Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **103**, 1–31; https://doi.org/10.14232/ejqtde.2018.1.103 www.math.u-szeged.hu/ejqtde/

Wave equation in higher dimensions – periodic solutions

Andrzej Nowakowski[✉] and **Andrzej Rogowski**[*]

Faculty of Mathematics and Computer Science, University of Lodz, Banacha 22, 90-238 Lodz, Poland

Received 11 June 2018, appeared 26 December 2018
Communicated by László Simon

Abstract. We discuss the solvability of the periodic-Dirichlet problem for the wave equation with forced vibrations $x_{tt}(t,y) - \Delta x(t,y) + l(t,y,x(t,y)) = 0$ in higher dimensions with sides length being irrational numbers and superlinear nonlinearity. To this effect we derive a new dual variational method.

Keywords: periodic-Dirichlet problem, semilinear equation of forced vibrating string, dual variational method.

2010 Mathematics Subject Classification: 35L05, 35L71.

1 Introduction

The aim of the paper is to look for solutions and their regularity to the following problem

$$\begin{aligned} x_{tt}(t,y) - \Delta x(t,y) + l(t,y,x(t,y)) = 0, \quad t \in \mathbb{R}, \quad y \in (0,\pi)^n, \\ x(t,y) = 0, \quad y \in \partial(0,\pi)^n, \quad t \in \mathbb{R}, \\ x(t+T,y) = x(t,y), \quad t \in \mathbb{R},\ y \in (0,\pi)^n. \end{aligned} \tag{1.1}$$

The study of time periodic solutions to (1.1), typically with $T = 2\pi$, has a long history. First papers (nonlinear l) concerned the case when $l = \epsilon f$ with $|\epsilon|$ sufficiently small and $f(t,y,\cdot)$ strongly monotone (see a survey [26] and also [4, 19]). To prove the existence results there a variant of the Lyapunov–Schmidt method together with the theory of monotone operators were used. The case when f is only monotone, using similar method and combining them with Schauder fixed point theorem was considered in [13]. In [31] Rabinowitz used his saddle point theorem in critical point theory together with a Galerkin argument to prove the existence of weak solution for nonlinearity l being of C^1 and sublinear at infinity. That paper has initiated a large literature devoted to the use of various techniques of modern critical point theory in the study of semilinear wave equations (see [15, 34] and the references therein).

✉ Corresponding author. Email: annowako@math.uni.lodz.pl
*Email: arogow@math.uni.lodz.pl

The strongly monotone and weakly monotone nonlinearities were considered also e.g. in [14, 21, 25]. In all the quoted papers the monotonicity assumption (strong or weak) is the key property for overcoming the lack of compactness in the infinite dimensional kernel of equation $x_{tt}(t,y) - x_{yy}(t,y) = 0$ (periodic-Dirichlet solutions). We underline that, in general, the weak solutions obtained in [32] are only continuous functions. Concerning regularity, Brézis and Nirenberg [14] proved – but only for strongly monotone nonlinearities – that any L^{∞}-solution of (1.1) is smooth, even in the nonperturbative case $\epsilon = 1$, whenever the nonlinearity l is smooth. On the other hand, very little is known about existence and regularity of solutions if we drop the monotonicity assumption on the forcing term l. Willem [38], Hofer [21] and Coron [16] have considered the class of equations (1.1) where $l(t,x,u) = g(u) + h(t,x)$ and $g(u)$ satisfies suitable linear growth conditions. In [16] for the autonomous case $h \equiv 0$, is proved, for the first time, existence of nontrivial solutions for non-monotone nonlinearities. The case of l being a difference of two convex nonautonomous functions is investigated in [3]: the nonlinearity $l \in C([0,\pi] \times \mathbb{R}^2, \mathbb{R})$ has the form $l(t,y,x) = \lambda g(t,y,x) + \mu h(t,y,x)$ with $\lambda, \mu \in \mathbb{R}$, g superlinear in x, h sublinear in x, and both g, h are 2π-periodic in t and nondecreasing in x. The solutions to (1.1) are obtained using variational method. The special form of l allows to control the levels of the weak limits of certain Palais–Smale sequences as the functional corresponding to (1.1) does not satisfy the Palais–Smale condition (compare also the references in [3]). In the paper [7] existence and regularity of solutions of (1.1) (with $l = \epsilon f$) are proved for a large class of non-monotone forcing terms $f(t,y,x)$, including, for example: $f(t,y,x) = \pm x^{2k} + x^{2k+1} + h(t,y)$, $f(t,y,x) = \pm x^{2k} + \tilde{f}(t,y,x)$ with $\tilde{f}_x(t,y,x) \geq \beta > 0$. The proof is based on a variational Lyapunov–Schmidt reduction, minimization arguments and a priori estimate methods.

It is interesting that arithmetical properties of the ratio $\alpha = T/\pi$ play an important role in the solvability of the periodic-Dirichlet problem (1.1) over $[0,T] \times (0,\pi)^n$. The main reason is that the nature of the spectrum of the corresponding linear problem

$$x_{tt}(t,y) - x_{yy}(t,y) + g(t,y) = 0 \tag{1.2}$$

depends in an essential way on the arithmetical nature of α. It has already been pointed out by Borel in [10] that there exist numbers α, satisfying some arithmetical conditions, such that the linear problem (1.2) need not have a solution in the class of analytic functions, if g is analytic. Later Novak [28] proved even more: that there exist irrationals α and functions g in L^2 that the equation (1.2) does not have any generalized periodic-Dirichlet solution. References on these questions can be found in [35]. The papers which treat the nonlinear problem of (1.1) consider in most cases only one dimensional space variable i.e. $n = 1$, autonomous nonlinearities ($l = l(x)$ or lastly some cases of $l = l(y,x)$) and in all cases only the irrational numbers with bounded partial quotients (see e.g. [2,9,17,18] and the references therein). Kuksin [22] (see also [23]) and Wayne [37] (compare also [36]), were able to find, extending in a suitable way KAM techniques, periodic solutions in some Hamiltonian PDE's in one spatial dimension under Dirichlet boundary conditions. As usual in KAM-type results, the periods of such persistent solutions satisfy a strong irrationality condition, as the classical Diophantine condition, so that these orbits exist only on energy levels belonging to some Cantor set of positive measure. The main limitation of this method is the fact that standard KAM-techniques require the linear frequencies to be well separated (non resonance between the linear frequencies). To overcome such difficulty a new method for proving the existence of small amplitude periodic solutions, based on the Lyapunov–Schmidt reduction, has been developed in [18]. Rather than attempting to make a series of canonical transformations which bring the Hamiltonian into some normal form, the solution is constructed directly. Making the ansatz that a periodic solution exists one writes this solution as a Fourier series and substitutes that series into the partial differential equation. In this way one is reduced to solve two equations: the so called (P) equation, which is infinite dimensional, where small denominators appear, and the finite dimensional (Q) equation, which corresponds to resonances. Due to the presence of small divisors the (P) equation is solved by a Nash–Moser Implicit Function Theorem. Later on, this method has been improved by Bourgain to show the persistence of periodic solutions in higher spatial dimensions [12]. The first results on the existence of small amplitude periodic solutions for some completely resonant PDE's as (1.1) have been given in [24], for the specific nonlinearity $l(x) = x^3$, and in [1] when $l(x) = x^3 + h.o.t.$ The approach of [1] is still based on the Lyapunov-Schmidt reduction. The (P) equation is solved, for the strongly irrational frequencies $\omega \in W_\gamma$, where $W_\gamma = \left\{\omega \in \mathbb{R} \,\middle|\, |\omega k - j| \geq \frac{\gamma}{k}, k \neq j\right\}$, through the Contraction Mapping Theorem. Next, the (Q) equation, infinite dimensional, is solved by looking for non degenerate critical points of a suitable functional and continuing them, by means of the Implicit Function Theorem, into families of periodic solutions of the nonlinear equation. The case of higher space dimension is investigated in [2]. In [8] is proved, assuming only that the nonlinearity l satisfies $l(0) = l'(0) = \cdots = l^{(p-1)}(0) = 0$, $l^{(p)}(0) = ap! \neq 0$ for some $p \in \mathbb{N}$, $p \geq 2$, the existence of a large number of small amplitude periodic solutions of (1.1) with fixed period.

The aim of this paper is to consider the case $n \geq 2$ with T being irrational numbers such that $\alpha = T/\pi$ has not necessary bounded partial quotients in its continued fraction and nonautonomous nonlinearity l. Moreover we show some relation between the type of number α, the regularity of nonlinearity of l and the regularity of the solution to (1.1), which is treated for the first time. To the knowledge of the authors, the above problem with α having unbounded partial quotients is also considered for the first time (except some special cases in [18]). To this effect we modify Theorem 6.3.1 from [35] to the case of higher dimension in (1.1). Next we develop our own critical point theorem basing on the type of irrational frequencies α to build a set on which the minimum of suitable functional is considered. That means first we define a functional of convex type (l is then monotone only) and using duality properties of convex analysis we prove existence and regularity of solution to (1.1) as a minimum of the functional on a suitable defined set depending on the type of irrational frequency. Next we consider similarly as in [3] l being the difference of two monotone functions but with different properties, and again the new functional corresponding to this l is considered on a new defined set depending on a new irrational frequency to which we apply the former result (with one monotone function!) and develop duality for that functional. We do not apply any known critical point tools. As the last step we investigate a certain form of l being a special combination of a finite number of increasing functions to which we apply induction method (with respect to the number of functions) and use the obtained result for difference of two monotone functions. Such an approach to (1.1) is different from all cited above. We would like to stress that the sets on which we minimize our functionals depend strictly on the type of an irrational frequency and the type of a nonlinearity. This means that for a given fixed irrational frequency and nonlinearity our theorems may not assert an existence to (1.1). They assert only that for some type of nonlinearity there exists an irrational frequency for which (1.1) has a solution.

More precisely we shall study (1.1) by variational method, i.e. we shall consider (1.1) as the critical points of the functional:

$$J(x) = \int_0^T \int_\Omega \left(\frac{1}{2} |\nabla x(t,y)|^2 - \frac{1}{2} |x_t(t,y)|^2 + L(t,y,x(t,y)) \right) dydt, \tag{1.3}$$

where $L_x = l$, $\Omega = (0,\pi)^n$, defined on $U^1 = H^1_{per}((0,T);H^1_0(\Omega))$. First we consider $L(t,y,\cdot)$ convex, next $L(t,y,\cdot)$ is a difference of convex functions (but more general case than in [3]) and lastly $L(t,y,\cdot)$ as a special finite combination of convex functions. Moreover we use different definition of α (see **T** below). Our purpose is to investigate (1.1) by studying critical points of functional (1.3) using in an essential way the form of l and the irrationality α. To this effect we apply approach which is based on ideas developed in [20] ($r = 2$ and $n = 1$, see below). Our aim is to find a nonlinear subsets $\hat{X}$ of U^1 and to study modifications of (1.3) just only on $\hat{X}$. The main difficulty in our approach is just the construction of the final set $\hat{X}$ which depend on the irrational frequency. Moreover we give clear relation between type of r, type of nonlinearity l and irrationality α (see below). We assume that

T *$T = \pi\alpha$, α is such that α^2 is irrational and satisfies $\left|\alpha^2 - p/d\right| \geq cd^{-r}$ for all $p, d \in \mathbb{N}$ with some constant $c > 0$ for a fixed $r \geq 2$.*

Remark 1.1. Note, this assumption implies that $\left|\alpha^2 - p^2/\left|q\right|^2\right| \geq c\left|q\right|^{-2r}$ *for all* $p \in \mathbb{N}$, $|q| = \sqrt{\sum_{i=1}^n q_i^2}$, $q_i \in \mathbb{N}$, $i = 1,\dots,n$ and just the last inequality we will use in the proof of Proposition 2.1.

We would like to stress that if $r > 2$ then we admit α being real algebraic number of degree greater than 2 as well as having unbounded partial quotients – on several properties of such numbers see e.g. [33]. We only mention that the case of $(0,\pi)^n$ being of dimension n is a little bit more complicated than $n = 1$, as some numbers $|q|$ are irrational. However even in the case of $n = 1$ the assumption **T** is interesting, usually it is assumed then that $|\alpha - p/d| \geq cd^{-2}$. We must underline that **T** means, in particular, that we do not consider irrationals of the type $\alpha = \sqrt{n}$, $n \in \mathbb{N}$ (see [27] for deep discussion on that case).

In order to give a reader an insight what does condition **T** mean let us recall some fundamental facts from number theory. Let $\alpha^2 = [a_0, a_1, a_2, \dots]$ ($a_0, a_1, a_2, \dots$ integers) be the continued fraction decomposition of the real number α^2 [33]. The integers $a_0, a_1, a_2, \dots$ are the partial quotients of α^2 and the rationals $\frac{p_n}{d_n} = [a_0, a_1, a_2, \dots, a_n]$ with p_n, d_n relatively prime integers, called the convergent of α^2, are such that $\frac{p_n}{d_n} \to \alpha^2$ as $n \to \infty$. An irrational number α^2 is badly approximated if there is a constant $c(\alpha)$ such that

$$\left|\alpha^2 - p/d\right| > c(\alpha)/d^2 \tag{1.4}$$

for every rational p/d, such a constant $c(\alpha)$ must satisfy $0 < c(\alpha) < 1/\sqrt{5}$. α^2 is badly approximated if and only if the partial quotients in its continued fraction expansion are bounded: $|a_n| \leq K(\alpha)$, $n = 0,1,2,\dots$ There are continuum many badly approximated numbers, and there exist continuum many numbers which are not badly approximated. The set of irrational numbers with bounded partial quotients coincides with the set of numbers of constant type, which are the numbers α^2 such that $d\left\|d\alpha^2\right\| \geq \frac{1}{r}$ for some real number $r \geq 1$ and all integers $d > 0$, where $\|b\|$ denotes the distance between the irrational number b and the closest integer.

By a classical theorem of Lagrange all real quadratic irrationals have bounded partial quotients. It follows from results of Borel [10] and Bernstein [6] that the set of all irrational numbers having bounded partial quotients is a dense, uncountable and null subset of the real line. Examples of transcendental numbers having bounded partial quotients are given by

$$f(n) = \sum_{i=0}^{\infty} \frac{1}{n^{2^i}},$$

for $n \geq 2$ an integer. Examples of transcendental numbers with unbounded partial quotients are given by

$$\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$$

or $\zeta(3)$. For π^2 we have

$$\left|\pi^2 - \frac{p}{d}\right| > \frac{1}{d^{\theta+\varepsilon}}, \qquad \theta = 11.85078\ldots, \tag{1.5}$$

for all $\varepsilon > 0$ and d sufficiently large.

