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Solutions to an anisotropic system via sub-supersolution method and Mountain Pass Theorem

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Abstract. We use the sub-supersolution method and the Mountain Pass Theorem in order to show existence and multiplicity of solutions for the quasilinear system given by

$$\begin{cases} -\left[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] = a_1(x)u + F_u(x,u,v) & \text{in } \Omega, \\ -\left[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left|\frac{p_i\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\right] = a_2(x)v + F_v(x,u,v) & \text{in } \Omega, \\ u,v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where a_j, $j = 1,2$ are functions in $L^\infty(\Omega)$ and and F_u and F_v are continuous functions on $\Omega \times \mathbb{R}^2$.

Keywords: anisotropic operator, sub-supersolution method, Mountain Pass Theorem.

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1 Introduction

In this paper we are concerned with existence and multiplicity of positive solutions for the following class of system nonlinear boundary value anisotropic problems given by

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big)\Big] = a_1(x)u + F_u(x,u,v) & \text{in } \Omega, \\ -\Big[\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial v}{\partial x_i}\Big|^{p_i-2}\frac{\partial v}{\partial x_i}\Big)\Big] = a_2(x)u + F_v(x,u,v) & \text{in } \Omega, \\ \quad u,v > 0 \text{ in } \Omega, \\ \quad u,v \in W_0^{1,\overrightarrow{p}}(\Omega), \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with smooth boundary, $N \geq 3$, $\overrightarrow{p} = (p_1, \ldots, p_N)$, $p_i > 1$, $\sum_{i=1}^{N} \frac{1}{p_i} > 1$, $p_1 < p_2 < \cdots < p_N < p^* := \frac{N\overline{p}}{N-\overline{p}}$. In this paper $\overline{p}$ denotes the harmonic mean

$$\overline{p} = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i}}.$$

For $j = 1, 2$, $a_j \geq 0$ is a nontrivial mensurable function. More precisely, we will suppose that the function a_j satisfy the following assumption:

(H) The function $a_j \in L^\infty(\Omega)$ with $a_j(x) > 0$.

In this paper F is a function on $\Omega \times \mathbb{R}^2$ of class C^1 satisfying

(H_1) There is $\delta > 0$ such that

$$F_s(x, s, t) \geq (1 - s)a_1(x), \quad \text{for every } 0 \leq s \leq \delta, \text{ a.e. in } \Omega,$$

and

$$F_t(x, s, t) \geq (1 - t)a_2(x), \quad \text{for every } 0 \leq t \leq \delta, \text{ a.e. in } \Omega.$$

(H_2) There is $1 < r < p^*$ such that

$$F_s(x, s, t) \leq a_1(x)(s^{r-1} + t^{r-1} + 1), \quad \text{for every } 0 \leq s,$$

and

$$F_t(x, s, t) \leq a_2(x)(s^{r-1} + t^{r-1} + 1), \quad \text{for every } 0 \leq t.$$

Thus, in order to show existence and multiplicity of solutions to problem (1.1), we define the Sobolev space $E = W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{p}}(\Omega)$ endowed with the norm

$$\|(u, v)\| = \|u\|_{1,\overrightarrow{p}} + \|v\|_{1,\overrightarrow{p}},$$

where

$$\|u\|_{1,\overrightarrow{p}} = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}.$$

We say that $u, v \in E$ is a positive weak solution of (1.1) if $u, v > 0$ in Ω and it verifies

$$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega a_1(x) u \varphi \, dx + \int_\Omega F_u(x, u, v) \varphi \, dx,$$

and

$$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_i} \, dx = \int_\Omega a_2(x) v \psi \, dx + \int_\Omega F_v(x, u, v) \psi \, dx,$$

for all $\varphi, \psi \in W_0^{1,\overrightarrow{p}}(\Omega)$.

In our first theorem we apply the sub-supersolution method to establish the existence of a weak solution for (1.1).

Theorem 1.1. *Assume that conditions (H), (H_1) and (H_2) hold. If $\|a_j\|_\infty$ is small, for $j = 1, 2$, then system (1.1) has a positive weak solution.*

In order to establish the existence of two solutions for problem (1.1), we also assume

(H_3) There are $s_0, t_0 > 0$ such that

$$0 < F(x,s,t) \leq \theta_s\, s\, F_s(x,s,t) + \theta_t\, t\, F_t(x,s,t), \text{ a.e in } \Omega, \text{ for all } t \geq t_0 \text{ and } s \geq s_0 \text{ in } \Omega,$$

where $\frac{1}{p^*} < \theta_s, \theta_t < \frac{1}{p_N}$.

