psi_i}(S)|\} + |B|$$

for $t, t_1, t_2 \in J$, $i = 1, \dots, n$ and $j \in \mathbb{N}$. Therefore there exists a convergent subsequence of $\{W(\lambda_j, x_j, a_j, b_j)\}$ by the Arzelà-Ascoli theorem and the Bolzano-Weierstrass theorem. Hence W is a compact operator.

Let $(\lambda, x, a, b), (\lambda, y, c, d) \in [0,1] \times \bar{\Omega}$. Then (cf. (H_1) and (6))

$$\|Z(\lambda, x, a, b) - Z(\lambda, y, c, d)\|_* \le \|P(x, b) - P(y, d)\| = \|L(\Pi(x, b)) - L(\Pi(y, d))\|$$

$$\le k\|\Pi(x, b) - \Pi(y, d)\|_1 = k\max\{\|\Pi(x, b) - \Pi(y, d)\|,\, \|x - y\|\}$$
$$\le k\max\{\|x - y\|T + |b - d|,\, \|x - y\|\}$$
$$\le k\mu(\|x - y\| + |b - d|) \le k\mu\|(x, a, b) - (y, c, d)\|_*.$$

Hence (ii) holds with $m = k\mu < \frac{1}{2}$.

Suppose (iii) was false. Then we could find $(\lambda_0, x_0, a_0, b_0) \in [0,1] \times \partial\Omega$ such that

$$W(\lambda_0, x_0, a_0, b_0) + Z(\lambda_0, x_0, a_0, b_0) = (x_0, a_0, b_0).$$

Then

$$x_0(t) = a_0 + \lambda_0\Big(-P(x_0, b_0)(t) + \int_0^t Q(x_0, b_0)(s)\,ds\Big) \quad \text{for } t \in J,$$

$$\varphi\Big(\int_0^t x_0(s)\,ds + b_0\Big) = A, \quad \psi(x_0) = B,$$

and consequently $x_0(t)$ is a solution of BVP $(8)_{(\lambda_0,a_0,b_0)}$, $(9)_{b_0}$, (10). By Lemma 2, $\|x_0\| < S$, $|a_0| < (1-k\mu)S \le S$ and $|b_0| < m_{\varphi A}+ST$, contrary to $(x_0,a_0,b_0) \in \partial\Omega$.

We have proved (34). Therefore there exists a fixed point of the operator $U+V$, say (u,a,b). It follows that

$$u(t) = a - P(u,b)(t) + \int_0^t Q(u,b)(s)\,ds \quad \text{for } t \in J, \tag{35}$$

$$\varphi\Big(\int_0^t u(s)\,ds + b\Big) = A, \quad \psi(u) = B. \tag{36}$$

Set $x(t) = \displaystyle\int_0^t u(s)\,ds + b$, $t \in J$. Then (cf. (5)-(7), (35) and (36))

$$x'(t) = a - L(x)(t) + \int_0^t F(x)(s)\,ds \quad \text{for } t \in J,$$

$$\varphi(x) = A, \quad \psi(x') = B,$$

and we see that $x(t)$ is a solution of BVP $(2), (3)$. $\square$

Example 3. Let $w_{ji} \in C^0(J;\mathbb{R})$, $\alpha_i, \beta_i, \gamma_i, \delta_i \in C^0(J;J)$ for $j = 1,2,\ldots,9$ and $i = 1,2$. Define $L_i : C^1(J;\mathbb{R}^2) \to C^0(J;\mathbb{R})$ $(i = 1,2)$ by

$$\begin{aligned} L_i(x)(t) &= w_{1i}(t)x_1(t) + w_{2i}(t)x_2(t) + w_{3i}(t)x_1(\alpha_i(t)) + w_{4i}(t)x_2(\beta_i(t)) \\ &+ w_{5i}(t)x_1'(t) + w_{6i}(t)x_2'(t) + w_{7i}(t)x_1'(\gamma_i(t)) + w_{8i}(t)x_2'(\delta_i(t)) + w_{9i}(t). \end{aligned}$$

