



## Asymptotic unboundedness of the norms of delayed matrix sine and cosine

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**Abstract.** In the paper, the asymptotic properties of recently defined special matrix functions called delayed matrix sine and delayed matrix cosine are studied. The asymptotic unboundedness of their norms is proved. To derive this result, a formula is used connecting them with what is called delayed matrix exponential with asymptotic properties determined by the main branch of the Lambert function.

**Keywords:** delay, delayed matrix functions, Lambert function, unboundedness.

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### 1 Introduction

Recently, a new formalization has been developed of the well-known method of steps [12, 13] for solving the initial-value problem for linear differential equations with constant coefficients and a single delay through special matrix functions called delayed matrix functions [6, 15, 20]. Using this method, representations have been found of solutions of homogeneous and non-homogeneous systems, and some stability and control problems were solved in [5, 16]. Also, a generalization has been developed to discrete systems and applied in [4, 21].


Let  $A$  be a nonzero  $n \times n$  constant matrix,  $\tau > 0$  and let  $\lfloor \cdot \rfloor$  be the floor function. The delayed matrix exponential, defined in [15], is a matrix polynomial on every interval  $[(k-1)\tau, k\tau)$ ,  $k = 0, 1, \dots$ , defined by

$$e_{\tau}^{At} = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} A^s \frac{(t - (s-1)\tau)^s}{s!}. \quad (1.1)$$

The delayed matrix exponential equals to zero matrix  $\Theta$  if  $t < -\tau$ , the unit matrix  $I$  on  $[-\tau, 0]$ , and is the fundamental matrix of a homogeneous linear system with a single delay

$$\dot{x}(t) = Ax(t - \tau). \quad (1.2)$$

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For the proof, we refer to [15]. In [15], too, a representation is derived of the solution of the Cauchy initial problem (1.2), (1.3), where

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0, \quad (1.3)$$

and  $\varphi: [-\tau, 0] \rightarrow \mathbb{R}^n$  is continuously differentiable.

Fundamental matrix (1.1) serves as a nice illustration of the general definition of a fundamental matrix to linear functional differential systems of delayed type [12,13]. For system (1.2), this definition reduces to (details are omitted)

$$X(t) = \begin{cases} A \int_{-\tau}^t X(u - \tau) du + I, & \text{for almost all } t \geq -\tau, \\ \Theta, & -2\tau \leq t < -\tau \end{cases} \quad (1.4)$$

and its step-by-step application gives

$$X(t) = e_{\tau}^{At}, \quad t \geq -2\tau.$$

With its usefulness, the delayed matrix exponential stimulated the search for other delayed matrix functions capable of simply expressing solutions of some linear differential systems with constant coefficients. In [6], solutions of a homogeneous second-order linear system with single delay

$$\ddot{x}(t) = -A^2 x(t - \tau). \quad (1.5)$$

are expressed through delayed matrix functions called the delayed matrix sine  $\text{Sin}_{\tau} At$  and delayed matrix cosine  $\text{Cos}_{\tau} At$  defined for  $t \in \mathbb{R}$  as

$$\text{Sin}_{\tau} At = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} (-1)^s A^{2s+1} \frac{(t - (s-1)\tau)^{2s+1}}{(2s+1)!} \quad (1.6)$$

and

$$\text{Cos}_{\tau} At = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} (-1)^s A^{2s} \frac{(t - (s-1)\tau)^{2s}}{(2s)!}. \quad (1.7)$$

Matrices (1.6) and (1.7) are related to the  $2n \times 2n$  fundamental matrix  $\mathcal{X}(t)$  of  $2n$ -dimensional system

$$\dot{y}(t) = \mathcal{A}y(t - \tau/2),$$

where

$$\mathcal{A} := \begin{pmatrix} \Theta & A \\ -A & \Theta \end{pmatrix}, \quad y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

equivalent with (1.5) through the substitution  $x(t) = y_1(t)$ . In much the same way as above, we can derive (for details we refer to [24])

$$\mathcal{X}(t) = e_{\tau/2}^{At} = \begin{pmatrix} \text{Cos}_{\tau} A(t - \tau/2) & \text{Sin}_{\tau} A(t - \tau) \\ -\text{Sin}_{\tau} A(t - \tau) & \text{Cos}_{\tau} A(t - \tau/2) \end{pmatrix}.$$

The paper aims to prove the asymptotic unboundedness of the norms of delayed matrix sine and delayed matrix cosine. This is done by utilizing relations between these functions and the delayed matrix exponential. The proof is based on the properties of the main branch of the Lambert function.

Therefore, we at first describe the necessary properties of the delayed exponential of a matrix and the Lambert function in Part 2. Then, in Part 3, the main result on the asymptotic properties of delayed matrix sine and delayed matrix cosine is proved.

## 2 Delayed matrix exponential and Lambert function

To explain clearly the relationship between delayed linear differential equations and Lambert function, we first consider the scalar case. Let  $n = 1$ ,  $A = (a)$ . Then, the fundamental matrix to the scalar case of the system (1.2), i.e., of

$$\dot{x}(t) = ax(t - \tau) \quad (2.1)$$

is defined by (1.1) as

$$e_{\tau}^{at} = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} a^s \frac{(t - (s-1)\tau)^s}{s!}.$$

and its values at nodes  $t = k\tau$ ,  $k = 0, 1, \dots$  are

$$\begin{aligned} e_{\tau}^{ak\tau} &= \sum_{s=0}^{k+1} a^s \frac{(k\tau - (s-1)\tau)^s}{s!} = \sum_{s=0}^k a^s \frac{(k+1-s)\tau^s}{s!} \\ &= 1 + a \frac{k\tau}{1!} + a^2 \frac{(k-1)^2 \tau^2}{2!} + \dots + a^{k-1} \frac{2^{k-1} \tau^{k-1}}{k!} + a^k \frac{\tau^k}{k!}. \end{aligned}$$