Theorem 1.2. *Assume that conditions (H), (H_1)–(H_3) hold. Then, problem (1.1) has two positive weak solutions if $\|a_j\|_\infty$ is small, for $j = 1, 2$.*

A considerable effort has been devoted during the last years to the study anisotropic problems. With no hope to be thorough, let us mention, for example [1, 2, 4–7, 9–14, 16, 20–22] and references therein.

In some sense our paper is a natural continuation of the studies initiated in [2] and it completes the results obtained there, because we study the existence and multiplicity of solutions for a system involving an anisotropic operator using subsolution & supersolution method. This paper seems to be the first to show results on an elliptic system involving an anisotropic operator.

When $p_i = 2$ we have $\Big[\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big)\Big] = \Delta u$ and when $p_i = p$ we have $\Big[\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big)\Big] = \Delta_p u$. Both cases are called isotropic cases or non-anisotropic cases and this kind of problem has been studied by many authors.

This paper is organized as follows. In the Section 2 we prove the unicity of solutions for the Linear anisotropic problem, a Comparison Principle and a regularity result for solutions to this class of problems. In the Section 3 we prove Theorem 1.1. Theorem 1.2 is proved in Section 4.

2 Technical results

We start proving a result of unicity of solution to the linear problem and a Comparison Principle of the anisotropic operator.

Lemma 2.1. *There is $u \in W_0^{1,\vec{p}}(\Omega)$ the unique solution of problem*

$$\begin{cases} -\Big[\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial w}{\partial x_i}\Big|^{p_i-2}\frac{\partial w}{\partial x_i}\Big)\Big] = a(x) \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

Proof. Consider the operator $T : W_0^{1,\vec{p}}(\Omega) \to (W_0^{1,\vec{p}}(\Omega))'$ such that $\langle Tu, \phi\rangle$ is given by

$$\langle Tu, \phi\rangle = \sum_{i=1}^N \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\,dx.$$

Since the inequality

$$C_i\Big|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\Big|^{p_i} \leq \Big\langle \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i} - \Big|\frac{\partial v}{\partial x_i}\Big|^{p_i-2}\frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\Big\rangle \tag{2.2}$$

is true for some $C_i > 0$ and for all $i = 1, \ldots, N$, we have that

$$\langle Tu - Tv, u - v\rangle > 0 \quad \text{for all } u, v \in W_0^{1,\vec{p}}(\Omega) \text{ with } u \neq v.$$

Moreover, if $\|u\| \to +\infty$, then, without loss of generality, we can assume that

$$\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \geq 1, \quad \text{for all } i = 1, 2, \ldots, N.$$

Hence, since $1 < p_1 \leq p_i$, for all $i = 1, 2, \ldots, N$, we have

$$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \geq \sum_{i=1}^{N} \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{p_1}{p_i}} \geq \frac{1}{N^{p_1 - 1}} \left(\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{L^{p_i}} \right)^{p_1},$$

which implies

$$\lim_{\|u\| \to \infty} \frac{\langle Tu, u \rangle}{\|u\|} = +\infty.$$

Thus, by Minty-Browder's Theorem [8, Théorème 5.16], there exists a unique $u \in W_0^{1,\vec{p}}(\Omega)$ that satisfies $Tu = a(x)$. □

Lemma 2.2. *If Ω is a bounded domain and if $u, v \in W_0^{1,\vec{p}}(\Omega)$ satisfy*

$$\begin{cases} -\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right) \right] \leq -\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p_i - 2} \frac{\partial v}{\partial x_i} \right) \right] & \text{in } \Omega, \\ u \leq v \quad \text{on } \partial\Omega, \end{cases}$$

then $u \leq v$ a.e. in Ω.

Proof. Taking $0 \leq \phi = \max\{u - v, 0\} \in W_0^{1,\vec{p}}(\Omega)$ as a test function, we obtain

$$\int_{\Omega \cap [u > v]} \sum_{i=1}^{N} \left\langle \left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{p_i - 2} \frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right\rangle dx \leq 0.$$

From inequality (2.2), we conclude that $\|(u - v)^+\| \leq 0$, this implies $u \leq v$ a.e. in Ω. □

Before proving the L^∞-regularity we enunciate an iteration lemma by Stampacchia that we will use.