The famous Roth's Theorem states that if α^2 is an algebraic number, i.e. a root of a polynomial $f(X) = a_e X^e + a_{e-1}X^{e-1} + \cdots + a_0$ (a_i integers), of degree $e \geq 2$, then for an arbitrary fixed $\varepsilon > 0$ and all rationals p/d with sufficiently large d the following inequality holds:

$$\left|\alpha^2 - \frac{p}{d}\right| > \frac{1}{d^{2+\varepsilon}}. \tag{1.6}$$

If α^2 is of degree 2 then by Liouville's Theorem we have inequality (1.4). For no single α^2 of degree ≥ 3 we do not know whether (1.4) holds. It is very likely (see [33]) that in fact (1.4) is false for every such α^2, i.e. that no such α^2 is badly approximated, or, put differently, that such α^2 has unbounded partial quotients in its continued fraction.

From the above we infer that the set of α satisfying **T** *is nonempty, in the following sense: there exists α irrational and $r \geq 2$ satisfying* **T** *with some constant $c > 0$ (compare (1.5))!*

2 Main results

We put $Q = (0,T) \times \Omega$ with $\Omega = (0,\pi)^n$ and $E = -\Delta$ for the Laplace operator with the domain $H^2(\Omega) \cap H_0^1(\Omega)$. We use the notation for the domain of the operator E^γ, $\gamma \geq 1$: $\mathcal{D}(E^\gamma) = {}^0H^{2\gamma}(\Omega)$, where ${}^0H^{2\gamma}(\Omega)$ is a Sobolev space of functions

$$\Big\{ x \in H^{2\gamma}(\Omega) : \frac{\partial x}{y_i^{2l}}(y_1,\ldots,y_{i-1},0,y_{i+1},\ldots,y_n) = \frac{\partial x}{y_i^{2l}}(y_1,\ldots,y_{i-1},\pi,y_{i+1},\ldots,y_n) = 0,$$
$$(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n) \in \Omega_i,\ l = 0,1,\ldots,\gamma-1,\ i = 1,\ldots,n \Big\},$$

where $\Omega_i = \{(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n) : (y_1,\ldots,y_{i-1},y_i,y_{i+1},\ldots,y_n) \in \Omega,\ y_i \in (0,\pi)\}$ (see [35]). By a solution of the problem (1.1) we mean a function $x \in U^{\frac{5}{2}r-1} = H_{per}^{\frac{5}{2}r-1}(\mathbb{R} \times \Omega) \cap H_{per}^0(\mathbb{R}; {}^0H^2(\Omega))$, that satisfies (1.1) almost everywhere, where $H_{per}^{\frac{5}{2}r-1}$ is the usual Sobolev space of periodic functions with respect to the first variable with period T. The exponent r is defined in **T.**

Let $\mathbf{L} \subset \mathbb{Z}^n$ be the lattice of the integers vectors $k = (k_1,\ldots,k_n)$ such that $k_i \geq 1$ for $i = 1,\ldots,n$. Put $|k| = \sqrt{\sum_{i=1}^n k_i^2}$, $|k|^2 = k_1^2 + \cdots + k_n^2$ and $\sum_{j,k} = \sum_{j=-\infty}^{+\infty} \sum_{k \in \mathbf{L}}$. Let

$\mathcal{H}^{\frac{9}{2}r-3}$=$H^0_{per}(\mathbb{R};H^{\frac{9}{2}r-3}(\Omega))$ be the usual Sobolev space. The norm $\|\cdot\|_{\mathcal{H}^{\frac{9}{2}r-3}}$ of $g\in\mathcal{H}^{\frac{9}{2}r-3}$ we define as square root of $\sum_{j,k}|k|^{9r-6}|g_{j,k}|^2$, i.e.

$$\|g\|_{\mathcal{H}^{\frac{9}{2}r-3}}=\left(\sum_{j,k}|k|^{9r-6}|g_{j,k}|^2\right)^{1/2},$$

where

$$g_{j,k}=\left(\frac{2^{n+1}}{\pi^nT}\right)^{1/2}\int_0^T\int_\Omega g(t,y)e^{-ij\frac{2\pi}{T}t}\sin k_1y_1\cdots\sin k_ny_ndydt. \tag{2.1}$$

To formulate our main results we need a modification of Theorem 6.3.1 from [35] for the case of higher dimension periodic-Dirichlet boundary conditions (1.1) and stronger regularity, i.e. the following

Proposition 2.1. *Let $g\in\mathcal{H}^{\frac{9}{2}r-3}$. Then there exists $\overline{x}\in U^{\frac{5}{2}r-1}$ being a unique solution to*

$$\begin{aligned} x_{tt}(t,y)-\Delta x(t,y)&=g(t,y), \\ x(t,y)&=0,\quad t\in\mathbb{R},\quad y\in\partial\Omega, \\ x(t+T,y)&=x(t,y),\quad t\in\mathbb{R},\quad y\in\Omega \end{aligned} \tag{2.2}$$

with

$$\overline{x}(t,y)=\left(\frac{2^{n+1}}{\pi^nT}\right)^{1/2}\sum_{j,k}(-j^24\alpha^{-2}+|k|^2)^{-1}g_{j,k}e^{ij\frac{2\pi}{T}t}\sin k_1y_1\cdots\sin k_ny_n, \tag{2.3}$$

$g_{j,k}$ is as in (2.1) and such that

$$\|\overline{x}\|_{U^{\frac{5}{2}r-1}}\le B\|g\|_{\mathcal{H}^{\frac{9}{2}r-3}}, \tag{2.4}$$

$$\|\overline{x}\|_{U^{\frac{5}{2}r-1}}\ge C\|g\|_{H^0} \tag{2.5}$$

*with $B^2=(2\alpha+1)^{5r-2}\alpha^4c^{-2}$ and $C^2=\frac{1}{9}\alpha^{5r-2}$ **independent of** g, where α and c are defined in* **T**.

Remark 2.2. Notice that in different way to one dimension case ($n=1$) the right hand side of (2.2) has to be more regular in space variable than existing solution of it – even for $r=2$. This fact will have influence for necessary regularity assumptions for our nonlinear equation (1.1).

Remark 2.3. Let us notice that constants B and C are determined by α and c. **Everywhere below constants** B and C **will always denote those occurring in** (2.4) and (2.5).

Assumptions M concerning equation (1.1).

M *Let F^1, $F^2,\dots,F^n$ of the variable (t,y,x) and a function G of the variable (t,y) be given. F^1, $F^2,\dots,F^n$ are measurable with respect to (t,y) in $[0,T]\times\Omega$ for all x in $\mathbb{R}$ and are continuously differentiable and convex with respect to x in $\mathbb{R}$ and satisfy*

$$F^i(t,y,x)\ge a_i(t,y)|x|^{\beta_i}+b_i(t,y),$$

for some $\beta_i>1$, $a_i>0$, $a_i,b_i\in L^2((0,T)\times\Omega)$, $i=1,\dots,n$, for all $(t,y)\in[0,T]\times\Omega$, $x\in\mathbb{R}$, $G(\cdot,\cdot)\in\mathcal{H}^{\frac{9}{2}r-3}$. Let $j_1,\dots,j_{n-1}$ be a sequence of numbers having values either -1 or $+1$. Assume that our original nonlinearity (see (1.1)) has the form

$$l=j_1F^1_x+j_2F^2_x+\cdots+j_{n-1}F^{n-1}_x+F^n_x+G. \tag{2.6}$$

Put $\bar{F}^n_x = F^n_x + G$, $\bar{F}^n = F^n + xG$. *For constants* D_{n-1}, E_{n-1}, $\mathcal{F}$ *there exists* $\hat{x} \in \mathcal{H}^{\frac{9}{2}r-3} \cap H^{\frac{5}{2}r-1}_{per}(\mathbb{R} \times \Omega) \cap H^0_{per}(\mathbb{R}; {}^0H^2(\Omega))$, $\|\hat{x}\|_{U^{\frac{5}{2}r-1}} \le B(E_{n-1} + \mathcal{F})$ *such that* $l_{n-1}(\hat{x})$, $F^n_x(\hat{x}) \in \mathcal{H}^{\frac{9}{2}r-3}$ *and* $\|l_{n-1}(\hat{x})\|_{\mathcal{H}^{\frac{9}{2}r-3}} \le E_{n-1}$, $\|l_{n-1}(\hat{x})\|_{H^0} \ge D_{n-1}$, $\|\bar{F}^n_x(\hat{x})\|_{\mathcal{H}^{\frac{9}{2}r-3}} \le \mathcal{F}$, $(K_x(h) = K_x(\cdot, \cdot, h(\cdot, \cdot)))$, *where* $l_{n-1} = j_1 F^1_x + j_2 F^2_x + ... + j_{n-1} F^{n-1}_x$.

Define the set

$$X^n_{FG} = \Big\{ x \in U^{\frac{5}{2}r-1} : \|x\|_{U^{\frac{5}{2}r-1}} \le B(E_{n-1} + \mathcal{F}),\ \|\bar{F}^n_x(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}} \le \mathcal{F}$$
$$\|l_{n-1}(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}} \le E_{n-1},\ \|l_{n-1}(x)\|_{H^0} \ge D_{n-1} \Big\}.$$

M' *In addition to* **Assumptions M** *assume that* $l_{n-1}(x), \bar{F}^n_x(x) \in \mathcal{H}^{\frac{5}{2}r-1}$ *for* $x \in \hat{X}^n_{FG}$ *and* $\|\bar{F}^n_x(v) + l_{n-1}(x)\|_{\mathcal{H}^{\frac{5}{2}r-1}} \ge D^0_n$ *for* $v, x \in \hat{X}^n_{FG}$, *where*

$$\hat{X}^n_{FG} = \Big\{ x \in U^{\frac{5}{2}r-1} : \|x\|_{U^{\frac{5}{2}r-1}} \le B(E_{n-1} + \mathcal{F}) \Big\}$$

and some D^0_n.

Remark 2.4. The **Assumptions M** and **M'** look very cumbersome. However the aim of them is only to ensure that the set $\hat{X}^n_{FG}$ is nonempty – we seek at it critical points (see theorem below). Of course, we could state less cumbersome assumptions but then they have to be much stronger to imply nonemptiness of $\hat{X}^n_{FG}$. In fact that is the price we pay for looking for new types of critical points.

Theorem 2.5 (Main theorem)**.** *Under* ***Assumptions M, M'*** *there exists* $\overline{x} \in \hat{X}^n_{FG}$ *such that* $J(\overline{x}) = \inf_{x \in \hat{X}^n_{FG}} J(x)$ *and* $\overline{x}$ *is a solution to* (1.1)*.*

Now we can formulate theorem which gives us additional informations on solutions to (1.1) important in classical mechanics. This theorem is absolutely new for problem (1.1).

Theorem 2.6. *Let* $\overline{x}$ *be such that* $J^{nFG}(\overline{x}) = \inf_{x \in \hat{X}^n_{FG}} J^{nFG}(x)$. *Then there exists* $(\bar{p}, \bar{q}) \in H^1((0,T) \times \Omega) \times H^1((0,T) \times \Omega)$ *such that for a.e.* $(t, y) \in (0,T) \times \Omega$,

$$\bar{p}(t,y) = \bar{x}_t(t,y), \tag{2.7}$$
$$\bar{q}(t,y) = \nabla \bar{x}(t,y), \tag{2.8}$$
$$\bar{p}_t(t,y) - \operatorname{div} \bar{q}(t,y) + l(t, y, \bar{x}(t,y)) = 0 \tag{2.9}$$

and

$$J^{nFG}(\bar{x}) = J^{nFG}_D(\bar{p}, \bar{q}, \bar{z}_1, \ldots, \bar{z}_{n-1}),$$

where

$$\bar{z}_i = -j_i F^i_x(t, y, \bar{x}(t,y)),\ i = 1, \ldots, n-1, \tag{2.10}$$

$$J^{nFG}(\bar{x}) = \int_0^T \int_\Omega \left(\frac{1}{2} |\nabla \bar{x}(t,y)|^2 - \frac{1}{2} |\bar{x}_t(t,y)|^2 \right) dy dt$$
$$+ \int_0^T \int_\Omega \Big(j_1 F^1(t, y, \bar{x}(t,y)) + j_2 F^2(t, y, \bar{x}(t,y)) + \cdots + \bar{F}^n(t, y, \bar{x}(t,y)) \Big) dy dt,$$

$$
\begin{aligned}
&J_D^{nFG}(\bar{p},\bar{q},\bar{z}_1,\ldots,\bar{z}_{n-1})\\
&\quad = -\int_0^T\int_\Omega \bar{F}^{n*}(t,y,-(\bar{p}_t(t,y)-\operatorname{div}\bar{q}(t,y)-\bar{z}_1(t,y)-\cdots-\bar{z}_{n-1}(t,y)))\,dydt\\
&\qquad -j_1\int_0^T\int_\Omega F^{1*}(t,y,\bar{z}_1(t,y))dydt-\cdots-j_{n-1}\int_0^T\int_\Omega F^{n-1*}(t,y,\bar{z}_{n-1}(t,y))dydt\\
&\qquad -\frac{1}{2}\int_0^T\int_\Omega |\bar{q}(t,y)|^2\,dydt+\frac{1}{2}\int_0^T\int_\Omega |\bar{p}(t,y)|^2\,dydt,
\end{aligned}
$$

F^{i*}, $\bar{F}^{n*}$ are the Fenchel conjugate of F^i, $\bar{F}^n$ with respect to the third variable. Moreover, $\bar{x}\in\hat{X}^n_{FG}$.

The proofs of the theorems are given in Sections 3, 4. They consist of several steps. First we prove Proposition 2.1. Next we prove Theorem 2.5 (Main theorem). First for the nonlinearity l consisting only of one function $j_1F^1_x+G$, next for the difference of two $F^1_x-F^2_x$ and then by an induction for the general case.

3 Proof of Proposition 2.1

We shall consider a more general case of Proposition 2.1, namely the case for $U^q = H^{q-2r+2}_{per}(\mathbf{R}\times\Omega)\cap H^0_{per}(\mathbb{R};{}^0H^2(\Omega))$ and $\mathcal{H}^q{=}H^0_{per}(\mathbb{R};H^q(\Omega))$, $q\geq\frac{9}{2}r-3$. The norm $\|\cdot\|_{\mathcal{H}^q}$ of $g\in\mathcal{H}^q$ we define as square root of $\sum_{j,k}|k|^{2q}\left|g_{j,k}\right|^2$, i.e.

$$
\|g\|_{\mathcal{H}^q}=\left(\sum_{j,k}|k|^{2q}\left|g_{j,k}\right|^2\right)^{1/2},
$$

where

$$
g_{j,k}=\left(\frac{2^{n+1}}{\pi^nT}\right)^{1/2}\int_0^T\int_\Omega g(t,y)e^{-ij\frac{2\pi}{T}t}\sin k_1y_1\cdots\sin k_ny_n dydt.
$$

We prove stronger regularity case, i.e. the following.