Assume that there exists a real solution  $c$  of a transcendental equation

$$c = ae^{-c\tau}, \quad (2.2)$$

i.e., that there exists a solution  $x(t) = e^{ct}$  of (2.1). Moreover, assume that, for a real root  $c$  of (2.2), we have

$$e_{\tau}^{ak\tau} \sim e^{ck\tau} = 1 + c \frac{k\tau}{1!} + c^2 \frac{k^2 \tau^2}{2!} + \dots + c^n \frac{k^n \tau^n}{n!} + \dots$$

when  $k \rightarrow \infty$ . Then,

$$\frac{e_{\tau}^{a(k+1)\tau}}{e_{\tau}^{ak\tau}} \sim \frac{e^{c(k+1)\tau}}{e^{ck\tau}} = e^{c\tau}, \quad k \rightarrow \infty. \quad (2.3)$$

Analyzing equation (2.3), provided it is valid, we can expect that, in a general case, the sequence of values of delayed matrix exponential at nodes  $t = k\tau$ ,  $k \rightarrow \infty$  is approximately represented by a “geometric progression” with the ordinary exponential of a constant matrix serving as a “quotient” factor.

It is reasonable to expect that such a constant matrix can be expressed by the principal branch of the Lambert function since (2.2) can be rewritten as

$$c\tau e^{c\tau} = a\tau \quad (2.4)$$

or as

$$c\tau = W(a\tau) \quad (2.5)$$

where  $W$  is the well-known Lambert  $W$ -function [3] (its properties given below are taken from this paper), defined as the inverse function to the function

$$z = f(w) = we^w, \quad (2.6)$$

i.e.,  $w = W(z)$ . If  $z = x + iy$  and  $w = u + iv$ , then (2.6) yields

$$x + iy = (u + iv)e^{u+iv} \quad (2.7)$$

and

$$x = e^u(u \cos v - v \sin v), \quad y = e^u(u \sin v + v \cos v). \quad (2.8)$$

The Lambert  $W$ -function is multi-valued (except for the point  $z = 0$ ). For real  $z = x > -1/e$  and  $w = u > -1$ , equation (2.6) defines a single-valued function  $w = W_0(x)$ . The function  $W_0(x)$  can be extended to the whole complex plane as a holomorphic function  $W_0(z)$  except for the values  $x < -1/e$  and  $y = 0$ . The extension  $w = W_0(z)$  is called the principal branch of the Lambert function.

The range of values of the principal branch  $W = W_0(z)$  is bounded by a parametric curve [3, p. 343]

$$\ell = \frac{-v}{\tan v} + iv, \quad -\pi < v < \pi \quad (2.9)$$

and equals to the domain

$$\mathcal{L} := \left\{ (u, v) \in \mathbb{C} : u \geq -1, |v| \leq |v^*| < \pi \text{ where } \frac{-v^*}{\tan v^*} = u \right\}.$$

For more details about the Lambert  $W$ -function, see [3].

The asymptotic properties of  $\exp(W_0(z))$  are, in principle, determined by the real part of  $W_0(z)$ . Let  $z = x + iy$  and

$$W_0(x + iy) = \operatorname{Re} W_0(x + iy) + i \operatorname{Im} W_0(x + iy) = u + iv.$$

The set of complex numbers  $z = x + iy$  such that  $\operatorname{Re} W_0(z) = u = 0$ , i.e., (see (2.7), (2.8)),

$$x + iy = iv \exp(iv)$$

is a closed curve  $\tilde{\ell}$ :

$$x = -v \sin v, \quad y = v \cos v \quad (2.10)$$

where, as it is clear from the definition of  $\mathcal{L}$ ,  $|v^*| = \pi/2$  for  $u = 0$  and  $|v| \leq \pi/2$ . We have (as a consequence of (2.8))

$$\operatorname{Re} W_0(z) < 0$$

if  $z$  lies within the interior of this curve and

$$\operatorname{Re} W_0(z) > 0 \quad (2.11)$$

for numbers  $z$  of its exterior. From (2.10) it follows easily that the exterior domain to  $\tilde{\ell}$  is specified by the inequality

$$|z| > -\arctan \left( \frac{\operatorname{Re} z}{|\operatorname{Im} z|} \right). \quad (2.12)$$

**Lemma 2.1.** For complex numbers  $z = x + iy$ ,  $z \neq 0$  with  $x \geq 0$ ,

$$|\operatorname{Im} W_0(z)| < \frac{\pi}{2}. \quad (2.13)$$

*Proof.* First, from (2.9) and definition of  $\mathcal{L}$ , we obtain inequality  $|v| = |\operatorname{Im} W_0(z)| < \pi$ , therefore,

$$v \sin v > 0. \quad (2.14)$$

Secondly, for  $w = u + iv = W_0(z)$ , the inequality  $u < 0$  implies  $|v| < \pi/2$  (see the definition of  $\mathcal{L}$ ) and, in this case, (2.13) holds. This guarantees that  $\operatorname{sign}(u \cos v) = \operatorname{sign} u$ . Applying (2.8) and the assumption that  $x$  is nonnegative, we obtain

$$e^u(u \cos v - v \sin v) = x \geq 0 \Rightarrow u \geq 0 \Rightarrow \arg W_0(z) \operatorname{Im} W_0(z) \geq 0.$$

This fact also implies

$$|\arg W_0(z) + \operatorname{Im} W_0(z)| = |\arg W_0(z)| + |\operatorname{Im} W_0(z)|. \quad (2.15)$$

