Infinitely many solutions for nonhomogeneous Choquard equations

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Abstract. In this paper, we study the following nonhomogeneous Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + f(x), \qquad x \in \mathbb{R}^N,$$

where $N \geq 3, \alpha \in (0, N), p \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right)$, I_α denotes the Riesz potential and $f \neq 0$. By using a critical point theorem for non-even functionals, we prove the existence of infinitely many virtual critical points for two classes of potential V. To the best of our knowledge, this result seems to be the first one for nonhomogeneous Choquard equation on the existence of infinity many solutions.

Keywords: Choquard equation, infinitely many solutions, non-even functional, variational methods.

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1   Introduction

In this paper, we are concerned with the following nonhomogeneous nonlocal problem

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + f(x), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3, \alpha \in (0, N), p \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right)$, and Riesz potential I_α is given by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}}$$

where Γ denotes the Gamma function. This equation arises in the study of nonlinear Choquard equations describing an electron trapped in its own hole, in a certain approximation to Hartree Fock theory of one component plasma [4].

When $f = 0$, the existence and qualitative properties of solutions for Choquard type equations (1.1) have been studied widely and intensively in literatures. See [1,3,6,9,13] and

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references therein for the existence of ground states, nodal solutions and multiple solutions to (1.1). For the results about qualitative properties such as regularity, symmetry, uniqueness and decay, one can refer to [8–11,14], for instance.

When $f \neq 0$, the authors in [16] (or [18]) proved that (1.1) has a ground state and bound state for f small enough via fibering mapping method. However, as we know, there is no result on the existence of infinitely many solutions of (1.1) with $f \neq 0$. Motivated by this, the main purpose of this paper is to consider the existence of infinitely many solutions.

For the potential V, we make the following assumptions that either

(V1) $V \in L^2_{\text{loc}}(\mathbb{R}^N)$ is such that $\operatorname{ess\,inf} V(x) > 0$, and $\int_{B(x)} \frac{1}{V(y)} dy \to 0$ as $|x| \to \infty$, where $B(x)$ is the unit ball in $\mathbb{R}^N$ centered at x,

or

(V2) V is a positive constant function.

Clearly, **(V1)** holds if V is a strictly positive continuous function in $\mathbb{R}^N$ and $V(x) \to \infty$ as $|x| \to \infty$. Let

$$H := \begin{cases} \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}, & \text{if } V \text{ satisfies } \textbf{(V1)}, \\ H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u \text{ is radial}\}, & \text{if } V \text{ satisfies } \textbf{(V2)}. \end{cases}$$

endowed with the inner product $(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) dx$ and norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx.$$

Let H^* be the duality space of H with norm $\| \cdot \|_{H^*}$, and $\langle \cdot, \cdot \rangle$ denotes duality pairing between H and H^*.

As usual, the corresponding energy functional of (1.1) is $E : H \to \mathbb{R}$ is

$$E(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx - \langle f, u \rangle.$$

It is easy to check that $E \in C^1(H, \mathbb{R})$, and the critical points of E are solutions of (1.1) in the weak sense. To state main results clearly, we consider the following equation which is related to (1.1)

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + \beta f(x), & x \in \mathbb{R}^N \\ \beta \in [-1, 1], \quad \langle f, u \rangle = 0. \end{cases} \tag{1.2}$$

It is well known that problem (1.2) can not be solved by looking for critical points of the functional E on $\langle f, u \rangle = 0$ with restriction condition $\beta \in [-1, 1]$, because β can not be viewed as a Lagrange multiplier.

Our main results are as follows.

