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Multiplicity of solutions for quasilinear elliptic problems involving Φ-Laplacian operator and critical growth

Xuewei Li and **Gao Jia** ✉

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China

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Abstract. In this paper, we study a class of quasilinear elliptic equations with Φ-Laplacian operator and critical growth. Using the symmetric mountain pass theorem and the concentration-compactness principle, we demonstrate that there exists $\lambda_i > 0$ such that our problem admits i pairs of nontrivial weak solutions provided $\lambda \in (0, \lambda_i)$.

Keywords: quasilinear elliptic equation, critical exponent, variational method.

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1 Introduction

In this paper, we discuss the existence of multiple solutions for the quasilinear elliptic problem

$$\begin{cases} -\Delta_\Phi u = \lambda |u|^{l^*-2}u + f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive parameter, $l^* = \frac{lN}{N-l}(1 < l < N)$ is the critical Sobolev exponent and $\Delta_\Phi u$ denotes the Φ-Laplacian operator, which is defined by $\Delta_\Phi u = \operatorname{div}(\phi(|\nabla u|)\nabla u)$. With respect to the function $\phi : (0, \infty) \to (0, \infty)$, we assume that it is C^1 and satisfies:

(ϕ_1) $\phi(t)t \to 0$ as $t \to 0$, $\phi(t)t \to \infty$ as $t \to \infty$;

(ϕ_2) $\phi(t)t$ is strictly increasing in $(0, \infty)$;

(ϕ_3) $0 < l - 1 := \inf_{t>0} \frac{(\phi(t)t)'}{\phi(t)} \leq \sup_{t>0} \frac{(\phi(t)t)'}{\phi(t)} =: m - 1 < N - 1.$

Throughout this paper we define

$$\Phi(t) = \int_0^t \phi(s)s ds, \qquad t \geq 0,$$

✉ Corresponding author. Email: gaojia89@163.com

which is extended as even function, $\Phi(t) = \Phi(-t)$, for all $t < 0$. In fact, under the assumptions (ϕ_1)–(ϕ_3), the equations like (1.1) may be allowed to possess complicated nonhomogeneous Φ-Laplacian operator. The examples are the following:

(i) p-Laplacian:$\phi(t) = pt^{p-2}$, for $1 < p < N$;

(ii) (p,q)-Laplacian:$\phi(t) = pt^{p-2} + qt^{q-2}$, for $1 < p < q < N$ and $q \in (p, p^*)$ with $p^* = \frac{pN}{N-p}$;

(iii) plasticity: $\phi(t) = pt^{p-2}(\log(1+t))^q + qt^{p-1}(1+t)^{-1}(\log(1+t))^{q-1}$, for $p \geq 1,\ q > 0$;

(iv) $p(x)$-Laplacian: $\phi(t) = p(x)t^{p(x)-2}$, for $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and $1 < p^- := \inf_{\mathbb{R}^N} p(x) \leq \sup_{\mathbb{R}^N} p(x) =: p^+ < N$.

In our discussion, we assume that the nonlinear term $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$ satisfies:

(f_1) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{l^*-1}} = 0$, uniformly $x \in \Omega$;

(f_2) there exist constants $\theta \in (m, l^*)$, $\sigma \in [0, l)$ and $C_0, C_1 > 0$, such that

$$F(x,t) - \frac{1}{\theta} f(x,t)t \leq C_0|t|^\sigma + C_1,$$

for $x \in \Omega$ and $t \in \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)ds$;

(f_3) there exist constants $\tau \in (m, l^*)$ and $C_2, C_3 > 0$ such that

$$F(x,t) \leq C_2|t|^\tau + C_3$$

for $x \in \Omega$ and $t \in \mathbb{R}$;

(f_4) there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$ such that

$$\liminf_{|t|\to\infty} \frac{F(x,t)}{|t|^m} = +\infty,$$

uniformly $x \in \Omega_0$;

(f_5) $f(x,0) = 0$ and $f(x,-t) = -f(x,t)$, for $x \in \Omega$ and $t > 0$.

Remark 1.1. It is easily seen that the following function satisfies hypotheses (f_1)–(f_4):

$$f(x,t) = |t|^{r-2}t, \quad \text{for } t > 0 \text{ and } r \in (m, l^*).$$

The equation (1.1), for $\Phi(t) = t^p$, is well known as the p-Laplacian equation involving critical growth $p^* = \frac{pN}{N-p}$. The boundary value problem

$$\begin{cases} -\Delta_p u = \mu|u|^{p^*-2}u + f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.2}$$

has been studied by B. Silva and Xavier [11]. The multiplicity of solutions for (1.2) is obtained by the variational method and the minimax critical point theorems. D. Silva improved the variational method and the concentration compactness principle to deal with the problem (cf. [12])

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda|u|^{q(x)-2}u + f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $0 < p(x) \leq q(x) \leq p^*(x) = \frac{p(x)N}{N-p(x)}$, $x \in \overline{\Omega}$. Further, one of the main motivations for the study of problem (1.1) is the following problem

$$-\operatorname{div}(\phi(|\nabla u|)\nabla u) = b(|u|)u + \lambda f(x,u), \qquad x \in \mathbb{R}^N, \tag{1.4}$$

where $N \geq 2$, $\lambda > 0$ and $b(|u|)u$ possesses critical growth. Fukagai, Ito and Narukawa [5] proved that problem (1.4) has a positive solution.

As is mentioned in [13], the problem (1.1) has many physical applications, for instance, in nonlinear elasticity, plasticity, generalized Newtonian fluids, etc. We refer the readers to the following related papers (cf. [2,4–6,9]) and references therein.

In this work we will propose a variant symmetric mountain pass theorem for solving the multiplicity of solutions for problem (1.1). This requires the functional associated with the problem (1.1) satisfies the $(PS)_c$ condition below a fixed level. Hence, it will allow us to use a more efficient concentration-compactness type principle than the problem (1.4), which just showed the weak limit u is positive in Fukagai, Ito and Narukawa [5].

The main difficulty in dealing with this class of problems is that the associated functional involves the critical growth term so that the embedding of $W_0^{1,\Phi}(\Omega)$ into $L^{l^*}(\Omega)$ is no longer compact. And another difficulty comes from the fact that Φ-Laplacian operator is nonhomogeneous, which requires some additional efforts to overcome the estimate. It is worthwhile mentioning that we exploit the compactness of the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow L_\Psi(\Omega)$, $\Phi \leq \Psi \ll \Phi_*$ and the existence of a Schauder basis for $W^{1,\Phi}(\Omega)$ to establish a lower bound for the minimax levels.

