Asymptotic behaviour of positive solutions of the model which describes cell differentiation

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1. Introduction

In this paper we will study the asymptotic behaviour of positive solutions to the system

$$(1) \qquad \left| \begin{array}{l} x_1'(t) = \frac{A(t)}{1+x_2^n(t)} - x_1(t) \\ x_2'(t) = \frac{B(t)}{1+x_1^n(t)} - x_2(t), \end{array} \right.$$

where A and B belong to $\mathcal{C}_+$ and $\mathcal{C}_+$ is the set of continuous functions $g : \mathcal{R} \longrightarrow \mathcal{R}$, which are bounded above and below by positive constants. n is fixed natural number. The system (1) describes cell differentiation, more precisely - its passes from one regime of work to other without loss of genetic information. The variables x_1 and x_2 make sense of concentration of specific metabolits. The parameters A and B reflect degree of development of base metabolism. The parameter n reflects the highest row of the repression's reactions. For more details on the interpretation of (1) one may see [1]. With $\mathcal{C}_\circ$ we denote the space of continuous and bounded functions $g : \mathcal{R} \longrightarrow \mathcal{R}$. For $g \in \mathcal{C}_\circ$ we define

$$g_L(\infty) = \lim\inf_{t\longrightarrow\infty} g(t), g_M(\infty) = \lim\sup_{t\longrightarrow\infty} g(t),$$

$$g_L = \inf\{g(t) : t \in \mathcal{R}\}, g_M = \sup\{g(t) : t \in \mathcal{R}\}.$$

2. Preliminary results

Here and further next lemmas will pay important role.

Lemma 1.[2]*Let $g : (\alpha, \infty) \longrightarrow \mathcal{R}$ be a bounded and differentiable function. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \longrightarrow_{n\longrightarrow\infty} \infty$, $g'(t_n) \longrightarrow_{n\longrightarrow\infty} 0$, $g(t_n) \longrightarrow_{n\longrightarrow\infty} g_M(\infty)$ (resp. $g(t_n) \longrightarrow_{n\longrightarrow\infty} g_L(\infty)$).*

Lemma 2.[2]*Let $g \in \mathcal{C}_\circ$ be a differentiable function. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $g'(t_n) \longrightarrow_{n\longrightarrow\infty} 0$, $g(t_n) \longrightarrow_{n\longrightarrow\infty} g_M$ (resp. $g(t_n) \longrightarrow_{n\longrightarrow\infty} g_L$).*

Proposition 1.*Let (x_1, x_2) be a positive solution of (1) and $A(t), B(t) \in \mathcal{C}_+$. Then*

$$\frac{A_L(\infty)}{1 + B_M^n(\infty)} \leq x_{1L}(\infty) \leq x_{1M}(\infty) \leq A_M(\infty),$$

$$\frac{B_L(\infty)}{1 + A_M^n(\infty)} \leq x_{2L}(\infty) \leq x_{2M}(\infty) \leq B_M(\infty).$$

Proof. From lemma 1 there exists a sequence $\{t_m\}_{m=1}^{\infty} \subset \mathcal{R}$ for which $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $x_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $x_1(t_m) \longrightarrow_{m\longrightarrow\infty} x_{1M}(\infty)$. Then from

$$x_1'(t_m) = \tfrac{A(t_m)}{1+x_2^n(t_m)} - x_1(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m\longrightarrow\infty} \frac{A(t_m)}{1 + x_2^n(t_m)} - x_{1M}(\infty) \leq A_M(\infty) - x_{1M}(\infty),$$

i. e.

$$x_{1M}(\infty) \leq A_M(\infty).$$

Let now $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $x_2'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $x_2(t_m) \longrightarrow_{m\longrightarrow\infty} x_{2L}(\infty)$. From

$$x_2'(t_m) = \frac{B(t_m)}{1 + x_1^n(t_m)} - x_2(t_m),$$

as $m \longrightarrow \infty$, we find that

$$0 = \lim_{m\longrightarrow\infty} \frac{B(t_m)}{1 + x_1^n(t_m)} - x_{2L}(\infty) \geq \frac{B_L(\infty)}{1 + A_M^n(\infty)} - x_{2L}(\infty)$$

or

$$x_{2L}(\infty) \geq \frac{B_L(\infty)}{1 + A_M^n(\infty)}.$$

Let $\{t_m\}_{m=1}^{\infty} \subset \mathcal{R}$ is susch that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $x_2'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $x_2(t_m) \longrightarrow_{m\longrightarrow\infty} x_{2M}(\infty)$. From

$$x_2'(t_m) = \frac{B(t_m)}{1 + x_1^n(t_m)} - x_2(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m\longrightarrow\infty} \frac{B(t_m)}{1+x_1^n(t_m)} - x_{2M}(\infty) \le B_M(\infty) - x_{2M}(\infty).$$

Consequently

$$x_{2M}(\infty) \le B_M(\infty).$$

Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $x_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $x_1(t_m) \longrightarrow_{m\longrightarrow\infty} x_{1L}(\infty)$. From equality

$$x_1'(t_m) = \frac{A(t_m)}{1+x_2^n(t_m)} - x_1(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m\longrightarrow\infty} \frac{A(t_m)}{1+x_2^n(t_m)} - x_{1L}(\infty) \ge \frac{A_L(\infty)}{1+B_M^n(\infty)} - x_{1L}(\infty)$$

or

$$x_{1L}(\infty) \ge \frac{A_L(\infty)}{1+B_M^n(\infty)}.$$

This completes the proof.

Remark. Proposition 1 shows that (1) is permanent, i. e. there exist positive constants α and β such that

$$0 < \alpha \le \liminf_{t\longrightarrow\infty} x_i(t) \le \limsup_{t\longrightarrow\infty} x_i(t) \le \beta < \infty, \quad i = 1, 2,$$

where $(x_1(t), x_2(t))$ is a positive solution of (1). In [3] was proved that permanence implies existence of positive periodic solutions of (1), when $A(t)$ and $B(t)$ are continuous positive periodic functions.