Proposition 3.1. *Let $g\in\mathcal{H}^q$. Then there exists $\overline{x}\in U^q$ being a unique solution to*

$$
\begin{aligned}
x_{tt}(t,y)-\Delta x(t,y)&=g(t,y),\\
x(t,y)&=0,\quad t\in\mathbb{R},\quad y\in\partial\Omega,\\
x(t+T,y)&=x(t,y),\quad t\in\mathbb{R},\quad y\in\Omega
\end{aligned}
$$

with

$$
\overline{x}(t,y)=\left(\frac{2^{n+1}}{\pi^nT}\right)^{1/2}\sum_{j,k}(-j^24\alpha^{-2}+|k|^2)^{-1}g_{j,k}e^{ij\frac{2\pi}{T}t}\sin k_1y_1\cdots\sin k_ny_n,
$$

$g_{j,k}$ is as in (2.1) and such that

$$
\|\overline{x}\|_{U^q}\leq B_q\|g\|_{\mathcal{H}^q}, \tag{3.1}
$$

$$
\|\overline{x}\|_{U^q}\geq C_q\|g\|_{H^0} \tag{3.2}
$$

*with $B_q^2=(2\alpha+1)^{2q-4r+4}\alpha^4c^{-2}$ and $C_q^2=\frac{1}{9}\alpha^{2q-4r+4}$ **independent of** g, where α and c are defined in* **T**.

Corollary 3.2. *Let $g \in \mathcal{H}^{\frac{5}{2}r-1}$ and let $\overline{x} \in U^{\frac{5}{2}r-1}$ be such that $\|\bar{x}\|_{U^{\frac{5}{2}r-1}} \leq B_{\frac{9}{2}r-3}W$, for some $W > 0$. Then there exists $\hat{x} \in U^{\frac{5}{2}r-1}$ being a unique solution to*

$$\begin{aligned} x_{tt}\,(t,y) &= \Delta\bar{x}(t,y) + g(t,y), \\ x(t,y) &= 0, \quad t \in \mathbb{R}, \quad y \in \partial\Omega, \\ x(t+T,y) &= x(t,y), \quad t \in \mathbb{R}, \quad y \in \Omega \end{aligned} \tag{3.3}$$

and such that

$$\|\hat{x}\|_{U^{\frac{5}{2}r-1}} \leq A_q \left(\|\bar{x}\|_{U^{\frac{5}{2}r-1}} - \|g\|_{\mathcal{H}^{\frac{5}{2}r-1}} \right) \tag{3.4}$$

with $A_q^2 = \frac{1}{16}$ independent of g and $\overline{x}$.

Proof of Proposition 3.1. Our reasoning is inspired by the proof of Theorem 6.3.1 from [35], but now for the case of higher dimension periodic-Dirichlet boundary conditions (1.1) and a stronger regularity result. We know that $x \in L^2((0,T);L^2(\Omega))$ belongs to U^q if and only if

$$\sum_{j,k} (|k| + |j|)^{2q-4r+4} \left| x_{j,k} \right|^2 < \infty, \tag{3.5}$$

where

$$x_{j,k} = \left(\frac{2^{n+1}}{\pi^n T} \right)^{1/2} \int_0^T \int_\Omega x(t,y) e^{-ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n dy dt.$$

Hence

$$x(t,y) = \left(\frac{2^{n+1}}{\pi^n T} \right)^{1/2} \sum_{j,k} x_{j,k} e^{ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n. \tag{3.6}$$

The square root of (3.5) defines a norm in U^q. Similarly for $g \in \mathcal{H}^q \subset L^2((0,T);L^2(\Omega))$ we have

$$g(t,y) = \left(\frac{2^{n+1}}{\pi^n T} \right)^{1/2} \sum_{j,k} g_{j,k} e^{ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n \tag{3.7}$$

with

$$\sum_{j,k} |k|^{2q} \left| g_{j,k} \right|^2 < \infty.$$

Substituting (3.6) and (3.7) in (2.2) gives

$$(-j^2 4\alpha^{-2} + |k|^2) x_{j,k} = g_{j,k}\,, \qquad j \in \mathbb{Z},\, k \in \mathbf{L}. \tag{3.8}$$

By our assumption **T** we can write a solution $\overline{x}$ of the problem (2.2) in the form (2.3). This function belongs to U^q since

$$\sum_{j,k} (|k| + |j|)^{2q-4r+4} (-j^2 4\alpha^{-2} + |k|^2)^{-2} \left| g_{j,k} \right|^2 \leq B_q^2 \|g\|^2_{\mathcal{H}^q} \tag{3.9}$$

with B_q some constant independent of g, defined later. This inequality is a direct consequence of the relation

$$\sup \left\{ (|k| + |j|)^{2q-4r+4} (-j^2 4\alpha^{-2} + |k|^2)^{-2} |k|^{-2q};\ (j,k) \in \mathbb{Z} \times \mathbf{L} \right\} < \infty.$$

To prove it let us put

$$\sum_1 = \left\{(j,k) \in \mathbb{Z} \times \mathbf{L};\, \alpha^{-1}|j| < |k|\right\},$$
$$\sum_2 = \left\{(j,k) \in \mathbb{Z} \times \mathbf{L};\; |k| \le \alpha^{-1}|j| \le 2|k|\right\},$$
$$\sum_3 = \left\{(j,k) \in \mathbb{Z} \times \mathbf{L};\, 2|k| < \alpha^{-1}|j|\right\}.$$

We confine ourselves to the estimation on the set $\sum_2$ (the other cases are more simply) – we apply assumption **T**, i.e. $\left(\alpha^2 - \frac{|2j|^2}{|k|^2}\right)^{-2} \le c^{-2}|k|^{4r}$, thus:

$$\begin{aligned}(|k| + |j|)^{2q-4r+4}&(-j^2 4\alpha^{-2} + |k|^2)^{-2}|k|^{-2q}\\ &\le (2\alpha+1)^{2q-4r+4}|k|^{2q-4r+4}\alpha^4|k|^{-4}\left(\alpha^2 - \frac{|2j|^2}{|k|^2}\right)^{-2}|k|^{-2q}\\ &\le (2\alpha+1)^{2q-4r+4}\alpha^4 c^{-2} < \infty.\end{aligned}$$

Hence we get also the estimation (3.1) with $B_q^2 = (2\alpha+1)^{2q-4r+4}\alpha^4 c^{-2}$. To obtain the estimation (3.2) we rewrite it for our case and show that:

$$\sum_{j,k}(|k| + |j|)^{2q-4r+4}(-j^2 4\alpha^{-2} + |k|^2)^{-2}|g_{j,k}|^2 \ge C_q^2\|g\|^2_{H^0}.$$

We note that it is true if

$$\inf\left\{(|k| + |j|)^{2q-4r+4}(-j^2 4\alpha^{-2} + |k|^2)^{-2}|k|^0;\; (j,k) \in \mathbb{Z} \times \mathbf{L}\right\} > 0.$$

Again we show that only in the set $\sum_{2,}$:

$$\begin{aligned}(|k| + |j|)^{2q-4r+4}(-j^2 4\alpha^{-2} + |k|^2)^{-2} &\ge |k|^{2q-4r+4}\alpha^{2q-4r+4}|k|^{-4}\frac{1}{9}\\ &= \frac{1}{9}\alpha^{2q-4r+4} > 0.\end{aligned}$$ □

Remark 3.3. In order to get Proposition 2.1 it is enough to put in Proposition 3.1 $q = \frac{9}{2}r - 3$.

Proof of Corollary 3.2. We follow the same way as in the proof of Proposition 3.1. We have for $\bar{x} \in U^{\frac{5}{2}r-1} \subset L^2((0,T);L^2(\Omega))$

$$\bar{x}(t,y) = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2}\sum_{j,k}\bar{x}_{j,k}e^{ij\frac{2\pi}{T}t}\sin k_1y_1\cdots\sin k_ny_n \tag{3.10}$$

with

$$\sum_{j,k}(|j|^{5r-2} + |k|^{5r-2})|\bar{x}_{j,k}|^2 < \infty.$$

Substituting (3.6) and (3.10) in (3.3) gives

$$j^2 4\alpha^{-2}x_{j,k} = |k|^2\bar{x}_{j,k} - g_{j,k}, \qquad j \in \mathbb{Z},\, k \in \mathbf{L}.$$

We can write a solution $\hat{x}$ of the problem (3.3) in the form

$$\hat{x}(t,y) = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \sum_{j,k} (j^2 4\alpha^{-2})^{-1} (|k|^2 \bar{x}_{j,k} - g_{j,k}) e^{ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n.$$

This function belongs to $U^{\frac{5}{2}r-1}$ since

$$\sum_{j,k} (|j|^{5r-2} + |k|^{5r-2})(j^2 4\alpha^{-2})^{-2} |k|^4 \left| \bar{x}_{j,k} - \frac{g_{j,k}}{|k|^2} \right|^2 \leq A_q^2 (\|\bar{x}\|_{U^{\frac{5}{2}r-1}} - \|g\|_{\mathcal{H}^{\frac{5}{2}r-3}})^2$$

with A_q some constant, defined later, independent of g and $\bar{x}$ such that $\|\bar{x}\|_{U^{\frac{5}{2}r-1}} \leq B_{\frac{9}{2}r-3} W$. This inequality is a direct consequence of the relation

$$\sup \left\{ (|j|^{5r-2} + |k|^{5r-2})(j^2 4\alpha^{-2})^{-2} |k|^4 (|j|^{5r-2} + |k|^{5r-2})^{-1};\ (j,k) \in \mathbb{Z} \times \mathbf{L} \right\} < \infty.$$

To prove it let us put

$$\begin{aligned} \sum_1 &= \left\{ (j,k) \in \mathbb{Z} \times \mathbf{L};\ \alpha^{-1} |j| < |k| \right\}, \\ \sum_2 &= \left\{ (j,k) \in \mathbb{Z} \times \mathbf{L};\ |k| \leq \alpha^{-1} |j| \leq 2|k| \right\}, \\ \sum_3 &= \left\{ (j,k) \in \mathbb{Z} \times \mathbf{L};\ 2|k| < \alpha^{-1} |j| \right\}. \end{aligned}$$

We confine ourselves to the estimation on the set $\sum_2$ (the other cases are more simply) – thus:

$$|k|^4 (j^2 4\alpha^{-2})^{-2} \leq \frac{1}{16} < \infty.$$

Hence we get also the estimation (3.4) with $A_q^2 = \frac{1}{16}$. □

4 Proof of the existence of solutions and their regularity for problem (1.1)

4.1 Simple case – one function: $l = F_x$

First consider another equation

$$\begin{aligned} x_{tt}(t,y) - \Delta x(t,y) + F_x(t,y,x(t,y)) &= 0, \\ x(t,y) &= 0, \quad y \in \partial\Omega, \quad t \in \mathbb{R}, \\ x(t+T,y) &= x(t,y), \quad t \in \mathbb{R}, \quad y \in \Omega \end{aligned} \tag{4.1}$$

and corresponding to it functional

$$J^F(x) = \int_0^T \int_\Omega \left(-\frac{1}{2} |\nabla x(t,y)|^2 + \frac{1}{2} |x_t(t,y)|^2 - F(t,y,x(t,y)) \right) dydt, \tag{4.2}$$

defined on $U^1 = H^1_{per}((0,T);H^1_0(\Omega))$. Observe, that (4.1) is the Euler–Lagrange equation for the action functional J^F. For that problem we assume the following hypotheses:

G1 *$F(t,y,x)$ is measurable with respect to (t,y) in $(0,T)\times\Omega$ for all x in $\mathbb{R}$, continuously differentiable and convex with respect to the third variable in $\mathbb{R}$ for a.e. $(t,y)\in(0,T)\times\Omega$. $(t,y)\to F(t,y,0)$ is integrable on $(0,T)\times\Omega$, $F_x(t,y,x)=F_x^1(t,y,x)+F^2(t,y)$, $(t,y,x)\in(0,T)\times\Omega\times\mathbb{R}$, $F^2(\cdot,\cdot)\in\mathcal{H}^{\frac{9}{2}r-3}$.*

G2 *There exist constants $E,\ D>0$ and $\check{x}\in\mathcal{H}^{\frac{9}{2}r-3}\cap H_{per}^{\frac{5}{2}r-1}(\mathbb{R}\times\Omega)\cap H^0_{per}(\mathbb{R};^0H^2(\Omega))$, $\|\check{x}\|_{U^{\frac{5}{2}r-1}}\le B(E+\|F^2(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}})$ such that $F_x^1(\check{x})\in\mathcal{H}^{\frac{9}{2}r-3}$ and*

$$\left\|F_x^1(\check{x})\right\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le E,\left\|F_x^1(\check{x})\right\|_{H^0}\ge D. \tag{4.3}$$

G3 *$F(t,y,x)\ge a(t,y)|x|^\beta+b(t,y)$, for some $\beta>1$, $a>0$, $a,b\in L^2((0,T)\times\Omega)$, for all $(t,y)\in(0,T)\times\Omega$, $x\in\mathbb{R}$.*

Let us put

$$X_F=\left\{x\in U^{\frac{5}{2}r-1}:\|x\|_{U^{\frac{5}{2}r-1}}\le B(E+\|F^2(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}),\left\|F_x^1(x)\right\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le E,\left\|F_x^1(x)\right\|_{H^0}\ge D\right\}.$$

By **G2** X_F is nonempty.

G2′ *$F_x^1(x)\in\mathcal{H}^{2\frac{1}{2}r-1}$ and $\|F_x(x)\|_{\mathcal{H}^{\frac{5}{2}r-1}}\ge D_0$ for $x\in\hat{X}_F$, where*

$$\hat{X}_F=\left\{x\in U^{\frac{5}{2}r-1}:\|x\|_{U^{\frac{5}{2}r-1}}\le B(E+\|F^2(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}})\right\}$$

and some $D_0>0$.

Remark 4.1. Let us notice that, except convexity, restrictions for F_x^1 are not strong, they are rather natural.

Remark 4.2. The convexity assumption of $F(t,y,\cdot)$ is strong. For example x^7 is nonconvex. To overcome that problem (at least partially) we study in last section the case of $l=j_1F_x^1+j_2F_x^2+\cdots+F_x^n+G$, where F^i are convex and j_i takes values $\{-1,+1\}$. Then $x^7=x^8+x^7+x^4-x^8-x^4$ is equal to difference of two convex functions $x^8+x^7+x^4$ and x^8+x^4. This case will be considered as a next step.

Exploiting the definition of the set X_F and Proposition 2.1 we prove the following lemma.