Our main result can be stated as follows.

Theorem 1.2. *Assume that (ϕ_1)–(ϕ_3) and (f_1)–(f_5) hold. Then for any given $i \in \mathbb{N}$, there exists $\lambda_i \in (0,\infty)$ such that for all $\lambda \in (0,\lambda_i)$, problem (1.1) possesses at least i pairs of nontrivial weak solutions.*

The organization of this paper is as follows. In Section 2, we set up the framework of Orlicz–Sobolev spaces and give some essential results of Φ-Laplacian. In Section 3, we present the functional associated with the problem (1.1) satisfies the Palais–Smale condition below a given level. Finally, in Section 4, we give some useful lemmas for our main result and the complete proof of the existence of multiple solutions for the problem (1.1).

2 Preliminaries

Due to the nature of the operator Δ_Φ we shall work in the framework of Orlicz–Sobolev spaces $W^{1,\Phi}(\Omega)$. For the sake of completeness, we recall some definitions and properties as follows.

The Orlicz space

$$L_\Phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} : \ u \text{ is measurable and } \int_\Omega \Phi(|u(x)|)dx < \infty \right\}$$

is a Banach space under the usual norm (Luxemburg norm)

$$\|u\|_\Phi = \inf_k \left\{ k > 0 \;\middle|\; \int_\Omega \Phi\left(\frac{|u(x)|}{k}\right) dx \leq 1 \right\}.$$

The Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$ is defined as the set of all weakly differentiable $u \in L_\Phi(\Omega)$ such that $D^\gamma u \in L_\Phi(\Omega)$ for all multi-indices $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ with $|\gamma| \leq 1$. The

Orlicz–Sobolev norm of $W^{1,\Phi}(\Omega)$ is defined as

$$\|u\|_{1,\Phi} = \|u\|_{\Phi} + \|\nabla u\|_{\Phi}.$$

We denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the Orlicz–Sobolev norm of $W^{1,\Phi}(\Omega)$.

If

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds < +\infty \quad \text{and} \quad \int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds = +\infty, \tag{2.1}$$

then the Sobolev conjugate N-function function Φ_* of Φ is given in [1] by

$$t \in (0,\infty) \mapsto \int_0^t \frac{\Phi^{-1}(s)}{s^{\frac{N+1}{N}}} ds.$$

Notice that Φ is N-function and (ϕ_3) guarantees (2.1) holds.

The dual $(L_\Phi(\Omega))^*$ is $L_{\widetilde{\Phi}}(\Omega)$ (cf. [6]), where $\widetilde{\Phi}$ is called the complement of Φ, given by

$$\widetilde{\Phi}(t) = \max_{s\geq 0}\{ts - \Phi(s)\}, \quad \text{for } t \geq 0. \tag{2.2}$$

By using of the assumptions (ϕ_1) and (ϕ_3), it turns out that Φ, Φ_* and $\widetilde{\Phi}$ are N-functions satisfying $\triangle_2$-condition (cf. [10]), namely there is a constant $C_4 > 0$ such that

$$\Phi(2t) \leq C_4\Phi(t), \quad \forall t > 0.$$

Meanwhile, the assumptions (ϕ_3) implies that

$(\phi_3)'$
$$1 < l := \inf_{t>0} \frac{\phi(t)t^2}{\Phi(t)} \leq \sup_{t>0} \frac{\phi(t)t^2}{\Phi(t)} =: m < N,$$

which ensures that $L_\Phi(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are separable and reflexive Banach spaces (cf. [10]).

Lemma 2.1. *Assume that (ϕ_1)–(ϕ_3) hold. Then for $t \geq 0$, we have*

$$\widetilde{\Phi}(\phi(t)t) = \phi(t)t^2 - \Phi(t) \leq \Phi(2t). \tag{2.3}$$

Proof. The convexity of $\Phi(t)$ implies that

$$\Phi(t) + \Phi'(t)(s-t) \leq \Phi(s),$$

for $s, t \geq 0$. By (ϕ_2) and $\Phi'(t) = \phi(t)t$, we have

$$\phi(t)ts - \Phi(s) \leq \phi(t)t^2 - \Phi(t),$$

for $s,\ t \geq 0$. Thus by (2.2), we obtain

$$\begin{aligned}
\widetilde{\Phi}(\phi(t)t) &= \max_{s\geq 0}\{\phi(t)ts - \Phi(s)\} \\
&\leq \phi(t)t^2 - \Phi(t) \\
&\leq \phi(t)t^2 \\
&\leq \int_t^{2t} \phi(z)z dz \\
&\leq \Phi(2t),
\end{aligned}$$

for $t \geq 0$. Hence, this shows (2.3). □

Remark 2.2. It is easy to see that $(\phi_3)'$ implies that

$(\phi_3)''$
$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m, \qquad t > 0$$

is verified.

It follows from the Poincaré inequality for Φ-Laplacian operator (cf. [7]) that there exists a constant $S_1 > 0$ such that
$$\|u\|_\Phi \leq S_1 \|\nabla u\|_\Phi,$$
for all $u \in W_0^{1,\Phi}(\Omega)$. As a consequence of this, the norm $\|\cdot\|_{1,\Phi}$ is equivalent to the norm
$$\|u\| := \|\nabla u\|_\Phi$$
on $W_0^{1,\Phi}(\Omega)$. In this paper, we will use $\|\cdot\|$ as the norm of $W_0^{1,\Phi}(\Omega)$.