Let X_1 be a positive solution of the equation

$$x'(t) = A(t) - x(t),$$

and X_2 be a positive solution of the equation

$$x'(t) = B(t) - x(t).$$

Proposition 2. *Let X_1, X_2 be as above and $A(t), B(t) \in \mathcal{C}_+$. Then*

$$A_L(\infty) \le X_{1L}(\infty) \le X_{1M}(\infty) \le A_M(\infty),$$

$$B_L(\infty) \le X_{2L}(\infty) \le X_{2M}(\infty) \le B_M(\infty).$$

Proof. From lemma 1 there exists a sequence $\{t_m\}_{m=1}^{\infty}$ of $\mathcal{R}$ for which $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $X_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $X_1(t_m) \longrightarrow_{m\longrightarrow\infty} X_{1L}(\infty)$. Then from

$$X_1'(t_m) = A(t_m) - X_1(t_m),$$

as $m \longrightarrow \infty$, we get

$$0 = \lim_{m\longrightarrow\infty} A(t_m) - X_{1L}(\infty) \geq A_L(\infty) - X_{1L}(\infty),$$

i. e.

$$X_{1L}(\infty) \geq A_L(\infty).$$

Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $X_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $X_1(t_m) \longrightarrow_{m\longrightarrow\infty} X_{1M}(\infty)$. From

$$X_1'(t_m) = A(t_m) - X_1(t_m),$$

as $m \longrightarrow \infty$, we find that

$$0 = \lim_{m\longrightarrow\infty} A(t_m) - X_{1M}(\infty) \leq A_M(\infty) - X_{1M}(\infty)$$

or

$$X_{1M}(\infty) \leq A_M(\infty).$$

In the same way we may prove other pair of inequalities.

3.Asymptotic behaviour of positive solutions

The results which are formulated and proved below are connected to (1) and to

$$(1_*) \qquad \left| \begin{array}{l} x_1'(t) = \frac{A_*(t)}{1+x_2^n(t)} - x_1(t) \\ x_2'(t) = \frac{B_*(t)}{1+x_1^n(t)} - x_2(t), \end{array} \right.$$

where $A_*, B_* \in \mathcal{C}_+$ and $A(t) - A_*(t) \longrightarrow_{t\longrightarrow\infty} 0$, $B(t) - B_*(t) \longrightarrow_{t\longrightarrow\infty} 0$.

We notice that every solution to (1)(resp. (1_*)) with positive initial data $x(t_\circ) = (x_1(t_\circ), x_2(t_\circ)) > 0$ $(x_*(t_\circ) = (x_{1*}(t_\circ), x_{2*}(t_\circ)) > 0)$ is defined and positive in $[t_\circ, \infty)$.

Theorem 1. *Let $A, B, A_*, B_* \in \mathcal{C}_+$ and*

$$A(t) - A_*(t) \longrightarrow_{t\longrightarrow\infty} 0, B(t) - B_*(t) \longrightarrow_{t\longrightarrow\infty} 0.$$

Let also

$$\frac{n^2 A_M^n(\infty) B_M^n(\infty)(1 + A_M^n(\infty))^{2n}(1 + B_M^n(\infty))^{2n}}{[A_L^n(\infty) + (1 + B_M^n(\infty))^n]^2[B_L^n(\infty) + (1 + A_M^n(\infty))^n]^2} < 1.$$

If $(x_1(t), x_2(t))$ *and* $(x_{1*}(t), x_{2*}(t))$ *are positive solutions respectively of (1) and* (1_*)*, then* $(x_1(t) - x_{1*}(t), x_2(t) - x_{2*}(t)) \longrightarrow_{t\longrightarrow\infty} (0,0)$*.*

Proof. Let $h_1(t) = x_1(t) - x_{1*}(t)$, $h_2(t) = x_2(t) - x_{2*}(t)$. We have

$$h_1'(t) = x_1'(t) - x_{1*}'(t) =$$

$$= \frac{A(t)}{1 + x_2^n(t)} - x_1(t) - \frac{A_*(t)}{1 + x_{2*}^n(t)} + x_{1*}(t) =$$

$$= -A(t)\frac{(x_2(t) - x_{2*}(t))(x_2^{n-1}(t) + x_2^{n-2}(t)x_{2*}(t) + \cdots + x_{2*}^{n-1}(t))}{(1 + x_2^n(t))(1 + x_{2*}^n(t))} -$$

$$-h_1(t) + \frac{A(t) - A_*(t)}{(1 + x_{2*}^n(t))}.$$

Let

$$\alpha(t) = A(t)\frac{x_2^{n-1}(t) + x_2^{n-2}(t)x_{2*}(t) + \cdots + x_{2*}^{n-1}(t)}{(1 + x_2^n(t))(1 + x_{2*}^n(t))}, \quad \beta(t) = \frac{A(t) - A_*(t)}{(1 + x_{2*}^n(t))}.$$

We notice that $\beta(t) \longrightarrow_{t\longrightarrow\infty} 0$. For $h_1(t)$ we get the equation

$$h_1'(t) = -h_1(t) - \alpha(t)h_2(t) + \beta(t).$$

On the other hand

$$h_2'(t) = x_2'(t) - x_{2*}'(t) =$$

$$= \frac{B(t)}{1 + x_1^n(t)} - x_2(t) - \frac{B_*(t)}{1 + x_{1*}^n(t)} + x_{2*}(t) =$$

$$= -B(t)\frac{(x_1(t) - x_{1*}(t))(x_1^{n-1}(t) + x_1^{n-2}(t)x_{1*}(t) + \cdots + x_{1*}^{n-1}(t))}{(1 + x_1^n(t))(1 + x_{1*}^n(t))} -$$

$$-h_2(t) + \frac{B(t) - B_*(t)}{(1 + x_{1*}^n(t))}.$$

Let

$$\gamma(t) = B(t)\frac{x_1^{n-1}(t) + x_1^{n-2}(t)x_{1*}(t) + \cdots + x_{1*}^{n-1}(t)}{(1 + x_1^n(t))(1 + x_{1*}^n(t))}, \quad \delta(t) = \frac{B(t) - B_*(t)}{(1 + x_{1*}^n(t))},$$

$\delta(t) \longrightarrow_{t\longrightarrow\infty} 0$. Then

$$h_2'(t) = -\gamma(t)h_1(t) - h_2(t) + \delta(t).$$

For $h_1(t)$ and $h_2(t)$ we find the system

$$\left| \begin{array}{l} h_1'(t) = -h_1(t) - \alpha(t)h_2(t) + \beta(t) \\ h_2'(t) = -\gamma(t)h_1(t) - h_2(t) + \delta(t). \end{array} \right.$$