Let $F_i : C^1(J;\mathbb{R}^2) \to L_1(J;\mathbb{R})$ $(i = 1,2)$ be continuous operators such that

$$|F_i(x)(t)| \le \widetilde{\omega}(t, \|x\|_1)$$

for a.e. $t \in J$ and each $x \in C^1(J;\mathbb{R}^2)$, where $\widetilde{\omega} \in \mathcal{K}(J \times [0,\infty);[0,\infty))$.

Consider BVP

$$\begin{aligned} (x_1'(t) + L_1(x)(t))' &= F_1(x)(t), \\ (x_2'(t) + L_2(x)(t))' &= F_2(x)(t), \end{aligned} \tag{37}$$

$$\varphi_1(x_1) = A_1, \quad \varphi_2(x_2) = A_2, \quad \psi_1(x_1') = B_1, \quad \psi_2(x_2') = B_2. \tag{38}$$

By Theorem 1, for each $\varphi_i, \psi_i \in \mathcal{A}_0$ and $A_i, B_i \in \mathbb{R}$ $(i = 1,2)$, BVP (37), (38) has a solution provided $\displaystyle\sum_{j=1}^{8} \|w_{ji}\|_0 < \frac{1}{2\mu}$ for $i = 1,2$.

Next Example 4 shows that for $T \le 1$ the condition $k \in [0, \frac{1}{2})$ in (H_1) is optimal and can not be replaced by $k \in [0, \frac{1}{2}]$. In the case of $T > 1$ we will show (see Example 5) that for each $k > \frac{1}{2T}$ in (H_1) there exists an unsolvable BVP of the type $(2), (3)$ satisfying (H_2).

Example 4. Let $T \le 1$. Consider BVP

$$\begin{aligned} &(x_1'(t) + \alpha(t)(x_1'(T) + x_2'(T)))' = 1,\\ &(x_2'(t) + \alpha(t)(x_1'(T) + x_2'(T)))' = 1, \end{aligned} \tag{39}$$

$$\begin{aligned} &\varphi_1(x_1) = A_1,\ \min\{x_1'(t) : t \in J\} = 0,\\ &\varphi_2(x_2) = A_2,\ \min\{x_2'(t) : t \in J\} = 0, \end{aligned} \tag{40}$$

where $\alpha \in C^0(J;\mathbb{R})$, $\|\alpha\|_0 = \frac{1}{4}$, $\alpha(0) = \frac{1}{4}$, $\alpha(T) = -\frac{1}{4}$, $\varphi_1, \varphi_2 \in \mathcal{A}_0$ and $A_1, A_2 \in \mathbb{R}$.

Let $L_i : C^1(J;\mathbb{R}^2) \to C^0(J;\mathbb{R})$, $L_i(x)(t) = \alpha(t)(x_1'(T) + x_2'(T))$ $(i = 1, 2)$. Then

$$\begin{aligned} \|L_i(x) - L_i(y)\|_0 &\le \|\alpha\|_0(|x_1'(T) - y_1'(T)| + |x_2'(T) - y_2'(T)|)\\ &\le \frac{1}{4}(\|x_1' - y_1'\|_0 + \|x_2' - y_2'\|_0) \le \frac{1}{2}\|x' - y'\| \le \frac{1}{2}\|x - y\|_1, \end{aligned}$$

and so $\|L(x) - L(y)\| \le \frac{1}{2}\|x - y\|_1$ for $x, y \in C^1(J;\mathbb{R}^2)$ where $L = (L_1, L_2)$. BVP (39), (40) satisfies (H_2) with $\omega(t, \varrho) = 1$ but in (H_1) we have $k = \frac{1}{2}$ $(= \frac{1}{2\mu})$.