Equation (2.6) yields

$$z = we^w = W_0(z)e^{W_0(z)}.$$

Therefore,

$$\arg z = \arg W_0(z) + \operatorname{Im} W_0(z)$$

and, due to relation, (2.15) we also have

$$|\arg z| = |\arg W_0(z)| + |\operatorname{Im} W_0(z)|. \quad (2.16)$$

For  $z \neq 0$  with non-negative real parts, we have  $\operatorname{Re} W_0(z) > 0$  by (2.11), from (2.14), we deduce  $\arg W_0(z) \neq 0$ ,  $\operatorname{Im} W_0(z) \neq 0$ , and, utilizing (2.16), we also have

$$\pi/2 \geq |\arg z| = |\arg W_0(z)| + |\operatorname{Im} W_0(z)| > |\operatorname{Im} W_0(z)|. \quad \square$$

Reverting to equation (2.3), we can expect that, in some cases, there exists a constant  $n \times n$  matrix  $C$  such that

$$\lim_{k \rightarrow \infty} e_\tau^{A(k+1)\tau} (e_\tau^{Ak\tau})^{-1} = e^{C\tau}, \quad (2.17)$$

provided that the matrices  $e_\tau^{Ak\tau}$  are nonsingular (this property will be assumed throughout the paper). One of such cases is analysed in [23] where the following is proved.

**Theorem 2.2.** *Let  $\lambda_j$ ,  $j = 1, \dots, n$  be the eigenvalues of the matrix  $A$  and let its Jordan canonical form be*

$$\operatorname{diag}(\lambda_1, \dots, \lambda_n) = D^{-1}AD \quad (2.18)$$

where  $D$  is a regular matrix. If

$$|\lambda_j| < 1/(\epsilon\tau),$$

$j = 1, \dots, n$ , then the sequence

$$e_\tau^{A(k+1)\tau} (e_\tau^{Ak\tau})^{-1}, \quad k \rightarrow \infty$$

converges, (2.17) holds and

$$e^{C\tau} = D \exp(\operatorname{diag}(W_0(\lambda_1\tau), \dots, W_0(\lambda_n\tau))) D^{-1}. \quad (2.19)$$

Note that from (2.19) we immediately get explicit form of  $C$  since

$$C\tau = D(\operatorname{diag}(W_0(\lambda_1\tau), \dots, W_0(\lambda_n, \tau))) D^{-1}$$

and

$$C = D \operatorname{diag}(W_0(\lambda_1\tau)/\tau, \dots, W_0(\lambda_n\tau)/\tau) D^{-1}.$$

### 3 Main result

The asymptotic properties of the delayed matrix sine and cosine can be deduced from the relations with the delayed exponential of a matrix. We give relevant formulas that are similar to the well-known Euler identity. Namely, for an arbitrary  $n \times n$  matrix  $A$  and  $t \in \mathbb{R}$ , we have

$$\operatorname{Sin}_\tau A(t - \tau) = \operatorname{Im} e_{\tau/2}^{iAt} = \frac{1}{2i} \left( e_{\tau/2}^{iAt} - e_{\tau/2}^{-iAt} \right) \quad (3.1)$$

and

$$\operatorname{Cos}_\tau A \left( t - \frac{\tau}{2} \right) = \operatorname{Re} e_{\tau/2}^{iAt} = \frac{1}{2} \left( e_{\tau/2}^{iAt} + e_{\tau/2}^{-iAt} \right). \quad (3.2)$$

Formulas (3.1), (3.2) can be proved directly using the definitions of  $e_{\tau/2}^{At}$ ,  $\operatorname{Sin}_\tau At$  and  $\operatorname{Cos}_\tau At$  given by formulas (1.1), (1.6) and (1.7) (for the proof we refer to [24]). Below, we use the spectral norm of a matrix defined as

$$\|A\|_S = \sqrt{\lambda_{\max}(A^*A)} \quad (3.3)$$

where  $A^*$  denotes the conjugate transpose of  $A$  and  $\lambda_{\max}$  is the largest eigenvalue of the matrix  $A^*A$ . The main result of the paper follows.

**Theorem 3.1.** *Let  $\lambda_j$ ,  $j = 1, \dots, n$  be the eigenvalues of the matrix  $A$  and let its Jordan canonical form be given by (2.18). If  $|\lambda_j| < 1/(\epsilon\tau)$ ,  $j = 1, \dots, n$  and there exists at least one  $j = j^* \in \{1, \dots, n\}$  such that  $\lambda_{j^*} \neq 0$ , then*

$$\limsup_{t \rightarrow \infty} \|\operatorname{Cos}_\tau At\|_S = \infty$$

and

$$\limsup_{t \rightarrow \infty} \|\operatorname{Sin}_\tau At\|_S = \infty.$$

*Proof.* We will only prove the assertion for  $\operatorname{Cos}_\tau At$  as the proof for  $\operatorname{Sin}_\tau At$  is analogous. Using equation (3.2), we derive the assertion of the theorem utilizing the asymptotic properties of the delayed exponential of matrix  $e_{\tau/2}^{iAt}$ . From the assumption (2.18), we readily get

$$(iA)^k = D \operatorname{diag}((i\lambda_1)^k, \dots, (i\lambda_n)^k) D^{-1}, \quad k \geq 0$$

and, using the associativity, we may express  $e_{\tau/2}^{iAk\tau/2}$  (with the aid of definition (1.1)) as

$$e_{\tau/2}^{iAk\tau/2} = D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 ik\tau/2}, \dots, e_{\tau/2}^{\lambda_n ik\tau/2} \right) D^{-1}. \quad (3.4)$$