Theorem 1.1. *Let $N \geq 3, \alpha \in (0, N)$. The following statements are true.*

  *(i) If $p \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right)$ and **(V1)** is satisfied, then for any $f \in H^*\backslash\{0\}$, either (1.1) or (1.2) has an unbounded sequence of solutions.*

  *(ii) If $p \in \left(\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right)$ and **(V2)** is satisfied, then for any radial $f \in H^*\backslash\{0\}$, either (1.1) or (1.2) has an unbounded sequence of radial solutions.*

Due to the difference of the action space H considered, the results and methods for cases **(V1)** and **(V2)** may be different. In view of Theorem 1.1, the range of p in part (ii) is smaller than in part (i). Indeed, for the case **(V2)**, the embedding $H \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in \left(2, \frac{2N}{N-2}\right)$ while for the case **(V1)**, the embedding $H \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in \left[2, \frac{2N}{N-2}\right)$, (see Lemma 2.3). So if $p = \frac{N+\alpha}{N}$ in (ii), we can not guarantee the compactness of nonlocal term, that is, we can not deduce that up to a subsequence, $\int_{\mathbb{R}^N}(I_\alpha * |u_n|^p)|u_n|^p \to \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p$ as $n \to \infty$ when u_n weakly converge to u in H. Furthermore, the compactness of nonlocal term is critical in the proof of Theorem 1.1 (ii). Therefore, $p \neq \frac{N+\alpha}{N}$ in part (ii). In addition, the proof of (i) makes use of the property of eigenvalues in H tending to infinity as in [17]. But this method does not work for the case **(V2)** because H does not have such a property. So we develop a new technique to overcome this problem by delicate asymptotic analysis of nonlocal term.

The remainder of this paper is organized as follows. In Section 2, some notations and preliminary results are presented. In Section 3, we are devoted to the proofs of our main results.

2   Preliminaries

In this section, some notations and elementary results are collected as follows.

- $\mathbb{N}$ is the set of all the positive integers.

- For $1 \leq s < \infty$, $L^s(\mathbb{R}^N)$ denotes the Lebesgue space with the norm $|u|_{L^s} = \left(\int_{\mathbb{R}^N}|u|^s \mathrm{d}x\right)^{\frac{1}{s}}$.

- Denote $\mathbb{D}(u) = \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p \mathrm{d}x$, and then for any $v \in H$,

$$\langle \mathbb{D}'(u), v \rangle = 2p \int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^{p-2}uv\mathrm{d}x.$$

- C denotes different positive constants line by line.

First let us recall the Hardy–Littlewood–Sobolev inequality.

Lemma 2.1 ([5, Theorem 4.3]). *Let $s, t > 1$ and $\alpha \in (0, N)$ with $\frac{1}{s} + \frac{1}{t} = 1 + \frac{\alpha}{N}$, $f \in L^s(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \alpha, s) > 0$ independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^{N-\alpha}}\,\mathrm{d}x\mathrm{d}y \leq C(N, \alpha, s)|f|_{L^s}|h|_{L^t}.$$

Here $C(N, \alpha, s)$ is a positive constant which depend only on N, α, s.

As a consequence of Lemma 2.1 and [7, Proposition 4.3], the following lemma holds true.

Lemma 2.2. *If u_n converges to some u in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, then*

$$\lim_{n\to\infty}\mathbb{D}(u_n) = \mathbb{D}(u),$$
$$\lim_{n\to\infty}\langle \mathbb{D}'(u_n), v \rangle = \langle \mathbb{D}(u), v \rangle$$

for any $v \in H$.

Next we give the property of the space H which plays a critical role in recovering the compactness.

Lemma 2.3 ([12, Proposition 2.1]). *The following statements are true.*

(i) *Under the assumption **(V1)**, the embedding $H \hookrightarrow H^1(\mathbb{R}^N)$ is continuous and H is a Hilbert space. Furthermore, the embedding $H \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in [2, \frac{2N}{N-2})$ and the spectrum of the self-adjoint operator of $-\Delta + V$ in $L^2(\mathbb{R}^N)$ is discrete, i.e. it consists of an increasing sequence $\{\lambda_n\}_{n \geq 1}$ of eigenvalues with finite multiplicity such that $\lambda_n \to \infty$ as $n \to \infty$ and $L^2(\mathbb{R}^N) = \sum_n M_n$, $M_n \perp M_{n'}$ for $n \neq n'$, where M_n is the eigenspace corresponding to λ_n.*

(ii) *Under the assumption **(V2)**, the embedding $H \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in (2, \frac{2N}{N-2})$.*

In the sequel, we list some definitions from the critical point theory.