Assume that $u(t) = (u_1(t), u_2(t))$ is a solution of BVP (39), (40). Then $u_1' = u_2'$. Indeed, since $(u_1'(t) - u_2'(t))' = 0$ for $t \in J$ there exists $c \in \mathbb{R}$ such that $u_1'(t) = u_2'(t) + c$ on J. From $\min\{u_1'(t) : t \in J\} = \min\{u_2'(t) : t \in J\} = 0$ we deduce that $u_1'(\nu) = 0$, $u_2'(\tau) = 0$ for some $\nu, \tau \in J$, and so $0 = u_1'(\nu) = u_2'(\nu) + c \ge c$. If $c < 0$ then $0 \le u_1'(\tau) = c$, a contradiction. Hence $c = 0$ and then

$$(u_1'(t) + 2\alpha(t)u_1'(T))' = 1 \quad \text{for } t \in J.$$

Using the equality $u_1'(\nu) = 0$ we have

$$u_1'(t) = 2(\alpha(\nu) - \alpha(t))u_1'(T) + t - \nu \quad \text{for } t \in J. \tag{41}$$

If $\nu = 0$ then (cf. (41) with $t = T$) $u_1'(T) = u_1'(T) + T$, which is impossible. Assume $\nu \in (0, T]$. Then (cf. (41) with $t = 0$)

$$u_1'(0) = 2\Big(\alpha(\nu) - \frac{1}{4}\Big)u_1'(T) - \nu \le -\nu$$

contrary to $u_1'(t) \ge 0$ for $t \in J$. It follows that BVP (39), (40) is unsolvable.

Example 5. Let $T > 1$ and $\varepsilon > 1$. Consider BVP

$$\begin{aligned} &(x_1'(t) + \alpha(t)(x_1(T) + x_2(T)))' = 1,\\ &(x_2'(t) + \alpha(t)(x_1(T) + x_2(T)))' = 1, \end{aligned} \tag{42}$$

$$\min\{x_i(t) : t \in J\} = 0, \quad \min\{x_i'(t) : t \in J\} = 0, \quad i = 1, 2, \tag{43}$$

where $\alpha \in C^0(J;\mathbb{R})$, $\|\alpha\|_0 = \frac{\varepsilon}{4T}$, $\int_0^T \alpha(s)\,ds = -\frac{1}{4}$, $\alpha(0) = \frac{1}{4T}$, $\alpha(T) = -\frac{\varepsilon}{4T}$ and $\alpha(t) \le \frac{1}{4T}$ for $t \in J$.

Let $L_i : C^1(J;\mathbb{R}^2) \to C^0(J;\mathbb{R})$, $(L_i x)(t) = \alpha(t)(x_1(T) + x_2(T))$ $(i = 1,2)$. Then

$$\begin{aligned}\|L_i(x) - L_i(y)\|_0 &\le \|\alpha\|_0(|x_1(T) - y_1(T)| + |x_2(T) - y_2(T)|)\\ &\le \frac{\varepsilon}{4T}(\|x_1 - y_1\|_0 + \|x_2 - y_2\|_0) \le \frac{\varepsilon}{2T}\|x - y\| \le \frac{\varepsilon}{2T}\|x - y\|_1,\end{aligned}$$

and so $\|Lx - Ly\| \le \frac{\varepsilon}{2T}\|x - y\|_1$ for $x, y \in C^1(J;\mathbb{R}^2)$ where $L = (L_1, L_2)$. Hence BVP (42), (43) satisfies (H_2) with $\omega(t,\varrho) = 1$ but in (H_1) we have $k = \frac{\varepsilon}{2T}$ $(> \frac{1}{2\mu})$.

