Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **81**, 1–7; https://doi.org/10.14232/ejqtde.2018.1.81 www.math.u-szeged.hu/ejqtde/

Bounded solutions for a class of Hamiltonian systems

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Received 8 December 2017, appeared 26 September 2018
Communicated by Gabriele Bonanno

Abstract. We obtain solutions bounded for all $t \in (-\infty, \infty)$ of systems of ordinary differential equations as limits of the solutions of the corresponding Dirichlet problems on $(-L, L)$, with $L \to \infty$. Using the variational approach, we derive a priori estimates for the corresponding Dirichlet problems, allowing passage to the limit, via a diagonal sequence.

Keywords: solutions bounded for all t, a priori estimates.

2010 Mathematics Subject Classification: 34B15.

1 Introduction

For $-\infty < t < \infty$, we consider the equation

$$u'' - a(t)u^3 = f(t)\,, \tag{1.1}$$

with continuous functions $a(t) > 0$ and $f(t)$. Clearly, "most" solutions of (1.1) blow up in finite time, for both increasing and decreasing t. By using two-dimensional shooting, S.P. Hastings and J.B. McLeod [2] showed that the equation (1.1) has a uniformly bounded on $(-\infty, \infty)$ solution, in case of constant $a(t)$ and uniformly bounded $f(t)$. Their proof used some non-trivial topological property of a plane. We use passage to the limit as in P. Korman and A. C. Lazer [4] to obtain the existence of a solution uniformly bounded on $(-\infty, \infty)$ for (1.1), and for similar equations. We produce a bounded solution as a limit of the solutions of the corresponding Dirichlet problems

$$u'' - a(t)u^3 = f(t) \qquad \text{for } t \in (-L, L), \quad u(-L) = u(L) = 0\,, \tag{1.2}$$

as $L \to \infty$. If $f(t)$ is bounded, it follows by the maximum principle that the solution of (1.2) satisfies a uniform in L a priori estimate, which allows passage to the limit.

Then we use a variational approach motivated by P. Korman and A.C. Lazer [4] (see also P. Korman, A. C. Lazer and Y. Li [5]), to get a similar result for a class of Hamiltonian systems. Again, we consider the corresponding Dirichlet problem on $(-L, L)$, which we solve by the minimization of the corresponding functional, obtaining in the process a uniform in L a priori estimate, which allows passage to the limit as $L \to \infty$.

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2 A model equation

Theorem 2.1. *Consider the equation (for $u = u(t)$)*

$$u'' - a(t)u^3 = f(t)\,, \tag{2.1}$$

where the given functions $a(t) \in C(\mathbb{R})$ and $f(t) \in C(\mathbb{R})$ are assumed to satisfy

$$|f(t)| \le M, \quad \text{for all } t \in \mathbb{R}, \text{ and some constant } M > 0\,,$$

and

$$a_0 \le a(t) \le a_1, \quad \text{for all } t \in \mathbb{R}, \text{ and some constants } a_1 \ge a_0 > 0\,.$$

Then the problem (2.1) has a classical solution uniformly bounded for all $t \in \mathbb{R}$, i.e., $|u(t)| \le K$ for all $t \in \mathbb{R}$, and some $K > 0$. Such a solution is unique.

Proof. We shall obtain a bounded solution as a limit of solutions to the corresponding Dirichlet problems

$$u'' - a(t)u^3 = f(t) \quad \text{for } t \in (-L, L), \quad u(-L) = u(L) = 0\,, \tag{2.2}$$

as $L \to \infty$. Large positive constants are supersolutions of this problem, while large negative constants provide subsolutions, which proves the existence of a solution, bounded uniformly in L, see e.g, L. Evans [1].

We claim that there is a *uniform in L bound* in $C^2[-L, L]$ for any solution of (2.2), i.e., there is a constant $K > 0$, so that for all $t \in [-L, L]$, and all $L > 0$,

$$|u(t)| \le K\,, \quad |u'(t)| \le K\,, \quad \text{and} \quad |u''(t)| \le K\,. \tag{2.3}$$

The first of these estimates is already established. From the equation (2.2) we get a uniform bound on $|u''(t)|$. Note that for all $t \in \mathbb{R}$, we can write

$$u(t+1) = u(t) + u'(t) + \int_t^{t+1} (t + 1 - \xi)u''(\xi)\, d\xi, \tag{2.4}$$

from which we immediately deduce a uniform bound on $|u'(t)|$.