For a natural number  $\ell$  we define

$$F_k^\ell(A) := e_{\tau/2}^{Ai(k+\ell)\tau/2} (e_{\tau/2}^{iAk\tau/2})^{-1}.$$

By Theorem 2.2 (formula (2.17)) and by (2.19), we have

$$\lim_{k \rightarrow \infty} F_k^1(A) = D \exp(\operatorname{diag}(W_0(\lambda_1 i\tau/2), \dots, W_0(\lambda_n i\tau/2))) D^{-1}. \quad (3.5)$$

From

$$F_k^\ell(a) = \prod_{l=1}^{\ell} F_{k-l-1}^1(A),$$

we obtain

$$\begin{aligned}\lim_{k \rightarrow \infty} F_k^\ell(A) &= \lim_{k \rightarrow \infty} \prod_{l=1}^{\ell} F_k^\ell(A) = \prod_{l=1}^{\ell} \lim_{k \rightarrow \infty} F_k^\ell(A) \\ &= \left( D \exp(\operatorname{diag}(W_0(\lambda_1 i \tau / 2), \dots, W_0(\lambda_n i \tau / 2))) D^{-1} \right)^\ell.\end{aligned}$$

Imagine, for a while, that the matrix  $A$  is a  $1 \times 1$  matrix, i.e.,  $A = (a)$ . Then, from (3.5) (with  $\lambda = a$ ,  $D := (1)$ ), we get

$$F_k^1(a) = (\exp(W_0(ai\tau/2))) (1 + v_a(k)) \quad (3.6)$$

where  $k$  is an arbitrary natural number and  $v = v_a(k)$  is a real discrete function such that

$$\lim_{k \rightarrow \infty} v_a(k) = 0. \quad (3.7)$$

Applying formula (3.6)  $\ell$  times, we obtain

$$F_k^\ell(A) = (\exp(W_0(ai\tau/2)))^\ell \prod_{l=1}^{\ell} (1 + v_a(k - 1 + l)).$$

Now we can derive a similar formula in the case of an  $n \times n$  matrix  $A$ . First, utilizing (3.6), we obtain:

$$\begin{aligned}F_k^1(A) &= D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 i(k+1)\tau/2}, \dots, e_{\tau/2}^{\lambda_n i(k+1)\tau/2} \right) D^{-1} \\ &\quad \times D \operatorname{diag} \left( \left( e_{\tau/2}^{\lambda_1 i k \tau / 2} \right)^{-1}, \dots, \left( e_{\tau/2}^{\lambda_n i k \tau / 2} \right)^{-1} \right) D^{-1} \\ &= D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 i(k+1)\tau/2} \left( e_{\tau/2}^{\lambda_1 i k \tau / 2} \right)^{-1}, \dots, e_{\tau/2}^{\lambda_n i(k+1)\tau/2} \left( e_{\tau/2}^{\lambda_n i k \tau / 2} \right)^{-1} \right) D^{-1} \\ &= D \operatorname{diag} \left( (\exp(W_0(\lambda_1 i \tau / 2))) (1 + v_{\lambda_1}(k)), \dots \right. \\ &\quad \left. \dots, (\exp(W_0(\lambda_n i \tau / 2))) (1 + v_{\lambda_n}(k)) \right) D^{-1} \\ &= D \operatorname{diag} \left( \exp(W_0(\lambda_1 i \tau / 2)), \dots, \exp(W_0(\lambda_n i \tau / 2)) \right) D^{-1} \\ &\quad \times D \operatorname{diag} \left( (1 + v_{\lambda_1}(k)), \dots, (1 + v_{\lambda_n}(k)) \right) D^{-1} \\ &= D \operatorname{diag} \left( \exp(W_0(\lambda_1 i \tau / 2)), \dots, \exp(W_0(\lambda_n i \tau / 2)) \right) D^{-1} M(k)\end{aligned} \quad (3.8)$$

where the matrix  $M(k)$  is defined as

$$M(k) := D \operatorname{diag}((1 + v_{\lambda_1}(k)), \dots, (1 + v_{\lambda_n}(k))) D^{-1}.$$

Denote

$$e^{W_0(iA)\tau/2} := D \operatorname{diag}(\exp(W_0(\lambda_1 i \tau / 2)), \dots, \exp(W_0(\lambda_n i \tau / 2))) D^{-1}.$$

This matrix commutes with  $M(k)$  since

$$\begin{aligned}e^{W_0(iA)\tau/2} M(k) &= D \operatorname{diag}(\exp(W_0(\lambda_1 i \tau / 2)), \dots, \exp(W_0(\lambda_n i \tau / 2))) D^{-1} \\ &\quad \times D \operatorname{diag}((1 + v_1(k)), \dots, (1 + v_n(k))) D^{-1} \\ &= D \operatorname{diag}((1 + v_1(k)), \dots, (1 + v_n(k))) D^{-1} \\ &\quad \times D \operatorname{diag}(\exp(W_0(\lambda_1 i \tau / 2)), \dots, \exp(W_0(\lambda_n i \tau / 2))) D^{-1} \\ &= M(k) e^{W_0(iA)\tau/2}.\end{aligned}$$