Assume that $u(t) = (u_1(t), u_2(t))$ is a solution of BVP (42), (43). Applying the same procedure as in Example 4, it is obvious that $u_1 = u_2$. Hence

$$(u_1'(t) + 2\alpha(t)u_1(T))' = 1 \quad \text{for } t \in J,$$

and since $\min\{u_1(t) : t \in J\} = 0$ and $\min\{u_1'(t) : t \in J\} = 0$ we have $u_1(t) \ge 0$, $u_1'(t) \ge 0$ on J and $u_1'(\nu) = 0$ for some $\nu \in J$. Therefore

$$u_1'(t) = 2(\alpha(\nu) - \alpha(t))u_1(T) + t - \nu \quad \text{for } t \in J. \tag{44}$$

Assume $\nu = 0$. Then

$$u_1'(t) = 2\Big(\frac{1}{4T} - \alpha(t)\Big)u_1(T) + t \ge t,$$

and so $u_1(t)$ is increasing on J and $\min\{u_1(t) : t \in J\} = 0$ implies $u_1(0) = 0$. Hence

$$u_1(t) = 2\Big(\frac{t}{4T} - \int_0^t \alpha(s)\,ds\Big)u_1(T) + \frac{t^2}{2} \quad \text{for } t \in J$$

and

$$u_1(T) = 2\Big(\frac{1}{4} - \int_0^T \alpha(s)\,ds\Big)u_1(T) + \frac{T^2}{2} = u_1(T) + \frac{T^2}{2},$$

which is impossible.

Let $\nu \in (0, T]$. Then (cf. (44))

$$u_1'(0) = 2\Big(\alpha(\nu) - \frac{1}{4T}\Big)u_1(T) - \nu \le -\nu,$$

contrary to $\min\{u_1'(t) : t \in J\} = 0$. We have proved that BVP (42), (43) is unsolvable.

The following example demonstrates that the condition $\lim\limits_{\varrho\to\infty} \int_0^T \omega(t,\varrho)\,dt = 0$ in (H_2) can not be replaced by $\limsup\limits_{\varrho\to\infty} \int_0^T \omega(t,\varrho)\,dt < \infty$.

Example 6. Consider BVP

$$x_1''(t) = 1 + \frac{2}{T^2}\|x\|_1, \quad x_2''(t) = 1 + \sqrt{\|x\|}, \tag{45}$$

$$\min\{x_1(t) : t \in J\} = 0, \ \varphi_1(x_2) = A, \ \min\{x_1'(t) : t \in J\} = 0, \ \varphi_2(x_2') = B, \tag{46}$$

where $\varphi_1, \varphi_2 \in \mathcal{A}_0$ and $A, B \in \mathbb{R}$. Assume that BVP (45), (46) is solvable and let $u(t) = (u_1(t), u_2(t))$ be its solution. Then $u_1''(t) \geq 1$ on J and the equality $\min\{u_1'(t) : t \in J\} = 0$ implies $u_1'(0) = 0$. Hence

$$u_1'(t) = \Big(1 + \frac{2}{T^2}\|u\|_1\Big)t \quad \text{for } t \in J, \tag{47}$$

and consequently $u_1(t)$ is increasing on J. From $\min\{u_1(t) : t \in J\} = 0$ we deduce that $u_1(0) = 0$ and then (cf. (47))

$$u_1(t) = \frac{1}{2}\Big(1 + \frac{2}{T^2}\|u\|_1\Big)t^2 \quad \text{for } t \in J.$$

Therefore

$$\|u_1\|_0 = \frac{T^2}{2} + \|u\|_1 \geq \frac{T^2}{2} + \|u_1\|_0.$$

which is impossible. Hence BVP (45), (46) is unsolvable.

We note that for (45) the inequality $|F(x)(t)| \leq \omega(t, \|x\|_1)$ in (H_2) is optimal with respect to the function ω for $\omega(t, \varrho) = 1 + \max\Big\{\frac{2}{T^2}\varrho, \sqrt{\varrho}\Big\}$ and we see that $\lim_{\varrho\to\infty} \frac{1}{\varrho}\int_0^T \omega(t, \varrho)\, dt = \frac{2}{T}$.

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