Let $h(t) = (h_1(t), h_2(t))$ and $|h|(t) = |h(t)|$. We assume that $|h_1|_M(\infty) > 0$. From lemma 1 there exists a sequence $\{t_m\}_{m=1}^{\infty}$ of $\mathcal{R}$ such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $h_1'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $|h_1|(t_m) \longrightarrow_{m\longrightarrow\infty} |h_1|_M(\infty)$. From

$$|h_1'(t_m)| = |-h_1(t_m) - \alpha(t_m)h_2(t_m) + \beta(t_m)|,$$

as $m \longrightarrow \infty$, we have

$$0 \geq |h_1|_M(\infty) - \alpha_M(\infty)|h_2|_M(\infty),$$

i. e.

$$|h_1|_M(\infty) \leq \alpha_M(\infty)|h_2|_M(\infty). \tag{2}$$

Since $|h_1|_M(\infty) > 0$ then $|h_2|_M(\infty) > 0$. Let now $\{t_m\}_{m=1}^{\infty} \subset \mathcal{R}$ is such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $h_2'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $|h_2|(t_m) \longrightarrow_{m\longrightarrow\infty} |h_2|_M(\infty)$. As $m \longrightarrow \infty$, from

$$|h_2'(t_m)| = |-\gamma(t_m)h_1(t_m) - h_2(t_m) + \delta(t_m)|,$$

we get

$$0 \geq |h_2|_M(\infty) - \gamma_M(\infty)|h_1|_M(\infty)$$

or

$$|h_2|_M(\infty) \leq \gamma_M(\infty)|h_1|_M(\infty).$$

From last inequality and (2) we find that

$$|h_1|_M(\infty)|h_2|_M(\infty) \leq \alpha_M(\infty)\gamma_M(\infty)|h_1|_M(\infty)|h_2|_M(\infty),$$

from where

$$1 \leq \alpha_M(\infty)\gamma_M(\infty).$$

Since

$$\alpha_M(\infty) = \left(A(t)\frac{x_2^{n-1}(t) + x_2^{n-2}(t)x_{2*}(t) + \cdots + x_{2*}^{n-1}(t)}{(1 + x_2^n(t))(1 + x_{2*}^n(t))} \right)_M (\infty) \leq$$

$$\leq A_M(\infty).\frac{n.B_M^{n-1}(\infty)}{\left(1 + \frac{B_L^n(\infty)}{(1+A_M^n(\infty))^n}\right)^2} = \frac{n.A_M(\infty).B_M^{n-1}(\infty)(1 + A_M^n(\infty))^{2n}}{[(1 + A_M^n(\infty))^n + B_L^n(\infty)]^2},$$

$$\gamma_M(\infty) = \left(B(t) \frac{x_1^{n-1}(t) + x_1^{n-2}(t) x_{1*}(t) + \cdots + x_{1*}^{n-1}(t)}{(1 + x_1^n(t))(1 + x_{1*}^n(t))} \right)_M (\infty) \leq$$

$$\leq B_M(\infty) . \frac{n.A_M^{n-1}(\infty)}{\left(1 + \frac{A_L^n(\infty)}{(1+B_M^n(\infty))^n}\right)^2} = \frac{n.B_M(\infty).A_M^{n-1}(\infty)(1 + B_M^n(\infty))^{2n}}{[(1 + B_M^n(\infty))^n + A_L^n(\infty)]^2}.$$

Therefore we get the contradiction

$$1 \leq \frac{n^2.A_M^n(\infty).B_M^n(\infty)(1 + A_M^n(\infty))^{2n}(1 + B_M^n(\infty))^{2n}}{[(1 + B_M^n(\infty))^n + A_L^n(\infty)]^2.[(1 + A_M^n(\infty))^n + B_L^n(\infty)]^2}.$$

The proof is complete.

Let

$$r_1 = \frac{A_M^2(\infty)}{A_L^2(\infty)}, \quad r_2 = \frac{B_M^2(\infty)}{B_L^2(\infty)},$$

$$p_1 = \frac{1}{r_1} \frac{1}{1 + B_M^n(\infty) r_2^n}, \quad p_2 = \frac{1}{r_2} \frac{1}{1 + A_M^n(\infty) r_1^n}.$$

Theorem 2. *Let $A, B, A_*, B_* \in \mathcal{C}_+$, $A(t) - A_*(t) \longrightarrow_{t \longrightarrow \infty} 0$, $B(t) - B_*(t) \longrightarrow_{t \longrightarrow \infty} 0$. If $(x_1(t), x_2(t))$ and $(x_{1*}(t), x_{2*}(t))$ are positive solutions respectively to (1) and (1_*) and*

$$\frac{n^2 r_1^n r_2^n A_M^n(\infty) B_M^n(\infty)}{(1 + A_L^n(\infty) p_1^n)^2 (1 + B_L^n(\infty) p_2^n)^2} < 1,$$

then $(x_1(t) - x_{1*}(t), x_2(t) - x_{2*}(t)) \longrightarrow_{t \longrightarrow \infty} (0, 0)$.

Proof. Let $x = \frac{x_1}{X_1}$, $y = \frac{x_2}{X_2}$, where X_1 and X_2 as in proposition 2. Then

$$x'(t) = \frac{1}{X_1(t)} . x_1'(t) - \frac{x_1(t)}{X_1^2(t)} . X_1'(t) =$$

$$= \frac{1}{X_1(t)} . \left[\frac{A(t)}{1 + x_2^n(t)} - x_1(t) \right] - \frac{x_1(t)}{X_1^2(t)} [A(t) - X_1(t)] =$$

$$= \frac{1}{X_1(t)} . \frac{A(t)}{1 + X_2^n(t) y^n(t)} - \frac{A(t)}{X_1(t)} . \frac{x_1(t)}{X_1(t)} = -\frac{A(t)}{X_1(t)} . x(t) + \frac{A(t)}{X_1(t)[1 + X_2^n(t) y^n(t)]},$$

i. e.