The embedding results below (cf. [1,3]) are used in this paper. First, we have
$$W_0^{1,\Phi}(\Omega) \hookrightarrow\hookrightarrow L_\Psi(\Omega), \tag{2.4}$$
if $\Phi \leq \Psi \ll \Phi_*$, where $\Psi \ll \Phi_*$ means that the function Ψ essentially grows more slowly than Φ_*. Furthermore,
$$W_0^{1,\Phi}(\Omega) \hookrightarrow L_{\Phi_*}(\Omega). \tag{2.5}$$
Define a constant $S_2 > 0$, such that for any $u \in W_0^{1,\Phi}(\Omega)$,
$$\|u\|_{\Phi_*} \leq S_2 \|u\|. \tag{2.6}$$
Besides this, it is worth mentioning that if (ϕ_1)–(ϕ_2) and $(\phi_3)''$ are satisfied, we have
$$L_\Phi(\Omega) \hookrightarrow L^l(\Omega),$$
$$L_{\Phi_*}(\Omega) \hookrightarrow L^{l^*}(\Omega).$$
Define a constant $S_3 > 0$, such that for any $u \in W_0^{1,\Phi}(\Omega)$,
$$\|u\|_{L^{l^*}(\Omega)} \leq S_3 \|u\|_{\Phi_*}. \tag{2.7}$$
Since
$$W_0^{1,\Phi}(\Omega) \hookrightarrow L^{l^*}(\Omega), \tag{2.8}$$
we can define a constant $S_4 > 0$, such that for any $u \in W_0^{1,\Phi}(\Omega)$,
$$\|u\|_{L^{l^*}(\Omega)} \leq S_4 \|u\|. \tag{2.9}$$

Lemma 2.3 ([5]). *Assume that (ϕ_1)–(ϕ_3) hold. For $t \geq 0$, set*
$$\eta_1(t) = \min\{t^l, t^m\}, \;\; \eta_2(t) = \max\{t^l, t^m\}.$$
Then Φ satisfies
$$\eta_1(t)\Phi(\rho) \leq \Phi(\rho t) \leq \eta_2(t)\Phi(\rho), \quad \text{for any } \rho,\ t > 0, \tag{2.10}$$
$$\eta_1(\|u\|_\Phi) \leq \int_\Omega \Phi(u)dx \leq \eta_2(\|u\|_\Phi), \quad \text{for } u \in L_\Phi(\Omega). \tag{2.11}$$

Let $\widetilde{\Phi}_*$ be the complement of Φ_*, we have

Lemma 2.4 ([5]). *Assume that* (ϕ_1)–(ϕ_3) *hold. For* $t \geq 0$, *set*

$$\eta_3(t) = \min\{t^{\widetilde{l}^*}, t^{\widetilde{m}^*}\}, \qquad \eta_4(t) = \max\{t^{\widetilde{l}^*}, t^{\widetilde{m}^*}\},$$

where $\widetilde{l}^* = \frac{l^*}{l^*-1}$ *and* $\widetilde{m}^* = \frac{m^*}{m^*-1}$. *Then* $\widetilde{\Phi}_*$ *satisfies*

$$\widetilde{m}^* \leq \frac{\widetilde{\Phi}'_*(t)t}{\widetilde{\Phi}_*(t)} \leq \widetilde{l}^*, \quad \text{for } t > 0,$$

$$\eta_3(t)\widetilde{\Phi}_*(\rho) \leq \widetilde{\Phi}_*(\rho t) \leq \eta_4(t)\widetilde{\Phi}_*(\rho), \quad \text{for any } \rho,\ t \geq 0, \tag{2.12}$$

$$\eta_3(\|u\|_{\widetilde{\Phi}_*}) \leq \int_\Omega \widetilde{\Phi}_*(u)dx \leq \eta_4(\|u\|_{\widetilde{\Phi}_*}), \quad \text{for } u \in L_{\widetilde{\Phi}_*}(\Omega). \tag{2.13}$$

Next, we recall the variational framework for problem (1.1). The functional I_λ: $W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ associated with our problem is given by

$$I_\lambda(u) = \int_\Omega \left(\Phi(|\nabla u|) - \frac{\lambda}{l^*}|u|^{l^*} - F(x,u)\right) dx, \qquad u \in W_0^{1,\Phi}(\Omega).$$

It is easy to verify that I_λ is well-defined and of class C^1 on $W_0^{1,\Phi}(\Omega)$. Hence finding weak solutions for the problem (1.1) is equivalent to find the critical points for the functional I_λ and the Gateaux derivative for I_λ has the following form:

$$\langle I'_\lambda(u), \psi\rangle = \int_\Omega (\phi(|\nabla u|)\nabla u \nabla \psi - \lambda |u|^{l^*-2}u\psi - f(x,u)\psi)dx,$$

for any $u,\ \psi \in W_0^{1,\Phi}(\Omega)$.

Definition 2.5. For given E a real Banach space and $I \in C^1(E,\mathbb{R})$, we say that I satisfies the Palais–Smale condition on the level $c \in \mathbb{R}$, denoted by $(PS)_c$ condition, if every sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in E.

In this article we will apply the following version of the symmetric mountain pass theorem (cf. [11]).

Lemma 2.6. *Let* $E = X \oplus Y$, *where* E *is a real Banach space and* X *is finite dimensional. Suppose* $I \in C^1(E,\mathbb{R})$ *is an even functional, satisfying* $I(0) = 0$ *and*

(I_1) *there exists a constant* $\rho > 0$ *such that* $I|_{\partial B_\rho \cap Y} > 0$;

(I_2) *there exist a subspace* W *of* E *with* $\dim X < \dim W < \infty$ *and* $M > 0$ *such that* $\max_{u\in W} I(u) < M$;

(I_3) *considering* $M > 0$ *given by* (I_2), I *satisfies* $(PS)_c$ *condition, for* $0 < c < M$.

Then I *possesses at least* $(\dim W - \dim X)$ *pairs of nontrivial critical points.*

3 The Palais–Smale condition

In this section, we will verify that the functional I_λ satisfies the $(PS)_c$ condition below a given level when $\lambda > 0$ is sufficiently small. In order to do this, we need some preliminary results.

First, we will show the Palais–Smale sequence $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ is bounded.

Lemma 3.1. *Assume that (ϕ_1)–(ϕ_3) and (f_1)–(f_2) hold. Then the $(PS)_c$ sequence $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ of I_λ is bounded.*

Proof. According to (f_2), $(\phi_3)''$ and Hölder's inequality, it follows that

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle &= \int_\Omega \left(\Phi(|\nabla u_n|) - \frac{1}{\theta}\phi(|\nabla u_n|)|\nabla u_n|^2\right) dx \\ &\quad + \lambda\left(\frac{1}{\theta} - \frac{1}{l^*}\right)\int_\Omega |u_n|^{l^*} dx - \int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n\right) dx \\ &\geq \left(\frac{1}{m} - \frac{1}{\theta}\right)\int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2 dx + \lambda\left(\frac{1}{\theta} - \frac{1}{l^*}\right)\|u_n\|^{l^*}_{L^{l^*}(\Omega)} \\ &\quad - C_0\|u_n\|^{\sigma}_{L^\sigma(\Omega)} - C_1|\Omega| \\ &\geq \lambda\left(\frac{1}{\theta} - \frac{1}{l^*}\right)\|u_n\|^{l^*}_{L^{l^*}(\Omega)} - C_0|\Omega|^{1-\frac{\sigma}{l^*}}\|u_n\|^{\sigma}_{L^{l^*}(\Omega)} - C_1|\Omega|. \end{aligned} \tag{3.1}$$

Moreover, by Young's inequality, we have

$$\|u_n\|^{\sigma}_{L^{l^*}(\Omega)} \leq \delta\|u_n\|^{l^*}_{L^{l^*}(\Omega)} + C_\delta, \tag{3.2}$$

where $\delta = \frac{\lambda\left(\frac{1}{\theta} - \frac{1}{l^*}\right)}{2C_0|\Omega|^{1-\frac{\sigma}{l^*}}}$ and $C_\delta = \frac{l^*-\sigma}{l^*}\left(\frac{\sigma}{\delta l^*}\right)^{\frac{\sigma}{l^*-\sigma}}$.