Lemma 4.3. *Let $x\in X_F$ and v be a solution of the periodic-Dirichlet problem for*

$$v_{tt}(t,y)-\Delta v(t,y)=-F_x(t,y,x(t,y))\quad \text{a.e. on } (0,T)\times\Omega. \tag{4.4}$$

Then

$$\left|\|F^2(\cdot,\cdot)\|_{H^0}-D\right|C\le\|v\|_{U^{\frac{5}{2}r-1}}\le B(E+\|F^2(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}).$$

Proof. Fix arbitrary $x\in X_F$, thus $F_x(x)\in\mathcal{H}^{\frac{9}{2}r-3}$. Hence by Proposition 2.1 there exists a unique solution $v\in U^{\frac{5}{2}r-1}$ of the periodic-Dirichlet problem for the equation (4.4) satisfying

$$C\|F_x(x)\|_{H^0}\le\|v\|_{U^{\frac{5}{2}r-1}}\le B\|F_x(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}}.$$

Next we get the following estimations

$$B\|F_x(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le B(E+\|F^2(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}),$$
$$C\left|\|F^2(\cdot,\cdot)\|_{H^0}-D\right|\le C\|F_x(x)\|_{H^0}.$$

Hence we get

$$C\left|\|F^2(\cdot,\cdot)\|_{H^0}-D\right|\le\|v\|_{U^{\frac{5}{2}r-1}}\le B(E+\|F^2(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}).$$ □

Define in X_F the map $X_F \ni x \to H(x) = v$ where v is a solution of the periodic-Dirichlet problem (4.4). By Lemma 4.3 and the definition of X_F, $H(X_F)$ is bounded in $U^{\frac{5}{2}r-1}$, it is contained in $\hat{X}_F$. Moreover the limit of weakly convergent sequence, in $U^{\frac{5}{2}r-1}$, from $H(X_F)$ belongs to $H(X_F)$. Note that, $H(X_F) \subset \mathcal{H}^{\frac{5}{2}r-1}$.

Put

$$X^F = \left\{ \check{x} \in U^{\frac{5}{2}r-1} : \ \check{x}_{tt}(t,y) = \Delta v(t,y) - F_x(t,y,v(t,y)), \text{ where } v \in \hat{X}_F \right\}.$$

Remark 4.4. Let us note that since $v \in \hat{X}_F$ thus $F_x(v) \in \mathcal{H}^{\frac{5}{2}r-1}$. Therefore by Corollary 3.2 X^F is nonempty and bounded in $U^{\frac{5}{2}r-1}$ by $A_q(\|v\|_{U^{\frac{5}{2}r-1}} - \|F_x(v)\|_{\mathcal{H}^{\frac{5}{2}r-3}})$ i.e. for all $v \in X^F$

$$\|\check{x}\|_{U^{\frac{5}{2}r-1}} \le A_q(\|v\|_{U^{\frac{5}{2}r-1}} - \|F_x(v)\|_{\mathcal{H}^{\frac{5}{2}r-3}}) \le A_q(B(E + \|F^2(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}) - D_0).$$

Corollary 4.5. $X^F \subset \hat{X}_F$.

Next define the set X^{Fd} : an element $(p,q) \in H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega)$ belongs to X^{Fd} provided that for each $x \in \hat{X}_F$ there exist $\hat{x} \in X^F$ such that for a.e. $(t,y) \in (0,T) \times \Omega$

$$p(t,y) = x_t(t,y) \quad \text{and} \quad p_t(t,y) - \operatorname{div} q(t,y) = -F_x(t,y,\hat{x}(t,y))$$
$$\text{with } q(t,y) = \nabla \hat{x}(t,y).$$

As the sets $\hat{X}_F$, X^F are nonempty therefore the set X^{Fd} is nonempty.

The dual functional to (4.2) is usually taken as

$$\begin{aligned} J_D^F(p,q) = & \int_0^T \int_\Omega F^*(t,y,-(p_t(t,y) - \operatorname{div}\ q(t,y)))\, dydt \\ & + \frac{1}{2}\int_0^T \int_\Omega |q(t,y)|^2\, dydt - \frac{1}{2}\int_0^T \int_\Omega |p(t,y)|^2\, dydt, \end{aligned} \tag{4.5}$$

where F^* is the Fenchel conjugate of F with respect to third variable and $J_D^F : X^{Fd} \to \mathbb{R}$.

We will look at relationships between the functional J^F and J_D^F on the set $X^F$and X^{Fd} respectively: Variational Principle at extreme points. It relates the critical values of both functionals and provides the necessary conditions that must be satisfied by the solution to problem (4.1).

Now we state the simple result of the paper which is existence theorem for particular case of (1.1) i.e. problem (4.1).

Theorem 4.6. *Assume **G1–G3**. Then there exists $\overline{x} \in \hat{X}_F$ such that*

$$\inf_{x \in \hat{X}_F} J^F(x) = J^F(\overline{x}),$$

Moreover, there exists $(\overline{p},\overline{q}) \in H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega)$ such that

$$J_D^F(\overline{p},\overline{q}) = J^F(\overline{x}) \tag{4.6}$$

and the following system holds

$$\overline{x}_t(t,y) = \overline{p}(t,y), \tag{4.7}$$
$$\nabla \overline{x}(t,y) = \overline{q}(t,y), \tag{4.8}$$
$$\overline{p}_t(t,y) - \operatorname{div} \overline{q}(t,y) = -F_x(t,y,\overline{x}(t,y)). \tag{4.9}$$

This result is new however it has strong assumption for nonlinearity F_x: convexity of $F(t,y,\cdot)$. Our aim is to relax them. But first we illustrate, by an example, a case of the above theorem.

Example 4.7. Assume $n=2$ and let α be such that α^2 is an algebraic number of degree 3. Thus α^2 satisfies (1.6) with e.g. $\varepsilon=1/2$. Then α^2 satisfies condition in **T** with $c=1$ and $r=5/2$. Let us consider $F^1(t,y,x)=x^6+x^5+x^2+1$. Of course $F^1(t,y,\cdot)$ is convex. Let $F^2(\cdot,\cdot)$ be any function in $(0,T)\times\Omega$ bounded in $\mathcal{H}^{8\frac{1}{4}}$ and such that $\int_0^T\int_\Omega \left|F^2(t,y)\right|^2 dydt \neq 0$. Take any $\check{x}\in\mathcal{H}^{100}$ such that $\left\|F_x^1(\check{x})\right\|_{\mathcal{H}^{8\frac{1}{4}}} \leq E=100$. Of course such an $\check{x}$ exists. Then assumptions **G1, G2, G2′, G3** are satisfied, so by the above theorem there exists $\overline{x}\in X^F$ being a solution to (4.1).

Example 4.8. We can consider also the case with $F(t,y,x)=x^6+x^5+x^2$. Then we take $F^1(t,y,x)=x^6+x^5+x^2+x$ and $F^2(t,y)=-1$. Thus the theorem for that case asserts that there exists nontrivial solution to (4.1).

4.1.1 The auxiliary results

By **G1–G3,** definition of $\hat{X}^F$, mean value theorem we get the following lemma.

Lemma 4.9. *There exists constant M_1 such that*

$$\int_0^T\int_\Omega F(t,y,x(t,y))dydt \geq M_1$$

for all $x\in\hat{X}_F$.

Lemma 4.10. *The functional J^F attains its infimum on $\hat{X}_F$ i.e. $\inf_{x\in\hat{X}_F} J^F(x)=J^F(\overline{x})$, where $\overline{x}\in\hat{X}_F$.*

Proof. By the definition of the set $\hat{X}_F$ and Lemma 4.9 we see that the functional J^F is bounded below on $\hat{X}_F$. We denote by $\{x^j\}$ a minimizing sequence for J^F in $\hat{X}^F$. This sequence has a subsequence which we denote again by $\{x^j\}$ converging weakly in $U^{\frac{5}{2}r-1}$ and strongly in U^1, hence also strongly in $L^2((0,T)\times\Omega;\mathbb{R})$ to a certain element $\overline{x}\in U^{\frac{5}{2}r-1}$. Moreover $\{x^j\}$ is also convergent almost everywhere. Thus by construction of the set $\hat{X}_F$, we observe that $\overline{x}\in\hat{X}_F$.

Hence

$$\liminf_{j\to\infty} J^F\left(x^j\right) \geq J^F(\bar{x}).$$

Thus

$$\inf_{x\in\hat{X}_F} J^F(x)=J^F(\bar{x}).$$

□

4.1.2 Proof of Theorem 4.6

Let $\bar{x}\in\hat{X}_F$ be such that $J^F(\bar{x})=\inf_{x\in\hat{X}_F} J^F(x)$. This means that there exists an $\hat{x}\in X^F\subset\hat{X}_F$ such that

$$\hat{p}(t,y)=\hat{x}_t(t,y) \tag{4.10}$$

and

$$\hat{p}_t(t,y)=\operatorname{div}\bar{q}(t,y)-F_x(t,y,\bar{x}(t,y)), \tag{4.11}$$

for a.a. $(t,y) \in (0,T) \times \Omega$ where $\bar{q}$ is given by

$$\bar{q}(t,y) = \nabla \bar{x}(t,y). \tag{4.12}$$

By the definitions of J^F, J_D^F, relations (4.10), (4.11) and the Fenchel–Young inequality it follows that

$$\begin{aligned}
J^F(\bar{x}) &= \int_0^T \int_\Omega \left(-\frac{1}{2} |\nabla \bar{x}(t,y)|^2 + \frac{1}{2} |\bar{x}_t(t,y)|^2 - F(t,y,\bar{x}(t,y)) \right) dydt \\
&\geq \int_0^T \int_\Omega \bar{x}_t(t,y)\, \hat{p}(t,y)\, dydt - \frac{1}{2} \int_0^T \int_\Omega |\hat{p}(t,y)|^2 dydt \\
&\quad - \int_0^T \int_\Omega \langle \nabla \bar{x}(t,y), \bar{q}(t,y) \rangle dydt + \frac{1}{2} \int_0^T \int_\Omega |\bar{q}(t,y)|^2 dydt \\
&\quad - \int_0^T \int_\Omega F(t,y,\bar{x}(t,y)) dydt \\
&\geq \int_0^T \int_\Omega F^*(t,y,-(\hat{p}_t(t,y) - div\, \bar{q}(t,y)))\, dydt \\
&\quad + \frac{1}{2} \int_0^T \int_\Omega |\bar{q}(t,y)|^2 dydt - \frac{1}{2} \int_0^T \int_\Omega |\hat{p}(t,y)|^2 dydt = J_D^F(\hat{p},\bar{q}).
\end{aligned}$$

Therefore we get that

$$J^F(\bar{x}) \geq J_D^F(\hat{p},\bar{q}).$$

Next observe that

$$\begin{aligned}
\inf_{x \in \hat{X}_F} J^F(x) &= J^F(\bar{x}) \leq J^F(\hat{x}) \\
&= \int_0^T \int_\Omega \left(-\hat{x}_t(t,y)\, \hat{p}(t,y)\, dydt + \frac{1}{2} |\hat{x}_t(t,y)|^2 \right) dydt \\
&\quad - \int_0^T \int_\Omega \left(\frac{1}{2} |\nabla \hat{x}(t,y)|^2 - \langle \nabla \hat{x}(t,y), \bar{q}(t,y) \rangle \right) dydt \\
&\quad - \int_0^T \int_\Omega \left(F(t,y,\hat{x}(t,y)) + \hat{x}(t,y)(\hat{p}_t(t,y) - div\, \bar{q}(t,y)) \right) dydt \\
&\leq \frac{1}{2} \left(- \int_0^T \int_\Omega |\hat{p}(t,y)|^2 dydt + \int_0^T \int_\Omega |\bar{q}(t,y)|^2 dydt \right) \\
&\quad \int_0^T \int_\Omega F^*(t,y,-(\hat{p}_t(t,y) - \operatorname{div} \bar{q}(t,y)))\, dydt = J_D^F(\hat{p},\bar{q})
\end{aligned}$$

and so

$$J^F(\bar{x}) \leq J_D^F(\hat{p},\bar{q}).$$

Thus we have equality $J^F(\bar{x}) = J_D^F(\hat{p},\bar{q})$. It implies

$$\begin{aligned}
&\int_0^T \int_\Omega F^*(t,y,-(\hat{p}_t(t,y) - \operatorname{div} \bar{q}(t,y)))\, dydt + \int_0^T \int_\Omega F(t,y,\bar{x}(t,y))\, dydt \\
&\quad + \frac{1}{2} \int_0^T \int_\Omega |\bar{q}(t,y)|^2 dydt + \frac{1}{2} \int_0^T \int_\Omega |\nabla \bar{x}(t,y)|^2 dydt \\
&= \frac{1}{2} \int_0^T \int_\Omega |\bar{x}_t(t,y)|^2 dydt + \frac{1}{2} \int_0^T \int_\Omega |\hat{p}(t,y)|^2 dydt.
\end{aligned}$$

The last means that

$$\begin{aligned}
&\int_0^T \int_\Omega F^*\left(t, y, -\left(\hat{p}_t\left(t, y\right) - \operatorname{div} \bar{q}\left(t, y\right)\right)\right) dydt + \int_0^T \int_\Omega F\left(t, y, \bar{x}\left(t, y\right)\right) dydt \\
&\qquad + \int_0^T \int_\Omega \bar{x}\left(t, y\right)\left(\hat{p}_t\left(t, y\right) - \operatorname{div} \bar{q}\left(t, y\right)\right) dydt \\
&\qquad + \frac{1}{2} \int_0^T \int_\Omega \left|\bar{q}\left(t, y\right)\right|^2 dydt + \frac{1}{2} \int_0^T \int_\Omega \left|\nabla \bar{x}\left(t, y\right)\right|^2 dydt - \int_0^T \int_\Omega \left\langle \nabla \bar{x}\left(t, y\right), \bar{q}\left(t, y\right)\right\rangle dydt \\
&= \frac{1}{2} \int_0^T \int_\Omega \left|\hat{p}\left(t, y\right)\right|^2 dydt + \frac{1}{2} \int_0^T \int_\Omega \left|\bar{x}_t\left(t, y\right)\right|^2 dydt - \int_0^T \int_\Omega \bar{x}_t\left(t, y\right) \hat{p}\left(t, y\right) dydt.
\end{aligned}$$

Hence and the standard convexity arguments and (4.11) we obtain two equalities

$$\begin{aligned}
\hat{p}(t, y) &= \bar{x}_t\left(t, y\right), \\
\hat{p}_t(t, y) &= \operatorname{div} \bar{q}\left(t, y\right) - F_x\left(t, y, \bar{x}\left(t, y\right)\right).
\end{aligned}$$

Thus, taking into account (4.12) we infer that

$$\bar{x}_{tt}(t, y) = \Delta \bar{x}\left(t, y\right) - F_x\left(t, y, \bar{x}\left(t, y\right)\right),$$

therefore there exist $\bar{p} = \bar{x}_t$ and $\bar{q} = \nabla \bar{x}$ i.e. (4.7), (4.8), (4.9) are satisfied and so we have the assertions of the theorem satisfied.