Lemma 2.3 (See [18]). *Assume that $\phi : [0, \infty) \to [0, \infty)$ is a nonincreasing function such that if $h > k > k_0$, for some $\alpha > 0, \beta > 1$, $\phi(h) \leq C(\phi(k))^\beta / (h - k)^\alpha$. Then $\phi(k_0 + d) = 0$, where $d^\alpha = c 2^{\frac{\alpha\beta}{\beta - 1}} \phi(k_0)^{\beta - 1}$.*

Lemma 2.4. *Let $v \in W_0^{1,\vec{p}}(\Omega)$ be a solution to problem*

$$\begin{cases} -\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial v}{\partial x_i} \right|^{p_i - 2} \frac{\partial v}{\partial x_i} \right) \right] = f & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

such that $f \in L^r(\Omega)$ with $r > p^*/(p^* - p_1)$. Then $v \in L^\infty(\Omega)$. In particular, if $\|f\|_r$ is small, then also $\|v\|_\infty$ is small.

Proof. Consider $v_k = \text{sign}(u)(|u|-k)^+$, then $v_k \in W_0^{1,\overrightarrow{p}}(\Omega)$ and $\frac{\partial v}{\partial x_i} = \frac{\partial v_k}{\partial x_i}$ in $A(k) = \{x \in \Omega : |u(x)| > k\}$. Let $|A(k)|$ be the Lebesgue measure of $A(k)$. Using v_k as test function and the Hölder inequality, we have

$$\sum_{i=1}^{N} \int_{A(k)} \left| \frac{\partial v_k}{\partial x_i} \right|^{p_i} dx = \int_\Omega f v_k dx \leq \left(\int_\Omega |v_k|^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_\Omega |f|^r dx \right)^{\frac{1}{r}} |A(k)|^{1-\left(\frac{1}{p^*}+\frac{1}{r}\right)}.$$

Let

$$0 < S = \inf_{u \in D^{1,\overrightarrow{p}}(\mathbb{R}^N), \|u\|_{p^*}=1} \left\{ \sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}^{p_i} \right\}, \quad \text{see [12]}.$$

Once that $p_i \geq p_1 > 1$, we have

$$S\left(\int_\Omega |u|^{p^*} dx \right)^{\frac{p_1}{p^*}} \leq \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx, \quad \text{for all } u \in W_0^{1,\overrightarrow{p}}(\Omega).$$

This implies

$$S\left(\int_{A(k)} |v_k|^{p^*} dx \right)^{\frac{p_1-1}{p^*}} \leq \left(\int_\Omega |f|^r dx \right)^{\frac{1}{r}} |A(k)|^{1-\left(\frac{1}{p^*}+\frac{1}{r}\right)}.$$

Note that if $0 < k < h$, $A(h) \subset A(k)$ and

$$|A(h)|^{\frac{1}{p^*}}(h-k) = \left(\int_{A(h)} (h-k)^{p^*} dx \right)^{\frac{1}{p^*}} \leq \left(\int_{A(k)} |v_k|^{p^*} dx \right)^{\frac{1}{p^*}},$$

then

$$|A(h)| \leq \frac{1}{(h-k)^{p^*}} \frac{1}{S^{\frac{p^*}{p_1-1}}} \|f\|_r^{\frac{p^*}{p_1-1}} |A(k)|^{\frac{p^*}{p_1-1}\left[1-\left(\frac{1}{p^*}+\frac{1}{r}\right)\right]}.$$

Since $r > \frac{p^*}{p^*-p_1}$, we have $\beta := \frac{p^*}{p_1-1}\left[1 - \left(\frac{1}{p^*} + \frac{1}{r}\right)\right] > 1$. Therefore, if we define

$$\phi(h) = |A(h)|, \qquad \alpha = p^*, \qquad \beta = \frac{p^*}{p_1 - 1}\left[1 - \left(\frac{1}{p^*} + \frac{1}{r}\right)\right], \qquad k_0 = 0,$$

we have that ϕ is a nonincreasing function and

$$\phi(h) \leq \frac{C}{(h-k)^\alpha} \phi(k)^\beta, \quad \text{for all } h > k > 0.$$