On the other hand, since $\{u_n\}$ is a $(PS)_c$ sequence, we have

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\theta}\langle I'_\lambda(u_n), u_n\rangle &\leq I_\lambda(u_n) + \frac{1}{\theta}\|I'_\lambda(u_n)\|_{W_0^{1,\tilde{\Phi}}(\Omega)}\|u_n\| \\ &\leq C_5 + C_6\|u_n\|, \end{aligned} \tag{3.3}$$

with some constants $C_5, C_6 > 0$.

Therefore, from (3.1), (3.2) and (3.3)), there exist constants $C_7, C_8 > 0$ such that

$$\|u_n\|^{l^*}_{L^{l^*}(\Omega)} \leq C_7 + C_8\|u_n\|. \tag{3.4}$$

Now, by (f_1), for given $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$|f(x,t)| \leq C_\epsilon + \epsilon|t|^{l^*-1}, \quad \text{for } x \in \Omega \text{ and } t \in \mathbb{R} \tag{3.5}$$

and

$$|F(x,t)| \leq C_\epsilon + \frac{\epsilon}{l^*}|t|^{l^*}, \quad \text{for } x \in \Omega \text{ and } t \in \mathbb{R}. \tag{3.6}$$

Consequently, by (3.4) and (3.6), we have

$$\begin{aligned} I_\lambda(u_n) &= \int_\Omega \Phi(|\nabla u_n|)dx - \frac{\lambda}{l^*}\int_\Omega |u_n|^{l^*}dx - \int_\Omega F(x,u_n)dx \\ &\geq \eta_1(\|u_n\|) - \frac{\lambda+\epsilon}{l^*}\|u_n\|^{l^*}_{L^{l^*}(\Omega)} - C_\epsilon|\Omega| \\ &\geq \eta_1(\|u_n\|) - \frac{\lambda+\epsilon}{l^*}C_8\|u_n\| - \frac{\lambda+\epsilon}{l^*}C_7 - C_\epsilon|\Omega| \end{aligned}$$

and

$$\eta_1(\|u_n\|) \leq \frac{\lambda+\epsilon}{l^*} C_8\|u_n\| + C(\epsilon). \tag{3.7}$$

This implies that $\{u_n\}$ is bounded. □

By (2.6), (2.7), (2.11) and Lemma 3.1, we obtain

Corollary 3.2. *If $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ is a $(PS)_c$ sequence of I_λ, then the sequences $\{\int_\Omega \Phi(|\nabla u_n|)dx\}$ and $\{\int_\Omega |u_n|^{l^*}dx\}$ are bounded.*

Next, we use the concentration-compactness type principle which is analogous to Lemma 4.2 of Fukagai, Ito and Narukawa [5]. This will be the keystone that enables us to verify that I_λ satisfies the $(PS)_c$ condition. First, we will recall a measure theory result as follows.

Let $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ be the $(PS)_c$ sequence. Lemma 3.1 and Corollary 3.2 show that $\{u_n\}$, $\{\int_\Omega \Phi(|\nabla u_n|)dx\}$ and $\{\int_\Omega |u_n|^{l^*}dx\}$ are bounded. Otherwise, we know that $L_\Phi(\Omega)$ and $L^{l^*}(\Omega)$ are reflexive Banach spaces. Then there exist two nonnegative measures $\mu, \nu \in \mathcal{M}(\overline{\Omega})$, the space of Radon measures and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\Phi(|\nabla u_n|) \rightharpoonup \mu, \quad \text{in } \mathcal{M}(\overline{\Omega}), \tag{3.8}$$

$$|u_n|^{l^*} \rightharpoonup \nu, \quad \text{in } \mathcal{M}(\overline{\Omega}). \tag{3.9}$$

Lemma 3.3. *Assume that (ϕ_1)–(ϕ_3) hold. Let $\{u_n\}$ of I_λ be a Palais–Smale sequence such that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$ and $\Phi(|\nabla u_n|) \rightharpoonup \mu$, $|u_n|^{l^*} \rightharpoonup \nu$ in $\mathcal{M}(\overline{\Omega})$, where μ, ν are two nonnegative measures on $\overline{\Omega}$. Then there exist an at most countable set J and a family $\{x_j\}_{j\in J}$ of distinct points in $\overline{\Omega}$ such that*

(i)
$$\nu = |u|^{l^*} + \sum_{j\in J} \nu_j \delta_{x_j},$$

where $\{\nu_j\}_{j\in J}$ is a family of positive constants and δ_{x_j} is the Dirac measure of mass 1 concentrated at x_j;

(ii)
$$\mu \geq \Phi(|\nabla u|) + \sum_{j\in J} \mu_j \delta_{x_j},$$

where $\{\mu_j\}_{j\in J}$ is a family of positive constants, satisfying $\nu_j \leq \max\{S_4^{l^*}\mu_j^{\frac{l^*}{l}}, S_4^{l^*}\mu_j^{\frac{l^*}{m}}\}$ for all $j \in J$.