We now take a sequence $L_j \to \infty$, and denote by $u_j(t) \in H_0^1(-\infty, \infty)$ the bounded solution of the problem (2.2) on the interval $(-L_j, L_j)$, extended as zero to the outside of the interval $(-L_j, L_j)$. For all $t_1 < t_2$, writing

$$|u_j(t_2) - u_j(t_1)| = \left| \int_{t_1}^{t_2} u_j'\, dt \right| \le \sqrt{t_2 - t_1} \left(\int_{t_1}^{t_2} \left(u_j'\right)^2 dt \right)^{1/2} \le K\,(t_2 - t_1)\,, \tag{2.5}$$

in view of (2.3), we conclude that the sequence $\{u_j(t)\}$ is equicontinuous and uniformly bounded on every interval $[-L_p, L_p]$. By the Arzelà–Ascoli theorem, it has a uniformly convergent subsequence on every $[-L_p, L_p]$. So let $\{u^1_{j_k}\}$ be a subsequence of $\{u_j\}$ that converges uniformly on $[-L_1, L_1]$. Consider this subsequence on $[-L_2, L_2]$ and select a further subsequence $\{u^2_{j_k}\}$ of $\{u^1_{j_k}\}$ that converges uniformly on $[-L_2, L_2]$. We repeat this procedure for all p, and then take the diagonal sequence $\{u^k_{j_k}\}$. It follows that it converges uniformly on any bounded interval to a function $u(t)$.

Expressing $\left(u_{j_k}^k\right)''$ from the equation (2.2), we conclude that the sequence $\left\{\left(u_{j_k}^k\right)''\right\}$, and then also $\left\{\left(u_{j_k}^k\right)'\right\}$ (in view of (2.4)), converge uniformly on bounded intervals. Denote $v(t) := \lim_{k\to\infty}\left(u_{j_k}^k\right)''(t)$. For t belonging to any bounded interval (a,b), similarly to (2.4), we write

$$u_{j_k}^k(t) = u_{j_k}^k(a) + (t-a)\left(u_{j_k}^k\right)'(a) + \int_a^t (t-\xi)\left(u_{j_k}^k\right)''(\xi)\, d\xi\,,$$

and conclude that $u(t) \in C^2(-\infty,\infty)$, and $u''(t) = v(t)$. Hence, we can pass to the limit in the equation (2.2), and conclude that $u(t)$ solves this equation on $(-\infty,\infty)$. We have $|u(t)| \le K$ on $(-\infty,\infty)$, proving the existence of a uniformly bounded solution.

Turning to the uniqueness, the difference $w(t)$ of any two bounded solutions $u(t)$ and $\tilde{u}(t)$ of (2.1) would be a bounded for all t solution of the linear equation

$$w'' - b(t)w = 0\,, \tag{2.6}$$

with $b(t) = a(t)(u^2 + u\tilde{u} + \tilde{u}^2) > 0$. It follows that $w(t)$ is convex when it is positive. If at some t_0, $w(t_0) > 0$ and $w'(t_0) > 0$ ($w'(t_0) < 0$), then $w(t)$ is unbounded as $t \to \infty$ ($t \to -\infty$), a contradiction. In case $w(t_0) < 0$ for some t_0, we observe that $-w(t)$ is also a solution of (2.6), and reach the same contradiction. Therefore, $w \equiv 0$. □

We now discuss the dynamical significance of the bounded solution, established in Theorem 2.1, let us call it $u_0(t)$. The difference of any two solutions of (2.1) satisfies (2.6). We see from (2.6) that any two solutions of (2.1) intersect at most once. Also from our analysis of the equation (2.6) above, we can expect $u_0(t)$ to have one-dimensional stable manifold as $t \to \pm\infty$, and any solution not on the stable manifold to become unbounded. It follows that $u_0(t)$ provides the only possible asymptotic form of the solutions that are bounded as $t \to \infty$ (or $t \to -\infty$), while all other solutions become unbounded.