Utilizing (3.4), (3.6), and (3.8), we derive

$$\begin{aligned}
F_k^\ell(A) &= e_{\tau/2}^{Ai(k+\ell)\tau/2} (e_{\tau/2}^{Ai(k+\ell-1)\tau/2})^{-1} \dots e_{\tau/2}^{Ai(k+2)\tau/2} (e_{\tau/2}^{Ai(k+1)\tau/2})^{-1} e_{\tau/2}^{Ai(k+1)\tau/2} (e_{\tau/2}^{Aik\tau/2})^{-1} \\
&= D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 i(k+\ell)\tau/2} (e_{\tau/2}^{\lambda_1 i(k+\ell-1)\tau/2})^{-1}, \dots, e_{\tau/2}^{\lambda_n i(k+\ell)\tau/2} (e_{\tau/2}^{\lambda_n i(k+\ell-1)\tau/2})^{-1} \right) D^{-1} \\
&\quad \times D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 i(k+\ell-1)\tau/2} (e_{\tau/2}^{\lambda_1 i(k+\ell-2)\tau/2})^{-1}, \dots \right. \\
&\quad \left. \dots, e_{\tau/2}^{\lambda_n i(k+\ell-1)\tau/2} (e_{\tau/2}^{\lambda_n i(k+\ell-2)\tau/2})^{-1} \right) D^{-1} \\
&\quad \dots \\
&\quad \times D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 i(k+1)\tau/2} (e_{\tau/2}^{\lambda_1 ik\tau/2})^{-1}, \dots, e_{\tau/2}^{\lambda_n i(k+1)\tau/2} (e_{\tau/2}^{\lambda_n ik\tau/2})^{-1} \right) D^{-1} \\
&= e^{W_0(iA)\tau/2} M(k+\ell-1) e^{W_0(iA)\tau/2} M(k+\ell-2) \dots e^{W_0(iA)\tau/2} M(k) \\
&= \left( e^{W_0(iA)\tau/2} \right)^{\ell} \prod_{l=0}^{\ell-1} M(k+l).
\end{aligned} \tag{3.9}$$

It is easy to see that the values of functions  $e_{\tau/2}^{\lambda_l ik\tau/2}$ ,  $\exp(\ell W_0(\lambda_l i\tau/2))$  ( $l = 1, \dots, n$ ) and the values of the same functions with complex conjugate arguments are complex conjugate too. Applying this fact to  $\operatorname{Cos}_\tau A((k+\ell-1)\tau/2) = \operatorname{Re}(e_{\tau/2}^{iA(k+\ell)\tau/2})$  (see (3.2)), we get (utilizing (3.4), (3.9)):

$$\begin{aligned}
\operatorname{Re} \left( e_{\tau/2}^{iA(k+\ell)\tau/2} \right) &= \frac{1}{2} \left( e_{\tau/2}^{iA(k+\ell)\tau/2} + e_{\tau/2}^{-iA(k+\ell)\tau/2} \right) \\
&= \frac{1}{2} \left( D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 ik\tau/2}, \dots, e_{\tau/2}^{\lambda_n ik\tau/2} \right) D^{-1} \left( e^{W_0(iA)\tau/2} \right)^{\ell} \prod_{l=0}^{\ell-1} M(k+l) \right. \\
&\quad \left. + D \operatorname{diag} \left( e_{\tau/2}^{-\lambda_1 ik\tau/2}, \dots, e_{\tau/2}^{-\lambda_n ik\tau/2} \right) D^{-1} \left( e^{W_0(-iA)\tau/2} \right)^{\ell} \prod_{l=0}^{\ell-1} M(k+l) \right) \\
&= \frac{1}{2} D \operatorname{diag} \left( e_{\tau/2}^{\lambda_1 ik\tau/2} \exp(\ell W_0(\lambda_1 i\tau/2)) \right. \\
&\quad \left. + e_{\tau/2}^{-\lambda_1 ik\tau/2} \exp(-\ell W_0(\lambda_1 i\tau/2)), \dots, e_{\tau/2}^{\lambda_n ik\tau/2} \exp(\ell W_0(\lambda_n i\tau/2)) \right. \\
&\quad \left. + e_{\tau/2}^{-\lambda_n ik\tau/2} \exp(-\ell W_0(\lambda_n i\tau/2)) \right) D^{-1} \prod_{l=0}^{\ell-1} M(k+l) \\
&= D \operatorname{diag} \left( \operatorname{Re} \left( e_{\tau/2}^{\lambda_1 ik\tau/2} \exp(\ell W_0(\lambda_1 i\tau/2)) \right), \dots \right. \\
&\quad \left. \dots, \operatorname{Re} \left( e_{\tau/2}^{\lambda_n ik\tau/2} \exp(\ell W_0(\lambda_n i\tau/2)) \right) \right) D^{-1} \prod_{l=0}^{\ell-1} M(k+l) \\
&= D \operatorname{diag} \left( \operatorname{Re} \left( e_{\tau/2}^{\lambda_1 ik\tau/2} \exp(\ell W_0(\lambda_1 i\tau/2)) \right) \prod_{l=0}^{\ell-1} (1 + v_{\lambda_1}(k+l)), \dots \right. \\
&\quad \left. \dots, \operatorname{Re} \left( e_{\tau/2}^{\lambda_n ik\tau/2} \exp(\ell W_0(\lambda_n i\tau/2)) \right) \prod_{l=0}^{\ell-1} (1 + v_{\lambda_n}(k+l)) \right) D^{-1}. \tag{3.10}
\end{aligned}$$

Now we use the well-known formula  $\operatorname{Re}(z_1 z_2) = |z_1| |z_2| \cos(\arg z_1 + \arg z_2)$  for complex numbers  $z_1, z_2$ . Set

$$z_1 = z_1(k, \lambda_l) := e_{\tau/2}^{\lambda_l ik\tau/2}, \quad z_2 = z_2(\lambda_l) := \exp(\ell W_0(\lambda_l i\tau/2)),$$



where  $l \in \{1, \dots, n\}$ , and denote

$$\begin{aligned}\alpha_1(k, \lambda_l) &:= \arg z_1(k, \lambda_l) = \arg \left( e_{\tau/2}^{\lambda_l i k \tau / 2} \right), \\ \alpha_2(\lambda_l) &:= \arg z_2(\lambda_l) = \arg \left( \exp(\ell W_0(\lambda_l i \tau / 2)) \right).\end{aligned}$$