$$x'(t) = -\frac{A(t)}{X_1(t)} . x(t) + \frac{A(t)}{X_1(t)[1 + X_2^n(t) y^n(t)]}.$$

$$y'(t) = \frac{1}{X_2(t)} . x_2'(t) - \frac{x_2(t)}{X_2^2(t)} . X_2'(t) =$$

$$= \frac{1}{X_2(t)} . \left[\frac{B(t)}{1 + x_1^n(t)} - x_2(t) \right] - \frac{x_2(t)}{X_2^2(t)} [B(t) - X_2(t)] =$$

$$= \frac{1}{X_2(t)} . \frac{B(t)}{1 + X_1^n(t) x^n(t)} - \frac{B(t)}{X_2(t)} . \frac{x_2(t)}{X_2(t)} = - \frac{B(t)}{X_2(t)} . y(t) + \frac{B(t)}{X_2(t)[1 + X_1^n(t) x^n(t)]},$$

i. e.

$$y'(t) = - \frac{B(t)}{X_2(t)} . y(t) + \frac{B(t)}{X_2(t)[1 + X_1^n(t) x^n(t)]}.$$

Consequently for $x(t)$ and $y(t)$ we get the system

$$(3) \qquad \left| \begin{array}{l} x'(t) = -\frac{A(t)}{X_1(t)} . x(t) + \frac{A(t)}{X_1(t)[1 + X_2^n(t) y^n(t)]} \\ y'(t) = -\frac{B(t)}{X_2(t)} . y(t) + \frac{B(t)}{X_2(t)[1 + X_1^n(t) x^n(t)]}. \end{array} \right.$$

Let us consider the system

$$(3_*) \qquad \left| \begin{array}{l} x'(t) = -\frac{A_*(t)}{X_{1*}(t)} . x(t) + \frac{A_*(t)}{X_{1*}(t)[1 + X_{2*}^n(t) y^n(t)]} \\ y'(t) = -\frac{B_*(t)}{X_{2*}(t)} . y(t) + \frac{B_*(t)}{X_{2*}(t)[1 + X_{1*}^n(t) x^n(t)]}, \end{array} \right.$$

where X_{1*} and X_{2*} be a positive solutions respectively to

$$x'(t) = A_*(t) - x(t),$$

$$x'(t) = B_*(t) - x(t).$$

Let $(x(t), y(t))$ be a positive solution to (3) and let $(x_*(t), y_*(t))$ be a positive solution to (3_*). At first, we will prove that

$$(x(t) - x_*(t), y(t) - y_*(t)) \longrightarrow_{t \longrightarrow \infty} (0, 0).$$

Let $h(t) = x(t) - x_*(t)$, $k(t) = y(t) - y_*(t)$. We have

$$h'(t) = x'(t) - x'_*(t) =$$

$$= - \frac{A(t)}{X_1(t)} . x(t) + \frac{A(t)}{X_1(t)[1 + X_2^n(t) y^n(t)]} + \frac{A_*(t)}{X_{1*}(t)} . x_*(t) - \frac{A_*(t)}{X_{1*}(t)[1 + X_{2*}^n(t) y_*^n(t)]} =$$

$$= - \frac{A(t)}{X_1(t)} . x(t) + \frac{A(t)}{X_1(t)} . x_*(t) - \frac{A(t)}{X_1(t)} . x_*(t) + \frac{A(t)}{X_{1*}(t)} . x_*(t) - \frac{A(t)}{X_{1*}(t)} . x_*(t) +$$

$$+ \frac{A_*(t)}{X_{1*}(t)} . x_*(t) + \frac{A_*(t)}{X_{1*}(t)[1 + X_{2*}^n(t) y^n(t)]} - \frac{A_*(t)}{X_{1*}(t)[1 + X_{2*}^n(t) y_*^n(t)]}$$

$$- \frac{A_*(t)}{X_{1*}(t)[1 + X_{2*}^n(t) y^n(t)]} + \frac{A(t)}{X_1(t)[1 + X_2^n(t) y^n(t)]} =$$

$$= - \frac{A(t)}{X_1(t)} [x(t) - x_*(t)] - A(t) x_*(t) \left[\frac{1}{X_1(t)} - \frac{1}{X_{1*}(t)} \right] - \frac{x_*(t)}{X_{1*}(t)} [A(t) - A_*(t)] +$$

$$+\frac{A(t)}{X_1(t)[1+X_2^n(t)y^n(t)]}-\frac{A_*(t)}{X_{1*}(t)[1+X_{2*}^n(t)y^n(t)]}-$$

$$-\frac{A_*(t)X_{2*}^n(t)[y(t)-y_*(t)][y^{n-1}(t)+y^{n-2}(t)y_*(t)+\cdots+y_*^{n-1}(t)]}{X_{1*}(t)[1+X_{2*}^n(t)y_*^n(t)][1+X_{2*}^n(t)y^n(t)]}.$$

Let

$$\alpha(t)=\frac{A(t)}{X_1(t)},$$

$$\beta(t)=\frac{A_*(t)X_{2*}^n(t)[y^{n-1}(t)+y^{n-2}(t)y_*(t)+\cdots+y_*^{n-1}(t)]}{X_{1*}(t)[1+X_{2*}^n(t)y_*^n(t)][1+X_{2*}^n(t)y^n(t)]},$$

$$\gamma(t)=A(t).x_*(t)\left[\frac{1}{X_1(t)}-\frac{1}{X_{1*}(t)}\right]+\frac{x_*(t)}{X_{1*}(t)}[A(t)-A_*(t)]-$$

$$-\frac{A(t)}{X_1(t)[1+X_2^n(t)y^n(t)]}+\frac{A_*(t)}{X_{1*}(t)[1+X_{2*}^n(t)y^n(t)]}.$$

Since $X_1(t)-X_{1*}(t)\longrightarrow_{t\longrightarrow\infty}0$, $A(t)-A_*(t)\longrightarrow_{t\longrightarrow\infty}0$, then $\gamma(t)\longrightarrow_{t\longrightarrow\infty}0$. For $h(t)$ we get the equation

$$h'(t)=-\alpha(t)h(t)-\beta(t)k(t)-\gamma(t).$$

In the same way we find the equation

$$k'(t)=-\delta(t)h(t)-\sigma(t)k(t)-\rho(t),$$

where

$$\sigma(t)=\frac{B(t)}{X_{2*}(t)},$$

$$\delta(t)=\frac{B_*(t)X_{1*}^n(t)[x^{n-1}(t)+x^{n-2}(t)x_*(t)+\cdots+x_*^{n-1}(t)]}{X_{2*}(t)[1+X_{1*}^n(t)x_*^n(t)][1+X_{1*}^n(t)x^n(t)]},$$

$$\rho(t)=B(t).y_*(t)\left[\frac{1}{X_2(t)}-\frac{1}{X_{2*}(t)}\right]+\frac{y_*(t)}{X_{2*}(t)}[B(t)-B_*(t)]-$$

$$-\frac{B(t)}{X_2(t)[1+X_1^n(t)x^n(t)]}+\frac{B_*(t)}{X_{2*}(t)[1+X_{1*}^n(t)x^n(t)]}.$$