Next we show that the conditions of this theorem cannot be completely removed. If $a(t) \equiv 0$, then for $f(t) = 1$, all solutions of (2.1) are unbounded as $t \to \pm\infty$. The same situation may occur in case $a(t) > 0$, if $f(t)$ is unbounded. Indeed, the equation

$$u'' - u^3 = 2\cos t - t\sin t - t^3 \sin^3 t \tag{2.7}$$

has a solution $u(t) = t\sin t$. Let $\tilde{u}(t)$ be any other solution of (2.7). Then $w(t) = u(t) - \tilde{u}(t)$ satisfies (2.6), with $b(t) = u^2 + u\tilde{u} + \tilde{u}^2 > 0$. Clearly, $w(t)$ cannot have points of positive local maximum, or negative local minimum. But then $\tilde{u}(t)$ cannot remain bounded as $t \to \pm\infty$, since in such a case the function $w(t)$ would be unbounded with points of positive local maximum and negative local minimum. It follows that all solutions of (2.7) are unbounded as $t \to \pm\infty$.

Remark. A similar result holds for the equation

$$u'' + h(t,u) = 0\,,$$

where $h \in C^{0,1}(\mathbb{R}\times\mathbb{R})$, provided that the corresponding Dirichlet problem on $(-L,L)$ has a supersolution and subsolution pair, uniformly in L.

3 Bounded solutions of Hamiltonian systems

We use variational approach to get a similar result for a class of Hamiltonian systems. We shall be looking for uniformly bounded solutions $u \in H^1(\mathbb{R}; \mathbb{R}^m)$ of the system

$$u_i'' - a(t)V_{z_i}(u_1, u_2, \ldots, u_m) = f_i(t), \qquad i = 1, \ldots, m. \tag{3.1}$$

Here $u_i(t)$ are the unknown functions, $a(t)$ and $f_i(t)$ are given functions on $\mathbb{R}$, $i = 1, \ldots, m$, and $V(z)$ is a given function on $\mathbb{R}^m$.

Theorem 3.1. *Assume that $a(t) \in C(\mathbb{R})$ satisfies $a(t) \geq a_0$ for all t, and some constant $a_0 > 0$. Assume that $f_i(t) \in C(\mathbb{R})$, with $|f_i(t)| \leq M$ for some $M > 0$ and all i and $t \in \mathbb{R}$. Also assume that $V(z) \in C^1(\mathbb{R}^m)$ satisfies*

$$V(z) \geq \alpha_1 |z|^2 - \alpha_2, \quad \text{for some } \alpha_1, \alpha_2 > 0, \text{ and all } z \in \mathbb{R}^m. \tag{3.2}$$

Assume also that $\int_{-\infty}^{\infty} f_i^2(t)\, dt < \infty$, for all i. Then the system (3.1) has a uniformly bounded solution $u_i(t) \in H^1(\mathbb{R})$, $i = 1, \ldots, m$ (i.e., for some constant $K > 0$, $|u_i(t)| < K$ for all $t \in \mathbb{R}$, and all i). This solution is in fact homoclinic, i.e., $\lim_{|t| \to \infty} u(t) = 0$.

Proof. We may assume that $\alpha_2 = 0$ in (3.2) (replacing V by $V + \alpha_2$). As in the previous section, we approximate solution of (3.1) by solutions of the corresponding Dirichlet problems ($i = 1, \ldots, m$)

$$u_i'' - a(t)V_{z_i}(u) = f_i(t), \quad \text{for } t \in (-L, L),\ u(-L) = u(L) = 0, \tag{3.3}$$

as $L \to \infty$. Solutions of (3.3) can be obtained as critical points of the corresponding variational functional $J(u) : \left[H_0^1(-L, L)\right]^m \to \mathbb{R}$ defined as

$$J(u) := \int_{-L}^{L} \left[\sum_{i=1}^{m} \left(\frac{1}{2} u_i'^2(t) + u_i(t) f_i(t) \right) + a(t) V(u(t)) \right] dt.$$

By (3.2), we have

$$J(u) \geq c_1 \sum_{i=1}^{m} \|u_i\|_{H^1(-L,L)} - c_2, \tag{3.4}$$

for some positive constants c_1 and c_2, uniformly in L, so that $J(u)$ is bounded from below, coercive and convex in u'. Hence, $J(u)$ has a minimizer in $\left[H_0^1(-L, L)\right]^m$, giving us a classical solution of (3.3), see e.g., L. Evans [1].