Proof. The proof of Lemma 3.3 is similar to Lemma 4.2 in Fukagai, Ito and Narukawa [5], we omit the details here. □

Lemma 3.4. *Assume that (ϕ_1)–(ϕ_3) and (f_1)–(f_2) hold. For a given $0 < \lambda < \infty$, let $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ be a Palais–Smale sequence of I_λ. Considering J given by Lemma 3.3, then for each $j \in J$, we have either $\nu_j = 0$ or*

$$\nu_j \geq \min\left\{\left(\frac{l}{\lambda S_4^l}\right)^{\frac{l^*}{l^*-l}}, \left(\frac{l}{\lambda S_4^m}\right)^{\frac{l^*}{l^*-m}}\right\}.$$

Proof. Let us first define $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi(x) = 1$ in $B(0,\frac{1}{2})$, $\operatorname{supp}(\psi) \subset B(0,1)$ and $0 \leq \psi(x) \leq 1$, $\forall x \in \mathbb{R}^N$. For each $j \in J$ and $\epsilon > 0$, let us define

$$\psi_\epsilon(x) = \psi\left(\frac{x-x_j}{\epsilon}\right), \qquad \forall x \in \mathbb{R}^N.$$

Then $\{u_n\psi_\epsilon(x)\} \subset W_0^{1,\Phi}(\Omega)$ is bounded in $W_0^{1,\Phi}(\Omega)$. From the fact that $I'_\lambda(u_n) \to 0$, it follows that

$$\langle I'_\lambda(u_n), u_n\psi_\epsilon\rangle = o_n(1),$$

i.e.,

$$\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n\psi_\epsilon) = \lambda \int_\Omega |u_n|^{l^*}\psi_\epsilon dx + \int_\Omega f(x,u_n)u_n\psi_\epsilon dx + o_n(1). \tag{3.10}$$

By $(\phi_3)''$, we obtain

$$\begin{aligned}\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n\psi_\epsilon)dx &= \int_\Omega \phi(|\nabla u_n|)|\nabla u_n|^2\psi_\epsilon dx + \int_\Omega \phi(|\nabla u_n|)(\nabla u_n \nabla\psi_\epsilon)u_n dx \\ &\geq l\int_\Omega \Phi(|\nabla u_n|)\psi_\epsilon dx + \int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\epsilon)u_n dx.\end{aligned} \tag{3.11}$$

It is obvious that

$$\begin{aligned}& l\int_\Omega \Phi(|\nabla u_n|)\psi_\epsilon dx + \int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\epsilon)u_n dx \\ &\quad \leq \lambda\int_\Omega |u_n|^{l^*}\psi_\epsilon dx + \int_\Omega f(x,u_n)u_n\psi_\epsilon dx + o_n(1).\end{aligned} \tag{3.12}$$

On the one hand, by Lemma 3.1, we know that the Palais–Smale sequence $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ of I_λ is bounded. Taking a subsequence of $\{u_n\}$ if necessary, we may suppose that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,\Phi}(\Omega), \tag{3.13}$$
$$u_n \to u \quad \text{in } L_\Phi(\Omega), \tag{3.14}$$
$$u_n \to u \quad \text{a.e. in } \Omega. \tag{3.15}$$

Moreover, from (2.3), (2.10) and (2.11) it is easy to see that

$$\int_\Omega \widetilde{\Phi}(\phi(|\nabla u_n|)\nabla u_n)dx \leq \int_\Omega \Phi(2|\nabla u_n|)dx \leq \eta_2(2)\int_\Omega \Phi(|\nabla u_n|)dx \leq \eta_2(2)\eta_2(\|u_n\|).$$

Clearly, the sequence $\{\phi(|\nabla u_n|)\nabla u_n\}$ is bounded in $L_{\widetilde{\Phi}}(\Omega)$. Thus, there exists a subsequence $\{u_n\}$ such that for some $\widetilde{\omega}_1 \in L_{\widetilde{\Phi}}(\Omega, \mathbb{R}^N)$

$$\phi(|\nabla u_n|)\nabla u_n \rightharpoonup \widetilde{\omega}_1 \quad \text{in } L_{\widetilde{\Phi}}(\Omega,\mathbb{R}^N). \tag{3.16}$$

Therefore, since $\operatorname{supp}(\nabla\psi_\epsilon) \subset B(x_j,\epsilon)$, (3.14) and (3.16), we have

$$\lim_{n\to\infty}\int_\Omega \phi(|\nabla u_n|)(\nabla u_n\nabla\psi_\epsilon)u_n dx = \int_\Omega (\widetilde{\omega}_1\nabla\psi_\epsilon)u dx. \tag{3.17}$$

On the other hand, we will prove

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)u_n\psi_\epsilon dx = \int_\Omega f(x,u)u\psi_\epsilon dx. \tag{3.18}$$

First, we show the following claim.

Claim 1 : $\{f(x,u_n)\}$ is bounded in $L_{\widetilde{\Phi}_*}(\Omega)$.

In fact, from (2.12), (3.5), Corollary 3.2, $\triangle_2$-condition and the convexity of $\widetilde{\Phi}_*$, there exist constants $C_9, C_{10} > 0$ such that

$$\begin{aligned}\int_\Omega \widetilde{\Phi}_*(f(x,u_n))dx &\le C_9 \int_\Omega \widetilde{\Phi}_*(|u_n|^{l^*-1})dx + C_{10}\int_\Omega \widetilde{\Phi}_*(C_\epsilon)dx \\ &\le C_9\widetilde{\Phi}_*(1)\int_{\{x\in\Omega;|u_n|\ge 1\}} |u_n|^{(l^*-1)\widetilde{l}_*}dx + C_9\int_{\{x\in\Omega;|u_n|<1\}} \widetilde{\Phi}_*(1)dx \\ &\quad + C_{10}\int_\Omega \widetilde{\Phi}_*(C_\epsilon)dx \\ &\le C_9\widetilde{\Phi}_*(1)\int_\Omega |u_n|^{l^*}dx + C_9\int_\Omega \widetilde{\Phi}_*(1)dx + C_{10}\int_\Omega \widetilde{\Phi}_*(C_\epsilon)dx \\ &< \infty.\end{aligned}$$

Therefore, the claim is proved.

By (3.5), (3.13)–(3.15) and Claim 1, we are now in a position to obtain (3.18).

Now, according to (3.8), (3.9), (3.17), (3.18) and letting $n \to \infty$ in (3.12), it follows that

$$l\int_\Omega \psi_\epsilon d\mu + \int_\Omega (\widetilde{\omega}_1 \nabla \psi_\epsilon)udx \le \lambda \int_\Omega \psi_\epsilon d\nu + \int_\Omega f(x,u)u\psi_\epsilon dx. \tag{3.19}$$

Next, we will prove that the second term of the left-hand side converges 0 as $\epsilon \to 0$.