4.2 Simple case – one function: $l = -F_x$

A similar theorem to Theorem 4.6 is true for the problem

$$\begin{aligned}
x_{tt}(t, y) - \Delta x(t, y) - F_x(t, y, x(t, y)) &= 0, \\
x(t, y) &= 0, \quad y \in \partial\Omega, \quad t \in \mathbb{R}, \\
x(t + T, y) &= x(t, y), \quad t \in \mathbb{R}, \quad y \in \Omega
\end{aligned} \tag{4.13}$$

and corresponding to it functional

$$J^{F-}\left(x\right) = \int_0^T \int_\Omega \left(-\frac{1}{2} |\nabla x(t, y)|^2 + \frac{1}{2} \left|x_t\left(t, y\right)\right|^2 + F(t, y, x(t, y)) \right) dydt, \tag{4.14}$$

defined on U^1 with same hypotheses **G1–G3** and the sets X_F, $\hat{X}_F$. Really, Lemmas 4.3–4.10 are still valid as sign of F does not change their proofs. The set

$$X^{-F} = \left\{ \check{x} \in U^{\frac{5}{2}r-1} : \ \Delta \check{x}\left(t, y\right) = v_{tt}\left(t, y\right) - F_x(t, y, v(t, y)), \text{ where } v \in \hat{X}_F \right\}.$$

Now by Corollary 3.2 X^{-F} is nonempty. The set X^{-Fd} is defined: an element $(p, q) \in H^1((0, T) \times \Omega) \times H^1((0, T) \times \Omega)$ belongs to X^{-Fd} provided that for each $x \in \hat{X}_F$, there exist $\hat{x} \in X^{-F}$ such that for a.e. $(t, y) \in (0, T) \times \Omega$

$$\begin{aligned}
q\left(t, y\right) = \nabla x\left(t, y\right) \quad \text{and} \quad p_t\left(t, y\right) - \operatorname{div} q\left(t, y\right) = -F_x(t, y, \hat{x}(t, y)) \\
\text{with } p\left(t, y\right) = \hat{x}_t\left(t, y\right).
\end{aligned}$$

As the set X^{-F} is nonempty therefore the set X^{-Fd} is nonempty. The dual functional to (4.14) is now

$$J_D^{F-}(p,q) = -\int_0^T\int_\Omega F^*(t,y,-(p_t(t,y)-\operatorname{div} q(t,y)))\,dydt + \frac{1}{2}\int_0^T\int_\Omega |q(t,y)|^2\,dydt - \frac{1}{2}\int_0^T\int_\Omega |p(t,y)|^2\,dydt,$$

Hence following the same way as in the proof of the above theorem we get for (4.13) the following theorem.

Theorem 4.11. *Let* $\inf_{x\in\hat{X}_F} J^{F-}(x) = J^{F-}(\bar{x})$. *Then there exists* $(\bar{p},\bar{q}) \in H^1((0,T)\times\Omega)\times H^1((0,T)\times\Omega)$ *such that for a.e.* $(t,y)\in(0,T)\times\Omega$,

$$\bar{p}(t,y) = \bar{x}_t(t,y), \tag{4.15}$$

$$\bar{q}(t,y) = \nabla\bar{x}(t,y), \tag{4.16}$$

$$\bar{p}_t(t,y) - \operatorname{div}\bar{q}(t,y) - F_x(t,y,\bar{x}(t,y)) = 0 \tag{4.17}$$

and

$$J^{F-}(\bar{x}) = J_D^{F-}(\bar{p},\bar{q}),$$

where

$$J_D^{F-}(\bar{p},\bar{q}) = -\int_0^T\int_\Omega F^*(t,y,-(\bar{p}_t(t,y)-\operatorname{div}\bar{q}(t,y)))\,dydt + \frac{1}{2}\int_0^T\int_\Omega |\bar{q}(t,y)|^2\,dydt - \frac{1}{2}\int_0^T\int_\Omega |\bar{p}(t,y)|^2\,dydt.$$

Moreover $\bar{x}\in\hat{X}_F$.

4.3 The case of nonlinearity $F - G$

Now we consider more complicated problem i.e.

$$\begin{aligned} x_{tt}(t,y) - \Delta x(t,y) - G_x(t,y,x(t,y)) + F_x(t,y,x(t,y)) &= 0, \\ x(t,y) &= 0, \quad y\in\partial\Omega, \quad t\in\mathbb{R}, \\ x(t+T,y) &= x(t,y), \quad t\in\mathbb{R}, \quad y\in\Omega \end{aligned} \tag{4.18}$$

and corresponding to it functional

$$J^{FG}(x) = \int_0^T\int_\Omega\left(-\frac{1}{2}|\nabla x(t,y)|^2 + \frac{1}{2}|x_t(t,y)|^2 + G(t,y,x(t,y)) - F(t,y,x(t,y))\right)dydt, \tag{4.19}$$

defined in U^1. The similar case of L ($l = L_x$) being a difference of two convex nonautonomous functions is investigated in [3]: the nonlinearity $l\in C([0,\pi]\times\mathbb{R}^2,\mathbb{R})$ has the form $l(t,y,x) = \lambda g(t,y,x) + \mu h(t,y,x)$ with $\lambda,\mu\in\mathbb{R}$, g superlinear in x, h sublinear in x and both g, h are 2π-periodic in t and nondecreasing in x.

For problem (4.18) we assume the following hypotheses:

GG1 *$F(t,y,x)$ and $G(t,y,x)$ are measurable with respect to (t,y) in $(0,T)\times\Omega$ for all x in $\mathbb{R}$ and continuously differentiable and convex with respect to the third variable in $\mathbb{R}$ for a.a. $(t,y)\in(0,T)\times\Omega$. $(t,y)\to F(t,y,0)-G(t,y,0)$ is integrable on $(0,T)\times\Omega$.*

GG2 *There exist constants D, E, $\mathcal{G}$ and $\check{x}\in\mathcal{H}^{\frac{9}{2}r-3}\cap H_{per}^{\frac{5}{2}r-1}(\mathbb{R}\times\Omega)\cap H_{per}^{0}(\mathbb{R};^{0}H^{2}(\Omega))$, $\|\check{x}\|_{U^{\frac{5}{2}r-1}}\le B(E+\mathcal{G})$ such that $F_x(\check{x})$, $G_x(\check{x})\in\mathcal{H}^{\frac{9}{2}r-3}$ and*

$$\|F_x(\check{x})\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le E,\qquad \|F_x(\check{x})\|_{H^0}\ge D,\qquad \|G_x(\check{x})\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le\mathcal{G}.$$

GG3 *$F(t,y,x)$ and $G(t,y.x)$* satisfy **G3.**

Define the set

$$X_{FG}=\Big\{x\in U^{\frac{5}{2}r-1}:\ \|x\|_{U^{\frac{5}{2}r-1}}\le B(E+\mathcal{G}),\ \|F_x(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le E,$$
$$\|F_x(x)\|_{H^0}\ge D,\ \|G_x(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le\mathcal{G}\Big\}.$$

By **GG2**, the set X_{FG} is nonempty.

GG2' *$F_x(x)$, $G_x(x)\in\mathcal{H}^{\frac{5}{2}r-1}$ and $\|F_x(v)-G_x(x)\|_{\mathcal{H}^{\frac{5}{2}r-1}}\ge D_1$ for all $v,x\in\hat{X}_{FG}$, where*

$$\hat{X}_{FG}=\Big\{x\in U^{\frac{5}{2}r-1}:\ \|x\|_{U^{\frac{5}{2}r-1}}\le B(E+\mathcal{G})\Big\}$$

and some $D_1>0$.

We have also the following lemma.

Lemma 4.12. *Define in X_{FG} the map $X_{FG}\ni x\to H(x)=v$ where v is a solution of the periodic-Dirichlet problem for*

$$v_{tt}(t,y)-\Delta v(t,y)+F_x(t,y,v(t,y)=G_x(t,y,x(t,y)))\quad a.e.\ on\ (0,T)\times\Omega.\tag{4.20}$$

Then $H(X_{FG})$ is bounded in $U^{\frac{5}{2}r-1}$ by $B(E+\mathcal{G})$.

Proof. Fix arbitrary $x\in X_{FG}$. It follows that $G_x(x)\in\mathcal{H}^{\frac{9}{2}r-3}$ and $\|G_x(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le\mathcal{G}$. Hence by Theorem 4.6 there exists a solution v of the periodic-Dirichlet problem for the equation (4.20) satisfying

$$\|v\|_{U^{\frac{5}{2}r-1}}\le B(E+\mathcal{G}).$$

Thus the last relation implies that for an arbitrary $x\in X_{FG}$ there exists $v=H(x)$ bounded in $U^{\frac{5}{2}r-1}$ by $B(E+\mathcal{G})$. □

Put

$$X^{FG}=\Big\{\hat{x}\in U^{\frac{5}{2}r-1}:\ \hat{x}_{tt}(t,y)-G_x(t,y,\hat{x}(t,y))$$
$$=\Delta v(t,y)-F_x(t,y,v(t,y)),\ \text{where } v\in\hat{X}_{FG}\Big\}.\tag{4.21}$$

Lemma 4.13. *For each $v\in\hat{X}_{FG}$ there exists a unique solution to*

$$\hat{x}_{tt}(t,y)-G_x(t,y,\hat{x}(t,y))=\Delta v(t,y)-F_x(t,y,v(t,y)),\tag{4.22}$$

i.e. the set X^{FG} is nonempty. Moreover, $X^{FG}\subset\hat{X}_{FG}$.

Proof. Let us fix any $v \in \hat{X}_{FG}$ and notice that $\hat{x} \in U^{\frac{5}{2}r-1}$ defining X^{FG} is a solution of the equation which is an Euler–Lagrange equation to the functional

$$\hat{J}(x) = \int_0^T \int_\Omega \left(\frac{1}{2} |x_t(t,y)|^2 + G(t,y,x(t,y)) + x(t,y)(\Delta v(t,y) - F_x(t,y,v(t,y))) \right) dydt.$$

Notice that $\hat{J}(x)$ is the strictly convex, thus weakly lower semicontinuous in $U^{\frac{5}{2}r-1}$and X^{FG} is weakly compact (since $\hat{X}_{FG}$ weakly compact) in $U^{\frac{5}{2}r-1}$. Thus $\hat{J}$ attains its unique minimum in $U^{\frac{5}{2}r-1}$ and the minimizer $\hat{x}^m$ of it belongs to X^{FG}. Next define a functional dual to $\hat{J}$ (in the sense of convex analysis):

$$\hat{J}_D(\hat{p}) = \int_0^T \int_\Omega \frac{1}{2}(|\hat{p}(t,y)|^2 + G^*(t,y,\hat{p}_t(t,y))dydt + \int_0^T \int_\Omega \psi(\hat{p}_t(t,y))dydt,$$

considered in $H^1((0,T)\times\Omega)$, where

$$\psi(\hat{p}_t(t,y)) \begin{cases} 0, & \text{if } \hat{p}_t(t,y) = \Delta v(t,y) - F_x(t,y,v(t,y)), \\ +\infty, & \text{if } \hat{p}_t(t,y) \neq \Delta v(t,y) - F_x(t,y,v(t,y)) \end{cases}$$

and $G^*(t,y,\cdot)$ is Fenchel conjugate of $G(t,y,\cdot)$. We see that $\hat{J}_D(\hat{p})$ attains its minimum at $\hat{p}^m$ such that

$$\hat{p}^m_t(t,y) = \Delta v(t,y) - F_x(t,y,v(t,y))$$

and using the standard tools of convex analysis we see next that $\hat{J}(\hat{x}^m) = -\hat{J}_D(\hat{p}^m)$. Moreover, minimizer $\hat{x}^m$ to $\hat{J}$ satisfies (4.22). Following in the same way as in the proof of Corollary 3.2 we can write a solution $\hat{x} \in X^{FG}$ in a form

$$\hat{x}(t,y) = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \sum_{j,k} (j^2 4\alpha^{-2})^{-1}(|k|^2 v_{j,k} - (f_{j,k} - g_{j,k})e^{ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n,$$

where

$$v_{j,k} = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \int_0^T \int_\Omega v(t,y)\, e^{-ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n dydt,$$

$$g_{j,k} = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \int_0^T \int_\Omega G_x(t,y,\hat{x}(t,y)) e^{-ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n dydt,$$

$$f_{j,k} = \left(\frac{2^{n+1}}{\pi^n T}\right)^{1/2} \int_0^T \int_\Omega F_x(t,y,v(t,y)) e^{-ij\frac{2\pi}{T}t} \sin k_1 y_1 \cdots \sin k_n y_n dydt.$$

Then since $\hat{x} \in U^{\frac{5}{2}r-1}$ we have also estimation as in Corollary 3.2, namely

$$\|\hat{x}\|_{U^{\frac{5}{2}r-1}} \leq A_q(\|v\|_{U^{\frac{5}{2}r-1}} - \|F_x(v) - G_x(\hat{x})\|_{\mathcal{H}^{\frac{5}{2}r-1}}) \leq A_q(B(E+\mathcal{G}) - D_1).$$

Thus $X^{FG} \subset \hat{X}_{FG}$. □

Next define the set X^{FGd}: an element $(p,q,z) \in H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega)$ belongs to X^{FGd} provided that there exist $x \in X^{FG}$, $\hat{x} \in \hat{X}_{FG}$ such that for a.e. $(t,y) \in (0,T)\times\Omega$

$$p(t,y) = x_t(t,y) \text{ and } p_t(t,y) - z(t,y) = \operatorname{div}\hat{q}(t,y) - F_x(t,y,\hat{x}(t,y))$$
$$\text{with } \hat{q}(t,y) = \nabla\hat{x}(t,y) \text{ and } z(t,y) = G_x(t,y,x(t,y)).$$

By Lemma 4.13 the set X^{FGd} is nonempty. The dual functional to (4.19) is then taken as

$$\begin{aligned} J_D^{FG}(p,q,z) = & \int_0^T \int_\Omega F^*(t,y,-(p_t(t,y) - \operatorname{div} q(t,y) - z(t,y)))\,dydt \\ & - \int_0^T \int_\Omega G^*(t,y,z(t,y))dydt + \frac{1}{2}\int_0^T \int_\Omega |q(t,y)|^2\,dydt \\ & - \frac{1}{2}\int_0^T \int_\Omega |p(t,y)|^2\,dydt, \end{aligned} \tag{4.23}$$

where G^* is the Fenchel conjugate of G with respect to third variable and

$$J_D^{FG} : H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega) \to \mathbb{R}.$$

Remark 4.14. Let us observe that if $\bar{x}$ is a solution to (4.18) then by Lemma 4.13 it has to belong to $\hat{X}_{FG}$.

Analogously as in the case of the functional J^F we prove the following lemma.

Lemma 4.15. *The functional J^{FG} attains its minimum on $\hat{X}_{FG}$ i.e. $\inf_{x\in\hat{X}_{FG}} J^{FG}(x) = J^{FG}(\overline{x})$, where $\overline{x} \in \hat{X}_{FG}$.*

Theorem 4.16. *Assume **GG1–GG3**. Let $J^{FG}(\overline{x}) = \inf_{x\in\hat{X}_{FG}} J^{FG}(x)$. Then there exists $(\bar{p},\bar{q}) \in H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega)$ such that for a.e. $(t,y) \in (0,T)\times\Omega$,*

$$\bar{p}(t,y) = \bar{x}_t(t,y), \tag{4.24}$$

$$\bar{q}(t,y) = \nabla\bar{x}(t,y), \tag{4.25}$$

$$\bar{p}_t(t,y) - \operatorname{div}\bar{q}(t,y) - G_x(t,y,\bar{x}(t,y)) + F_x(t,y,\bar{x}(t,y)) = 0 \tag{4.26}$$

and

$$J^{FG}(\bar{x}) = J_D^{FG}(\bar{p},\bar{q},\bar{z}),$$

where $\bar{z}(t,y) = G_x(t,y,\bar{x}(t,y))$.