We now take a sequence $L_j \to \infty$, and denote by $u_j(t) \in H^1(\mathbb{R}; \mathbb{R}^m)$ a vector solution of the problem (3.3) on the interval $(-L_j, L_j)$, extended as zero vector to the outside of the interval $(-L_j, L_j)$. The crucial observation (originated from [4]) is that the variational method provides a uniform in L and j bound on $\|u_j(t)\|_{H^1(-L,L)}$. Indeed, we have $H_0^1(-L, L) \subset H_0^1(-\tilde{L}, \tilde{L})$ for $\tilde{L} > L$. If we now denote by M_L the minimum value of $J(u)$ on $\left[H_0^1(-L, L)\right]^m$, then M_L is non-increasing in L (there are more competing functions for larger L), and in particular, using (3.4),

$$c_1 \sum_{i=1}^{m} \|u_{j,i}\|_{H^1(-L,L)} - c_2 \leq J(u_j) \leq M_1, \tag{3.5}$$

if $L_j > 1$. By Sobolev's embedding in one dimension, we conclude a L^∞ bound on $u_{j,i}$, uniformly in L. Indeed, we write $w = u_{j,i}$, and let t_0 be the point of maximum of $|w(t_0)|$. Then it follows from (3.5) that

$$w^2(t_0) = 2\int_{-L}^{t_0} w(t)w'(t)\,dt \le 2\left(\int_{-L}^{L} w^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_{-L}^{L} w'^2(t)\,dt\right)^{\frac{1}{2}} \le C$$

for some constant C independent of L. The formula (3.5) also provides us with a uniform in j bound on $\int_{-L_j}^{L_j}\sum_{i=1}^{m}\left(u'_{j,i}(t)\right)^2 dt$, from which we conclude that the sequence $\{u_j(t)\}$ is equicontinuous on every bounded interval (as in (2.5) above). With the sequence $\{u_j(t)\}$ equicontinuous and uniformly bounded on every interval $[-L_p, L_p]$, it converges uniformly to some $u \in C(\mathbb{R};\mathbb{R}^m)$ on $[-L_p, L_p]$. From the equation (3.3), we have uniform convergence of $\{u''_j\}$ on bounded intervals, and hence uniform convergence of $\{u'_j\}$ follows from (2.4). We complete the existence proof by going to the limit via diagonal sequence, as in the proof of Theorem 2.1.

The solution obtained is in fact homoclinic, as follows from the inequality (for continuous $u : \mathbb{R} \to \mathbb{R}^m$ with $u' \in L^2_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$)

$$|u(t)| \le \sqrt{2}\left(\int_{t-1/2}^{t+1/2}\left(|u(s)|^2 + |u'(s)|^2\right)\,ds\right)^{1/2}$$

derived on p. 385 of [3], in view of the uniform in L estimate on $\|u\|_{[H^1(-L,L)]^m}$ that we obtained above. However, it is not clear if $\lim_{|t|\to\infty} u'(t) = 0$. □

Example 3.2. Consider the case $m = 2$, $V(z) = \frac{1}{2}\left(z_1^2 + z_1z_2 + z_2^2\right) + h(z_1,z_2)$, with $h(z_1,z_2) > 0$, continuously differentiable and bounded. We consider the system

$$\begin{cases} u_1'' - a(t)\left(u_1 + \frac{1}{2}u_2 + h_{z_1}(u_1,u_2)\right) = f_1(t), \\ u_2'' - a(t)\left(\frac{1}{2}u_1 + u_2 + h_{z_2}(u_1,u_2)\right) = f_2(t), \end{cases}$$

where the functions $a(t), f_1(t), f_2(t)$ satisfy the assumptions of Theorem 3.1. We conclude the existence of a uniformly bounded for all $t \in \mathbb{R}$ homoclinic solution. Observe that we may replace the condition $f_i(t) \in L^2(-\infty,\infty)$ by $|f_i(t)| \le M$ for some $M > 0$ and all $t \in \mathbb{R}$, $i = 1,2$. The estimate (3.4) still follows if we add a large positive constant to V.

