Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **68**, 1–21; https://doi.org/10.14232/ejqtde.2018.1.68 www.math.u-szeged.hu/ejqtde/

Multiple positive solutions for Schrödinger problems with concave and convex nonlinearities

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Received 19 November 2017, appeared 5 August 2018
Communicated by Petru Jebelean

Abstract. In this paper, we consider the multiplicity of positive solutions for a class of Schrödinger equations involving concave-convex nonlinearities in the whole space. With the help of the Nehari manifold, Ekeland variational principle and the theory of Lagrange multipliers, we prove that the Schrödinger equation has at least two positive solutions, one of which is a positive ground state solution.

Keywords: Schrödinger problem, Nehari manifold, Ekeland variational principle.

2010 Mathematics Subject Classification: 35A01, 35A15.

1 Introduction and main results

This paper concerns the multiplicity of positive solutions for the following Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.1}$$

where $1 < q < 2 < p < 2^*$ ($2^* = \infty$ if $N = 1,2$ and $2^* = 2N/(N-2)$ if $N \geq 3$) and $V(x), f(x), g(x)$ satisfy suitable conditions.

There are many works on nonlinearity of concave-convex type under various conditions on potential $V(x)$. When $V(x) \equiv 0$, Equation (1.1) is considered in a bounded domain. This problem can date back to the famous work of Ambrosetti–Brezis–Cerami in [1], where the authors considered the following problem

$$\begin{cases} -\triangle u = \lambda |u|^{q-2}u + |u|^{p-2}u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < q < 2 < p \leq 2^*$. They proved that Equation (1.2) has at least two positive solutions for suffciently small $\lambda > 0$. In this case, the compact embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ $(p \in [2, 2^*))$ plays an important role; for more general results

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in bounded domains see [4,5,8,13,18,24,27] and their references. In the whole space $\mathbb{R}^N$ some authors concerned Equation (1.1) with $V(x)$ satisfying suitable conditions such that the embedding

$$X := \left\{u \in H^1\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V(x)|u(x)|^2 dx < +\infty\right\} \hookrightarrow L^p\left(\mathbb{R}^N\right), \qquad p \in [2, 2^*), \tag{1.3}$$

is compact. For example, Bartsch and Wang [2] first introduced the following weaker condition

$(\overline{V})$ $V(x) \in C\left(\mathbb{R}^N, \mathbb{R}\right), V_0 := \inf_{\mathbb{R}^N} V(x) > 0$ and for any $M > 0$, there exists a constant $r_0 > 0$ such that $\text{meas}\left(\{x \in B_{r_0}(y) : V(x) \leq M\}\right) \to 0$ as $|y| \to +\infty$, where $B_{r_0}(y)$ denotes the ball centered at y with radius r_0 and mea s the Lebesgue measure in $\mathbb{R}^N$.

For some results in this area, we also refer to [14,21].

If the potential function $V(x)$ is bounded, the embedding (1.3) is not compact; in the case of the constant potential, i.e., $V(x)$ is a positive constant in Equation (1.1), we can refer to [25,26,28]. However, we do not know any results for Equation (1.1) with both $V(x)$ and $g(x)$ bounded functions. A direct extension to the case $V(x)$ and $g(x)$ bounded functions is faced with difficulties. On the one hand, because the nonlinearity is a combination of the concave and convex terms, estimating the critical value by suitable autonomous equation becomes complex. On the other hand, since both $V(x)$ and $g(x)$ are bounded functions, the proof of the (PS) condition satisfied for the critical value in suitable range becomes delicate. In this paper, we are concerned about Equation (1.1) with both $V(x)$ and $g(x)$ bounded functions on the basis of variational arguments. If $V(x)$, $f(x)$ and $g(x)$ satisfy the suitable conditions, we prove multiple positive solutions for equation (1.1) under the quantitative assumption. Up to now, there is a lot of papers considered different problems and obtained the relevant results under the quantitative assumption, see [6,7,12,29] for Kirchhoff problems, [15,26,27] for Schrödinger problems and [16] for Schrödinger–Maxwell problems. For example, Wu [27] considered the following Schrödinger problem:

$$\begin{cases} -\triangle u = f(x)|u|^{q-2}u + (1 - g(x))|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases}$$

where $1 < q < 2, 2^* = 2N/(N-2) (N \geq 3)$, $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a bounded domain with smooth boundary and the weight functions $f, g \in C(\overline{\Omega})$ satisfy the suitable conditions. Then there exists $\lambda_0 > 0$ such that if $\|f^+\|_{L^{q^*}} < \lambda_0$, this problem has three positive solutions, where $q^* = 2^*/(2^* - q)$ and $f^+ = \max\{f, 0\} \neq 0$.

To state our main result, we introduce precise conditions on $V(x)$, $f(x)$ and $g(x)$:

(V) $V(x) \in C\left(\mathbb{R}^N, \mathbb{R}\right), 0 < V_0 := \inf_{x \in \mathbb{R}^N} V(x) \leq V(x) \leq V_\infty := \lim_{|x| \to +\infty} V(x) < +\infty$,

(f) f is positive, continuous and belongs to $L^{q^*}\left(\mathbb{R}^N\right)$, where q^* is conjugate to p/q (i.e. $q^* = p/(p-q)$),

(g) $g(x) \in C\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right), 0 < g_\infty := \lim_{|x| \to +\infty} g(x) \leq g(x) \leq \sup_{x \in \mathbb{R}^N} g(x) < +\infty.$

Our main result is as follows.

Let $\sigma := (p-2)(2-q)^{(2-q)/(p-2)} \left(\frac{S_p}{p-q}\right)^{(p-q)/(p-2)}$ and $0 < \sigma^* = q\sigma/2 < \sigma$, where S_p is the best Sobolev constant described in the following Lemma 2.2.

Theorem 1.1. *Under the assumptions* (V), (f) *and* (g), *if* $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, *Equation* (1.1) *has at least two positive solutions, which correspond to negative energy and positive energy, respectively; in particular, the one with negative energy is a positive ground state solution.*

The combined effects of a sub-linear and a super-linear terms change the structure of the solution set. According to the behaviour of nonlinearities and to the results we want to prove, the method of the decomposition of Nehari manifold turns out to be more appropriate. With the help of suitable autonomous equation, the Ekeland variational principle and the theory of Lagrange multipliers, we can prove that Equation (1.1) has at least two positive solutions, one of which is a positive ground state solution. In addition, the condition (V) can be replaced by other forms.

Remark 1.2. Assume that $(\overline{V})$, (f) and (g), if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}$ is sufficiently small, then Theorem 1.1 still holds.

Remark 1.3. Assume that $V(x) \equiv C$, (f) and (g), if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)}$ is sufficiently small, then Theorem 1.1 still holds, where C is a positive constant.

The rest of this paper is organized as follows: Section 2 is dedicated to our variational framework and some preliminary results. Section 3 concerns with the proof of Theorem 1.1.

Throughout this paper, C and C_i denote distinct constants. $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the standard norm $|u|_p = \left(\int_{\mathbb{R}^N}|u|^p dx\right)^{1/p}$ for $1 \le p < \infty$ and $|u|_\infty = \sup_{x\in\mathbb{R}^N}|u(x)|$ for $p = \infty$. When it causes no confusion, we still denote by $\{u_n\}$ a subsequence of the original sequence $\{u_n\}$.

2 Preliminary results

With the fact that the problem (1.1) has a variational structure, the proof is based on the variational approach and the use of the Nehari manifold technique. So, we will first recall some preliminaries and establish the variational setting for our problem in this section.