Definition 2.4 ($(\mathrm{P.S.})_c$ condition). A sequence $\{u_n\}_{n \geq 1}$ is a Palais–Smale sequence of the functional E at level c ($(\mathrm{P.S.})_c$ sequence for short): if $E(u_n) \to c$ and $E'(u_n) \to 0$. E is said to satisfy the $(\mathrm{P.S.})_c$ condition if any $(\mathrm{P.S.})_c$ sequence $\{u_n\}_{n \geq 1}$ has a convergent subsequence.

Definition 2.5 ($(\mathrm{sP.S.})_c$ condition, see [2]). The functional E is said to satisfy the symmetrized Palais–Smale condition at level c ($(\mathrm{sP.S.})_c$ condition for short): if E satisfies $(\mathrm{P.S.})_c$ condition and any sequence $\{u_n\}_{n \geq 1}$ is relatively compact in H whenever it satisfies the following conditions

$$\lim_{n \to \infty} E(u_n) = \lim_{n \to \infty} E(-u_n) = c \tag{2.1}$$

and

$$\lim_{n \to \infty} \|E'(u_n) - \mu_n E'(-u_n)\| = 0 \quad \text{for some positive sequence of reals } \mu_n. \tag{2.2}$$

Denote the set of critical points at level c by $\mathcal{K}_c = \{u \in H : E(u) = c, \ E'(u) = 0\}$.

Definition 2.6 ([2]). Denote the set of $\mathbb{Z}_2$-resonant points at level c by

$$\mathcal{K}_c^f = \{u \in H : E(u) = E(-u) = c, \ E'(u) = \lambda E'(-u), \ \lambda > 0\},$$

and the set of virtual critical points at level c by $I_c = \mathcal{K}_c^f \bigcup \mathcal{K}_c$. The corresponding value c is called virtual critical values.

Theorem 2.7 ([2, Proposition 2.1]). *Let E be a C^1 functional satisfying $(sP.S.)_c$ condition on a Hilbert space $H = X \bigoplus Y$ with $\dim(X) < \infty$. Assume that $E(0) = 0$ as well as the following conditions:*

(i) *there is $\rho > 0$ and $\alpha \geq 0$ such that $\inf E(S_\rho(Y)) \geq \alpha$, where $S_\rho(Y) = \{u \in Y : \|u\| = \rho\}$;*

(ii) *there exists an increasing sequence $\{X_n\}_{n \geq 1}$ of finite dimensional subspace of H, all containing X such that $\lim_{n \to \infty} \dim(X_n) = \infty$ and for each n, $\sup E(S_{R_n}(X_n)) \leq 0$ for some $R_n > \rho$.*

Then E has an unbounded sequence of virtual critical values.

Throughout the paper, we are devoted to the proof of our main result by verifying Theorem 2.7. Therefore, Theorem 1.1 can be restated as follows.

Theorem 2.8. *Under the same assumptions of Theorem 1.1, problem (1.1) has an unbounded sequence of virtual critical values.*

3 Proof of the main results

In this section, we prove Theorem 1.1 (i) in Subsection 3.1 and (ii) in Subsection 3.2.