4 Bounded solutions for a class of systems

In this section we provide a generalization of Theorem 2.1 to systems that are not necessarily Hamiltonian, and thus do not necessarily have variational structure. We begin with an a priori estimate. By $\|z\|$ we denote the Euclidean norm of $z \in \mathbb{R}^m$.

Lemma 4.1. *Let $u = (u_1, ..., u_m) \in C^2(\mathbb{R};\mathbb{R}^m)$ be a classical solution of*

$$u_i''(x) - \lambda H_i\left(u(x)\right) = \lambda f_i(x) \quad \text{for } x \in (-L,L)\,, \quad u_i(-L) = u_i(L) = 0\,, \tag{4.1}$$

$i = 1,2,\ldots,m$, where $\lambda \in [0,1]$ is a parameter. Assume that the functions $H_i(z) \in C(\mathbb{R}^m,\mathbb{R})$ satisfy

$$\lim_{\|z\|\to\infty}\frac{\sum_{i=1}^{m} z_iH_i(z)}{\sum_{i=1}^{m}|z_i|} = \infty\,. \tag{4.2}$$

Assume also that for some $M > 0$ the functions $f_i(x) \in C(\mathbb{R}, \mathbb{R})$ satisfy

$$|f_i(x)| \leq M, \quad \text{for all } i, \text{ and } x \in \mathbb{R}. \tag{4.3}$$

Then there is a constant $K_0 > 0$ such that for all i, and $x \in [-L, L]$ we have

$$|u_i(x)| \leq K_0, \quad \text{uniformly in } L > 1, \text{ and } \lambda \in [0,1]. \tag{4.4}$$

Proof. Let x_0 denote any point of maximum of $q(x) := \sum_{i=1}^m u_i^2(x)$ on $[-L, L]$. If $x_0 = \pm L$, then the estimate (4.4) holds trivially, so assume that $x_0 \in (-L, L)$, and then

$$0 \geq \frac{1}{2} q''(x_0) = \sum_{i=1}^m \left(u_i(x_0) u_i''(x_0) + u_i'^2(x_0) \right) \geq \sum_{i=1}^m u_i(x_0) u_i''(x_0). \tag{4.5}$$

By (4.2) we can fix K_0 so that $\sum_{i=1}^m z_i H_i(z) > M \sum_{i=1}^m |z_i|$ for all $\|z\| > K_0$. We claim that $|q(x_0)| = \|u(x_0)\|^2 \leq K_0^2$, which provides the desired a priori estimate. Indeed, if one assumes that $|q(x_0)| > K_0^2$, then using (4.5) and (4.1), we get

$$\begin{aligned} 0 &\geq \sum_{i=1}^m u_i(x_0) u_i''(x_0) = \lambda \sum_{i=1}^m \left[u_i(x_0) H_i(u(x_0)) + f_i(x_0) u_i(x_0) \right] \\ &\geq \lambda \sum_{i=1}^m \left[u_i(x_0) H_i(u(x_0)) - M |u_i(x_0)| \right] > 0, \end{aligned}$$

for $\lambda \in (0, 1]$, which is a contradiction. At $\lambda = 0$, the estimate holds trivially. □

Theorem 4.2. *Assume that the continuous functions $H_i(u)$ and $f_i(x)$ satisfy the conditions (4.2) and (4.3). Then the system*

$$u_i''(x) - H_i(u_1(x), \ldots, u_m(x)) = f_i(x), \quad i = 1, 2, \ldots, m \tag{4.6}$$

has a classical solution, uniformly bounded for all $x \in \mathbb{R}$, i.e., $|u_i(x)| \leq K$ for all $x \in \mathbb{R}$, $1 \leq i \leq m$, and some $K > 0$.

Proof. We obtain a bounded solution as a limit of the solutions of the corresponding Dirichlet problems (4.1) at $\lambda = 1$. Existence of such solutions follows by Schaefer's fixed point theorem, see e.g., [1], in view of the a priori estimate given by Lemma 4.1. Using arguments similar to those in Theorem 2.1, we obtain estimates that are similar to (2.3). This enables us to take the limit as $L \to \infty$ and carry out the diagonal argument as in Theorem 2.1 to obtain a bounded solution to the system (4.6). □

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