Define

$$E := \left\{ u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} V(x)|u|^2\,dx < +\infty \right\}$$

with the associate norm

$$\|u\| = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx\right)^{\frac{1}{2}}.$$

Under the assumption (V), we know that the norm $\|\cdot\|$ is equivalent to the usual norm in $H^1(\mathbb{R}^N)$. The energy functional corresponding to Equation (1.1) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)|u|^2\right)\,dx - \frac{1}{q}\int_{\mathbb{R}^N} f(x)|u|^q\,dx - \frac{1}{p}\int_{\mathbb{R}^N} g(x)|u|^p\,dx, \qquad u \in E. \tag{2.1}$$

Lemma 2.1. *If* (V), (f) *and* (g) *hold, then the functional* $I \in C^1(E,\mathbb{R})$ *and for any* $u, v \in E$

$$\begin{aligned}\left\langle I'(u), v\right\rangle = &\int_{\mathbb{R}^N}\nabla u\nabla v\,dx + \int_{\mathbb{R}^N} V(x)uv\,dx \\ &- \int_{\mathbb{R}^N} f(x)|u|^{q-2}uv\,dx - \int_{\mathbb{R}^N} g(x)|u|^{p-2}uv\,dx.\end{aligned} \tag{2.2}$$

Furthermore, I' *is weakly sequentially continuous in* E.

Proof. The proof is a direct computation. Here we omit details and refer to [23]. □

Lemma 2.2 ([23]). *Under the assumption (V), the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $p \in [2, 2^*]$. Let*

$$S_p = \inf_{u \in E \setminus \{0\}} \frac{\|u\|^2}{\left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{2/p}} > 0,$$

then

$$|u|_p \le S_p^{-\frac{1}{2}} \|u\|, \qquad \forall u \in E.$$

It is well-known that seeking a weak solution of Equation (1.1) is equivalent to finding a critical point of the corresponding functional I. In the following, we are devoted to finding the critical point of the corresponding functional I.

As usual, some energy functional such as I in (2.1) is not bounded from below on E but, as we will see, is bounded from below on an appropriate subset of E and a minimizer on this set (if it exists) may give rise to a solution of corresponding differential equation (see [22]). A good exemplification for an appropriate subset of E is the so-called Nehari manifold

$$\mathcal{N} := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle I'(u), u \rangle = 0 \right\},$$

where $\langle \ , \ \rangle$ denotes the usual duality between E and E^*. It is clear to see that $u \in \mathcal{N}$ if and only if for $u \in H^1(\mathbb{R}^N) \setminus \{0\}$,

$$\|u\|^2 = \int_{\mathbb{R}^N} f(x)|u|^q \, dx + \int_{\mathbb{R}^N} g(x)|u|^p \, dx. \tag{2.3}$$

Obviously, $\mathcal{N}$ contains all nontrivial solutions of Equation (1.1). Below, we shall use the Nehari manifold methods to find critical points for the functional I.

The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of functions of the form $K_u : t \to I(tu)$ for $t > 0$. Such maps are known as fibering maps, which were introduced by Drábek and Pohozaev in [9]. For $u \in E$, let

$$\begin{aligned}
K_u(t) = I(tu) &= \frac{1}{2} t^2 \|u\|^2 - \frac{1}{q} t^q \int_{\mathbb{R}^N} f(x)|u|^q \, dx - \frac{1}{p} t^p \int_{\mathbb{R}^N} g(x)|u|^p \, dx; \\
K_u'(t) &= t\|u\|^2 - t^{q-1} \int_{\mathbb{R}^N} f(x)|u|^q \, dx - t^{p-1} \int_{\mathbb{R}^N} g(x)|u|^p \, dx; \\
K_u''(t) &= \|u\|^2 - (q-1) t^{q-2} \int_{\mathbb{R}^N} f(x)|u|^q \, dx - (p-1) t^{p-2} \int_{\mathbb{R}^N} g(x)|u|^p \, dx.
\end{aligned}$$

Lemma 2.3. *Let $u \in E$ and $t > 0$. Then $tu \in \mathcal{N}$ if and only if $K_u'(t) = 0$, that is, the critical points of $K_u(t)$ correspond to the points on the Nehari manifold. In particular, $u \in \mathcal{N}$ if and only if $K_u'(1) = 0$.*

Proof. The result is an immediate consequence of the fact:

$$K_u'(t) = \langle I'(tu), u \rangle = \frac{1}{t} \langle I'(tu), tu \rangle. \qquad \square$$

Thus, it is natural to split $\mathcal{N}$ into three parts corresponding to local minima, points of inflection and local maxima. Accordingly, we define

$$\mathcal{N}^+ = \{u \in \mathcal{N} \mid K_u''(1) > 0\}, \quad \mathcal{N}^0 = \{u \in \mathcal{N} \mid K_u''(1) = 0\} \quad \text{and} \quad \mathcal{N}^- = \{u \in \mathcal{N} \mid K_u''(1) < 0\}.$$

It is easy to see that

$$K_u''(1) = \|u\|^2 - (q-1)\int_{\mathbb{R}^N} f(x)|u|^q\,dx - (p-1)\int_{\mathbb{R}^N} g(x)|u|^p\,dx. \tag{2.4}$$

Define

$$\Psi(u) = K_u'(1) = \langle I'(u), u\rangle = \|u\|^2 - \int_{\mathbb{R}^N} f(x)|u|^q\,dx - \int_{\mathbb{R}^N} g(x)|u|^p\,dx. \tag{2.5}$$

Then for $u \in \mathcal{N}$,

$$\begin{aligned}\left(\frac{d}{dt}\Psi(tu)\right)\Big|_{t=1} &= \langle \Psi'(u), u\rangle = \langle \Psi'(u), u\rangle - \langle I'(u), u\rangle = K_u''(1)\\ &= \|u\|^2 - (q-1)\int_{\mathbb{R}^N} f(x)|u|^q\,dx - (p-1)\int_{\mathbb{R}^N} g(x)|u|^p\,dx.\end{aligned}$$

For each $u \in \mathcal{N}$, $\Psi(u) = K_u'(1) = 0$. Thus, for each $u \in \mathcal{N}$, we have

$$K_u''(1) = K_u''(1) - (q-1)\Psi(u) = (2-q)\|u\|^2 - (p-q)\int_{\mathbb{R}^N} g(x)|u|^p\,dx \tag{2.6}$$

and

$$K_u''(1) = K_u''(1) - (p-1)\Psi(u) = (2-p)\|u\|^2 + (p-q)\int_{\mathbb{R}^N} f(x)|u|^q\,dx. \tag{2.7}$$

In order to ensure the Nehari manifold $\mathcal{N}$ to be a C^1-manifold, we need the following proposition.

Proposition 2.4. *Let $\sigma := (p-2)(2-q)^{(2-q)/(p-2)}\left(\frac{S_p}{p-q}\right)^{(p-q)/(p-2)}$, where S_p is the best Sobolev constant described in Lemma 2.2. If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then the set $\mathcal{N}^0 = \emptyset$.*

Proof. Suppose, on the contrary, there exists a $u \in \mathcal{N}$ such that $K_u''(1) = 0$. By Lemma 2.2,

$$\int_{\mathbb{R}^N} g(x)|u|^p\,dx \le |g|_\infty S_p^{-\frac{p}{2}}\|u\|^p. \tag{2.8}$$

Noting that $2 < p < 2^*$, from (2.6) we have

$$(2-q)\|u\|^2 \le (p-q)|g|_\infty S_p^{-\frac{p}{2}}\|u\|^p,$$

so

$$\|u\| \ge \left(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\right)^{\frac{1}{p-2}}. \tag{2.9}$$

Moreover, by the Hölder inequality and Lemma 2.2, we have

$$\int_{\mathbb{R}^N} f(x)|u|^q\,dx \le \left(\int_{\mathbb{R}^N} |f(x)|^{q*}\,dx\right)^{\frac{1}{q^*}}\left(\int_{\mathbb{R}^N} |u|^p\,dx\right)^{\frac{q}{p}} = |f|_{q*}|u|_p^q \le |f|_{q*}S_p^{-\frac{q}{2}}\|u\|^q. \tag{2.10}$$

From (2.7) we have

$$(p-2)\|u\|^2 \le (p-q)|f|_{q*}S_p^{-\frac{q}{2}}\|u\|^q,$$

which implies that

$$\|u\| \le \left(\frac{(p-q)|f|_{q^*}}{(p-2)S_p^{\frac{q}{2}}}\right)^{\frac{1}{2-q}}. \tag{2.11}$$