Since $X_2(t)-X_{2*}(t)\longrightarrow_{t\longrightarrow\infty}0$, $B(t)-B_*(t)\longrightarrow_{t\longrightarrow\infty}0$, then $\rho(t)\longrightarrow_{t\longrightarrow\infty}0$.

Therefore for $h(t)$ and $k(t)$ we get the system

$$\left|\begin{array}{rcl} h'(t) &=& -\alpha(t)h(t)-\beta(t)k(t)-\gamma(t) \\ k'(t) &=& -\delta(t)h(t)-\sigma(t)k(t)-\rho(t). \end{array}\right.$$

We assume that $|h|_M(\infty) > 0$. From lemma 1 there exists a sequence $\{t_m\}_{m=1}^{\infty}$ of $\mathcal{R}$ such that $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $|h'(t_m)| \longrightarrow_{m\longrightarrow\infty} 0$, $|h|(t_m) \longrightarrow_{m\longrightarrow\infty} |h|_M(\infty)$. From

$$|h'(t_m)| = |-\alpha(t_m)h(t_m) - \beta(t_m)k(t_m) - \gamma(t_m)|,$$

as $m \longrightarrow \infty$, we find that

$$0 \geq |h|_M(\infty) - \left(\frac{\beta}{\alpha}\right)_M(\infty)|k|_M(\infty),$$

i. e.

$$|h|_M(\infty) \leq \left(\frac{\beta}{\alpha}\right)_M(\infty)|k|_M(\infty). \tag{4}$$

Consequently $|k|_M(\infty) > 0$. Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ for which $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $|k'(t_m)| \longrightarrow_{m\longrightarrow\infty} 0$, $|k|(t_m) \longrightarrow_{m\longrightarrow\infty} |k|_M(\infty)$. As $m \longrightarrow \infty$ in

$$|k'(t_m)| = |-\delta(t_m)h(t_m) - \sigma(t_m)k(t_m) - \rho(t_m)|,$$

we get

$$0 \geq |k|_M(\infty) - \left(\frac{\delta}{\sigma}\right)_M(\infty)|h|_M(\infty),$$

i. e.

$$|k|_M(\infty) \leq \left(\frac{\delta}{\sigma}\right)_M(\infty)|h|_M(\infty). \tag{5}$$

From here and from (4) we find that

$$1 \leq \left(\frac{\beta}{\alpha}\right)_M(\infty)\left(\frac{\delta}{\sigma}\right)_M(\infty). \tag{6}$$

Lemma 3. *Let (x, y) be a positive solution of (3). The following inequalities hold*

$$p_1 \leq x_L(\infty) \leq x_M(\infty) \leq r_1,$$

$$p_2 \leq y_L(\infty) \leq y_M(\infty) \leq r_2.$$

Proof of Lemma 3. Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ for which $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $x'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $x(t_m) \longrightarrow_{m\longrightarrow\infty} x_M(\infty)$. From the equality

$$x'(t_m) = -\frac{A(t_m)}{X_1(t_m)}x(t_m) + \frac{A(t_m)}{X_1(t_m)[1 + X_2^n(t_m)y^n(t_m)]},$$

as $m \longrightarrow \infty$, we get

$$0 \leq -\frac{A_L(\infty)}{X_{1M}(\infty)} x_M(\infty) + \frac{A_M(\infty)}{X_{1L}(\infty)},$$

i. e.

$$x_M(\infty) \leq \frac{A_M^2(\infty)}{A_L^2(\infty)}$$

or $x_M(\infty) \leq r_1$. In the same way we get $y_M(\infty) \leq r_2$. Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of $\mathcal{R}$ for which $t_m \longrightarrow_{m\longrightarrow\infty} \infty$, $x'(t_m) \longrightarrow_{m\longrightarrow\infty} 0$, $x(t_m) \longrightarrow_{m\longrightarrow\infty} x_L(\infty)$. From

$$x'(t_m) = -\frac{A(t_m)}{X_1(t_m)} x(t_m) + \frac{A(t_m)}{X_1(t_m)[1 + X_2^n(t_m) y^n(t_m)]},$$

as $m \longrightarrow \infty$, we find that

$$0 \geq -\frac{A_M(\infty)}{X_{1L}(\infty)} x_L(\infty) + \frac{A_L(\infty)}{X_{1M}(\infty)} . \frac{1}{1 + X_{2M}^n(\infty) y_M^n(\infty)},$$

i. e.

$$x_L(\infty) \geq \frac{A_L(\infty)}{A_M(\infty)} . \frac{X_{1L}(\infty)}{X_{1M}(\infty)} \frac{1}{1 + X_{2M}^n(\infty) y_M^n(\infty)} \geq$$

$$\geq \left(\frac{A_L(\infty)}{A_M(\infty)} \right)^2 . \frac{1}{1 + B_M^n(\infty) r_2^n} = p_1,$$

or $x_L(\infty) \geq p_1$. In the same way we get $y_L(\infty) \geq p_2$.