3.1 Case $V \neq$ const

Lemma 3.1. *The functional E satisfies the* $(sP.S.)_c$ *condition.*

Proof. We first show that E satisfies $(P.S.)_c$ condition. Let $\{u_n\}_{n\geq 1}$ be a sequence such that $E(u_n) \to c$ and $E'(u_n) \to 0$. Then

$$
\begin{aligned}
c + C\|u_n\| &\geq E(u_n) - \frac{1}{2p}E'(u_n)u_n + \left(1 - \frac{1}{2p}\right)\langle f, u_n \rangle \\
&= \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|^2,
\end{aligned}
\tag{3.1}
$$

which implies that $\{u_n\}_{n\geq 1}$ is bounded. Up to a subsequence, $u_n \rightharpoonup u_0$ in H and $u_n \to u_0$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. By Lemma 2.2, it follows that for any $v \in H$, $\langle E'(u_n), v \rangle \to \langle E'(u_0), v \rangle$ and hence $E'(u_0) = 0$. Note that $\langle E'(u_n), u_n \rangle \to 0$ and $\langle E'(u_0), u_0 \rangle = 0$. By using Lemma 2.2 again, it follows that $\|u_n\| \to \|u_0\|$ and then $u_n \to u_0$ in H. Furthermore, this yields that $E(u_0) = c$, $E'(u_0) = 0$ and $u_0 \neq 0$ due to $f \neq 0$.

Next, we prove that if a sequence $\{v_n\}_{n\geq 1} \subset H$ satisfies (2.1) and (2.2), then $\{v_n\}_{n\geq 1}$ is relatively compact. Indeed, we can conclude from (2.1) that $\langle f, v_n \rangle \to 0$, and from (2.2) that

$$
(1 + \mu_n)\left[\int_{\mathbb{R}^N} \nabla v_n \nabla \phi + V(x)v_n\phi - \frac{1}{2p}\mathbb{D}'(v_n)\phi\right] - (1 - \mu_n)\langle f, \phi \rangle \to 0, \quad \text{for any } \phi \in H.
$$

This means that

$$
\langle E_0'(v_n), \phi \rangle - \frac{1 - \mu_n}{1 + \mu_n}\langle f, \phi \rangle \to 0,
\tag{3.2}
$$

where $E_0(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx$. Note that $\frac{1-\mu_n}{1+\mu_n} \in [-1, 1]$. Then by the definition of operator norm and (3.2), there exists $C_1 > 0$ independent of n such that

$$
\|E_0'(v_n)\|_{H^*} = \sup_{\|\phi\|=1} \langle E_0'(v_n), \phi \rangle \leq C_1.
$$

This implies that $\{v_n\}_{n\geq 1}$ is bounded. So it follows from (2.2) that $\langle E'(v_n) - \mu_n E'(-v_n), v_n \rangle \to 0$, that is,

$$
\langle E_0'(v_n), v_n \rangle - \frac{1 - \mu_n}{1 + \mu_n}\langle f, v_n \rangle \to 0.
$$

Without loss of generality, we assume that, up to a subsequence, $v_n \rightharpoonup v_0$ in H and then $v_n \to v_0$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ as $n \to \infty$. Since $\langle f, v_0 \rangle = \lim_{n\to\infty}\langle f, v_n \rangle = 0$, it follows

$$
\langle E_0'(v_n), v_n \rangle \to 0, \quad \text{as } n \to \infty.
\tag{3.3}
$$

On the other hand, by Lemma 2.2, we deduce from (3.2) that

$$
\langle E_0'(v_0), v_0 \rangle = \lim_{n\to\infty}\langle E_0'(v_n), v_0 \rangle = 0, \quad \text{as } n \to \infty.
$$

This, together with (3.3) yields that

$$
\begin{aligned}
\lim_{n\to\infty}(\|v_n\|^2 - \|v_0\|^2) &= \lim_{n\to\infty}\left\{\langle E_0'(v_n), v_n \rangle - \langle E_0'(v_0), v_0 \rangle + \frac{1}{2p}\langle \mathbb{D}'(v_n), v_n \rangle - \frac{1}{2p}\langle \mathbb{D}'(v_0), v_0 \rangle\right\} \\
&= 0.
\end{aligned}
$$

Hence $v_n \to v_0$ in H.