From the facts that the spectral radius is less or equal any matrix norm, the following inequality for the spectral norm holds

$$\begin{aligned}\|\text{Cos}_{\tau} A((k + \ell - 1)\tau / 2)\|_S &\geq \rho(\text{Cos}_{\tau} A((k + \ell - 1)\tau / 2)) \\ &= \rho\left(\text{Re}\left(e_{\tau/2}^{iA(k+\ell)\tau/2}\right)\right) = \rho_{k+\ell}.\end{aligned}\quad (3.11)$$

The similar matrices have same spectra and the spectral radii. The spectrum of diagonal matrix consists to elements of the diagonal and using (3.10), we obtain

$$\begin{aligned}\rho_k &= \max_{j=1, \dots, n} \left\{ \left| \text{Re} \left( e_{\tau/2}^{\lambda_j i k \tau / 2} \exp(\ell W_0(\lambda_j i \tau / 2)) \prod_{l=0}^{\ell-1} (1 + v_{\lambda_j}(k + l)) \right) \right| \right\} \\ &\geq (1 + v^*(k))^\ell \max_{j=1, \dots, n} \left\{ \left| \text{Re} \left( e_{\tau/2}^{\lambda_j i k \tau / 2} \exp(\ell W_0(\lambda_j i \tau / 2)) \right) \right| \right\}\end{aligned}\quad (3.12)$$

where

$$v^*(k) := \min_{j=1, \dots, n; l=0, \dots, \ell-1} \left\{ v_{\lambda_j}(k + l) \right\}$$

and, by (3.7),

$$\lim_{k \rightarrow \infty} v^*(k) = 0. \quad (3.13)$$

Applying (3.11) and (3.12) we obtain the inequality

$$\begin{aligned}\|\text{Cos}_{\tau} A((k + \ell - 1)\tau / 2)\|_S &\geq (1 + v^*(k))^\ell \max_{j=1, \dots, n} \left\{ \left| \text{Re} \left( e_{\tau/2}^{\lambda_j i k \tau / 2} \exp(\ell W_0(\lambda_j i \tau / 2)) \right) \right| \right\} \\ &\geq (1 + v^*(k))^\ell \max_{j=1, \dots, n} \left\{ \left| e_{\tau/2}^{\lambda_j i k \tau / 2} \right| \left| \exp(\ell W_0(\lambda_j i \tau / 2)) \right| \left| \cos(\alpha_1(k, \lambda_l) + \alpha_2(\lambda_l)) \right| \right\}.\end{aligned}$$

Assume that  $j = j^* \in \{1, \dots, n\}$  is fixed and that the eigenvalue  $\lambda_{j^*} \neq 0$  of the matrix  $A$  is real. Then, the number  $z^* = i\lambda_{j^*}\tau/2$  lies in the exterior domain of  $\tilde{\ell}$  since inequality (2.12) holds, i.e.,

$$|z^*| = |i\lambda_{j^*}\tau/2| > -\arctan\left(\frac{\text{Re } z^*}{|\text{Im } z^*|}\right) = -\arctan 0 = 0 \quad (3.14)$$

and, by (2.11),

$$\text{Re } W_0(z^*) = \text{Re } W_0(i\lambda_{j^*}\tau/2) > 0. \quad (3.15)$$

Now assume that  $j = j^* \in \{1, \dots, n\}$  is fixed and that the eigenvalue  $\lambda_{j^*} \neq 0$  of the matrix  $A$  is a complex number. Since  $\bar{\lambda}_{j^*}$  is an eigenvalue of  $A$  as well, we can assume that  $\lambda_{j^*} = x - iy$  where  $y > 0$ . Then, the number  $z^* = i\lambda_{j^*}\tau/2$  lies in the exterior domain of  $\tilde{\ell}$  since inequality (2.12) holds, i.e.,

$$|z^*| = |i\lambda_{j^*}\tau/2| = \frac{\tau}{2}|ix + y| = \frac{\tau}{2}\sqrt{x^2 + y^2} > -\arctan\left(\frac{\text{Re } z^*}{|\text{Im } z^*|}\right) = -\arctan\left(\frac{y}{|x|}\right)$$

where  $\arctan(y/|x|) > 0$ . Then, by (2.11),

$$\text{Re } W_0(z^*) = \text{Re } W_0(i\lambda_{j^*}\tau/2) > 0. \quad (3.16)$$

From (3.15) and (3.16), it follows that there exists an eigenvalue  $\lambda_{j^*}$  of  $A$  and a constant  $\tilde{C}$  such that

$$\operatorname{Re} W_0(i\lambda_{j^*}\tau/2) > \tilde{C} > 0. \quad (3.17)$$

Utilizing (3.1), (3.2) (where  $A := (\lambda_{j^*})$  and  $t = k\tau/2$ ) we derive

$$e^{\frac{\lambda_{j^*}ik\tau/2}{\tau/2}} = \operatorname{Cos}_\tau \lambda_{j^*}(k-1)\tau/2 + i \operatorname{Sin}_\tau \lambda_{j^*}(k/2-1)\tau. \quad (3.18)$$