This with (2.9) and (2.11) implies that

$$|f|_{q^*}|g|_\infty^{\frac{2-q}{p-2}} > \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q}S_p^{\frac{q}{2}} = (p-2)(2-q)^{\frac{2-q}{p-2}}\left(\frac{S_p}{p-q}\right)^{\frac{p-q}{p-2}} = \sigma,$$

which contradicts with the condition. □

Proposition 2.5. *Suppose that $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$ and $u \in E$. Then, there are unique t^+ and t^- with $0 < t^+ < t_{\max} < t^-$ such that $t^+u \in \mathcal{N}^+$, $t^-u \in \mathcal{N}^-$ and*

$$I(t^+u) = \inf_{0\le t\le t_{\max}} I(tu), \quad I(t^-u) = \sup_{t\ge t_{\max}} I(tu).$$

Proof. Let

$$h(t) = t^{2-q}\|u\|^2 - t^{p-q}\int_{\mathbb{R}^N} g(x)|u|^p\,dx,$$

then we have

$$K_u'(t) = t^{q-1}\left(h(t) - \int_{\mathbb{R}^N} f(x)|u|^q\,dx\right). \tag{2.12}$$

Clearly, $h(0) = 0$ and $h(t) \to -\infty$ as $t \to \infty$. From $1 < q < 2 < p < 2^*$ and

$$h'(t) = t^{p-q-1}\left((2-q)t^{2-p}\|u\|^2 - (p-q)\int_{\mathbb{R}^N} g(x)|u|^p\,dx\right) = 0,$$

we can infer that there is a unique $t_{\max} > 0$ such that $h(t)$ achieves its maximum at $t_{\max}$, increasing for $t \in [0, t_{\max})$ and decreasing for $t \in (t_{\max}, \infty)$ with $\lim_{t\to\infty} h(t) = -\infty$, where

$$t_{\max} = \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} g(x)|u|^p\,dx}\right)^{\frac{1}{p-2}}.$$

It follows

$$\begin{aligned} h(t_{\max}) &= \|u\|^q\left(\frac{\|u\|^p}{\int_{\mathbb{R}^N} g(x)|u|^p dx}\right)^{\frac{2-q}{p-2}}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\ &\ge \|u\|^q\left(\frac{\|u\|^p}{|g|_\infty S_p^{-\frac{p}{2}}\|u\|^p}\right)^{\frac{2-q}{p-2}}\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} \\ &= \|u\|^q\left(\frac{(2-q)S_p^{\frac{p}{2}}}{|g|_\infty(p-q)}\right)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} > 0. \end{aligned} \tag{2.13}$$

From $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, (2.10) and (2.13) we also have

$$\int_{\mathbb{R}^N} f(x)|u|^q\,dx < \|u\|^q\left(\frac{(2-q)S_p^{\frac{p}{2}}}{|g|_\infty(p-q)}\right)^{\frac{2-q}{p-2}}\frac{p-2}{p-q} < h(t_{\max}). \tag{2.14}$$

Moreover, for $tu \in \mathcal{N}$, $K_u'(t) = 0$. By (2.12) we obtain that

$$K_u''(t) = t^{q-1}h'(t).$$

By (2.12) and (2.14) we know there are unique t^+ and t^- with $0 < t^+ < t_{\max} < t^-$ such that $K'_{t^+u}(1) = 0$, $K'_{t^-u}(1) = 0$, that is $t^+u, t^-u \in \mathcal{N}$. From $K''_u(t) = t^{q-1}h'(t)$ and $h'(t^+) > 0 > h'(t^-)$, one arrives at the conclusion. □

The forthcoming lemma is to obtain the minimizing sequence of the energy functional I on the Nehari manifold $\mathcal{N}$.

Lemma 2.6. *The energy functional I is coercive and bounded from below on $\mathcal{N}$.*

Proof. For $u \in \mathcal{N}$, then, by the Hölder inequality and Lemma 2.2,

$$\begin{aligned} I(u) &= I(u) - \frac{1}{p}\langle I'(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N} f(x)|u|^q\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)|f|_{q^*} S_p^{-\frac{q}{2}}\|u\|^q. \end{aligned}$$

This completes the proof. □

Lemma 2.7. *Under the assumptions (V), (f) and (g), the following results hold.*

(i) If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then $c_1 = \inf_{u\in\mathcal{N}^+} I(u) < 0$;

(ii) If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma^*)$, then $c_2 = \inf_{u\in\mathcal{N}^-} I(u) > 0$, where $\sigma^* = q\sigma/2$ and σ described in Proposition 2.4.

Proof. (i) For each $u \in \mathcal{N}^+$, $K''_u(1) > 0$. From (2.7), we have

$$(p-q)\int_{\mathbb{R}^N} f(x)|u|^q\,dx > (p-2)\|u\|^2.$$

If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then

$$\begin{aligned} I(u) = I(u) - \frac{1}{p}\langle I'(u), u\rangle &= \frac{p-2}{2p}\|u\|^2 - \frac{p-q}{pq}\int_{\mathbb{R}^N} f(x)|u|^q\,dx \\ &< \frac{p-2}{2p}\|u\|^2 - \frac{p-2}{pq}\|u\|^2 = \frac{(p-2)(q-2)}{2pq}\|u\|^2 < 0. \end{aligned} \tag{2.15}$$

Thus, $\inf_{u\in N^+} I(u) < 0$.

(ii) For each $u \in \mathcal{N}^-$, $K''_u(1) < 0$. From (2.9) and (2.10), we have if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma^*)$, then

$$\begin{aligned} I(u) &= I(u) - \frac{1}{p}\langle I'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N} f(x)|u|^q\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right)|f|_{q_*} S_p^{-\frac{q}{2}}\|u\|^q \\ &= \|u\|^q\left(\left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^{2-q} - \left(\frac{1}{q} - \frac{1}{p}\right)|f|_{q_*} S_p^{-\frac{q}{2}}\right) \\ &\geq \left(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\right)^{\frac{q}{p-2}}\left(\left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\right)^{\frac{2-q}{p-2}} - \left(\frac{1}{q} - \frac{1}{p}\right)|f|_{q^*} S_p^{-\frac{q}{2}}\right) > 0. \end{aligned}$$

□

Lemma 2.8. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then the set $\mathcal{N}^-$ is closed in E.*

Proof. Let $\{u_n\} \subset \mathcal{N}^-$ such that $u_n \to u$ in E. In the following we prove $u \in \mathcal{N}^-$. Indeed, for any $u \in \mathcal{N}^-$, from (2.6) we have

$$(2-q)\|u\|^2 < (p-q)\int_{\mathbb{R}^N} g(x)|u|^p\,dx.$$

Similar to the proof of (2.9), we have

$$\|u\| \geq \left(\frac{(2-q)S_p^{\frac{p}{2}}}{(p-q)|g|_\infty}\right)^{\frac{1}{p-2}}. \tag{2.16}$$

Hence $\mathcal{N}^-$ is bounded away from 0.

By $\langle I'(u_n), u_n\rangle = 0$ and Lemma 2.1, we have $\langle I'(u), u\rangle = 0$. (2.6) implies that $K''_{u_n}(1) \to K''_u(1)$. From $K''_{u_n}(1) < 0$, we have $K''_u(1) \leq 0$. By Proposition 2.4 we know, if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then $K''_u(1) < 0$. Thus we deduce $u \in \mathcal{N}^-$. □

The following lemma is used to extract a $(PS)_{c_1}$ (or $(PS)_{c_2}$) sequence from the minimizing sequence of the energy functional I on the Nehari manifold $\mathcal{N}^+$ (or $\mathcal{N}^-$).