Moreover, let $\beta_n = \frac{1-\mu_n}{1+\mu_n}$ and up to a subsequence, $\beta = \lim_{n\to\infty} \beta_n \in [-1, 1]$. Then by (3.2), we see that v_0 is a nontrivial weak solution of

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u + \beta f(x).$$

The proof is completed. $\qquad\square$

In view of Lemma 2.3, let $\{e_k\}_{k\geq 1} \subset H$ be an orthonormal basis of eigenvectors of the operator $-\Delta + V$. Let $X = \text{span}\{e_1, e_2, \cdots, e_{k_0}\}$ and Y be the orthogonal complement of X. Clearly, $\dim X = k_0$, where $\dim X$ denotes the dimension of the space X.

Lemma 3.2. *For $k_0 = \dim X$ large enough, there exist $\theta > 0$ and $\rho > 0$ such that $E(u) \geq \theta$ for all $u \in Y$ with $\|u\| = \rho$.*

Proof. By Lemma 2.3, it follows that for any $u \in Y$,

$$\|u\|^2 > \lambda_{k_0}|u|^2_{L^2}. \tag{3.4}$$

We finish our proof by distinguishing two cases.

(i) When $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$, we have $\frac{2Np}{N+\alpha} \in (2, \frac{2N}{N-2})$. By interpolation inequality, we conclude that

$$|u|_{L^{\frac{2Np}{N+\alpha}}} \leq |u|_{L^2}^{\frac{s(N+\alpha)}{Np}} |u|_{L^{\frac{2N}{N-2}}}^{\frac{(1-s)(N+\alpha)}{(N-2)p}} \tag{3.5}$$

where $s \in (0, 1)$ and $2s + \frac{2N}{N-2}(1-s) = \frac{2Np}{N+\alpha}$. Then by (3.4) and Sobolev inequality, we have

$$
\begin{aligned}
E(u) &\geq \frac{1}{2}\|u\|^2 - C|u|_{L^{\frac{2Np}{N+\alpha}}}^{2p} - \|f\|_{H^*}\|u\| \\
&\geq \frac{1}{2}\|u\|^2 - C|u|_{L^{\frac{2Np}{N+\alpha}}}^{2p-2}\|u\|^2 - \|f\|_{H^*}\|u\| \\
&\geq \|u\|^2 \left(\frac{1}{2} - C|u|_{L^2}^{\frac{s(N+\alpha)}{N}\cdot\frac{2p-2}{p}} |u|_{L^{\frac{2N}{N-2}}}^{\frac{(1-s)(N+\alpha)}{N-2}\cdot\frac{2p-2}{p}}\right) - \|f\|_{H^*}\|u\| \\
&\geq \|u\|^2 \left(\frac{1}{2} - C\lambda_{k_0}^{-\frac{s(N+\alpha)}{2N}\cdot\frac{2p-2}{p}}\|u\|^{\frac{s(N+\alpha)}{N}\cdot\frac{2p-2}{p}} |u|_{L^{\frac{2N}{N-2}}}^{\frac{(1-s)(N+\alpha)}{N-2}\cdot\frac{2p-2}{p}}\right) - \|f\|_{H^*}\|u\| \\
&\geq \|u\|^2 \left(\frac{1}{2} - C\lambda_{k_0}^{-\frac{s(N+\alpha)}{2N}\cdot\frac{2p-2}{p}}\|u\|^{2p-2}\right) - \|f\|_{H^*}\|u\|.
\end{aligned}
\tag{3.6}
$$

Note that there exists $\rho > 0$ such that $\theta := \frac{1}{4}\rho^2 - \rho\|f\|_{H^*} > 0$. Since $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $\lambda_k \to \infty$ as $k \to \infty$, we can find k_0 sufficiently large such that

$$\frac{1}{2} - C\lambda_{k_0}^{-\frac{s(N+\alpha)}{2N}\cdot\frac{2p-2}{p}}\rho^{2p-2} \leq \frac{1}{4}.$$

Thus, for any $u \in Y$ with $\|u\| = \rho$, we conclude from (3.6) that $E(u) \geq \theta$.