By $I'_\lambda(u_n) \to 0$, we have for any $v \in W_0^{1,\Phi}(\Omega)$

$$\langle I'_\lambda(u_n), v\rangle = \int_\Omega (\phi(|\nabla u_n|)\nabla u_n \nabla v - \lambda |u_n|^{l^*-2}u_n v - f(x,u_n)v)dx = o_n(1). \tag{3.20}$$

Moreover, from Claim 1, there is a subsequence $\{u_n\}$ such that

$$\lambda |u_n|^{l^*-1} + f(x,u_n) \rightharpoonup \widetilde{\omega}_2 \quad \text{in } L_{\widetilde{\Phi}_*}(\Omega), \tag{3.21}$$

for some $\widetilde{\omega}_2 \in L_{\widetilde{\Phi}_*}(\Omega)$. Hence, by (3.16), (3.20) and (3.21), we conclude

$$\int_\Omega (\widetilde{\omega}_1 \nabla v - \widetilde{\omega}_2 v)dx = 0,$$

for any $v \in W_0^{1,\Phi}(\Omega)$. Substituting $v = u\psi_\epsilon$, we have

$$\int_\Omega (\widetilde{\omega}_1 \nabla(u\psi_\epsilon) - \widetilde{\omega}_2 u\psi_\epsilon)dx = 0,$$

i.e.,

$$\int_\Omega (\widetilde{\omega}_1 \nabla \psi_\epsilon)udx = -\int_\Omega (\widetilde{\omega}_1 \nabla u - \widetilde{\omega}_2 u)\psi_\epsilon dx.$$

Noting $\widetilde{\omega}_1 \nabla u - \widetilde{\omega}_2 u \in L^1(\Omega)$, we see that the right-hand side tends to 0 as $\epsilon \to 0$. Evidently,

$$\lim_{\epsilon\to 0}\int_\Omega (\widetilde{\omega}_1 \nabla \psi_\epsilon)udx = 0. \tag{3.22}$$

Furthermore, by (3.5) and Lemma 3.1, we have

$$\int_\Omega |f(x,u)u|dx \le C_\epsilon \int_\Omega |u|dx + \epsilon\int_\Omega |u|^{l^*}dx \le C_\epsilon \|u\|_{L^1(\Omega)} + \epsilon S_4 \|u\|^{l^*} < \infty.$$

This implies that

$$\lim_{\epsilon\to 0}\int_\Omega f(x,u)u\psi_\epsilon dx = 0. \tag{3.23}$$

Consequently, by (3.22) and (3.23), letting $\epsilon \to 0$ in (3.19), we obtain for each $j \in J$

$$l\mu_j \le \lambda \nu_j.$$

By Lemma 3.3, we get

$$\min\{S_4^{-l^* l}\nu_j^l, S_4^{-l^* m}\nu_j^m\} \le \mu_j^{l^*} \le \left(\frac{\lambda}{l}\right)^{l^*} \nu_j^{l^*},$$

i.e., $\nu_j = 0$ or

$$\nu_j \ge \min\left\{\left(\frac{l}{\lambda S_4^l}\right)^{\frac{l^*}{l^*-l}}, \left(\frac{l}{\lambda S_4^m}\right)^{\frac{l^*}{l^*-m}}\right\}. \qquad \square$$

Lemma 3.5. *Assume that* (ϕ_1)*–*(ϕ_3) *and* (f_1)*–*(f_2) *hold. Let* $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ *be a* $(PS)_c$ *sequence of* I_λ*. Then, given* $M > 0$*, there exists* $\lambda^* > 0$ *such that* I_λ *satisfies* $(PS)_c$ *condition for all* $0 < c < M$*, provided* $0 < \lambda < \lambda^*$*.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence of I_λ and $0 < c < M$, taking $n \to \infty$ in (3.1), we obtain

$$\begin{aligned} \lambda\left(\frac{1}{\theta} - \frac{1}{l^*}\right)\int_\Omega d\nu &\le c + C_1|\Omega| + C_0|\Omega|^{1-\frac{\sigma}{l^*}}\left(\int_\Omega d\nu\right)^{\frac{\sigma}{l^*}} \\ &< M + C_1|\Omega| + C_0|\Omega|^{1-\frac{\sigma}{l^*}}\left(\int_\Omega d\nu\right)^{\frac{\sigma}{l^*}}. \end{aligned} \tag{3.24}$$

Therefore, if we choose

$$\lambda^* = \min\left\{lS_4^{-\frac{1}{l}},\ lS_4^{-\frac{1}{m}},\ \left(\frac{d_1}{M+d_2}\right)^{\frac{l^*-l}{l-\sigma}} S_4^{-\frac{l(l^*-\sigma)}{l-\sigma}},\ \left(\frac{d_1}{M+d_2}\right)^{\frac{l^*-m}{m-\sigma}} S_4^{-\frac{m(l^*-\sigma)}{m-\sigma}}\right\},$$

where $d_1 = l^{\frac{l^*-\sigma}{l-\sigma}}\left(\frac{1}{\theta} - \frac{1}{l^*}\right)^{\frac{l^*-l}{l-\sigma}}$ and $d_2 = C_1|\Omega| + C_0|\Omega|^{1-\frac{\sigma}{l^*}}$, then we have from (3.24)

$$\int_\Omega d\nu < \min\left\{\left(\frac{l}{\lambda S_4^l}\right)^{\frac{l^*}{l^*-l}}, \left(\frac{l}{\lambda S_4^m}\right)^{\frac{l^*}{l^*-m}}\right\}, \tag{3.25}$$

for all $0 < \lambda < \lambda^*$.

As a consequence of this fact and Lemma 3.4, we conclude that for each $j \in J$, $\nu_j = 0$ and

$$\lim_{n\to\infty}\int_\Omega |u_n|^{l^*}dx = \int_\Omega |u|^{l^*}dx.$$

Thus, there exists $u \in W_0^{1,\Phi}(\Omega)$ such that, up to subsequence,

$$u_n \to u \quad \text{in } L^{l^*}(\Omega). \tag{3.26}$$

Next, from $\langle I'_\lambda(u_n), (u_n - u)\rangle = o_n(1)$, we have

$$\lim_{n\to\infty}\int_\Omega (\phi(|\nabla u_n|)\nabla u_n\nabla(u_n-u) - \lambda|u_n|^{l^*-2}u_n(u_n-u) - f(x,u_n)(u_n-u))dx = 0. \tag{3.27}$$

Hence, we can derive from (3.13)–(3.15), (3.18), (3.26) and (3.27) that

$$\lim_{n\to\infty}\int_\Omega \phi(|\nabla u_n|)\nabla u_n\nabla(u_n-u)dx = 0.$$

Moreover, by (3.13) and Lemma 5 in [8], we conclude that

$$u_n \to u \quad \text{in } W_0^{1,\Phi}(\Omega). \qquad \square$$

4 Proof of Theorem 1.2

In order to verify Theorem 1.2, we need to prove that Lemma 2.6 is applicable in our situation, namely the functional I_λ on $W_0^{1,\Phi}(\Omega)$ satisfies the hypotheses (I_1) and (I_2).