Lemma 2.9. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then for every $u \in \mathcal{N}^+$, there exist $\epsilon > 0$ and a differentiable function $\varphi^+ : B_\epsilon(0) \to \mathbb{R}_+ := (0,+\infty)$ such that*

$$\varphi^+(0) = 1, \quad \varphi^+(w)(u-w) \in \mathcal{N}^+, \quad \forall w \in B_\epsilon(0)$$

and

$$\langle(\varphi^+)'(0), w\rangle = L(u,w)/K''_u(1), \tag{2.17}$$

where

$$L(u,w) = 2\langle u, w\rangle - q\int_{\mathbb{R}^N} f(x)|u|^{q-2}uw\,dx - p\int_{\mathbb{R}^N} g(x)|u|^{p-2}uw\,dx.$$

Moreover, for any $C_1, C_2 > 0$, there exists $C > 0$ such that if $C_1 \leq \|u\| \leq C_2$, then

$$\left|\left\langle \left(\varphi^+\right)'(0), w\right\rangle\right| \leq C\|w\|.$$

Proof. We define $F : \mathbb{R} \times E \to \mathbb{R}$ by

$$F(t,w) = K'_{u-w}(t),$$

it is easy to see that F is differentiable. Since $F(1,0) = 0$ and

$$F_t(1,0) = K''_u(1) > 0,$$

we apply the implicit function theorem at point $(1,0)$ to obtain the existence of $\epsilon > 0$ and differentiable function $\varphi^+ : B_\epsilon(0) \to \mathbb{R}_+ := (0,+\infty)$ such that

$$\varphi^+(0) = 1, \qquad F\left(\varphi^+(w), w\right) = 0, \qquad \forall w \in B_\epsilon(0).$$

Thus,

$$\varphi^+(w)(u-w) \in \mathcal{N}, \qquad \forall w \in B_\epsilon(0).$$

Next, we prove for any $w \in B_\epsilon(0)$, $\varphi^+(u-w) \in \mathcal{N}^+$. Indeed, by $u \in \mathcal{N}^+$ and the set $\mathcal{N}^- \cup \mathcal{N}^0$ is closed, we know $\operatorname{dist}\left(u, \mathcal{N}^- \cup \mathcal{N}^0\right) > 0$. Since $\varphi^+(w)(u-w)$ is continuous with respect to w, we know when ϵ is small enough, for $w \in B_\epsilon(0)$, then

$$\left\|\varphi^+(w)(u-w) - u\right\| < \frac{1}{2}\operatorname{dist}\left(u, \mathcal{N}^- \cup \mathcal{N}^0\right),$$

so

$$\begin{aligned}\left\|\varphi^+(w)(u-w) - \mathcal{N}^- \cup \mathcal{N}^0\right\| &\geq \operatorname{dist}\left(u, \mathcal{N}^- \cup \mathcal{N}^0\right) - \operatorname{dist}\left(\varphi^+(w)(u-w), u\right) \\ &> \frac{1}{2}\operatorname{dist}\left(u, \mathcal{N}^- \cup \mathcal{N}^0\right) > 0.\end{aligned}$$

Thus, for $w \in B_\epsilon(0)$, then $\varphi^+(w)(u-w) \in \mathcal{N}^+$.

Besides, by the differentiability of implicit function theorem, we have

$$\left\langle (\varphi^+)'(0), w \right\rangle = -\frac{\langle F_w(1,0), w\rangle}{F_t(1,0)}.$$

Note that $L(u,w) = -\langle F_w(1,0), w\rangle$ and $K_u''(1) = F_t(1,0)$. Therefore (2.17) holds.

In the following we prove that there exists $\delta > 0$ such that $K_u''(1) \geq \delta > 0$ with $C_1 \leq \|u\| \leq C_2$, $u \in \mathcal{N}^+$, where $C_1, C_2 > 0$. On the contrary, if there exists a sequence $\{u_n\} \in \mathcal{N}^+$ with $C_1 \leq \|u_n\| \leq C_2$, such that for any δ_n sufficiently small, $K_{u_n}''(1) \leq \delta_n$, $\delta_n \to 0$ as $n \to \infty$. From (2.6) we have

$$(2-q)\left\|u_n\right\|^2 = (p-q)\int_{\mathbb{R}^N} g(x)\left|u_n\right|^p\,dx + O(\delta_n),$$

where $O(\delta_n) \to 0$ as $\delta_n \to 0$. Noting that $1 < q < 2 < p < 2^*$, $C_1 \leq \|u_n\| \leq C_2$ and (2.8), we have

$$(2-q)\left\|u_n\right\|^2 \leq (p-q)|g|_\infty S_p^{-\frac{p}{2}}\left\|u_n\right\|^p + O\left(\delta_n\right),$$

and so

$$\left\|u_n\right\| \geq \left(\frac{(2-q)S_p^{p/2}}{(p-q)|g|_\infty}\right)^{1/(p-2)} + O\left(\delta_n\right). \tag{2.18}$$

From (2.7) we also have

$$(p-2)\left\|u_n\right\|^2 = (p-q)\int_{\mathbb{R}^N} f(x)\left|u_n\right|^q\,dx + O\left(\delta_n\right).$$

In view of (2.10), we have

$$(p-2)\left\|u_n\right\|^2 \leq (p-q)|f|_{q*}S_p^{-\frac{q}{2}}\left\|u_n\right\|^q + O\left(\delta_n\right),$$

which implies that

$$\left\|u_n\right\| \leq \left(\frac{(p-q)|f|_{q*}}{(p-2)S_p^{q/2}}\right)^{1/(2-q)} + O\left(\delta_n\right). \tag{2.19}$$

Let $n \to \infty$, from (2.18) and (2.19) we deduce a contradiction.

Thus if $C_1 \leq \|u\| \leq C_2$, then there exists $C > 0$ such that

$$\left|\left\langle \left(\varphi^+\right)'(0), w\right\rangle\right| \leq C\|w\|.$$

This completes the proof. □

Similarly, we establish the following lemma.

Lemma 2.10. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then for every $u \in \mathcal{N}^-$, there exist $\epsilon > 0$ and a differentiable function $\varphi^- : B_\epsilon(0) \to \mathbb{R}_+ := (0,+\infty)$*

$$\varphi^-(0) = 1, \qquad \varphi^-(w)(u-w) \in \mathcal{N}^-, \qquad \forall w \in B_\epsilon(0)$$

and

$$\left\langle (\varphi^-)'(0), w \right\rangle = L(u,w)/K_u''(1),$$

where

$$L(u,w) = 2\langle u,w\rangle - q\int_{\mathbb{R}^N} f(x)|u|^{q-2}uw\,dx - p\int_{\mathbb{R}^N} g(x)|u|^{p-2}uw\,dx.$$

Moreover, for any $C_1, C_2 > 0$, there exists $C > 0$ such that if $C_1 \le \|u\| \le C_2$,

$$|\langle (\varphi^-)'(0), w\rangle| \le C\|w\|.$$

From above, we can extract a $(PS)_{c_1}$ (or $(PS)_{c_2}$) sequence from the minimizing sequence of the energy functional I on the Nehari manifold $\mathcal{N}^+$ (or $\mathcal{N}^-$).

Lemma 2.11. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, then the minimizing sequence $\{u_n\} \subset \mathcal{N}^+$ is the $(PS)_{c_1}$ sequence in E.*

Proof. By Lemma 2.10 and the Ekeland Variational Principle [10,23] on $\mathcal{N}^+ \cup \mathcal{N}^0$, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}^+ \cup \mathcal{N}^0$ such that

$$\inf_{u\in\mathcal{N}^+\cup\mathcal{N}^0} I(u) \le I(u_n) < \inf_{u\in\mathcal{N}^+\cup\mathcal{N}^0} I(u) + \frac{1}{n}, \tag{2.20}$$

$$I(u_n) - \frac{1}{n}\|v - u_n\| \le I(v), \quad \forall v \in \mathcal{N}^+ \cup \mathcal{N}^0. \tag{2.21}$$

From Proposition 2.5, we know for each $u \in E\backslash\{0\}$, there is a unique t^+ such that $t^+u \in \mathcal{N}^+$, then $\inf_{u\in\mathcal{N}^+} I \le I(t^+u)$. By Lemma 2.7 and $I(0) = 0$, we get that $\inf_{u\in\mathcal{N}^+\cup\mathcal{N}^0} I(u) = \inf_{u\in\mathcal{N}^+} I(u) = c_1$. Thus we may assume $u_n \in \mathcal{N}^+$, $I(u_n) \to c_1 < 0$. By Lemma 2.9, since $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma)$, we can find $\epsilon_n > 0$ and differentiable function $\varphi_n^+ = \varphi_n^+(w) > 0$ such that

$$\varphi_n^+(w)(u_n - w) \in \mathcal{N}^+, \qquad \forall w \in B_{\epsilon_n}(0).$$