(ii) When $p = \frac{N+\alpha}{N}$, it follows that $\frac{2Np}{N+\alpha} = 2$. By Lemmas 2.1 and 2.3,

$$
\begin{aligned}
E(u) &\geq \frac{1}{2}\|u\|^2 - C|u|_{L^2}^{2p} - \|f\|_{H^*}\|u\| \\
&\geq \|u\|^2 \left(\frac{1}{2} - C|u|_{L^2}^{2p-2}\right) - \|f\|_{H^*}\|u\| \\
&\geq \|u\|^2 \left(\frac{1}{2} - C\lambda_{k_0}^{p-1}\|u\|^{2p-2}\right) - \|f\|_{H^*}\|u\|.
\end{aligned}
\tag{3.7}
$$

Then by similar arguments as those in (i), there also exist $\rho > 0$ and $\theta > 0$ such that $E(u) \geq \theta$ for all $u \in Y$ with $\|u\| = \rho$.

To sum up, the proof is completed. □

Lemma 3.3. *Let ρ be defined in Lemma 3.2, and $\{X_n\}_{n \geq 1} \subset H$ containing X be an increasing sequence of finite dimensional subspace with $\lim_{n \to \infty} \dim X_n = \infty$. Then for each n, there exists $R_n > \rho$ such that*

$$\sup_{u \in X_n, \|u\| = R_n} E(u) \leq 0.$$

Proof. For each n, define $\mathcal{S}_n = \{u \in X_n : \|u\| = 1\}$ and $d_n = \min_{u \in \mathcal{S}_n} \mathbb{D}(u)$. Since X_n is a finite dimensional subspace, the set $\mathcal{S}_n$ is compact and by Lemma 2.2, d_n can be achieved and $d_n > 0$. Then for any $R > 0$ and $u \in X_n$ with $\|u\| = R$, it holds that

$$E(u) \leq \frac{1}{2}\|u\|^2 - \frac{1}{2p}\mathbb{D}(u) + \|f\|_{H^*}\|u\| \leq \frac{1}{2}R^2 - \frac{1}{2p}d_n R^{2p} + \|f\|_{H^*}R.$$

Therefore, there is $R_n > \rho$ large enough such that $E(u) \leq 0$ for $\|u\| = R_n$. The lemma follows. □

Proof of Theorem 1.1 (i). It is a direct consequence of Lemmas 3.1–3.3 and Theorem 2.7. So equation (1.1) has a sequence of virtual critical values, and Theorem 1.1 (i) follows. □

3.2 Case $V = \text{const}$

Let $V(x) \equiv V_0 > 0$ be a constant function. In order to prove Theorem 2.8, we first show the following lemma. Denote by $\{v_i\}_i$ the orthogonal basis of H.

Lemma 3.4. *Let $X_k = \text{span}\{v_1, v_2, \cdots, v_k\} \subset H$ with $\dim X_k < \infty$ and Y_k be the orthogonal complement of X_k in H. Then*

$$\limsup_{k \to \infty} \sup_{u \in Y_k} \frac{\mathbb{D}(u)}{\|u\|^{2p}} = 0.$$

Proof. To this end, we denote $\gamma_k = \sup_{u \in Y_k, \|u\| = 1} \mathbb{D}(u)$. Then

(i) $\infty > \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k \geq \gamma_{k+1} \geq \cdots > 0$;

(ii) For each $k \geq 1$, there exists $u_k \in Y_k$ with $\|u_k\| = 1$ such that $\gamma_k = \mathbb{D}(u_k)$.