Example 4.17. We show, how to use, the above theorem to solve the nonconvex superlinear problem:

$$\begin{aligned} x_{tt} - \Delta x + 5x^4 + \cos\frac{1}{2}y = 0, & \\ x(t,y) = 0, \quad y \in \partial\Omega,\ t \in \mathbb{R}, & \\ x(t+T,y) = x(t,y), \quad t \in \mathbb{R},\ y \in \Omega. & \end{aligned} \tag{4.27}$$

Assume $n = 2$ and let α be such that α^2 is an algebraic number of degree 3. Thus α^2 satisfies (1.6) with e.g. $\varepsilon = 1/2$. Then α^2 satisfies condition in **T** with $c = 1$ and $r = 5/2$. Let us put $x^5 = x^6 + x^5 + x^2 - x^6 - x^2$. Then assume $F(t,y,x) = x^6 + x^5 + x^2 + x\cos\frac{1}{2}y$ and $G(t,y,x) = x^6 + x^2$. $F(t,y,\cdot)$ and $G(t,y,\cdot)$ are convex. Thus take for $\hat{x}$ such an $\hat{x}(\cdot,\cdot) > 0$ that $\hat{x} \in \mathcal{H}^{100}$ and $\|F_x(\check{x})\|_{\mathcal{H}^{8\frac{1}{4}}} \le E = 100$ and $\|G_x(\check{x})\|_{\mathcal{H}^{8\frac{1}{4}}} \le \mathcal{G} = 100$, for D_1 take $\left(\int_0^T \int_\Omega \left|\cos\frac{1}{2}y\right|^2 dydt\right)^{1/2}$ then assumptions **GG1–GG3** are satisfied, so by the above theorem there exists $\overline{x} \in \hat{X}_{FG}$ (nontrivial) being a solution to (4.27).

Proof of the theorem. Let $\bar{x} \in \hat{X}_{FG}$ be such that $J^{FG}(\bar{x}) = \inf_{x \in \hat{X}_{FG}} J^{FG}(x)$. As $\bar{x} \in \hat{X}_{FG}$, therefore there exists an $\hat{x} \in X^{FG} \subset \hat{X}_{FG}$ such that

$$\hat{p}(t,y) = \hat{x}_t(t,y) \tag{4.28}$$

and

$$\hat{p}_t(t,y) = \operatorname{div} \bar{q}(t,y) - F_x(t,y,\bar{x}(t,y)) + \hat{z}(t,y), \tag{4.29}$$

for a.a. $(t,y) \in (0,T) \times \Omega$ where $\bar{q}$ is given by

$$\bar{q}(t,y) = \nabla \bar{x}(t,y), \tag{4.30}$$

$$\hat{z}(t,y) = G_x(t,y,\hat{x}(t,y)), \qquad \bar{z}(t,y) = G_x(t,y,\bar{x}(t,y)). \tag{4.31}$$

By the definitions of J^{FG}, J_D^{FG}, relations (4.28), (4.31) and the Fenchel–Young inequality it follows that

$$\begin{aligned}
J^{FG}(\bar{x}) &= \int_0^T \int_\Omega \left(-\frac{1}{2}|\nabla \bar{x}(t,y)|^2 + \frac{1}{2}|\bar{x}_t(t,y)|^2 + G(t,y,\bar{x}(t,y)) - F(t,y,\bar{x}(t,y)) \right) dydt \\
&\geq \int_0^T \int_\Omega \bar{x}_t(t,y)\,\hat{p}(t,y)\,dydt - \frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,dydt \\
&\quad - \int_0^T \int_\Omega \langle \nabla \bar{x}(t,y), \bar{q}(t,y) \rangle\,dydt + \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2\,dydt \\
&\quad + \int_0^T \int_\Omega \langle \bar{x}(t,y), \hat{z}(t,y) \rangle\,dydt \\
&\quad - \int_0^T \int_\Omega G^*(t,y,\hat{z}(t,y))dydt - \int_0^T \int_\Omega F(t,y,\bar{x}(t,y))dydt \\
&\geq \int_0^T \int_\Omega F^*(t,y,-(\hat{p}_t(t,y) - \operatorname{div}\bar{q}(t,y) - \hat{z}(t,y)))\,dydt + \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2\,dydt \\
&\quad - \frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,dydt - \int_0^T \int_\Omega G^*(t,y,\hat{z}(t,y))dydt = J_D^{FG}(\hat{p},\bar{q},\hat{z}).
\end{aligned}$$

Hence we infer

$$J^{FG}(\hat{x}) \geq J^{FG}(\bar{x}) \geq J_D^{FG}(\hat{p},\bar{q},\hat{z}).$$

Next observe that

$$\begin{aligned}
&\inf_{x \in \hat{X}_{FG}} J^{FG}(x) = J^{FG}(\bar{x}) \leq J^{FG}(\hat{x}) \\
&\quad = \int_0^T \int_\Omega \left(-\hat{x}_t(t,y)\,\hat{p}(t,y) + \frac{1}{2}|\hat{x}_t(t,y)|^2 \right) dydt \\
&\quad + \int_0^T \int_\Omega \left(-\frac{1}{2}|\nabla \hat{x}(t,y)|^2 + \langle \nabla \hat{x}(t,y), \bar{q}(t,y) \rangle \right) dydt \\
&\quad - \int_0^T \int_\Omega (\hat{x}(t,y)\,\hat{z}(t,y) - G(t,y,\hat{x}(t,y)))\,dydt \\
&\quad - \int_0^T \int_\Omega (F(t,y,\hat{x}(t,y)) - \hat{x}(t,y)(-\hat{p}_t(t,y) + \operatorname{div}\bar{q}(t,y) + \hat{z}(t,y)))\,dydt \\
&\quad \leq \frac{1}{2}\left(-\int_0^T \int_\Omega |\hat{p}(t,y)|^2\,dydt + \int_0^T \int_\Omega |\bar{q}(t,y)|^2\,dydt \right) - \int_0^T \int_\Omega G^*(t,y,\hat{z}(t,y))dydt \\
&\quad + \int_0^T \int_\Omega F^*(t,y,-(\hat{p}_t(t,y) - \operatorname{div}\bar{q}(t,y) - \hat{z}(t,y)))\,dydt = J_D^{FG}(\hat{p},\bar{q},\hat{z})
\end{aligned}$$

and so

$$J^{FG}(\bar{x}) \leq J^{FG}(\hat{x}) \leq J_D^{FG}(\hat{p},\bar{q},\hat{z}).$$

Thus we have equality $J^{FG}(\hat{x}) = J_D^{FG}(\hat{p},\bar{q},\hat{z})$. It implies

$$\begin{aligned}
&\int_0^T \int_\Omega F^*\left(t,y,-\left(\hat{p}_t(t,y) - \operatorname{div}\bar{q}(t,y) - \hat{z}(t,y)\right)\right) dydt + \int_0^T \int_\Omega F(t,y,\bar{x}(t,y))\, dydt \\
&\quad + \frac{1}{2}\int_0^T \int_\Omega |\nabla \bar{x}(t,y)|^2\, dydt + \frac{1}{2}\int_0^T \int_\Omega |\bar{q}(t,y)|^2\, dydt \\
&= \frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\, dydt + \frac{1}{2}\int_0^T \int_\Omega |\bar{x}_t(t,y)|^2\, dydt \\
&\quad + \int_0^T \int_\Omega G(t,y,\bar{x}(t,y))\, dydt + \int_0^T \int_\Omega G^*(t,y,\hat{z}(t,y)) dydt.
\end{aligned}$$

Therefore by (4.29), (4.30) and standard convexity arguments

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_\Omega |\hat{p}(t,y)|^2\, dydt + \frac{1}{2}\int_0^T \int_\Omega |\bar{x}_t(t,y)|^2\, dydt - \int_0^T \int_\Omega \bar{x}_t(t,y)\,\hat{p}(t,y)\, dydt \\
&\quad + \int_0^T \int_\Omega G(t,y,\bar{x}(t,y))\, dydt + \int_0^T \int_\Omega G^*(t,y,\hat{z}(t,y)) dydt - \int_0^T \int_\Omega \bar{x}(t,y)\,\hat{z}(t,y) dydt \\
&= 0.
\end{aligned}$$

and as a consequence we have the equalities

$$\hat{p}(t,y) = \bar{x}_t(t,y),\ \hat{z}(t,y) = G_x(t,y,\bar{x}(t,y)). \tag{4.32}$$

Hence, by (4.31), (4.32) $\hat{z} = \bar{z}$ and $\bar{x}_t = \hat{x}_t$. We have also equality $J^{FG}(\bar{x}) = J_D^{FG}(\hat{p},\bar{q},\bar{z})$ which similarly implies

$$\hat{p}_t(t,y) - div\,\bar{q}(t,y) - \bar{z}(t,y) = -F_x(t,y,\bar{x}(t,y))$$

and so we have the assertions of the theorem satisfied. □

4.4 More general case – proof of the main theorem

Let us consider now a sequence of convex (with respect to third variable) functions F^1, $F^2,\ldots,F^n$ of the variables (t,y,x) and a function G of the variable (t,y). Let $j_1,\ldots,j_n$ be a sequence of numbers having values either -1 or $+1$. Let us assume that our original nonlinearity (see (1.1)) has the form

$$l = j_1F_x^1 + j_2F_x^2 + \cdots + j_{n-1}F_x^{n-1} + F_x^n + G.$$

To prove the existence of solution to (1.1) with nonlinearity l just defined we use an induction argument. To this effect let us put $l_{n-1} = j_1F_x^1 + j_2F_x^2 + \cdots + j_{n-1}F_x^{n-1}$ and consider the problem

$$\begin{aligned}
x_{tt}(t,y) - \Delta x(t,y) + l_{n-1}(t,y,x(t,y)) + G(t,y) = 0, \quad t \in \mathbf{R},\ y \in \Omega,& \\
x(t,y) = 0, \quad y \in \partial\Omega,\ t \in \mathbb{R},& \\
x(t+T,y) = x(t,y), \quad t \in \mathbb{R},\ y \in \Omega.&
\end{aligned} \tag{4.33}$$

For that problem we assume the following hypotheses:

$G_{n-1}1$ *F^1, $F^2, \ldots, F^{n-1}$ are measurable with respect to (t,y) in $(0,T)\times\Omega$ for all x in $\mathbb{R}$ and continuously differentiable and convex with respect to the third variable in $\mathbb{R}$ for a.e. $(t,y)\in(0,T)\times\Omega$. $(t,y)\to l_{n-1}(t,y,0)+G(t,y)$ is integrable on $(0,T)\times\Omega$, $G(\cdot,\cdot)\in\mathcal{H}^{\frac{9}{2}r-3}$.*

$G_{n-1}2$ *There exist constants D_{n-1}, E_{n-1}, $\mathcal{F}$ and $\hat{x}\in\mathcal{H}^{\frac{9}{2}r-3}\cap H_{per}^{\frac{5}{2}r-1}(\mathbb{R}\times\Omega)\cap H^0_{per}(\mathbb{R};^0 H^2(\Omega))$, $\|\hat{x}\|_{U^{\frac{5}{2}r-1}}\le B(E_{n-1}+\|G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}})$ such that $l_{n-1}(\hat{x})\in\mathcal{H}^{\frac{9}{2}r-3}$ and*

$$\|l_{n-1}(\hat{x})\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le E_{n-1},\ \|l_{n-1}(\hat{x})\|_{H^0}\ge D_{n-1},\ \|G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le\mathcal{F}.$$

$G_{n-1}3$ *F^1, $F^2, \ldots, F^{n-1}$ satisfy* **G3.**

Define the following set

$$X_{n-1}=\Big\{x\in U^{\frac{5}{2}r-1}:\ \|x\|_{U^{\frac{5}{2}r-1}}\le B(E_{n-1}+\|G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}),\\ \|l_{n-1}(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}}\le E_{n-1}, \|l_{n-1}(x)\|_{H^0}\ge D_{n-1}\Big\}.$$

By **$G_{n-1}2$** the set X_{n-1} is nonempty.

$G_{n-1}2'$ *$l_{n-1}(x)\in\mathcal{H}^{\frac{5}{2}r-1}$ and $\|l_{n-1}(x)+G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{5}{2}r-1}}\ge D^0_{n-1}$ for $x\in\hat{X}_{n-1}$, where*

$$\hat{X}_{n-1}=\Big\{x\in U^{\frac{5}{2}r-1}:\|x\|_{U^{\frac{5}{2}r-1}}\le B(E_{n-1}+\mathcal{F})\Big\}$$

and some D^0_{n-1}.

Analogously as Lemma 4.3 one can prove the following lemma.

Lemma 4.18. *Let $x\in X_{n-1}$ and v be a solution of the periodic-Dirichlet problem for*

$$v_{tt}(t,y)-\Delta v(t,y)=-\left(l_{n-1}(t,y,x(t,y))+G(t,y)\right)\quad \text{a.e. on } (0,T)\times\Omega. \tag{4.34}$$

Then

$$|\|G(\cdot,\cdot)\|_{\mathcal{H}^0}-D_{n-1}|\,C\le\|v\|_{U^{\frac{5}{2}r-1}}\le B(E_{n-1}+\|G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}}).$$

Define in X_{n-1} the map $X_{n-1}\ni x\to H(x)=v$ where v is a solution of the periodic-Dirichlet problem (4.34). Similarly as in the former cases we notice that $H(X_{n-1})\subset\mathcal{H}^{\frac{5}{2}r-1}$.