By the continuity of $\varphi_n^+(w)$ and $\varphi_n^+(0) = 1$, without loss of generality, we can assume ϵ_n is sufficiently small such that $1/2 \le \varphi_n^+(w) \le 3/2$ for $\|w\| < \epsilon_n$. From $\varphi_n^+(w)(u_n - w) \in \mathcal{N}^+$ and (2.21) , we have

$$I\left(\varphi_n^+(w)(u_n - w)\right) \ge I(u_n) - \frac{1}{n}\left\|\varphi_n^+(w)(u_n - w) - u_n\right\|,$$

which implies that

$$\left\langle I'(u_n), \varphi_n^+(w)(u_n - w) - u_n\right\rangle + o\left(\left\|\varphi_n^+(w)(u_n - w) - u_n\right\|\right) \ge -\frac{1}{n}\left\|\varphi_n^+(w)(u_n - w) - u_n\right\|.$$

Consequently,

$$\begin{aligned}&\varphi_n^+(w)\left\langle I'(u_n), w\right\rangle + \left(1 - \varphi_n^+(w)\right)\left\langle I'(u_n), u_n\right\rangle \\ &\quad \le \frac{1}{n}\left\|\left(\varphi_n^+(w) - 1\right)u_n - \varphi_n^+(w)w\right\| + o\left(\left\|\varphi_n^+(w)(u_n - w) - u_n\right\|\right).\end{aligned}$$

By the choice of ϵ_n and $1/2 \le \varphi_n^+(w) \le 3/2$, we infer that there exists $C_3 > 0$ such that

$$\left|\left\langle I'(u_n), w\right\rangle\right| \le \frac{1}{n}\left\|\left\langle (\varphi_n^+)'(0), w\right\rangle u_n\right\| + \frac{C_3}{n}\|w\| + o\left(\left|\left\langle (\varphi_n^+)'(0), w\right\rangle\right| (\|u_n\| + \|w\|)\right).$$

Below we prove for $\{u_n\} \subset \mathcal{N}^+$, $\inf_n \|u_n\| \ge C_1 > 0$, where C_1 is a constant. Indeed, if not, then $I(u_n)$ would converge to zero, which contradicts $I(u_n) \to c_1 < 0$. Moreover, by Lemma 2.6 we know that I is coercive on $\mathcal{N}^+$, $\{u_n\}$ is bounded in E. Thus, there exists $C_2 > 0$ such that $0 < C_1 \le \|u_n\| \le C_2$. From Lemma 2.9, $\left|\left\langle (\varphi_n^+)'(0), w\right\rangle\right| \le C\|w\|$. So

$$\left|\left\langle I'(u_n), w\right\rangle\right| \le \frac{C}{n}\|w\| + \frac{C}{n}\|w\| + o(\|w\|)$$

and

$$\begin{aligned} \left\|I'(u_n)\right\| &= \sup_{w \in E\setminus\{0\}} \frac{|\langle I'(u_n), w\rangle|}{\|w\|} \le \frac{C}{n} + o(1), \\ \left\|I'(u_n)\right\| &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.22}$$

Thus, $\{u_n\} \subset \mathcal{N}^+$ is $(PS)_{c_1}$ for I in E. □

Lemma 2.12. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma)$, then the minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ is the $(PS)_{c_2}$ sequence in E.*

Proof. From Lemma 2.8, $\mathcal{N}^-$ is closed in E. By Lemma 2.6, we know I is coercive on $\mathcal{N}^-$. So we use the Ekeland Variational Principle [23] on $\mathcal{N}^-$ to obtain a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ such that

$$\begin{aligned} \inf_{u \in \mathcal{N}^-} I(u) \le I(u_n) &< \inf_{u \in \mathcal{N}^-} I(u) + \frac{1}{n}, \\ I(u_n) - \frac{1}{n}\|v - u_n\| &\le I(v), \qquad \forall v \in \mathcal{N}^-. \end{aligned}$$

In view of (2.15) and Lemma 2.6, we know that there exist $C_1, C_2 > 0$ such that

$$0 < C_1 \le \|u_n\| \le C_2.$$

Hence by Lemma 2.10, in the same way as Lemma 2.11, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$ is the $(PS)_{c_2}$ sequence in E. □

The following lemmas aims at obtaining the critical points of I on the whole space from the critical points of $I|_{\mathcal{N}^+}$ and $I|_{\mathcal{N}^-}$, respectively.

Lemma 2.13. *Suppose that u is a local minimizer for I on $\mathcal{N}^+$. Then $I'(u) = 0$.*

Proof. If $u \ne 0$, u is a local minimizer for I on $\mathcal{N}^+$, then u is a nontrivial solution of the optimization problem

$$\text{minimize } I \text{ subject to } \Psi(u) = 0,$$

where $\Psi(u)$ is described in (2.5). Then, $u \in \mathcal{N}^+ \subset \mathcal{N}$ such that

$$I(u) = c_1 = \inf_{u \in \mathcal{N}^+} I(u) = \inf_{u \in \mathcal{N}} I(u).$$

Note that $\Psi'(u) \neq 0$ and $\mathcal{N}^+$ is a local differential manifold. So by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $I'(u) = \mu\Psi'(u)$. Thus

$$\langle I'(u), u \rangle = \mu \langle \Psi'(u), u \rangle .$$

Since $u \in \mathcal{N}^+$, we have $\langle I'(u), u\rangle = 0$ and $\langle \Psi'(u), u\rangle = K''_u(1) \neq 0$. Hence, $\mu = 0$ and $I'(u) = 0$. □

Lemma 2.14. *Suppose that u is a nontrivial critical point of $I|_{\mathcal{N}^-}$, then it is a nontrivial critical point of I in E, i.e., $I'(u) = 0$.*

Proof. If u is a nontrivial critical point of $I|_{\mathcal{N}^-}$, i.e., $u \in \mathcal{N}^- \backslash \{0\}$ and $(I|_{\mathcal{N}^-})'(u) = 0$. Note that $\mathcal{N}^-$ is a local differential manifold and $\Psi'(u) \neq 0$, where $\Psi(u)$ is described in (2.5). So by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $I'(u) = \mu\Psi'(u)$. Thus

$$\langle I'(u), u \rangle = \mu \langle \Psi'(u), u \rangle .$$

Since $u \in \mathcal{N}^-$, we have $\langle I'(u), u\rangle = 0$ and $\langle \Psi'(u), u\rangle = K''_u(1) \neq 0$. Hence, $\mu = 0$ and $I'(u) = 0$. Thus the proof is complete. □

3 Proof of Theorem 1.1

In order to obtain the nontrivial solutions, we bring in the following lemma.

Lemma 3.1 (Lions [19,20,23])**.** *Let $r > 0, q \in [2, 2^*)$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u|^q \, dx = 0,$$

then we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$. Here $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 1, 2$.

Lemma 3.2. *Let $\{u_n\} \subset E$ be a bounded $(PS)_c$ sequence for I. Then either*

(i) $u_n \to 0$ in E, or

(ii) there exist a sequence $\{y_n\} \in \mathbb{R}^N$ and constants r, $\delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \, dx \geq \delta > 0.$$

Proof. Suppose the condition (ii) is not satisfied, i.e. for any $r > 0$, we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2 \, dx = 0.$$

Then by Lemma 3.1, $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$. Therefore,

$$0 \leq \left| \int_{\mathbb{R}^N} f(x) |u_n|^q \, dx + \int_{\mathbb{R}^N} g(x) |u_n|^p \, dx \right| \leq |f|_{q^*} |u_n|_p^q + |g|_\infty |u_n|_p^p \to 0.$$

Since $\{u_n\} \subset E$ is a bounded $(PS)_c$ sequence for I, we have

$$o(1) = I'(u_n) u_n = \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x) |u_n|^2 \right) dx - \left(\int_{\mathbb{R}^N} f(x) |u_n|^q \, dx + \int_{\mathbb{R}^N} g(x) |u_n|^p \, dx \right),$$

as $n \to \infty$. It follows that $u_n \to 0$ in E as $n \to \infty$, i.e., the condition (i) is satisfied. Thus, the proof is complete. □

To recover the compactness, we need to evaluate the critical value of Equation (1.1) through the critical value of a autonomous equation. Now, we consider the following autonomous equation

$$\begin{cases} -\triangle u + V_\infty u = g_\infty |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{3.1}$$

where $2 < p < 2^*(2^* = \infty$ if $N = 1, 2$ and $2^* = 2N/(N-2)$ if $N \geq 3)$. The corresponding functional and the corresponding manifold are

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_\infty |u|^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^N} g_\infty |u|^p \, dx$$

and

$$\mathcal{N}_\infty = \left\{ u \in H^1\left(\mathbb{R}^N\right) \setminus \{0\} \;\middle|\; \langle I'_\infty(u), u\rangle = 0 \right\}.$$

Let w_0 be the unique radially symmetric solution of Equation (3.1) such that $I_\infty(w_0) = c_\infty$, where $c_\infty = \inf_{u \in \mathcal{N}_\infty} I_\infty(u)$ (see [3,17]).