By Lemma 2.3, we have $\phi(d) = 0$ for $d = c\|f\|_r^{\frac{1}{p_1-1}} |\Omega|^{\frac{\beta-1}{\alpha}} / S^{\frac{1}{p_1-1}}$, then

$$\|u\|_\infty \leq \frac{c\|f\|_r^{\frac{1}{p_1-1}} |\Omega|^{\frac{\beta-1}{\alpha}}}{S^{\frac{1}{p_1-1}}}. \qquad \square$$

3 Proof of Theorem 1.1

We say that $[(\underline{u}, \underline{v}), (\overline{u}, \overline{u})]$ is a pair of sub and supersolution for the problem (1.1), respectively, if $\underline{u}, \underline{v} \in E \cap L^\infty(\Omega)$, $\overline{u}, \overline{u} \in E \cap L^\infty(\Omega)$

a) $\underline{u} \leq \overline{u}, \underline{v} \leq \overline{v}$ *in* Ω and $\underline{u} = 0 \leq \overline{u}, \underline{v} = 0 \leq \overline{v}$ on $\partial\Omega$,

b) Given φ, ψ, with $\varphi, \psi \geq 0$, we have

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \leq \int_\Omega a_1(x)\underline{u}\varphi + \int_\Omega F_u(x, \underline{u}, w)\, \varphi\, dx \text{ for all } w \in [\underline{v}, \overline{v}] \\ \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \underline{v}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{v}}{\partial x_i} \frac{\partial \psi}{\partial x_i} \leq \int_\Omega a_2(x)\underline{v}\psi + \int_\Omega F_v(x, w, \underline{v})\, \psi\, dx \text{ for all } w \in [\underline{u}, \overline{u}] \end{cases} \tag{3.1}$$

$$\begin{cases} \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \overline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \geq \int_\Omega a_1(x)\overline{u}\, \varphi + \int_\Omega F_u(x, \overline{u}, w)\, \varphi\, dx \text{ for all } w \in [\underline{v}, \overline{v}] \\ \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \overline{v}}{\partial x_i} \right|^{p_i-2} \frac{\partial \overline{v}}{\partial x_i} \frac{\partial \psi}{\partial x_i} \geq \int_\Omega a_2(x)\overline{v}\, \psi + \int_\Omega F_v(x, w, \overline{v})\, \psi\, dx \text{ for all } w \in [\underline{u}, \overline{u}] \end{cases} \tag{3.2}$$

Lemma 3.1. *Assume that* (H), (H_1) *and* (H_2) *hold. If* $\|a_j\|_\infty$ *is small, for* $j = 1, 2$, *then there exist* $\underline{u}, \underline{v}, \overline{u}, \overline{v} \in E \bigcap L^\infty(\Omega)$ *such that*

i) $\|(\underline{u}, \underline{v})\|_\infty \leq \delta$, *where* δ *is the constant that appeared in the hypothesis* (H_1).

ii) $0 < \underline{u}(x) \leq u(x)$ *a.e in* Ω *and* $0 < \underline{v}(x) \leq \overline{v}(x)$ *a.e in* Ω.

iii) $(\underline{u}, \underline{v})$ *is a subsolution and* $(\overline{u}, \overline{v})$ *is a supersolution of* (1.1).

Proof. By Lemma 2.1, there is a unique positive solution $\underline{u} \in W_0^{1,\overrightarrow{p}}(\Omega)$ satisfying the problem below

$$\begin{cases} -\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} \right) \right] = a_1(x) \quad \text{in } \Omega, \\ \underline{u} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Similary, there exists a unique positive solution $\underline{v} \in W_0^{1,\overrightarrow{p}}(\Omega)$ satisfying

$$\begin{cases} -\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \underline{v}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{v}}{\partial x_i} \right) \right] = a_2(x) \quad \text{in } \Omega, \\ \underline{v} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.4, $\underline{u}, \underline{v} \in L^\infty(\Omega)$ and there exist $C_1, C_2 > 0$ such that $\|\underline{u}\|_\infty \leq C_1\|a\|_\infty$ and $\|\underline{v}\|_\infty \leq C_2\|a\|_\infty$. Now we fix $\|a_j\|_\infty$, with $j = 1, 2$ so that

$$\|\underline{u}\|_\infty \leq \frac{\delta}{2} \quad \text{and} \quad \|\underline{v}\|_\infty \leq \frac{\delta}{2},$$

which ends the proof of the condition (i).