Put

$$X^{n-1}=\Big\{\hat{x}\in U^{\frac{5}{2}r-1}:\ \hat{x}_{tt}(t,y)+l_{n-1}(t,y,v(t,y))=\Delta v(t,y)-G(t,y),\ v\in\hat{X}_{n-1}\Big\}.$$

Remark 4.19. Let us note that since $v\in\hat{X}_{n-1}$ thus $l_{n-1}(v)\in\mathcal{H}^{\frac{5}{2}r-1}$. Therefore by Corollary 3.2 X^{n-1} is nonempty and bounded in $U^{\frac{5}{2}r-1}$ by $A_q(\|v\|_{U^{\frac{5}{2}r-1}}-\|l_{n-1}(v)+G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{5}{2}r-1}})$ i.e. for all $v\in\hat{X}_{n-1}$

$$\begin{aligned}\|\hat{x}\|_{U^{\frac{5}{2}r-1}}&\le A_q(\|v\|_{U^{\frac{5}{2}r-1}}-\|l_{n-1}(v)+G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{5}{2}r-1}})\\&\le A_q(B(E_{n-1}+\|G(\cdot,\cdot)\|_{\mathcal{H}^{\frac{9}{2}r-3}})-D^0_{n-1}).\end{aligned}$$

Induction hypothesis:

IH *Under hypotheses* **$G_{n-1}1$, $G_{n-1}2$, $G_{n-1}2'$, $G_{n-1}3$** *problem (4.33) has a solution in $\hat{X}_{n-1}$.*

Consider now our problem (see (1.1)) with l defined by (2.6). Put $\bar{F}^n_x = F^n_x + G$, $\bar{F}^n = F^n + xG$. We assume the following hypotheses:

$\mathbf{G_n1}$ *Assume* **$\mathbf{G_{n-1}1}$–$\mathbf{G_{n-1}3}$.** *Let F^n is measurable with respect to (t,y) in $(0,T)\times\Omega$ for all x in $\mathbb{R}$ and continuously differentiable and convex with respect to the third variable in $\mathbb{R}$ for a.e. $(t,y) \in (0,T)\times\Omega$.*

$\mathbf{G_n2}$ *For D_{n-1}, E_{n-1}, $\mathcal{F}$ there exists $\hat{x} \in \mathcal{H}^{\frac{9}{2}r-3} \cap H^{\frac{5}{2}r-1}_{per}(\mathbb{R}\times\Omega) \cap H^0_{per}(\mathbb{R};^0 H^2(\Omega))$, $\|\hat{x}\|_{U^{\frac{5}{2}r-1}} \leq B(E_{n-1}+\mathcal{F})$ such that $l_{n-1}(\hat{x})$, $F^n_x(\hat{x}) \in \mathcal{H}^{\frac{9}{2}r-3}$ and $\|l_{n-1}(\hat{x})\|_{\mathcal{H}^{\frac{9}{2}r-3}} \leq E_{n-1}$, $\|l_{n-1}(\hat{x})\|_{H^0} \geq D_{n-1}$, $\|\bar{F}^n_x(\hat{x})\|_{\mathcal{H}^{\frac{9}{2}r-3}} \leq \mathcal{F}$.*

$\mathbf{G_n3}$ *F^n satisfies* **G3.**

Define the set

$$X^n_{FG} = \Big\{ x \in U^{\frac{5}{2}r-1} : \|x\|_{U^{\frac{5}{2}r-1}} \leq B(E_{n-1}+\mathcal{F}),\ \|l_{n-1}(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}} \leq E_{n-1}, \\ \|l_{n-1}(x)\|_{H^0} \geq D_{n-1}, \|\bar{F}^n_x(x)\|_{\mathcal{H}^{\frac{9}{2}r-3}} \leq \mathcal{F} \Big\}. \quad (4.35)$$

By **$\mathbf{G_n2}$** the set X^n_{FG} is nonempty.

$\mathbf{G_n2'}$ *$l_{n-1}(x)$, $\bar{F}^n_x(x) \in \mathcal{H}^{\frac{5}{2}r-1}$ and $\|\bar{F}^n_x(v) + l_{n-1}(x)\|_{\mathcal{H}^{\frac{5}{2}r-1}} \geq D^0_n$ for $v, x \in \hat{X}^n_{FG}$, where*

$$\hat{X}^n_{FG} = \Big\{ x \in U^{\frac{5}{2}r-1} : \|x\|_{U^{\frac{5}{2}r-1}} \leq B(E_{n-1}+\mathcal{F}) \Big\}$$

and some D^0_n.

Similarly as in Lemma 4.12, using now hypothesis **IH**, one can prove the following lemma.

Lemma 4.20. *Define in X^n_{FG} the map $X^n_{FG} \ni x \to H(x) = v$ where v is a solution of the periodic-Dirichlet problem for*

$$v_{tt}(t,y) - \Delta v(t,y) + l_{n-1}(t,y,v(t,y))) = -\bar{F}^n_x(t,y,x(t,y)) \quad \textit{a.e. on } (0,T)\times\Omega.$$

Then $H(X^n_{FG})$ is bounded in $U^{\frac{5}{2}r-1}$ by $B(E_{n-1}+\mathcal{F})$.

Let us denote by $l^+_{n-1}(t,y,x)$ the sum of those F^i_x for which $j_i = -1$, we assume (after renumbering) that they are the first m of $j_1F^1_x + j_2F^2_x + \cdots + j_{n-1}F^{n-1}_x$ i.e. $l^+_{n-1} = \sum_{i=1}^m F^i_x$ and by $l^-_{n-1} = \sum_{i=m+1}^{n-1} F^i_x$, thus $l_{n-1}(t,y,x) = -l^+_{n-1}(t,y,x) + l^-_{n-1}(t,y,x)$, and respectively $L^+_{n-1} = \sum_{i=1}^m F^i$, $L^-_{n-1} = \sum_{i=m+1}^{n-1} F^i$. Define

$$X^{nFG} = \Big\{ \hat{x} \in U^{\frac{5}{2}r-1} :\ \hat{x}_{tt}(t,y) - l^+_{n-1}(t,y,\hat{x}(t,y)) \\ = \Delta v(t,y) - l^-_{n-1}(t,y,v(t,y)) - \bar{F}^n_x(t,y,v(t,y)) \text{ where } v \in \hat{X}^n_{FG} \Big\} \quad (4.36)$$

Again similarly as Lemma 4.13 one can prove the following lemma.

Lemma 4.21. *For each $v \in \hat{X}^n_{FG}$ there exists a solution to*

$$\hat{x}_{tt}(t,y) - l^+_{n-1}(t,y,\hat{x}(t,y)) = \Delta v(t,y) - l^-_{n-1}(t,y,v(t,y)) - \bar{F}^n_x(t,y,v(t,y)),$$

i.e. the set X^{nFG} is nonempty. Moreover, $X^{nFG} \subset \hat{X}^n_{FG}$.

Example 4.22. Let us assume the same n and α as in Examples 4.7 and 4.17. Consider the nonlinearity $f = 7x^6 - 5x^4 \cos x + x^5 \sin x + (1/4)x^3$ and the following problem

$$\begin{aligned} x_{tt}(t,y) - \Delta x(t,y) + f(t,y,x(t,y)) = 0, \quad t \in \mathbb{R}, \quad y \in (0,\pi)^n, \\ x(t,y) = 0, \quad y \in \partial(0,\pi)^n, \quad t \in \mathbb{R}, \\ x(t+T,y) = x(t,y), \quad t \in \mathbb{R}, \quad y \in (0,\pi)^n. \end{aligned} \tag{4.37}$$

For this nonlinearity f we form the function $l = F_x^1 + F_x^2 - F_x^3$, where $F_x^1 = 10x^9 + 7x^6 + 23x$, $F_x^2 = 1/4x^3$, $F_x^3 = 10x^9 + 5x^4 \cos x - x^5 \sin x + 23x$. Then corresponding F^1, F^2, F^3 satisfy $\mathbf{G_n}$ **1–**$\mathbf{G_n}$**3** for some suitable chosen *constants* D_n^0, E_{n-1}, $\mathcal{F}$. Therefore equation (1.1) with that l has a solution by the above theorem and in a consequence equation (4.37) has a solution.

Remark 4.23. The last example shows that enough large class of nonlinear functions l can be treated by the method presented in the paper. Note that Example 4.22 can not be treated by Theorem 4.16.

Define functional corresponding to problem (see (1.1)) with l defined by (2.6):

$$\begin{aligned} J^{nFG}(x) = &\int_0^T \int_\Omega \left(-\frac{1}{2} |\nabla x(t,y)|^2 + \frac{1}{2} |x_t(t,y)|^2 \right) dydt \\ &+ \int_0^T \int_\Omega \left(L_{n-1}^+(t,y,x(t,y)) - L_{n-1}^-(t,y,x(t,y)) - \bar{F}^n(t,y,x(t,y)) \right) dydt. \end{aligned} \tag{4.38}$$

Next define the set X^{nFGd} : an element $(p,q,z_1,\dots,z_{n-1}) \in H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega) \times \dots \times H^1((0,T)\times\Omega)$ belongs to X^{nFGd} provided that there exists $x \in X^{nFG}$, $\hat{x} \in \hat{X}_{FG}^n$ such that for a.e. $(t,y) \in (0,T) \times \Omega$

$$p_t(t,y) - \operatorname{div} \hat{q}(t,y) - l_{n-1}^+(t,y,x(t,y)) = -l_{n-1}^-(t,y,\hat{x}(t,y)) - \bar{F}_x^n(t,y,\hat{x}(t,y))$$

and

$$p(t,y) = x_t(t,y) \text{ with } \hat{q}(t,y) = \nabla \hat{x}(t,y),$$

$$l_{n-1}^+(t,y,x(t,y)) = \sum_{i=1}^m z_i(t,y),\ l_{n-1}^-(t,y,\hat{x}(t,y)) = \sum_{i=m+1}^{n-1} \hat{z}_i(t,y).$$

The dual functional to (4.38) is then taken as

$$\begin{aligned} &J_D^{nFG}(p,q,z_1,\dots,z_{n-1}) \\ &= \int_0^T \int_\Omega^{1*} \bar{F}^{n*} \left(t,y, -\left(p_t(t,y) - \operatorname{div} q(t,y) - \sum_{i=1}^m z_i(t,y) + \sum_{i=m+1}^{n-1} z_i(t,y) \right) \right) dydt \\ &- \int_0^T \int_\Omega \sum_{i=1}^m L_i^{+*}(t,y,z_i(t,y)) dydt + \int_0^T \int_\Omega \sum_{i=m+1}^{n-1} L_i^{-*}(t,y,z_i(t,y)) dydt \\ &+ \frac{1}{2} \int_0^T \int_\Omega |q(t,y)|^2 dydt - \frac{1}{2} \int_0^T \int_\Omega |p(t,y)|^2 dydt, \end{aligned} \tag{4.39}$$

where L_i^{+*} is the Fenchel conjugate F^{i*} $i = 1,\dots,m$ and L_i^{-*} is the Fenchel conjugate F^{i*} $i = m+1,\dots,n-1$ and $J_D^{nFG} : H^1((0,T)\times\Omega) \times H^1((0,T)\times\Omega) \times \dots \times H^1((0,T)\times\Omega) \to \mathbb{R}$.

Analogously as in the case of functional J^{FG} one can prove the following.

Lemma 4.24. *The functional* J^{nFG} *attains its infimum on* $\hat{X}^n_{FG}$ *i.e.* $\inf_{x\in\hat{X}^n_{FG}} J^{nFG}(x) = J^{nFG}(\overline{x})$, *where* $\overline{x} \in \hat{X}^n_{FG}$.

Now we are in position to prove Theorem 2.6 and in consequence the main theorem of the paper.

Proof of the main theorem. Let $\bar{x} \in \hat{X}^n_{FG}$ be such that $J^{nFG}(\bar{x}) = \inf_{x\in\hat{X}^n_{FG}} J^{nFG}(x)$. Thus $\bar{x} \in \hat{X}^n_{FG}$ implies: there exists an $\hat{x} \in X^{nFG} \subset \hat{X}^n_{FG}$ such that

$$\hat{p}(t,y) = \hat{x}_t(t,y) \tag{4.40}$$

and

$$\hat{p}_t(t,y) - l^+_{n-1}(t,y,\hat{x}(t,y)) = \operatorname{div}\bar{q}(t,y) - l^-_{n-1}(t,y,\bar{x}(t,y)) - \bar{F}^n_x(t,y,\bar{x}(t,y)),$$

for a.a. $(t,y) \in (0,T)\times\Omega$ where $\bar{q}$ is given by

$$\bar{q}(t,y) = \nabla\bar{x}(t,y)$$

and

$$\sum_{i=1}^{m}\hat{z}_i(t,y) = l^+_{n-1}(t,y,\hat{x}(t,y)), \qquad \sum_{i=m+1}^{n-1}\bar{z}_i(t,y) = l^-_{n-1}(t,y,\bar{x}(t,y)). \tag{4.41}$$

By the definitions of J^{nFG}, J^{nFG}_D, relations (4.40), (4.41) and the Fenchel–Young inequality, it follows that

$$\begin{aligned}
J^{nFG}(\bar{x}) = & \int_0^T\int_\Omega\left(-\frac{1}{2}|\nabla\bar{x}(t,y)|^2 + \frac{1}{2}|\bar{x}_t(t,y)|^2 + L^+_{n-1}(t,y,\bar{x}(t,y)) - L^-_{n-1}(t,y,\bar{x}(t,y))\right)dydt\\
& - \int_0^T\int_\Omega \bar{F}^n(t,y,\bar{x}(t,y))\,dydt\\
\geq & \int_0^T\int_\Omega \bar{x}_t(t,y)\,\hat{p}(t,y)\,dydt - \frac{1}{2}\int_0^T\int_\Omega|\hat{p}(t,y)|^2\,dydt\\
& - \int_0^T\int_\Omega\langle\nabla\bar{x}(t,y),\bar{q}(t,y)\rangle\,dydt + \frac{1}{2}\int_0^T\int_\Omega|\bar{q}(t,y)|^2\,dydt\\
& - \int_0^T\int_\Omega\sum_{i=1}^{m}\bar{x}(t,y)\,\hat{z}_i(t,y)\,dydt + \int_0^T\int_\Omega L^+_{n-1}(t,y,\bar{x}(t,y))dydt\\
& + \int_0^T\int_\Omega\sum_{i=m+1}^{n-1}\bar{x}(t,y)\,\bar{z}_i(t,y)\,dydt - \int_0^T\int_\Omega L^-_{n-1}(t,y,\bar{x}(t,y))dydt\\
& - \int_0^T\int_\Omega\bar{F}^n(t,y,\bar{x}(t,y))dydt\\
\geq & \int_0^T\int_\Omega\bar{F}^{n*}\left(t,y,-\left(\hat{p}_t(t,y) - div\,\bar{q}(t,y) - \sum_{i=1}^{m}\hat{z}_i(t,y) + \sum_{i=m+1}^{n-1}\bar{z}_i(t,y)\right)\right)dydt\\
& + \frac{1}{2}\int_0^T\int_\Omega|\bar{q}(t,y)|^2\,dydt - \frac{1}{2}\int_0^T\int_\Omega|\hat{p}(t,y)|^2\,dydt\\
& - \int_0^T\int_\Omega\left(\sum_{i=1}^{m}L^{+*}_i(t,y,\hat{z}_i(t,y))\right)dydt + \int_0^T\int_\Omega\left(\sum_{i=m+1}^{n-1}L^{-*}_i(t,y,\bar{z}_i(t,y))\right)dydt,
\end{aligned}$$

and so

$$J^{nFG}(\hat{x}) \geq J^{nFG}(\bar{x}) \geq J^{nFG}_D(\hat{p},\bar{q},\hat{z}_1,\ldots,\hat{z}_m,\bar{z}_{m+1},\ldots,\bar{z}_{n-1}).$$