In the following, we prove that when the critical value of Equation (1.1) is contained in the suitable range, $(PS)_c$ condition holds.

Proposition 3.3. *Let the assumptions of (V), (f) and (g) be satisfied, if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma^*)$, then each $(PS)_c$ sequence $\{u_n\} \subset \overline{\mathcal{N}}$ ($\overline{\mathcal{N}} = \mathcal{N}^+$ or $\mathcal{N}^-$) for I in E with $c < c_1 + c_\infty$ has a strongly convergent subsequence, where c_1 is described in Lemma 2.7.*

Proof. Let $\{u_n\} \subset \overline{\mathcal{N}}$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

From Lemma 2.6 we know that the $(PS)_c$ sequence $\{u_n\} \subset \overline{\mathcal{N}}$ for I in E is bounded. Then, going if necessary to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } E, \\ u_n &\to u && \text{in } L^r_{\text{loc}}(\mathbb{R}^N),\ r \in [2, 2^*), \\ u_n &\to u && \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.2}$$

Set $v_n := u_n - u$, then there exists $C > 0$ such that $\|v_n\| < C$. It is sufficient to prove that $v_n \to 0$ in E as $n \to \infty$.

Note that

$$\left||u_n|^s - |u|^s\right| \leq |u_n - u|^s \quad \text{for } s > 1, \tag{3.3}$$

we can infer that

$$\int_{\mathbb{R}^N} f(x)\,|u_n|^q \, dx \to \int_{\mathbb{R}^N} f(x)|u|^q \, dx \quad \text{and} \quad \int_{\mathbb{R}^N} f(x)\,|v_n|^q \, dx \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Indeed, from the condition (f), we have that for any $\epsilon > 0$, there exists R sufficiently large such that

$$\left(\int_{|x|>R} |f(x)|^{q^*} dx\right)^{1/q^*} < \epsilon.$$

And from $\{u_n\} \subset \overline{\mathcal{N}}$ in E is bounded, we can infer that $\left(\int_{|x|>R} |u_n - u|^p \, dx\right)^{q/p}$ is bounded. These facts with (3.2), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \left| f(x) \left(|u_n|^q - |u|^q \right) \right| dx \leq & \int_{\mathbb{R}^N} f(x) \, |u_n - u|^q \, dx \\
= & \int_{|x|\leq R} f(x) \, |u_n - u|^q \, dx + \int_{|x|>R} f(x) \, |u_n - u|^q \, dx \\
\leq & \left(\int_{|x|\leq R} |f(x)|^{q^*} dx \right)^{1/q^*} \left(\int_{|x|\leq R} |u_n - u|^p \, dx \right)^{q/p} \\
& + \left(\int_{|x|>R} |f(x)|^{q^*} dx \right)^{1/q^*} \left(\int_{|x|>R} |u_n - u|^p \, dx \right)^{q/p} \\
\to & \, 0 \quad \text{as } n \to \infty.
\end{aligned}$$

From (3.2) and Brézis–Lieb lemma in [23], we can deduce that

$$\begin{aligned}
I(v_n) &= I(u_n - u) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla (u_n - u)|^2 + V(x) \, |u_n - u|^2 \right) dx \\
&\quad - \frac{1}{p} \int_{\mathbb{R}^N} g(x) \, |u_n - u|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^N} f(x) \, |u_n - u|^q \, dx \\
&= I(u_n) - I(u) + o(1)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
I'(v_n) \, v_n &= I'(u_n - u)(u_n - u) \\
&= \int_{\mathbb{R}^N} \left(|\nabla (u_n - u)|^2 + V(x) \, |u_n - u|^2 \right) dx \\
&\quad - \int_{\mathbb{R}^N} f(x) \, |u_n - u|^q \, dx - \int_{\mathbb{R}^N} g(x) \, |u_n - u|^p \, dx. \\
&= I'(u_n) \, u_n - I'(u)u + o(1).
\end{aligned} \tag{3.6}$$

By Lemma 2.1, I' is weakly sequentially continuous in E, so $I'(u) = 0$. Therefore if $u \neq 0$ and $I'(u)u = 0$, then $u \in \mathcal{N}^+$ or $u \in \mathcal{N}^-$. According to Lemma 2.7, no matter $u \in \mathcal{N}^+$ or $u \in \mathcal{N}^-$, we all have $I(u) \geq c_1$. If $u = 0$, then $I(u) = I(0) = 0 > c_1$. So

$$I(v_n) = I(u_n) - I(u) + o(1) \leq c - c_1 + o(1) \tag{3.7}$$

and

$$I'(v_n) \, v_n = o(1). \tag{3.8}$$

Indeed, if $v_n \nrightarrow 0$ in E, we choose $(t_n) \subset (0, \infty)$ such that $\{t_n v_n\} \subset \mathcal{N}_\infty$. We will prove that the case of $\limsup_{n\to\infty} t_n > 1$, $\limsup_{n\to\infty} t_n < 1$ and $\limsup_{n\to\infty} t_n = 1$ cannot happen. Then we obtain a contradiction and $v_n \to 0$ in E. To do this, we distinguish the following three cases:

(i) $\limsup_{n\to\infty} t_n > 1$.

In this case, we may suppose there exist $\sigma > 0$ and a subsequence still denoted by $\{t_n\}$ such that $t_n \geq 1 + \sigma$ for all $n \in \mathbb{N}$. From (3.6) and (3.8), we have

$$I'(v_n) \, v_n = \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x) \, |v_n|^2 \right) dx - \int_{\mathbb{R}^N} f(x) |v_n|^q \, dx - \int_{\mathbb{R}^N} g(x) \, |v_n|^p \, dx = o(1). \tag{3.9}$$

Moreover, since $\{t_n v_n\} \subset \mathcal{N}_\infty$, then we have

$$I'_\infty(t_n v_n) t_n v_n = t_n^2 \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V_\infty |v_n|^2\right) dx - t_n^p \int_{\mathbb{R}^N} g_\infty |v_n|^p \, dx = 0. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain that

$$\int_{\mathbb{R}^N} (V_\infty - V(x)) |v_n|^2 \, dx + \int_{\mathbb{R}^N} (g(x) - g_\infty) |v_n|^p \, dx + \int_{\mathbb{R}^N} f(x) |v_n|^q \, dx$$
$$= \int_{\mathbb{R}^N} g_\infty \left(t_n^{p-2} - 1\right) |v_n|^p \, dx + o(1).$$

By conditions (V) and (g), for any $\epsilon > 0$, there exists $R = R(\epsilon) > 0$ such that

$$V(x) \geq V_\infty - \epsilon \quad \text{and} \quad g_\infty \geq g(x) - \epsilon \quad \text{for any } |x| > R. \tag{3.11}$$

This with (3.2) and (3.4) implies that

$$\left((1+\sigma)^{p-2} - 1\right) \int_{\mathbb{R}^N} g_\infty |v_n|^p \, dx \leq C\varepsilon + o(1). \tag{3.12}$$

By $v_n \nrightarrow 0$ in E and (3.9), similar to Lemma 3.2, we can prove that there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $r,\ \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |v_n|^2 \, dx \geq \delta > 0. \tag{3.13}$$

If we set $w_n(x) = v_n(x + y_n)$, then there exists a function w and a subsequence still denoted by $\{w_n\}$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^s_{loc}(\mathbb{R}^N)$ where $s \in [2, 6)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Moreover, by (3.13) there exists a subset $\Lambda \subset \mathbb{R}^N$ with positive measure such that $w \neq 0$ a.e. in Λ. It follows from (3.12) that

$$0 < \left((1+\sigma)^{p-2} - 1\right) \int_\Lambda g_\infty |w|^p \, dx \leq C\varepsilon + o(1),$$

where $\epsilon > 0$ is arbitrary. This is impossible.