Let  $k = k^*$  be such that

$$\operatorname{Cos}_\tau \lambda_{j^*}(k^*-1)\tau/2 \neq 0. \quad (3.19)$$

It is easy to see that such a  $k^*$  always exists and note that it can be assumed greater than an arbitrarily given sufficiently large positive integer. Then (3.18), implies

$$\alpha_1(k^*, \lambda_{j^*}) \neq \pm \frac{\pi}{2}. \quad (3.20)$$

By (2.13), we have  $|\alpha_2(\lambda_{j^*})| < \pi/2$ . With regard to  $\alpha_2(\lambda_{j^*})$ , we consider two cases below:

**a)** Let  $\alpha_2(\lambda_{j^*}) \neq 0$ . Then, each interval  $[\pi/2 + 2s\pi, \pi/2 + 2s\pi + \pi]$ , where  $s = 0, 1, \dots$ , contains at least two elements of an equidistant sequence

$$\{\alpha_1(k^*, \lambda_{j^*}) + n\alpha_2(\lambda_{j^*})\}_{n=-\infty}^{\infty}$$

and, in each interval, there exists an element of this sequence  $\alpha^s$  such that

$$|\alpha^s - \pi/2| > \frac{\pi}{4}, \quad |\alpha^s - \pi/2 - \pi| > \frac{\pi}{4}$$

and

$$|\cos(\alpha^s)| > \sqrt{2}/2. \quad (3.21)$$

**b)** Let  $\alpha_2(\lambda_{j^*}) = 0$ . Then, (3.20) implies

$$|\cos \alpha^s| = |\cos \alpha_1(k^*, \lambda_{j^*})| \neq 0. \quad (3.22)$$

Therefore, in both cases **a)** and **b)**, there exists a sequence of positive integers  $\{\ell_l\}_{l=1}^{\infty}$  such that  $\lim_{l \rightarrow \infty} \ell_l = \infty$  and (due to (3.17), (3.21) and (3.22)) for all sufficiently large  $\ell_l$

$$|\exp(\ell_l W_0(i\lambda_{j^*}\tau/2))| |\cos(\alpha_1(k^*, \lambda_{j^*}) + \ell_l \alpha_2(\lambda_{j^*}))| > M \exp(\ell_l C\tau/2) \quad (3.23)$$

where

$$M := \begin{cases} \frac{\sqrt{2}}{2}, & \text{if } \alpha_2(\lambda_{j^*}) \neq 0, \\ |\cos \alpha_1(k^*, \lambda_{j^*})|, & \text{if } \alpha_2(\lambda_{j^*}) = 0 \end{cases}$$

and  $C$  is a constant satisfying  $0 < C < \tilde{C}$ . Moreover, from (3.13), it follows that, for every sufficiently large  $k$ , there exists a constant  $C_0$  satisfying  $0 < C_0 < C$  such that

$$1 + v^*(k) > \exp(-C_0\tau/2). \quad (3.24)$$

From (3.12), (3.23), (3.24), we can derive

$$\begin{aligned} \|\text{Cos}_\tau A((k^* + \ell_l - 1)\tau/2)\|_S &\geq (1 + v^*(k^*))^{\ell_l} \left| e_{\tau/2}^{\lambda_{j^*} i k^* \tau/2} \right| \\ &\quad \times \left| \exp(\ell_l W_0(\lambda_{j^*} i \tau/2)) \right| \left| \cos(\alpha_1(k^*, \lambda_{j^*}) + \alpha_2(\lambda_{j^*})) \right| \\ &\geq \exp(-\ell_l C_0 \tau/2) \left| e_{\tau/2}^{\lambda_{j^*} i k^* \tau/2} \right| M \exp(\ell_l C \tau/2) \\ &= M \left| e_{\tau/2}^{\lambda_{j^*} i k^* \tau/2} \right| \exp(\ell_l (C - C_0) \tau/2). \end{aligned}$$

Finally, we conclude

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|\text{Cos}_\tau A t\|_S &\geq \lim_{l \rightarrow \infty} \|\text{Cos}_\tau A((k^* + \ell_l - 1)\tau/2)\|_S \\ &\geq \lim_{l \rightarrow \infty} M \left| e_{\tau/2}^{\lambda_{j^*} i k^* \tau/2} \right| \exp(\ell_l (C - C_0) \tau/2) \\ &= \infty. \end{aligned}$$

An analogous assertion can also be obtained for  $\text{Sin}_\tau A t$ . The scheme of the proof in this case remains the same with the following minor modifications. In (3.10) the imaginary parts of the complex expressions considered is used instead of their real parts. The relation (3.10) turns into

$$\begin{aligned} \text{Sin}_\tau A((k + \ell - 2)\tau/2) &= D \text{diag} \left( \text{Im} \left( e_{\tau/2}^{\lambda_1 i k \tau/2} \exp(\ell W_0(\lambda_1 i \tau/2)) \right) \prod_{l=0}^{\ell-1} (1 + v_{\lambda_1}(k+l)), \dots \right. \\ &\quad \left. \dots, \text{Im} \left( e_{\tau/2}^{\lambda_n i k \tau/2} \exp(\ell W_0(\lambda_n i \tau/2)) \right) \prod_{l=0}^{\ell-1} (1 + v_{\lambda_n}(k+l)) \right) D^{-1} \end{aligned}$$

and the estimation (3.12) has the form

$$\begin{aligned} \|\text{Sin}_\tau A((k + \ell - 2)\tau/2)\|_S &\geq (1 + v^*(k))^\ell \max_{j=1, \dots, n} \left\{ \left| e_{\tau/2}^{\lambda_j i k \tau/2} \right| \left| \exp(\ell W_0(\lambda_j i \tau/2)) \right| \left| \sin(\alpha_1(k, \lambda_j) + \alpha_2(\lambda_j)) \right| \right\}. \end{aligned}$$

In (3.19),  $\text{Sin}_\tau$  instead of  $\text{Cos}_\tau$  is used and the constant  $M$  must be redefined as

$$M := \begin{cases} \frac{\sqrt{2}}{2}, & \text{if } \alpha_2(\lambda_{j^*}) \neq 0, \\ \left| \sin \alpha_1(k^*, \lambda_{j^*}) \right|, & \text{if } \alpha_2(\lambda_{j^*}) = 0. \end{cases}$$

□

## 4 Concluding remarks

In this part, we discuss some connections with previous results and facts. The author is grateful to the referee for drawing attention to several topics which are discussed below.