Now we will continue with the proof of Theorem 2. We have

$$\left(\frac{\beta}{\alpha} \right)_M (\infty) \leq \frac{\beta_M(\infty)}{\alpha_L(\infty)}.$$

Using the lemma 3 we get

$$\left(\frac{\beta}{\alpha} \right)_M (\infty) \leq \left(\frac{A_M(\infty)}{A_L(\infty)} \right)^2 . \frac{n r_2^{n-1}(\infty) B_M^n(\infty)}{(1 + B_L^n(\infty) p_2^n)^2}.$$

On the other hand

$$\delta_M(\infty) \leq \frac{B_M(\infty) n r_1^{n-1} A_M^n(\infty)}{B_L(\infty)[1 + A_L^n(\infty) p_1^n]^2},$$

$$\sigma_L(\infty) \geq \frac{B_L(\infty)}{X_{2M}(\infty)} \geq \frac{B_L(\infty)}{B_M(\infty)}.$$

Consequently

$$\left(\frac{\delta}{\sigma} \right)_M (\infty) \leq \left(\frac{B_M(\infty)}{B_L(\infty)} \right)^2 . \frac{n r_1^{n-1} A_M^n(\infty)}{[1 + A_L^n(\infty) p_1^n]^2}.$$

From (6) we get the contradiction

$$1 \leq \frac{n^2 r_1^n r_2^n A_M^n(\infty) B_M^n(\infty)}{[1 + A_L^n(\infty) p_1^n]^2 [1 + B_L^n(\infty) p_2^n]^2}$$

with the condition of Theorem 2. Consequently $|h|_M(\infty) = 0$. From here $x(t) - x_*(t) \longrightarrow_{t \longrightarrow \infty} 0$. We assume that $|k|_M(\infty) > 0$. From (5) we find that $|h|_M(\infty) > 0$, which is contradiction. Consequently $|k|_M(\infty) = 0$. From here $y(t) - y_*(t) \longrightarrow_{t \longrightarrow \infty} 0$. Since

$$x(t) - x_*(t) = \frac{x_1(t)}{X_1(t)} - \frac{x_{1*}(t)}{X_{1*}(t)} \longrightarrow_{t \longrightarrow \infty} 0$$

and $X_1(t) - X_{1*}(t) \longrightarrow_{t \longrightarrow \infty} 0$, then $x_1(t) - x_{1*}(t) \longrightarrow_{t \longrightarrow \infty} 0$. From

$$y(t) - y_*(t) = \frac{x_2(t)}{X_2(t)} - \frac{x_{2*}(t)}{X_{2*}(t)} \longrightarrow_{t \longrightarrow \infty} 0$$

and $X_2(t) - X_{2*}(t) \longrightarrow_{t \longrightarrow \infty} 0$, therefores $x_2(t) - x_{2*}(t) \longrightarrow_{t \longrightarrow \infty} 0$.

Theorem 3. *Let $A, B \in \mathcal{C}_+$, and*

$$\frac{n^2 A_M^n B_M^n (1 + A_M^n)^{2n} (1 + B_M^n)^{2n}}{[A_L^n + (1 + B_M^n))^n]^2 [B_L^n + (1 + A_M^n)^n]^2} < 1.$$

Then the system (1) has exactly one positive solution $(x_{10}(t), x_{20}(t))$ such that $x_{10}(t) \in \mathcal{C}_+$, $x_{20}(t) \in \mathcal{C}_+$. In particular

$$(x_1(t) - x_{10}(t), x_2(t) - x_{20}(t)) \longrightarrow_{t \longrightarrow \infty} (0, 0),$$

for every positive solution $(x_1(t), x_2(t))$ of (1).

Proof. In $\mathcal{C}_+$ we define the metric

$$\|g - f\| = \sup\{|g(t) - f(t)| : t \in \mathcal{R}\} \quad for \quad f, g \in \mathcal{C}_+.$$

Let

$$X_{x_1} = \{x(t) \in \mathcal{C}_+ : \frac{A_L}{1 + B_M^n} \leq x(t) \leq A_M\},$$

$$X_{x_2} = \{x(t) \in \mathcal{C}_+ : \frac{B_L}{1 + A_M^n} \leq x(t) \leq B_M\}.$$

Let also $x_2(t) \in X_{x_2}$ be fixed and $x_{1x_2}(t)$ is the unique solution to the problem

$$u'(t) = \frac{A(t)}{1 + x_2^n(t)} - u(t), \quad u \in \mathcal{C}_+.$$

If $x_{1x_2}(t)$ attains its maximum when $t = t_1$, then $x'_{1x_2}(t_1) = 0$. From

$$0 = x'_{1x_2}(t_1) = \frac{A(t_1)}{1 + x_2^n(t_1)} - x_{1x_2}(t_1),$$

we get

$$x_{1x_2}(t) \leq A_M \quad \forall t \in \mathcal{R}.$$

If $x_{1x_2}(t)$ attains its minimum in $t = t_1$, then $x'_{1x_2}(t_1) = 0$ and

$$0 = x'_{1x_2}(t_1) = \frac{A(t_1)}{1 + x_2^n(t_1)} - x_{1x_2}(t_1) \geq \frac{A_L}{1 + B_M^n} - x_{1x_2}(t_1).$$

Consequently

$$x_{1x_2}(t) \geq \frac{A_L}{1 + B_M^n}, \quad \forall t \in \mathcal{R},$$

i. e. $x_{1x_2}(t) \in X_{x_1}$, $\forall t \in \mathcal{R}$. From here, the map

$$P : X_{x_2} \longrightarrow X_{x_1}, \quad Px_2 = x_{1x_2},$$

is well defined. Analogously, the map

$$Q : X_{x_1} \longrightarrow X_{x_2}, \quad Qx_1(t) = x_{2x_1}(t),$$

where $x_{2x_1}(t)$ is the unique positive bound solution of

$$u'(t) = \frac{B(t)}{1 + x_1^n(t)} - u(t), \quad u \in \mathcal{C}_+,$$

is well defined. Let $v_0(t), v_1(t) \in X_{x_2}$. Then

$$[P(v_0) - P(v_1)]' = P'(v_0) - P'(v_1) =$$

$$= \frac{A(t)}{1 + v_0^n(t)} - P(v_0) - \frac{A(t)}{1 + v_1^n(t)} + P(v_1) =$$

$$= \frac{A(t)[v_1^n(t) - v_0^n(t)]}{[1 + v_0^n(t)][1 + v_1^n(t)]} - [P(v_0) - P(v_1)] =$$

$$= -\frac{A(t)[v_1^{n-1}(t) + v_1^{n-2}(t)v_0(t) + \cdots + v_0^{n-1}(t)]}{[1 + v_0^n(t)][1 + v_1^n(t)]}(v_0(t) - v_1(t)) - [P(v_0) - P(v_1)].$$