In order to prove *ii)*, we invoke Lemma 2.1 one more time to show that there exists a unique positive solution $\overline{u} \in W_0^{1,\overrightarrow{p}}(\Omega) \bigcap L^\infty(\Omega)$

$$\begin{cases} -\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial \overline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i} \right) \right] = 1 + a_1(x) \quad \text{in } \Omega, \\ \overline{u} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.3}$$

and there exists a unique positive solution $\overline{v} \in W_0^{1,\overrightarrow{p}}(\Omega)$

$$\begin{cases} -\left[\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial \overline{v}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{v}}{\partial x_i}\right)\right] = 1 + a_2(x) \quad \text{in } \Omega, \\ \overline{v} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Note that, for all $0 \leq \varphi, \psi \in W_0^{1,\overrightarrow{p}}(\Omega)$, we have

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\, dx = \int_\Omega [a_1(x)+1]\varphi\, dx \geq \int_\Omega a_1(x)\,\varphi\, dx = \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\, dx$$

and

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \overline{v}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i}\, dx = \int_\Omega [a_2(x)+1]\psi\, dx \geq \int_\Omega a_2(x)\,\psi\, dx = \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \underline{v}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i}\, dx.$$

Then, from Lemma 2.2 we conclude that $\underline{u}(x) \leq \overline{u}(x)$ a.e. in Ω and $\underline{v}(x) \leq \overline{v}(x)$ a.e. in Ω, which proves the condition *ii)*.

Our final task is to check that the condition *iii)* holds. First, we use the maximum principle in [9, Corollary 4.4] and conclude that $\underline{u}, \underline{v} > 0$. Now using the definition of $\underline{u}$, $\underline{v}$ and (H_1), we obtain, for each $\varphi, \psi \geq 0$,

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} - \int_\Omega a_1(x)\underline{u}\,\varphi dx - \int_\Omega F_u(x,\underline{u},v)\,\varphi\, dx \\ &\quad \leq \int_\Omega a_1(x)\varphi dx - \int_\Omega a_1(x)\,\underline{u}\,\varphi dx - \int_\Omega (1-\underline{u})a_1(x)\varphi\, dx \\ &\quad = 0 \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \underline{v}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} - \int_\Omega a_2(x)\underline{v}\,\psi dx - \int_\Omega F_v(x,u,\underline{v})\,\psi\, dx \\ &\quad \leq \int_\Omega a_2(x)\varphi dx - \int_\Omega a_2(x)\,\underline{v}\,\psi dx - \int_\Omega (1-\underline{v})a_1(x)\psi\, dx \\ &\quad = 0 \end{aligned}$$

Then, $(\underline{u}, \underline{v})$ is a subsolution for problem (1.1).

Now, we use (H_2), (3.3) and (3.4) we have for $\|a_j\|_\infty$ sufficiently small such that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \varphi}{\partial x_i} - \int_\Omega a_1(x)\overline{u}\,\varphi - \int_\Omega F_u(x,\overline{u},v)\,\varphi\, dx \\ &\quad \geq \left(1 - \|a_1\|_\infty\|\overline{u}\|_\infty - \|a_1\|_\infty - \|a_1\|_\infty\|\overline{u}\|_\infty^{r-1} - \|a_1\|_\infty\|v\|_\infty^{r-1}\right)\int_\Omega \varphi\, dx > 0 \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial \overline{v}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \psi}{\partial x_i} - \int_\Omega a_2(x)\overline{v}\,\psi - \int_\Omega F_v(x,u,\overline{v})\,\psi\, dx \\ &\quad \geq \left(1 - \|a_2\|_\infty\|\overline{u}\|_\infty - \|a_2\|_\infty - \|a_2\|_\infty\|u\|_\infty^{r-1} - \|a_2\|_\infty\|\overline{v}\|_\infty^{r-1}\right)\int_\Omega \psi\, dx > 0 \end{aligned}$$