Next observe that

$$\begin{aligned}
&\inf_{x\in\hat{X}^n_{FG}} J^{nFG}(x) = J^{nFG}(\bar{x}) \le J^{nFG}(\hat{x}) \\
&\le \int_0^T\int_\Omega \left(-\hat{x}_t(t,y)\,\hat{p}(t,y) + \frac{1}{2}|\hat{x}_t(t,y)|^2\right) dydt \\
&\quad - \int_0^T\int_\Omega \left(\frac{1}{2}|\nabla\hat{x}(t,y)|^2 - \langle\nabla\hat{x}(t,y),\bar{q}(t,y)\rangle\right) dydt \\
&\quad + \int_0^T\int_\Omega \sum_{i=m+1}^{n-1}\left(\hat{x}(t,y)\,\bar{z}_i(t,y) - L^-_{n-1}(t,y,\hat{x}(t,y))\right) dydt \\
&\quad - \int_0^T\int_\Omega \sum_{i=1}^{m}\left(\hat{x}(t,y)\,\hat{z}_i(t,y) - L^+_{n-1}(t,y,\hat{x}(t,y))\right) dydt - \int_0^T\int_\Omega \{\bar{F}^n(t,y,\hat{x}(t,y)) \\
&\quad -\hat{x}(t,y)\left(-\hat{p}_t(t,y) + div\,\bar{q}(t,y) + \sum_{i=1}^{m}\hat{z}_i(t,y) - \sum_{i=m+1}^{n-1}\bar{z}_i(t,y)\right)\Bigg\}\, dydt \\
&= \frac{1}{2}\left(-\int_0^T\int_\Omega |\hat{p}(t,y)|^2\,dydt + \int_0^T\int_\Omega |\bar{q}(t,y)|^2\,dydt\right) \\
&\quad - \int_0^T\int_\Omega\left(\sum_{i=1}^{m} L_i^{+*}(t,y,\hat{z}_i(t,y))\right)dydt + \int_0^T\int_\Omega\left(\sum_{i=m+1}^{n-1} L_i^{-*}(t,y,\bar{z}_i(t,y))\right)dydt \\
&\quad - \int_0^T\int_\Omega \bar{F}^{n*}\left(t,y,-\left(\hat{p}_t(t,y) - div\,\bar{q}(t,y) - \sum_{i=1}^{m}\hat{z}_i(t,y) + \sum_{i=m+1}^{n-1}\bar{z}_i(t,y)\right)\right)dydt \\
&= J_D^{nFG}(\hat{p},\bar{q},\hat{z}_1,\ldots,\hat{z}_m,\bar{z}_{m+1},\ldots,\bar{z}_{n-1})
\end{aligned}$$

and so

$$J^{nFG}(\hat{x}) \le J_D^{nFG}(\hat{p},\bar{q},\hat{z}_1,\ldots,\hat{z}_m,\bar{z}_{m+1},\ldots,\bar{z}_{n-1})\,.$$

Thus we have equality $J^{nFG}(\hat{x}) = J_D^{nFG}(\hat{p},\bar{q},\hat{z}_1,\ldots,\hat{z}_m,\bar{z}_{m+1},\ldots,\bar{z}_{n-1})$ which implies similarly as in former cases

$$\begin{aligned}
&\frac{1}{2}\int_0^T\int_\Omega |\hat{p}(t,y)|^2\,dydt + \frac{1}{2}\int_0^T\int_\Omega |\bar{x}_t(t,y)|^2\,dydt - \int_0^T\int_\Omega \langle\bar{x}_t(t,y),\hat{p}(t,y)\rangle\,dydt \\
&\quad + \int_0^T\int_\Omega\left(\sum_{i=1}^{m} L_i^{+*}(t,y,\hat{z}_i(t,y))\right)dydt + \int_0^T\int_\Omega L^+_{n-1}(t,y,\bar{x}(t,y))dydt \\
&\quad - \int_0^T\int_\Omega \sum_{i=1}^{m}\bar{x}(t,y)\,\hat{z}_i(t,y)\,dydt \\
&\quad = 0
\end{aligned}$$

and as a consequence the equality

$$\hat{p}(t,y) = \hat{x}_t(t,y)\,, \qquad \hat{z}_i(t,y) = F_x^i(t,y,\bar{x}(t,y))\,, \qquad i = 1,\ldots,m. \tag{4.42}$$

Hence, by (4.41), (4.42) $\hat{z}_i = \bar{z}_i$, $i = 1,\ldots,m$ and $\bar{x}_t = \hat{x}_t$. We have also equality $J^{nFG}(\bar{x}) = J_D^{nFG}(\hat{p},\bar{q},\hat{z}_1,\ldots,\hat{z}_m,\bar{z}_{m+1},\ldots,\bar{z}_{n-1})$ which similarly implies

$$\hat{p}_t(t,y) - \mathrm{div}\,\bar{q}(t,y) - l^+_{n-1}(t,y,\bar{x}(t,y)) + l^-_{n-1}(t,y,\bar{x}(t,y)) = -\bar{F}_x^n(t,y,\bar{x}(t,y))$$

and $\nabla\hat{x} = \nabla\bar{x}$, $i = 1,\ldots,n-1$, thus we have the assertions of theorem satisfied. □

5 Conclusions

The nonlinear terms in problems of type (1.1), at the beginning of investigations, were monotone functions or sublinear at infinity (see a survey [26]). Next step was the paper of [3] where nonlinear term was a difference of two monotone functions. In this paper we extend nonlinearity to be a finite linear combination of monotone functions. In many papers concerning problem (1.1) we can observe that monotonicity of nonlinearity is essential to prove existence of solution to (1.1). The open problem appears: whether the nonlinearity l can be of the form $l = f + g$ where f is monotone function and g only continuous. It was already pointed in [10] that arithmetical properties of the ratio $\alpha = T/\pi$ play an important role in a solvability of the periodic-Dirichlet problem (1.1) (see also interesting discussion on that problem in [35]). There is only a few papers which treat that problem in case when T is irrational number such that $\alpha = T/\pi$ has not necessary bounded partial quotients in its continued fraction with nonlinear l. The case with spatial dimension $n \geq 2$ and $r > 2$ has not been almost investigated. We have proved, for $n \geq 2$ and $r \geq 2$, that if α satisfies assumption **T** then with l being a finite linear combination of monotone functions, problem (1.1) has a solution in $H_{per}^{\frac{5}{2}r-1}(\mathbb{R} \times \Omega)$ i.e. a strong solution. Moreover we proved that this solution satisfies variational and duality principle.

References

[1] D. Bambusi, S. Paleari, Families of periodic solutions of resonant PDEs, *J. Nonlinear Sci.* **11**(2001), 69–87. https://doi.org/10.1007/s003320010010; MR1819863; Zbl 0994.37040

[2] D. Bambusi, S. Paleari, Families of periodic orbits for some PDE's in higher dimensions, *Commun. Pure Appl. Anal.* **1**(2002), 269–279. https://doi.org/10.3934/cpaa.2002.1.269; MR1938615; Zbl 1034.35081

[3] T. Bartsch, Y. H. Ding, C. Lee, Periodic solutions of a wave equation with concave and convex nonlinearities, *J. Differential Equations* **153**(1999), 121–141. https://doi.org/10.1006/jdeq.1998.3551; MR1682275; Zbl 0926.35013

[4] A. K. Ben-Naoum, J. Mawhin, The periodic-Dirichlet problems for some semilinear wave equations, *J. Differential Equations* **96**(1992), 340–354. https://doi.org/10.1016/0022-0396(92)90158-J; MR1156665; Zbl 0779.35068

[5] J. Berkovits, J. Mawhin, Diophantine approximation, Bessel functions and radially symmetric periodic solutions of semilinear wave equations in a ball, *Trans. Amer. Math. Soc.* **353**(2001) 5041–5055. https://doi.org/10.1090/S0002-9947-01-02875-6; MR1852093; Zbl 0986.35070

[6] F. Bernstein, Über eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem (in German), *Math. Ann.* **71**(1911), 417–439. https://doi.org/10.1007/BF01456856; MR1511668

[7] M. Berti, L. Biasco, Forced vibrations of wave equations with non-monotone nonlinearities, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**(2006), 439–474. https://doi.org/10.1016/j.anihpc.2005.05.004; MR2245752; Zbl 1103.35076

[8] M. Berti, P. Bolle, Periodic solutions of nonlinear wave equations with general nonlinearities, *Commun. Math., Phys.* **243**(2003), 315–328. https://doi.org/10.1007/s00220-003-0972-8; MR2021909; Zbl 1072.35015

[9] M. Berti, P. Bolle, Multiplicity of periodic solutions of nonlinear wave equations, *Nonlinear Anal.* **56**(2004), 1011–1046. https://doi.org/10.1016/j.na.2003.11.001; MR2038735; Zbl 1064.35119

[10] É. Borel, Sur les équations aux dérivées partielles à coefficients constants et les fonctions non analytiques (in French), *C. R. Acad. Sci. Paris Ser. A–B.* **121**(1895), 933–935. Zbl 26.0384.03

[11] J. Bourgain, Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations, *Ann. of Math.* **148**(1998), 363–439. https://doi.org/10.2307/121001; MR1668547; Zbl 0928.35161

[12] J. Bourgain, Construction of periodic solutions of nonlinear wave equations in higher dimension, *Geom. Funct. Anal.* **5**(1995), 629–639. https://doi.org/10.1007/BF01902055; MR1345016; Zbl 0834.35083

[13] H. Brézis, L. Nirenberg, Characterizations of the range of some nonlinear operarors and applications to boundary value problems, *Ann. Scuola Norm. Sup. Pisa* **5**(1978), 225–326. MR0513090; Zbl 0386.47035

[14] H. Brézis, L. Nirenberg, Forced vibrations for nonlinear wave equation, *Comm. Pure Appl. Math.* **31**(1978), 1–30. https://doi.org/10.1002/cpa.3160310102; MR0470377; Zbl 0378.35040

[15] H. Brézis, Periodic solutions of nonlinear vibrating strings and duality principles, *Bull. Amer. Math. Soc.* **8**(1983), 409–426. https://doi.org/10.1090/S0273-0979-1983-15105-4; MR693957; Zbl 0515.35060

[16] J.-M. Coron, Periodic solutions of a nonlinear wave equation without assumption of monotonicity, *Math. Ann.* **262**(1983), 273–285. https://doi.org/10.1007/BF01455317; MR0690201; Zbl 0489.35061

[17] W. Craig, *Problèmes de petits diviseurs dans les équations aux dérivées partielles* (in French), Société Mathématique de France, Paris, 2000. MR1804420; Zbl 0977.35014

[18] W. Craig, E. Wayne, Newton's method and periodic solutions of nonlinearwave equation. *Commun. Pure and Appl. Math.* **46**(1993), 1409–1498. https://doi.org/10.1002/cpa.3160461102; MR1239318; Zbl 0794.35104

[19] A. Fonda, J. Mawhin, Iterative and variational methods for the solvability of some semilinear equations in Hilbert spaces, *J. Differential Equations* **98**(1992), 355–375. https://doi.org/10.1016/0022-0396(92)90097-7; MR1170475; Zbl 0761.34047

[20] M. Galewski, A. Nowakowski, The existence of solutions for a nonlinear wave equation, *Top. Meth. Non. Anal.* **28**(2006), 385–399. MR2289692; Zbl 1134.35078

[21] W. S. Hall, On the existence of periodic solutions for the equations $D_{tt}u + (-1)^p D_x^{2p} u = \epsilon f(\cdot,\cdot,u)$, *J. Differential Equations* **7**(1970), 509–526. https://doi.org/10.1016/0022-0396(70)90098-7; MR0265738; Zbl 0198.14002

[22] S. B. Kuksin, Perturbation of conditionally periodic solutions of in nite-dimensional Hamiltonian systems, *Izv. Akad. Nauk SSSR, Ser. Mat.* **52**(1988), 41–63. https://doi.org/10.1070/IM1989v032n01ABEH000733; MR0936522; Zbl 0662.58036

[23] S. Kuksin, *Analysis of Hamiltonian PDEs*, Oxford Lecture Series in Mathematics and its Applications, Vol. 19, 2000. MR1857574; Zbl 0960.35001

[24] B. V. Lidskij, E. I. Shulman, Periodic solutions of the equation $u_{tt} - u_{xx} + u^3 = 0$, *Funct. Anal. Appl.* **22**(1988), 332–333. https://doi.org/10.1007/BF01077432; MR0977006; Zbl 0837.35012

[25] H. Lovicarová, Periodic solutions of a weakly nonlinear wave equation in one dimension, *Czechoslovak Math. J.* **19**(1969) 324–342. MR0247249; Zbl 0181.10901

[26] J. Mawhin, Periodic solutions of some semilinear wave equations and systems: a survey, *Chaos Solitons Fractals* **5**(1995), 1651–1669. https://doi.org/10.1016/0960-0779(94)00169-Q; MR96m:34074; Zbl 1079.35532

[27] P. J. McKenna, On solutions of a nonlinear wave equation when the ration of the period to the length of the interval is irrational, *Proc. Amer. Math. Soc.* **93**(1985), 59–64. https://doi.org/10.1090/S0002-9939-1985-0766527-X; MR766527; Zbl 0576.35081

[28] M. Novak, Remark on periodic solutions of linear wave equation in one dimension. *Comment. Math. Univ. Carolinae* **15**(1974), 513–519. MR0355354; Zbl 0301.35053

[29] A. Nowakowski, A new variational principle and duality for periodic solutions of Hamilton's equations, *J. Differential Equations* **97**(1992), 174–188. https://doi.org/10.1016/0022-0396(92)90089-6; MR1161317; Zbl 0759.34039

[30] A. Nowakowski, I. Nowakowska, Dirichlet problem for semilinear hyperbolic equation, *Nonlinear Anal.* **63**(2005) e43–e52. https://doi.org/10.1016/j.na.2004.12.024; Zbl 1153.35362

[31] P. Rabinowitz, Some minimax theorems and applications to nonlinear partial differential equations, in: *Nonlinear Analysis, A Tribute to E. H. Rothe, edited by Cesari, Kannan, Weinberg*, (1978), 161–178. MR0501092; Zbl 0466.58015

[32] P. Rabinowitz, Time periodic solutions of nonlinear wave equations, *Manuscripta Math.* **5**(1971), 165–194. https://doi.org/10.1007/BF01325026; MR0326179; Zbl 0219.35062

[33] W. Schmidt, *Diophantine approximation*, Springer-Verlag, Berlin, 1980. MR0568710; Zbl 0421.10019

[34] M. Struwe, *Variational methods*, Springer, Berlin, 1990. MR1078018; Zbl 0746.49010

[35] O. Vejvoda, L. Hermann, V. Lovicar, M. Sova, I. Straškraba, M. Štědrý, *Partial differential equations: time-periodic solutions*, Sijthoff & Noordhoff, Alphen aan den Rijn, 1981. MR0653987; Zbl 0501.35001

[36] C. E. Wayne, Periodic solutions of nonlinear partial differential equations, *Notices Amer. Math. Soc.* **44**(1997), 895–902. MR1467653; Zbl 0912.35016

[37] C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Commun. Math. Phys.* **127**(1990), 479–528. https://doi.org/10.1007/BF02104499; MR1040892; Zbl 0708.35087

[38] M. Willem, Density of the range of potential operators, *Proc. Amer. Math. Soc.* **83**(1981), 341–344. https://doi.org/10.1090/S0002-9939-1981-0624926-7; MR624926; Zbl 0478.49012