From lemma 2 there exists a sequence $\{t_m\}_{m=1}^{\infty}$ of $\mathcal{R}$ for which $[P(v_0) - P(v_1)]'(t_m) \longrightarrow_{m \longrightarrow \infty} 0$, $|P(v_0) - P(v_1)|(t_m) \longrightarrow_{m \longrightarrow \infty} \|P(v_0) - P(v_1)\|$. From

$$|[P(v_0) - P(v_1)]'(t_m)| =$$

$$= | - \frac{A(t_m)[v_1^{n-1}(t_m) + v_1^{n-2}(t_m)v_0(t_m) + \cdots + v_0^{n-1}(t_m)]}{[1+v_0^n(t_m)][1+v_1^n(t_m)]}(v_0(t_m) - v_1(t_m)) -$$
$$-[P(v_0) - P(v_1)](t_m)|,$$

as $m \longrightarrow \infty$, we get

$$\|P(v_0) - P(v_1)\| \leq \frac{n.A_M.B_M^{n-1}(1+A_M^n)^{2n}}{[B_L^n + (1+A_M^n)^n]^2}\|v_0 - v_1\|.$$

In the same way, we find that for $v_0(t), v_1(t) \in X_{x_1}$

$$\|Q(v_0) - Q(v_1)\| \leq \frac{n.B_M.A_M^{n-1}(1+B_M^n)^{2n}}{[A_L^n + (1+B_M^n)^n]^2}\|v_0 - v_1\|.$$

Let $R = Q.P : X_{x_2} \longrightarrow X_{x_2}$. The map R is well defined and we have for it($v_0, v_1 \in X_{x_2}$)

$$\|R(v_0) - R(v_1)\| = \|Q.P(v_0) - Q.P(v_1)\| \leq$$
$$\leq \frac{n.B_M.A_M^{n-1}(1+B_M^n)^{2n}}{[A_L^n + (1+B_M^n)^n]^2}\|P(v_0) - P(v_1)\| \leq$$
$$\leq \frac{n^2.A_M^n.B_M^n(1+A_M^n)^{2n}.(1+B_M^n)^{2n}}{[A_L^n + (1+B_M^n)^n]^2.[B_L^n + (1+A_M^n)^n]^2}\|v_0 - v_1\| < \|v_0 - v_1\|.$$

Consequently R is a contraction map. Let $x_{20} \in X_{x_2}$ be a unique fixed point of R. Then (Px_{20}, x_{20}) is a solution to (1) in $\mathcal{C}_+ \bigotimes \mathcal{C}_+$. If (x_1, x_2) is another solution to (1) in $\mathcal{C}_+ \bigotimes \mathcal{C}_+$, then $x_2 \in X_{x_2}$ and $x_1 \in X_{x_1}$. But $P(x_2) = x_1$, $Q(x_1) = x_2$ and $R(x_2) = x_2$. From here $x_2 = x_{20}$ and $x_1 = x_{10}$. Therefore (1) has exactly one solution in $\mathcal{C}_+ \bigotimes \mathcal{C}_+$. The rest of the proof follows from theorem 1 since its conditions are hold.

Theorem 4. *Let* $A, B, A_*, B_* \in \mathcal{C}_+$,

$$A(t) - A_*(t) \longrightarrow_{t \longrightarrow \infty} 0, B(t) - B_*(t) \longrightarrow_{t \longrightarrow \infty} 0$$

and

$$\frac{nA_M^{n-1}(\infty)B_M(\infty)(1+A_M^n(\infty))(1+B_M^n(\infty))^{2n}}{B_L(\infty)((1+B_M^n(\infty))^n + A_L^n(\infty))^2} < \frac{A_L(\infty)}{A_M^2(\infty)(1+B_M^n(\infty))},$$

$$\frac{nB_M^{n-1}(\infty)A_M(\infty)(1+B_M^n(\infty))(1+A_M^n(\infty))^{2n}}{A_L(\infty)((1+A_M^n(\infty))^n + B_L^n(\infty))^2} < \frac{B_L(\infty)}{B_M^2(\infty)(1+A_M^n(\infty))}.$$

Then, if $(x_1(t), x_2(t))$ *and* $(x_{1*}(t), x_{2*}(t))$ *are positive solutions respectively to (1) and* (1_*)*, then* $(x_1(t) - x_{1*}(t), x_2(t) - x_{2*}(t)) \longrightarrow_{t \longrightarrow \infty} (0, 0)$.

Proof. Let $(x_1(t), x_2(t))$, $(x_{1*}(t), x_{2*}(t))$ be positive solutions respectively to (1) and (1_*), which are defined in $(T,\infty)(T>0)$. For $t>T$ we define

$$r(t) = \left| ln\frac{x_{1*}(t)}{x_1(t)} \right| + \left| ln\frac{x_{2*}(t)}{x_2(t)} \right|.$$

There exists $X \subset (T,\infty)$ such that $r(t)$ is differentiable in X and for $t \in X$ we have

$$r'(t) = sg\ln\frac{x_{1*}(t)}{x_1(t)}\left(\ln\frac{x_{1*}(t)}{x_1(t)}\right)' + sg\ln\frac{x_{2*}(t)}{x_2(t)}\left(\ln\frac{x_{2*}(t)}{x_2(t)}\right)' =$$

$$= sg\ln\frac{x_{1*}(t)}{x_1(t)}.\frac{x_1(t)}{x_{1*}(t)}\frac{x'_{1*}(t)x_1(t) - x_{1*}(t)x'_1(t)}{x_1^2(t)} +$$

$$+ sg\ln\frac{x_{2*}(t)}{x_2(t)}.\frac{x_2(t)}{x_{2*}(t)}\frac{x'_{2*}(t)x_2(t) - x_{2*}(t)x'_2(t)}{x_2^2(t)} =$$

$$= sg(x_{1*}(t) - x_1(t))\left[\frac{x'_{1*}(t)}{x_{1*}(t)} - \frac{x'_1(t)}{x_1(t)}\right] + sg(x_{2*}(t) - x_2(t))\left[\frac{x'_{2*}(t)}{x_{2*}(t)} - \frac{x'_2(t)}{x_2(t)}\right] =$$

$$= sg(x_{1*}(t) - x_1(t))\Big[\frac{A_*(t)}{x_{1*}(t)[1+x_{2*}^n(t)]} - 1 - \frac{A(t)}{x_1(t)[1+x_2^n(t)]} + 1\Big] +$$