Clearly, since $Y_{k+1} \subset Y_k$ for $k \geq 1$, (i) is trivial. In addition, (ii) follows by using minimizing method. In fact, for each $k \geq 1$, there exists a sequence $\{u_j^k\}_{j \geq 1} \in Y_k$ such that $\|u_j^k\| = 1$ and $\mathbb{D}(u_j^k) \to \gamma_k$ as $j \to \infty$. Up to a subsequence, $u_j^k \rightharpoonup u_k$ in H and then $u_j^k \to u_k$ in $L^s(\mathbb{R}^N)$ with $s \in \left(2, \frac{2N}{N-2}\right)$ as $j \to \infty$. By Lemma 2.2, we conclude that $\|u_k\| \leq 1$ and $\mathbb{D}(u_k) = \gamma_k$. It suffices to prove $\|u_k\| = 1$. Otherwise $\|u_k\| < 1$. By scaling, set $u_k = \lambda u_k^*$ with $\|u_k^*\| = 1$. Then $\lambda < 1$. Thus

$$\gamma_k = \mathbb{D}(u_k) = \mathbb{D}(\lambda u_k^*) = \lambda^{2p}\mathbb{D}(u_k^*) < \gamma_k,$$

a contradiction. So $\|u_k\| = 1$ and (ii) holds.

In view of (ii), we can choose a subsequence $\{u_{n_k}\}$ of $\{u_k\}_{k \geq 1}$ with $\|u_{n_k}\| = 1$ such that

$$\gamma_{n_k} = \mathbb{D}(u_{n_k}) \quad \text{and} \quad X_{n_k} \oplus \{u_{n_k}\} \subset X_{n_{k+1}}.$$

Clearly, $(u_{n_i}, u_{n_j}) = 0$ for each $i \neq j$. Note that up to a subsequence, $u_{n_k} \rightharpoonup u^*$ in H as $k \to \infty$. We further claim that $u^* = 0$. In fact, suppose by contradiction that $u^* \neq 0$. Then $\lim_{k \to \infty}(u^*, u_{n_k}) = \|u^*\|^2 > 0$. However according to the Parseval identity,

$$\infty > \|u^*\|^2 \geq \sum_{k \geq 1} \|(u^*, u_{n_k})u_{n_k}\|^2 = \sum_{k \geq 1} |(u^*, u_{n_k})|^2 \to \sum_{k \geq 1} \|u^*\|^2 = +\infty,$$

which yields a contradiction. Thus, the claim holds and then $u_{n_k} \to 0$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. By Lemma 2.2,

$$\gamma_{n_k} = \mathbb{D}(u_{n_k}) \to 0.$$

Therefore by (i), $\lim_{k \to \infty} \gamma_k = \lim_{k \to \infty} \gamma_{n_k} = 0$. The proof is completed. $\square$

Proof of Theorem 1.1 (ii). When V is a constant function, Lemmas 3.1 and 3.3 are also valid. According to Theorem 2.7, it suffices to prove Lemma 3.2. Let $\rho > 0$ be such that $\theta := \frac{1}{4}(\rho^2 - \rho\|f\|_{H^*}) > 0$. Note that for any $u \in Y$,

$$E(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2p}\mathbb{D}(u) - \|f\|_{H^*}\|u\|$$

$$= \|u\|^2 \left(\frac{1}{2} - \frac{1}{2p}\frac{\mathbb{D}(u)}{\|u\|^{2p}}\|u\|^{2p-2} \right) - \|f\|_{H^*}\|u\|.$$

According to Lemma 3.4, there is $k_0 \in \mathbb{N}$ sufficiently large such that for any $k \geq k_0$,

$$\frac{1}{2} - \frac{1}{2p}\frac{\mathbb{D}(u)}{\|u\|^{2p}}\|u\|^{2p-2} \leq \frac{1}{4} \quad \text{if } u \in Y_k \text{ with } \|u\| = \rho.$$

Thus, by Theorem 2.7 again and symmetric criticality principle [15, Theorem 1.28], equation (1.1) has a sequence of virtual critical values, and Theorem 1.1 (ii) follows. $\square$

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References

[1] S. Cingolani, M. Clapp, S. Secchi, Multiple solutions to a magnetic nonlinear Choquard equation, *Z. Angew. Math. Phys.* **63**(2012), No. 2, 233–248. https://doi.org/10.1007/s00033-011-0166-8; MR2912346