**i) Relationship with a linear ordinary non-delayed system.** In the paper, properties of delayed matrix exponential and the Lambert  $W$ -function are used to prove that spectral norms of delayed matrix sine and delayed matrix cosine are unbounded for  $t \rightarrow \infty$ . This property is proved under the assumption that the spectral radius  $\rho(A)$  of the matrix  $A$  is less than  $1/(e\tau)$ .

Many papers bring results on so-called special solutions of delayed differential systems (we refer, e.g., to [1, 2, 7–11, 14, 17–19, 22] and to the references therein) approximating, in a certain sense, all solutions of a given system. One of the conditions guaranteeing the existence of special solutions is often (restricted to system (1.2)) the inequality

$$\|A\| < 1/(e\tau)$$

where  $\|\cdot\|$  is an arbitrary norm. The totality of all special solutions is only an  $n$ -parameter family where  $n$  equals the number of equations of the system. Moreover, it is often stated that, in such a case, some properties (such as stability properties) of solutions of the initial system are the same as those for solutions of a corresponding system of ordinary differential equations.

Because of the well-known inequality  $\rho(A) \leq \|A\|$ , it is generally not possible from an assumed inequality  $\rho(A) < 1/(e\tau)$  to deduce  $\|A\| < 1/(e\tau)$ . Nevertheless, for the spectral norm (3.3) used in the paper, we get (under the conditions of Theorem 3.1),

$$\rho(A) = \|A\|_S < 1/(e\tau).$$

It means that, in a way, the properties of solutions of (1.2) are close, in a meaning, to properties of an ordinary differential system and (1.2) is asymptotically ordinary. I.e., every solution of system (1.2) is asymptotically close to a solution of a system of ordinary differential equations.

The construction of such a linear non-delayed system is described, e.g., in [1, Theorem 2.4] (see also the Summary part in [17]). However, to find such a system is, in general, not an easy task. The formula defining the matrix of ordinary differential system ([1, formula (2.8)] or [17, formula (2.10)]) is a series of recurrently defined matrices and to find its sum is not always possible (we refer to [7, Theorem 1.2], [17, part 4]).

In the case of a constant matrix, the fundamental matrix  $X_o(t)$  of the corresponding ordinary differential system equals an ordinary matrix exponential  $X_o(t) = \exp(\Lambda_0 t)$  where the matrix  $\Lambda_0$  is a unique solution of the matrix equation

$$\Lambda = A \exp(-\Lambda\tau)$$

such that  $\|\Lambda_0\| \tau < 1$  (see the proof of statement (i) of the Theorem in [17]). So, an analysis of the asymptotic behavior of the solutions of system (1.2) reduces, in a meaning, to an analysis of the asymptotic behavior of solutions of a system of ordinary differential equations  $x' = \Lambda_0 x$ , i.e., analysis of the properties of the matrix  $\Lambda_0$ . Tracing the proof of Theorem 3.1, we can assert that the investigation of properties of the matrix  $\Lambda_0$  is, in our case, performed by using the properties of Lambert  $W$ -function defined in Part 2 (see also the motivation example (2.1) and formulas (2.2)–(2.5)).

**ii) Existence of a root of characteristic equation with positive real part.** Let  $n = 1$  and  $A = (a)$  in (1.5). Then, the characteristic equation (derived by substituting  $x = \exp(\lambda t)$ ) equals

$$\lambda^2 = -a^2 \exp(-\tau\lambda) \tag{4.1}$$

and is equivalent with

$$\frac{\lambda\tau}{2} \exp\left(\frac{\lambda\tau}{2}\right) = \pm \frac{ia\tau}{2}.$$

Utilizing the Lambert  $W$ -function, the last equation can be written as (see (2.4), (2.5))

$$\frac{\lambda\tau}{2} = W\left(\pm \frac{ia\tau}{2}\right),$$

therefore, all roots of (4.1) are values of the Lambert function. For

$$z = z_{\pm} = \pm i\tau/2,$$

inequality (2.12), which determines the domain of the points for which the principal branch of the Lambert function  $W_0$  has positive real parts (inequality (2.11)), holds (see also (3.14), (3.15)). Thus, we conclude that the unboundedness of the delayed matrix sine and cosine is related to the existence of a root of characteristic equation with positive real part.

**iii) Asymptotic behavior of the fundamental matrix solution by using the characteristic equation.** As noted in the Introduction, the general definition of a fundamental matrix to linear functional differential systems of delayed type in [12,13] yields (in the simple case of the matrix of the system with single delay being a constant matrix) a delayed matrix exponential by formula (1.4). Delayed matrix sine and cosine can be expressed through delayed matrix exponential by formulas (3.1), (3.2). Therefore, both Theorem 2.2 and Theorem 3.1, formulate the asymptotic properties of the relevant fundamental matrix solutions depending on the properties of the eigenvalues of the matrix  $A$  and, consequently, through the properties of the roots of the characteristic equation described by the Lambert  $W$ -function. It is an open question if the method used in the paper can be extended to matrices  $A$  with Jordan canonical forms different from (2.18) in order to get further results on the behavior of the fundamental matrix solution.

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