$$+ sg(x_{2*}(t) - x_2(t))\Big[\frac{B_*(t)}{x_{2*}(t)[1+x_{1*}^n(t)]} - 1 - \frac{B(t)}{x_2(t)[1+x_1^n(t)]} + 1\Big] =$$

$$= sg(x_{1*}(t) - x_1(t))\Big[\frac{A_*(t)}{x_{1*}(t)[1+x_{2*}^n(t)]} - \frac{A_*(t)}{x_{1*}(t)[1+x_2^n(t)]} + \frac{A_*(t)}{x_{1*}(t)[1+x_2^n(t)]} -$$

$$- \frac{A_*(t)}{x_1(t)[1+x_2^n(t)]} + \frac{A_*(t)}{x_1(t)[1+x_2^n(t)]} - \frac{A(t)}{x_1(t)[1+x_2^n(t)]}\Big] +$$

$$+ sg(x_{2*}(t) - x_2(t))\Big[\frac{B_*(t)}{x_{2*}(t)[1+x_{1*}^n(t)]} - \frac{B_*(t)}{x_{2*}(t)[1+x_1^n(t)]} + \frac{B_*(t)}{x_{2*}(t)[1+x_1^n(t)]} -$$

$$- \frac{B_*(t)}{x_2(t)[1+x_1^n(t)]} + \frac{B_*(t)}{x_2(t)[1+x_1^n(t)]} - \frac{B(t)}{x_2(t)[1+x_1^n(t)]}\Big] =$$

$$= sg(x_{1*}(t) - x_1(t))\Big[\frac{A_*(t)}{x_{1*}(t)}.\frac{x_2^n(t) - x_{2*}^n(t)}{(1+x_{2*}^n(t))(1+x_2^n(t)} +$$

$$+ \frac{A_*(t)(x_1(t) - x_{1*}(t))}{x_1(t)x_{1*}(t))(1+x_2^n(t))} + \frac{A_*(t) - A(t)}{x_1(t)(1+x_2^n(t))}\Big] +$$

$$+ sg(x_{2*}(t) - x_2(t))\Big[\frac{B_*(t)}{x_{2*}(t)}.\frac{x_1^n(t) - x_{1*}^n(t)}{(1+x_{1*}^n(t))(1+x_1^n(t)} +$$

$$+\frac{B_*(t)(x_2(t)-x_{2*}(t))}{x_2(t)x_{2*}(t))(1+x_1^n(t))}+\frac{B_*(t)-B(t)}{x_2(t)(1+x_1^n(t))}\Big]\le$$

$$\le-\frac{A_L(\infty)}{A_M^2(\infty)(1+B_M^n(\infty))}|x_1(t)-x_{1*}(t)|-\frac{B_L(\infty)}{B_M^2(\infty)(1+A_M^n(\infty))}|x_2(t)-x_{2*}(t)|+$$

$$+\frac{B_M(\infty)}{\frac{B_L(\infty)}{1+A_M^n(\infty)}}.\frac{n.A_M^{n-1}(\infty)}{\left(1+\frac{A_L^n(\infty)}{(1+B_M^n(\infty))^n}\right)^2}|x_1(t)-x_{1*}(t)|+$$

$$+\frac{A_M(\infty)}{\frac{A_L(\infty)}{1+B_M^n(\infty)}}.\frac{n.B_M^{n-1}(\infty)}{\left(1+\frac{B_L^n(\infty)}{(1+A_M^n(\infty))^n}\right)^2}|x_2(t)-x_{2*}(t)|+\epsilon(t),$$

where $\epsilon(t)\longrightarrow_{t\longrightarrow\infty}0$. From the conditions of theorem 4 there exists a positive constant α for which

$$r'(t)\le-\alpha(|x_1(t)-x_{1*}(t)|+|x_2(t)-x_{2*}(t)|)+\epsilon(t),\quad t\in X.$$

From the mean value theorem there exists a positive constant $m>0$ such that

$$|lnx_1(t)-lnx_{1*}(t)|\le m^{-1}|x_1(t)-x_{1*}(t)|,\quad |lnx_2(t)-lnx_{2*}(t)|\le m^{-1}|x_2(t)-x_{2*}(t)|$$

and we get

$$r'(t)\le-\alpha mr(t)+\epsilon(t),\quad t\in X.$$

Therefore $r(t)\longrightarrow_{t\longrightarrow\infty}0$, from where

$$x_1(t)-x_{1*}(t)\longrightarrow_{t\longrightarrow\infty}0,\quad x_2(t)-x_{2*}(t)\longrightarrow_{t\longrightarrow\infty}0.$$

Remark. Theorems 1, 2, 3, 4 show that the system (1) is globally stable, i. e. any two positive solutions attract each other as $t\longrightarrow\infty$. We notice that the condition of theorem 3 is stronger than the condition in [3].

4. Examples

Systems

$$(7)\qquad \left|\begin{array}{l} x'(t)=\frac{2+\sin^2\sqrt{t+1}}{5(1+y^3(t))}-x(t)\\ y'(t)=\frac{2+\cos^2\sqrt{t+1}}{5(1+x^3(t))}-y(t),\end{array}\right.$$

$$(7_*)\qquad \left|\begin{array}{l} x'(t)=\frac{2+\sin^2\sqrt{t}}{5(1+y^3(t))}-x(t)\\ y'(t)=\frac{2+\cos^2\sqrt{t}}{5(1+x^3(t))}-y(t),\end{array}\right.$$

satisfy all conditions of theorems 1, 2, 3.

Systems

$$\left| \begin{array}{l} x_1'(t) = \frac{\frac{1}{3}+\frac{1}{t^2+1}}{1+x_2^2(t)} - x_1(t) \\ x_2'(t) = \frac{\frac{1}{3}+\frac{1}{t^2+1}}{1+x_1^2(t)} - x_2(t) \end{array} \right.$$

and

$$\left| \begin{array}{l} x_1'(t) = \frac{\frac{1}{3}}{1+x_2^2(t)} - x_1(t) \\ x_2'(t) = \frac{\frac{1}{3}}{1+x_1^2(t)} - x_2(t) \end{array} \right.$$

satisfy conditions of theorem 4.

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