First, since $E = W_0^{1,\Phi}(\Omega)$ is a separable and reflexive Banach space, then there exist a Schauder basis $\{e_i\}_{i\in\mathbb{N}} \subset E$ and $\{e_j^*\}_{j\in\mathbb{N}} \subset E^*$ such that

$$(e_i, e_j^*) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$E = \overline{\operatorname{span}\{e_i | i \in \mathbb{N}\}}, \qquad E^* = \overline{\operatorname{span}\{e_j^* | j \in \mathbb{N}\}}.$$

Now, fixing a Schauder basis $\{e_i\}_{i\in\mathbb{N}}$ of $W_0^{1,\Phi}(\Omega)$, we set

$$X_k := \operatorname{span}\{e_1, \cdots, e_k\}, \qquad Y_k := \bigcap_{j=1}^{k} \operatorname{Ker} e_j^*, \tag{4.1}$$

in such way that $E = W_0^{1,\Phi}(\Omega) = X_k \oplus Y_k$, for $k \in \mathbb{N}$.

Lemma 4.1. *Assume that (ϕ_1)–(ϕ_3) hold. If $\Phi \le \Psi \ll \Phi_*$, setting*

$$S_{k,\Psi} := \sup\{\|u\|_{L_\Psi(\Omega)} : \|u\| = 1,\ u \in Y_k,\ k \in \mathbb{N}\},$$

then $\lim_{k\to\infty} S_{k,\Psi} = 0$.

Proof. It is clear that $0 \le S_{k+1,\Psi} \le S_{k,\Psi}$. Thus we have $S_{k,\Psi} \to S_\Psi \ge 0$, as $k \to \infty$. And for every $k \ge 0$, there exists $u_k \in Y_k$ such that $\|u_k\| = 1$ and

$$\|u_k\|_{L_\Psi(\Omega)} > \frac{S_{k,\Psi}}{2}. \tag{4.2}$$

By definition of Y_k, $u_k \rightharpoonup 0$ in $W_0^{1,\Phi}(\Omega)$, as $k \to \infty$. By (2.4), we have $u_k \to 0$ in $L_\Psi(\Omega)$, as $k \to \infty$. Using (4.2), we obtain $S_{k,\Psi} \to 0$, as $k \to \infty$. Hence we have proved that $S_\Psi = 0$. □

Lemma 4.2. *Assume that (ϕ_1)–(ϕ_3) and (f_1)–(f_3) hold. Then there exist constants k, ρ, $\widetilde{\lambda} > 0$ and $\alpha > 0$, such that for any $u \in Y_k$ with $\|u\| = \rho$ and $0 < \lambda < \widetilde{\lambda}$,*

$$I_\lambda|_{\partial B_\rho \cap Y_k} \ge \alpha.$$

Proof. From (f_3), (2.9), (2.11) and Hölder's inequality, there exists a constant $S_4 > 0$ such that

$$\begin{aligned} I_\lambda(u) &= \int_\Omega \left(\Phi(|\nabla u|) - \frac{\lambda}{l^*}|u|^{l^*} - F(x,u) \right) dx \\ &\ge \eta_1(\|u\|) - \frac{\lambda}{l^*} S_4^{l^*} \|u\|^{l^*} - C_2 \int_\Omega |u|^\tau dx - C_3|\Omega| \\ &\ge \eta_1(\|u\|) - \frac{\lambda}{l^*} S_4^{l^*} \|u\|^{l^*} - C_2 |\Omega|^{1-\frac{\tau}{l^*}} \|u\|^\tau_{L^{l^*}(\Omega)} - C_3|\Omega|. \end{aligned} \tag{4.3}$$

By (2.7), Lemma 3.5 and Lemma 4.1, considering S_{k,Φ_*} to be chosen posteriorly, for all $u \in Y_k$ and $\|u\| = \rho > 1$, we have

$$\begin{aligned} I_\lambda(u) &\geq \eta_1(\|u\|) - \frac{\lambda}{l^*} S_4^{l^*} \|u\|^{l^*} - C_2 |\Omega|^{1-\frac{\tau}{l^*}} S_3^\tau \|u\|^\tau_{L_{\Phi_*}(\Omega)} - C_3|\Omega| \\ &\geq \rho^l - \frac{\lambda}{l^*} S_4^{l^*} \rho^{l^*} - C_2|\Omega|^{1-\frac{\tau}{l^*}} S_3^\tau S_{k,\Phi_*}^\tau \rho^\tau - C_3|\Omega| \\ &\geq \rho^l (1 - C_2|\Omega|^{1-\frac{\tau}{l^*}} S_3^\tau S_{k,\Phi_*}^\tau \rho^{\tau - l}) - C_3|\Omega| - \frac{\lambda}{l^*} S_4^{l^*} \rho^{l^*}. \end{aligned}$$

Now, by Lemma 4.1 again and taking k sufficiently large, there exists sufficiently small S_{k,Φ_*} such that $C_2|\Omega|^{1-\frac{\tau}{l^*}} S_3^\tau S_{k,\Phi_*}^\tau \rho^{\tau-l} \leq \frac{1}{2}$, $\frac{1}{2}\rho^l - C_3|\Omega| \geq \frac{1}{4}\rho^l$ and $\rho = \rho(S_{k,\Phi_*}) > 1$.

Consequently, for every $u \in Y_k$ with $\|u\| = \rho > 1$ and k sufficiently large, there exist sufficiently small $\widetilde{\lambda} > 0$ and a constant $\alpha > 0$ such that

$$I_\lambda(u) \geq \frac{1}{4}\rho^l - \frac{\lambda}{l^*} S_4^{l^*} \rho^{l^*} > \alpha > 0$$

for $0 < \lambda < \widetilde{\lambda}$. Hence, we complete the proof of Lemma 4.2. □

Lemma 4.3. *Assume that (ϕ_1)–(ϕ_3) and (f_4) hold. Then for given $q \in \mathbb{N}$, there exist a subspace W of $W_0^{1,\Phi}(\Omega)$ and a constant $M_q > 0$, independent of λ, such that* $\dim W = q$ *and* $\max_{u \in W} I_\lambda(u) < M_q$.