Then $\overline{u}, \overline{v}$ is a supersolution of (1.1). □

Consider the functions

$$G_s(x,s,t)=\begin{cases} a_1(x)\overline{u}(x)+F_s(x,\overline{u}(x),t), & s>\overline{u}(x)\\ a_1(x)s+F_s(x,s,t), & \underline{u}(x)\le s\le \overline{u}(x)\\ a_1(x)\underline{u}(x)+F_s(x,\underline{u}(x),t), & s<\underline{u}(x),\end{cases} \tag{3.5}$$

and

$$G_t(x,s,t)=\begin{cases} a_2(x)\overline{v}(x)+F_t(x,s,\overline{v}(x)), & t>\overline{v}(x)\\ a_2(x)t+F_t(x,s,t), & \underline{v}(x)\le t\le \overline{v}(x)\\ a_2(x)\underline{v}(x)+F_t(x,s,\underline{v}(x)), & t<\underline{v}(x),\end{cases} \tag{3.6}$$

and the auxiliary problem

$$\begin{cases} -\left[\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\right]=G_u(x,u,v) \text{ in } \Omega,\\ \quad u>0 \text{ in } \Omega,\\ -\left[\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\right)\right]=G_v(x,u,v) \text{ in } \Omega,\\ \quad u,v>0 \text{ in } \Omega,\\ \quad u,v\in W_0^{1,\overrightarrow{p}}(\Omega).\end{cases} \tag{3.7}$$

We define the functional $\Phi : E\to\mathbb{R}$ by

$$\Phi(u,v)=\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i=1}^{N}\int_\Omega \frac{1}{p_i}\left|\frac{\partial v}{\partial x_i}\right|^{p_i}dx-\int_\Omega G(x,u,v)dx. \tag{3.8}$$

We have $\Phi\in C^1(E,\mathbb{R})$ with

$$\begin{aligned}\Phi'(u,v)(\varphi,\psi)=&\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}\,dx+\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx\\ &-\int_\Omega G_u(x,u,v)\,\psi\,dx-\int_\Omega G_v(x,u,v)\,\varphi\,dx,\end{aligned}$$

for all $u,v,\psi,\varphi\in E$.

From (H_2) and definition of G_s and G_t, we have that

$$|G_s(x,s,t)|\le K_1, \quad \text{for some } K_1>0, \quad \text{a.e. in } \Omega \tag{3.9}$$

and

$$|G_t(x,s,t)|\le K_2, \quad \text{for some } K_2>0, \quad \text{a.e. in } \Omega. \tag{3.10}$$

From (3.9) and (3.10) , we have that Φ is coercive. Then, we can obtain that (u_n,v_n) is a bounded sequence in E such that

$$\Phi(u_n,v_n)\to c=\inf_{\mathcal{M}}\Phi,$$

where

$$\mathcal{M}=\left\{(u,v)\in E:\underline{u}\le u\le\overline{u} \text{ a.e. in } \Omega \text{ and } \underline{v}\le v\le\overline{v} \text{ a.e. in } \Omega\right\}.$$

Hence, up to subsequence, we have

$$\begin{cases} (u_n, v_n) \rightharpoonup (u,v) \text{ in } E, \\ (u_n, v_n) \to v \text{ in } L^s(\Omega) \times L^s(\Omega),\ 1 \le s < p^*, \\ (u_n(x), v_n(x)) \to (u(x), v(x)) \text{ a.e in } \Omega. \end{cases} \tag{3.11}$$

Now, note that $\mathcal{M}$ is closed and convex in E. By [19, Therem 1.2], the restriction $\Phi|_{\mathcal{M}}$ attains its infimum at a point (u,v) in $\mathcal{M}$. Using the same argument as in the proof of [19, Therem 2.4], we see that (u,v) weakly solves (3.7). Since $G_s(x,s,t) = a_1(x)s + F_s(x,s,t)$ for $s \in [\underline{u}, \overline{u}]$ and $G_t(x,s,t) = a_2(x)t + F_t(x,s,t)$ for $t \in [\underline{v}, \overline{v}]$ then (u,v) is a positive weak solution of (1.1).

4 Proof of Theorem 1.2

Let $(\underline{u}, \underline{v}) \in E \cap L^\infty(\Omega)$ the subsolution of Problema (1.1). In our next result we prove that the functional Φ satisfies the geometric hypotheses of the Mountain Pass Theorem (to see [3]).