(ii) $\limsup_{n\to\infty} t_n < 1$.

In this case, without loss of generality, we suppose that $t_n < 1$ for all $n \in \mathbb{N}$. From (3.2), (3.4), (3.7), (3.9), (3.10) and (3.11), we can deduce that

$$\begin{aligned}
c_\infty &\leq I_\infty(t_n v_n) = I_\infty(t_n v_n) - \frac{1}{p}\left\langle I'_\infty(t_n v_n), t_n v_n \right\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \left(|\nabla t_n v_n|^2 + V_\infty |t_n v_n|^2\right) dx < \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V_\infty |v_n|^2\right) dx \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x) |v_n|^2 + (V_\infty - V(x)) |v_n|^2\right) dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x) |v_n|^2\right) dx - \frac{1}{p} \int_{\mathbb{R}^N} g(x) |v_n|^p \, dx \\
&\quad + \left(\frac{1}{2} - \frac{1}{p}\right) (V_\infty - V(x)) |v_n|^2 \, dx + o(1) \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + V(x) |u_n|^2\right) dx - \frac{1}{p} \int_{\mathbb{R}^N} g(x) |u_n|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^N} f(x) |u_n|^q \, dx \\
&\quad - \left(\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \, dx - \frac{1}{p} \int_{\mathbb{R}^N} g(x)|u|^p \, dx - \frac{1}{q} \int_{\mathbb{R}^N} f(x)|u|^q \, dx\right) + C\epsilon + o(1) \\
&= I(u_n) - I(u) + C\epsilon + o(1) \leq c - c_1 + C\epsilon + o(1).
\end{aligned}$$

Let $n \to \infty$, we get $c \geq c_1 + c_\infty$. This contradicts $c < c_1 + c_\infty$.

(iii) $\limsup_{n\to\infty} t_n = 1$.

In this case, there exists a subsequence, still denoted by $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$. Note that

$$I(v_n) - I_\infty(t_n v_n) = \frac{1}{2}\int_{\mathbb{R}^N} (1-t_n^2)\,|\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\,|v_n|^2\,dx - \frac{t_n^2}{2}\int_{\mathbb{R}^N} V_\infty\,|v_n|^2\,dx$$
$$- \frac{1}{p}\int_{\mathbb{R}^N} g(x)\,|v_n|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N} g_\infty\,|t_n v_n|^p\,dx - \frac{1}{q}\int_{\mathbb{R}^N} f(x)\,|v_n|^q\,dx.$$

From (3.2), (3.4) and (3.11), we can infer that

$$I(v_n) \geq I_\infty(t_n v_n) - C\epsilon + o(1) \geq c_\infty - C\epsilon + o(1).$$

This with (3.7) implies that $c \geq c_1 + c_\infty$, which contradicts $c < c_1 + c_\infty$. □

Lemma 3.4. *If $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, Equation (1.1) has at least one positive solution.*

Proof. From Lemma 2.11, we know if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, then there is minimizing sequence $\{u_n\} \subset \mathcal{N}^+$ which is a $(PS)_{c_1}$ sequence in E. Obviously, $c_1 < c_1 + c_\infty$, so from Proposition 3.3, there is a strongly convergent subsequence still denoted by $\{u_n\}$ such that $u_n \to u_1$ in E. From Lemma 2.11 we know there exist $C_1, C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$, then $0 < C_1 \leq \|u_1\| \leq C_2$. Thus $u_1 \neq 0$.

Next we prove $u_1 \in \mathcal{N}^+$. Indeed, By (2.6), it follows that $K''_{u_n}(1) \to K''_{u_1}(1)$. From $K''_{u_n}(1) > 0$, we have $K''_{u_1}(1) \geq 0$. By Proposition 2.4 and $u_1 \neq 0$ we know, if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, then $K''_{u_1}(1) > 0$. Thus

$$u_1 \in \mathcal{N}^+, \quad I(u_1) = \lim_{n\to\infty} I(u_n) = \inf_{u\in\mathcal{N}^+} I(u).$$

We recall (see [11]) that $\int_{\mathbb{R}^N} |\nabla|u||^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx$, therefore $I(u_1) = I(|u_1|)$ and $|u_1| \in \mathcal{N}^+$, then, without loss of generality, we may assume that u_1 is positive. This with Lemma 2.13 implies the desired result. □

In the following, motivated by the arguments in [26], we will prove $c_2 < c_1 + c_\infty$. Let

$$w_l(x) = w_0(x + le), \quad \text{for } l \in \mathbb{R} \text{ and } e \in \mathbb{S}^{N-1},$$

where $\mathbb{S}^{N-1} = \left\{x \in \mathbb{R}^N : |x| = 1\right\}$. Then, $w_l(x)$ is also a positive solution of limit equation (3.1) and $I'_\infty(w_l)\,w_l = 0$, $c_\infty = I_\infty(w_l)$.

Lemma 3.5. *Under the assumptions of Proposition 3.3, if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0,\sigma^*)$, then*

$$c_2 < c_1 + c_\infty.$$

Proof. We prove this result in the following two steps.

Step 1: For all $l \in \mathbb{R}$, $\sup_{t\geq 0} I(u_1 + t w_l) < c_1 + c_\infty$.

Since

$$I(u_1 + t w_l) \to I(u_1) = c_1 < 0 \quad \text{as } t \to 0$$

and

$$I(u_1 + t w_l) \to -\infty \quad \text{as } t \to \infty,$$

then there exist $t_2 > t_1 > 0$ such that $I(u_1 + tw_l) < c_1 + c_\infty$ for all $t \in [0, t_1] \cup [t_2, \infty)$. It is sufficient to prove that $\sup_{t_1 \le t \le t_2} I(u_1 + tw_l) < c_1 + c_\infty$. Indeed, by Willem [23], we know that

$$I_\infty(tw_l) \le c_\infty \quad \text{for all } l \in \mathbb{R}.$$

Note that

$$(u+v)^p - u^p - v^p - pu^{p-1}v \ge 0 \quad \text{for } (u,v) \in [0,\infty) \times [0,\infty) \text{ and } p > 2.$$

Furthermore, since u_1 is one of positive solution of Equation (1.1), $w_l(x)$ is a positive solution of limit equation (3.1), $t_1 \le t \le t_2$ and the conditions (V), (f), (g), we can infer that

$$\begin{aligned}
I(u_1 + tw_l) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(u_1 + tw_l)|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\,|u_1 + tw_l|^2\, dx \\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N} f(x)\,|u_1 + tw_l|^q\, dx - \frac{1}{p}\int_{\mathbb{R}^N} g(x)\,|u_1 + tw_l|^p\, dx \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_1|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla tw_l|^2\, dx + \int_{\mathbb{R}^N} t\nabla u_1 \nabla w_l\, dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\,|u_1|^2\, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty\,|tw_l|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^N} (V(x) - V_\infty)\,|tw_l|^2\, dx \\
&\quad + \int_{\mathbb{R}^N} tV(x)u_1 w_l\, dx - \frac{1}{q}\int_{\mathbb{R}^N} f(x)\,|u_1|^q\, dx \\
&\quad - \frac{1}{p}\left(\int_{\mathbb{R}^N} g(x)\,|u_1|^p\, dx + \int_{\mathbb{R}^N} g_\infty\,|tw_l|^p\, dx + +p\int_{\mathbb{R}^N} g(x)\,|u_1|^{p-1}\, tw_l\, dx\right) \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^N} (g(x) - g_\infty)\,|tw_l|^p\, dx \\
&< I(u_1) + I_\infty(tw_l) \le c_1 + c_\infty.
\end{aligned}$$

Step 2: There exist $t_0 > 0$, $s_l \in (0,1)$ such that $u_1 + s_l t_0 w_l \in \mathcal{N}^-$, then combining Step 1, we obtain $c_2 < c_1 + c_\infty$.