Proof. First, from (f_4), let $x_0 \in \Omega_0$ and $r_0 > 0$ be such that $\overline{B(x_0, r_0)} \subset \Omega_0$ and $0 < |\overline{B(x_0, r_0)}| < \frac{|\Omega_0|}{2}$. We take $u_1 \in C_0^\infty(\Omega)$ with $\operatorname{supp}(u_1) = \overline{B(x_0, r_0)}$. Considering $\Omega_1 := \Omega_0 \setminus \overline{B(x_0, r_0)}$, we have $|\Omega_1| > \frac{|\Omega_0|}{2} > 0$. Next, let $x_1 \in \Omega_1$ and $r_1 > 0$ be such that $\overline{B(x_1, r_1)} \subset \Omega_1$ and $0 < |\overline{B(x_1, r_1)}| < \frac{|\Omega_1|}{2}$. We take $u_2 \in C_0^\infty(\Omega)$ with $\operatorname{supp}(u_2) = \overline{B(x_1, r_1)}$. After a finite number of steps, we get $u_1, u_2, \ldots, u_q$ such that $\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \emptyset$ and $|\operatorname{supp}(u_i)| > 0$, for all i, $j \in \{1, 2, \ldots, q\}$ and $i \neq j$.

Let $W = \operatorname{span}\{u_1, u_2, \ldots, u_q\}$. For every $u \in W \setminus \{0\}$, we have $\int_{\Omega_0} |u|^m dx > 0$, $u = t_u v = tv$ and $v \in \partial B(0,1) \cap W$. By (2.10) and (2.11), we obtain

$$\begin{aligned} \max_{u \in W \setminus \{0\}} I_\lambda(u) &= \max_{\substack{v \in \partial B(0,1) \cap W \\ t > 0}} \int_\Omega \left(\Phi(t|\nabla v|) - \frac{\lambda}{l^*}|tv|^{l^*} - F(x, tv) \right) dx \\ &\leq \max_{\substack{v \in \partial B(0,1) \cap W \\ t > 0}} \left(\eta_2(t) \int_\Omega \Phi(|\nabla v|) dx - \int_\Omega F(x, tv) dx \right) \\ &\leq \max_{\substack{v \in \partial B(0,1) \cap W \\ t > 0}} \left(\eta_2(t) \eta_2(\|v\|) - \int_\Omega F(x, tv) dx \right) \\ &= \max_{\substack{v \in \partial B(0,1) \cap W \\ t > 0}} \left(\eta_2(t) \left(1 - \frac{1}{\eta_2(t)} \int_\Omega F(x, tv) dx \right) \right). \end{aligned} \tag{4.4}$$

Next, in order to prove the lemma, it suffices to show that

$$\lim_{|t| \to \infty} \frac{1}{|t|^m} \int_\Omega F(x, tv) dx > 1 \tag{4.5}$$

uniformly for $v \in \partial B(0,1) \cap W$.

In fact, by (f_4), for some positive constant K, there is a constant $C_K > 0$ such that

$$F(x, s) \geq K|s|^m - C_K,$$

for any $x \in \Omega_0$ and every $s \in \mathbb{R}$. Evidently, for $t > 0$ and $v \in \partial B(0,1) \cap W$ with $\int_{\Omega_0} |v|^m dx > 0$, we have

$$\int_\Omega F(x,tv)dx = \int_{\Omega_0} F(x,tv)dx \geq Kt^m \int_{\Omega_0} |v|^m dx - C_K|\Omega_0|.$$

Moreover, since W is finite dimensional, there exist constants $a_1, a_2 > 0$ such that for any $v \in \partial B(0,1) \cap W$

$$a_1 \leq \|v\|_{L^m(\Omega_0)} \leq a_2.$$

It is easy to see that

$$\int_\Omega F(x,tv)dx \geq Kt^m a_1^m - C_K|\Omega_0| \tag{4.6}$$

and

$$\lim_{|t|\to\infty} \frac{1}{|t|^m} \int_\Omega F(x,tv)dx \geq Ka_1^m.$$

This implies that the inequality (4.5) is obtained by taking $K > \frac{1}{a_1^m}$.

Furthermore, by (4.4) and (4.6), we have

$$\begin{aligned} \max_{\substack{v\in\partial B(0,1)\cap W \\ t>0}} I_\lambda(tv) &\leq \max_{\substack{v\in\partial B(0,1)\cap W \\ t>0}} \left(\eta_2(t)\eta_2(\|v\|) - \int_\Omega F(x,tv)dx\right) \\ &\leq \max_{t>0}(\eta_2(t) - K|t|^m a_1^m + C_K|\Omega_0|). \end{aligned}$$

Hence we obtain

$$\lim_{|t|\to 0} I_\lambda(tv) \leq C_K|\Omega_0|$$

uniformly for $v \in \partial B(0,1) \cap W$.

Therefore, for given $q \in \mathbb{N}$, there exists a constant $M_q > 0$, independent of λ, such that $\max_{u\in W_q} I_\lambda(u) < M_q$. □

Proof of Theorem 1.2. Firstly, we will apply Lemma 2.6. We recall that $W_0^{1,\Phi}(\Omega) = X_k \oplus Y_k$, where X_k and Y_k are defined in (4.1). Invoking Lemma 4.2, there exist $k \in \mathbb{N}$ and $\widetilde{\lambda} > 0$ such that for all $0 < \lambda < \widetilde{\lambda}$, I_λ satisfies (I_1). Secondly, by Lemma 4.3 we obtain $W_{i+k} \subset W_0^{1,\Phi}(\Omega)$ with $\dim W_{i+k} = i + k = i + \dim X_k (i \in \mathbb{N})$ and such that for all $0 < \lambda < \widetilde{\lambda}$, I_λ satisfies (I_2). Thirdly, by Lemma 3.5, denoting $\lambda_i = \min\{\widetilde{\lambda}, \lambda^*\}$, we have that for all $0 < \lambda < \lambda_i$, I_λ satisfies (I_3). Consequently, by (f_5), we have $I_\lambda(0) = 0$ and $I_\lambda(u)$ is even. Hence, we can apply Lemma 2.6 to conclude that I_λ possesses at least i pairs of nontrivial solutions for $\lambda_i > 0$. □

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