[2] I. Ekeland, N. Ghoussoub, Z_2-equivariant Ljusternik–Schnirelman theory for non-even functionals, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15**(1998), No. 3, 341–370. https://doi.org/10.1016/s0294-1449(98)80121-2; MR1629353

[3] M. Ghimenti, J. Van Schaftingen, Nodal solutions for the Choquard equation, *J. Funct. Anal.* **271**(2016), No. 1, 107–135. https://doi.org/10.1016/j.jfa.2016.04.019; MR3494244

[4] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard nonlinear equation, *Stud. Appl. Math.* **57**(1976/77), No. 2, 93–105. https://doi.org/10.1007/978-3-642-55925-9_37; MR0471785

[5] E. H. Lieb, M. Loss, *Analysis*, Graduate Studies in Mathematics, Vol. 14, American Mathematical Society, Providence, Rhode Island, 2001.

[6] P. L. Lions, The Choquard equation and related questions, *Nonlinear Anal.* **4**(1980), No. 6, 1063–1073. https://doi.org/10.1016/0362-546X(80)90016-4; MR591299

[7] C. Mercuri, V. Moroz, J. Van Schaftingen, Groundstates and radial solutions to nonlinear Schrödinger–Poisson–Slater equations at the critical frequency, *Calc. Var. Partial Differential Equations* **55**(2016), No. 6, 1–58. https://doi.org/10.1007/s00526-016-1079-3; MR3568051

[8] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, *Trans. Amer. Math. Soc.* **367**(2015), No. 9, 6557–6579. https://doi.org/10.1090/s0002-9947-2014-06289-2; MR3356947

[9] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay estimates, *J. Funct. Anal.* **265**(2013), No. 2, 153–184. https://doi.org/10.1016/j.jfa.2013.04.007; MR3056699

[10] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, *Arch. Ration. Mech. Anal.* **195**(2010), No. 2, 455–467. https://doi.org/10.1007/s00205-008-0208-3; MR2592284

[11] D. Ruiz, J. Van Schaftingen, Odd symmetry of least energy nodal solutions for the Choquard equation, *J. Differential Equations* **264**(2018), No. 2, 1231–1262. https://doi.org/10.1016/j.jde.2017.09.034; MR3720843

[12] A. Salvatore, Some multiplicity results for a superlinear elliptic problem in $\mathbb{R}^N$, *Topol. Meth. Nonlinear Anal.* **21**(2003), No. 1, 29–39. https://doi.org/10.12775/TMNA.2003.002; MR1980134

[13] J. Van Schaftingen, J. K. Xia, Choquard equations under confining external potentials, *NoDEA Nonlinear Differential Equations Appl.* **24**(2017), No. 1, 1–24. https://doi.org/10.1007/s00030-016-0424-8; MR3582827

[14] T. Wang, T. S. Yi, Uniqueness of positive solutions of the Choquard type equations, *Appl. Anal.* **96**(2017), No. 3, 409–417. https://doi.org/10.1080/00036811.2016.1138473; MR3592922

[15] M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007

[16] T. Xie, L. Xiao, J. Wang, Existence of multiple positive solutions for Choquard equation with perturbation, *Adv. Math. Phys.* **2015**(2015), Art. No. 760157, 1–10. https://doi.org/10.1155/2015/760157; MR3407033

[17] M. B. Yang, B. R. Li, Solitary waves for non-homogeneous Schrödinger–Maxwell system, *Appl. Math. Comput.* **215**(2009), No. 1, 66–70. https://doi.org/10.1016/j.amc.2009.04.038; MR2568305

[18] H. ZHANG, J. X. XU, F. B. ZHANG, Bound and ground states for a concave–convex generalized Choquard equation, *Acta Appl. Math.* **147**(2017), No. 1, 81–93. https://doi.org/10.1007/s10440-016-0069-y; MR3592796