First, we prove that

$$\mathcal{N}^- = \left\{ u \in E\backslash\{0\} : \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right) = 1 \right\}.$$

Indeed, for $u \in \mathcal{N}^-$, set $v = u/\|u\|$, then by Proposition 2.5, there is a unique $t^-(v) > 0$ such that $t^-(v)v \in \mathcal{N}^-$ i.e. $t^-(u/\|u\|)\,u/\|u\| \in \mathcal{N}^-$. Because of the uniqueness, $t^-(u/\|u\|)\,1/\|u\| = 1$ is proved. For $u \in E\backslash\{0\}$ with $t^-(u/\|u\|)\,1/\|u\| = 1$, set $v = u/\|u\| \in E\backslash\{0\}$, then by Proposition 2.5, there is a unique $t^-(v) > 0$ such that $t^-(v)v = t^-(u/\|u\|)\,u/\|u\| \in \mathcal{N}^-$, so $u \in \mathcal{N}^-$. Let

$$U_1 = \left\{ u \in E : \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right) > 1 \right\} \cup \{0\}$$

and

$$U_2 = \left\{ u \in E : \frac{1}{\|u\|} t^-\left(\frac{u}{\|u\|}\right) < 1 \right\}.$$

Then $\mathcal{N}^-$ separates E into two connected components U_1 and U_2, that is $E \setminus \mathcal{N}^- = U_1 \cup U_2$.

Define a path $\gamma_l(s) = u_1 + st_0 w_l$ for $s \in [0,1]$, then $\gamma_l(0) = u_1$ and $\gamma_l(1) = u_1 + t_0 w_l$. If we can prove $\gamma_l(0) = u_1 \in U_1$ and $\gamma_l(1) = u_1 + t_0 w_l \in U_2$, the continuity of $t(u)$ as in [23] yields that there exists $s_l \in (0,1)$ such that $u_1 + s_l t_0 w_l \in \mathcal{N}^-$. Thus, it is sufficient to prove that (i) $\gamma_l(0) = u_1 \in U_1$ and (ii) $\gamma_l(1) = u_1 + t_0 w_l \in U_2$.

(i) $\gamma_l(0) = u_1 \in U_1$.

Indeed, $u_1 \in \mathcal{N}^+$, if $\mathcal{N}^+ \subset U_1$, then $u_1 \in U_1$. In the following we prove $\mathcal{N}^+ \subset U_1$. For any $u \in \mathcal{N}^+ \subset E$, there is unique $t^+(u)$ such that $t^+(u)u \in \mathcal{N}^+$. From the uniqueness, we obtain $t^+(u) = 1$. By Proposition 2.5, we have $1 = t^+(u) < t_{\max}(u) < t^-(u)$. Since $t^-(u) = t^-\left(u/\|u\|\right)/\|u\|$, then $1 < t^-\left(u/\|u\|\right)/\|u\|$, that is $\mathcal{N}^+ \subset U_1$.

(ii) $\gamma_l(1) = u_1 + t_0 w_l \in U_2$.

Indeed, for any $u_n \in E \setminus \{0\}$, there exists $t_n^- := t^-(u_n)$ such that $\{t_n^- u_n\} \subset \mathcal{N}^-$, we fist prove $\{t_n^-\}$ is bounded. Suppose on the contrary that there exists a subsequence, we still denote $\{t_n^-\}$, such that $t_n^- \to \infty$. Then $I(t_n^- u_n) \to -\infty$, this contradicts Lemma 2.6 I is bounded from below on $\mathcal{N}^-$. So there exists $M > 0$ such that $t^-\left((u_1 + t_0 w_l)/\|u_1 + t_0 w_l\|\right) < M$. Let

$$t_0 = \left(\frac{p-2}{pc_\infty}\left|M^2 - \|u_1\|^2\right|\right)^{\frac{1}{2}},$$

where

$$c_\infty = I_\infty(w_l) = I_\infty(w_l) - \frac{1}{p}I'_\infty(w_l)\,w_l = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N}\left(|\nabla w_l|^2 + V_\infty |w_l|^2\right)dx.$$

Since $w_l(x) = w_0(x + le) \rightharpoonup 0$, $\nabla w_l = \nabla w_0(x + le) \rightharpoonup 0$ as $l \to \infty$ and V is a positive bounded function, we have $\int_{|x|\le R}(V_\infty - V(x))|w_l|^2\,dx \to 0$ and $\int_{|x|>R}(V_\infty - V(x))|w_l|^2\,dx \to 0$ as $l \to \infty$ due to (3.11). Then

$$\begin{aligned}
\|u_1 + t_0 w_l\|^2 &= \|u_1\|^2 + t_0^2\|w_l\|^2 + 2t_0\int_{\mathbb{R}^N}(\nabla u_1 \nabla w_l + V(x)u_1 w_l)\,dx \\
&= \|u_1\|^2 + \frac{p-2}{pc_\infty}\|w_l\|^2\left|M^2 - \|u_1\|^2\right| + 2t_0\int_{\mathbb{R}^N}(\nabla u_1 \nabla w_l + V(x)u_1 w_l)\,dx \\
&= \|u_1\|^2 + \frac{p-2}{pc_\infty}\|w_l\|^2\left|M^2 - \|u_1\|^2\right| + o(1) \quad \text{as } l \to \infty.
\end{aligned}$$

and

$$\begin{aligned}
c_\infty = I_\infty(w_l) &= I_\infty(w_l) - \frac{1}{p}I'_\infty(w_l)\,w_l \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N}\left(|\nabla w_l|^2 + V_\infty|w_l|^2\right)dx \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N}\left(|\nabla w_l|^2 + V(x)|w_l|^2 + (V_\infty - V(x))|w_l|^2\right)dx \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^N}\left(|\nabla w_l|^2 + V(x)|w_l|^2\right)dx + o(1) \quad \text{as } l \to \infty.
\end{aligned}$$

From above, we deduce that

$$\begin{aligned}
\|u_1 + t_0 w_l\|^2 &= \|u_1\|^2 + t_0^2\|w_l\|^2 + o(1) \\
&= \|u_1\|^2 + \frac{p-2}{pc_\infty}\|w_l\|^2\left|M^2 - \|u_1\|^2\right| + o(1) \\
&> \|u_1\|^2 + \left|M^2 - \|u_1\|^2\right| + o(1) > M^2 + o(1) \\
&> \left(t^-\left(\frac{u_1 + t_0 w_l}{\|u_1 + t_0 w_l\|}\right)\right)^2 + o(1) \quad \text{as } l \to \infty.
\end{aligned}$$

Thus $t^-\left((u_1 + t_0 w_l)/\|u_1 + t_0 w_l\|\right)/\|u_1 + t_0 w_l\| < 1$, $u_1 + t_0 w_l \in U_2$. □

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 2.12, we know if $|f|_{q^*}|g|_\infty^{(2-q)/(p-2)} \in (0, \sigma^*)$, then there is a minimizing sequence $\{u_n\} \subset \mathcal{N}^-$, which is a $(PS)_{c_2}$ sequence in E. By Lemma 3.5, $c_2 < c_1 + c_\infty$, so from Proposition 3.3, there is a strongly convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_2$ in E as $n \to \infty$. By Lemma 2.8 the set $\mathcal{N}^-$ is closed, we know $u_2 \in \mathcal{N}^-$. Thus, $I(u_2) = \lim_{n\to\infty} I(u_n) = \inf_{u\in\mathcal{N}^-} I(u)$. Since $I(u_2) = I(|u_2|)$ and $|u_2| \in \mathcal{N}^-$, then, without loss of generality, we may assume that u_2 is positive. Lemma 2.14 implies that u_2 is a positive solution of Equation (1.1). This with Lemmas 2.7 and 3.4 completes the proof of Theorem 1.1. □

Acknowledgements

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version. This work is partially supported by Natural Science Foundation of China (Nos: 11371090, 11571140, 11601204, 11671077). The first author is also supported by Natural Science Foundation of Jiangsu Education Committee (No. 18KJB110002).

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