Consider the functions

$$\widehat{G}_s(x,s,t) = \begin{cases} a_1(x)s + F_s(x,s,t), & s > \underline{u}(x) \\ a_1(x)\underline{u}(x) + F_s(x, \underline{u}(x), t), & s \le \underline{u}(x), \end{cases} \tag{4.1}$$

$$\widehat{G}_t(x,s,t) = \begin{cases} a_2(x)t + F_t(x,s,t), & t > \underline{v}(x) \\ a_2(x)\underline{v}(x) + F_t(x, \underline{v}(x), t), & t \le \underline{v}(x) \end{cases} \tag{4.2}$$

and define the functional $\widehat{\Phi} : W_0^{1,\overrightarrow{p}}(\Omega) \to \mathbb{R}$ by

$$\widehat{\Phi}(u,v) = \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx - \int_\Omega \widehat{G}(x,u,v) dx. \tag{4.3}$$

Note that by (H_2),(4.1) and (4.2) , we have

$$\widehat{G}_s(x,s,t) \le \widetilde{C_1}|t| + a_1(x)|s|^r + a_1(x)s|t|^r, \quad \text{for all } s \ge 0, \tag{4.4}$$

and

$$\widehat{G}_t(x,s,t) \le \widetilde{C_2}|t| + a_2(x)|t|^r + a_1(x)t|s|^r, \quad \text{for all } t \ge 0, \tag{4.5}$$

for some constants $\widetilde{C_1}, \widetilde{C_2} > 0$.

Lemma 4.1. *The functional $\widehat{\Phi}$ satisfies the (PS)-condition for every $c \in \mathbb{R}$.*

Proof. Let $(u_n, v_n) \subset E$ be a sequence such that

$$\widehat{\Phi}(u_n, v_n) \to c \ \text{ and } \ \widehat{\Phi}'(u_n, v_n) \to 0. \tag{4.6}$$

Using (H_3) and Sobolev's embedding, there are $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1 + \|(u_n, v_n)\| &\ge \widehat{\Phi}(u_n, v_n) - \left[\theta_{u_n} \hat{\Phi}'(u_n, v_n)(u_n, 0) + \theta_{v_n} \hat{\Phi}'(u_n, v_n)(0, v_n)\right] \\ &\ge C_2 \|(u_n, v_n)\|^{p_1}, \end{aligned} \tag{4.7}$$

where get that (u_n, v_n) is a bounded sequence in E and hence, up to subsequence, we have

$$\begin{cases} (u_n, v_n) \rightharpoonup (u, v) & \text{in } E, \\ (u_n, v_n) \to (u, v) & \text{in } L^s(\Omega) \times L^s(\Omega),\ 1 \leq s < p^*, \\ (u_n(x), v_n(x)) \to (u(x), v(x)) & \text{a.e. in } \Omega. \end{cases} \tag{4.8}$$

Using (4.6), (4.8), (2.2), the Lebesgue dominated convergence theorem and standard arguments, up to subsequence, we obtain

$$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right|^{p_i} \leq o_n(1),$$

and

$$\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^{p_i} \leq o_n(1),$$

which implies $(u_n, v_n) \to (u, v)$ in E. □

Lemma 4.2. *Assume that (H), (H_1)–(H_3) hold. Then for $\|a_j\|_{L^\infty}$ small, for $j = 1, 2$, $\widehat{\Phi}$ satisfies:*

i) There are $R > \|(\underline{u}, \underline{v})\|$ and $\beta > 0$, such that

$$\widehat{\Phi}(\underline{u}, \underline{v}) < 0 < \beta \leq \inf_{(u,v) \in \partial B_R(0)} \widehat{\Phi}(u, v).$$

ii) There are $e \in W_0^{1,\overrightarrow{p}}(\Omega) \setminus B_{2R}(0)$ such that $\widehat{\Phi}(e) < \beta$.

Proof. Since $(\underline{u}, \underline{v})$ is a subsolution of (1.1), $\widehat{G_s}(x, \underline{u}, t) = \big(a_1(x)\underline{u} + F_s(x, \underline{u}, t)\big)\underline{u}$ and $\widehat{G_t}(x, s, \underline{v}) = \big(a_2(x)\underline{v} + F_t(x, s, \underline{v})\big)\underline{v}$, with $p_i > 1$, for $i = 1, 2, \ldots N$, we have

$$\begin{aligned} \widehat{\Phi}(\underline{u}, \underline{v}) = & \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i} dx + \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \underline{v}}{\partial x_i} \right|^{p_i} dx \\ & - \int_\Omega (a_1(x)\underline{u} + F_s(x, \underline{u}, t))\underline{u} dx - \int_\Omega (a_2(x)\underline{v} + F_s(x, s, \underline{v}))\underline{v}\, dx\,. \end{aligned} \tag{4.9}$$

Now, let $\|(u, v)\| = R > 1$, without loss of generality, we can assume that

$$\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \geq 1, \quad \text{for all } i = 1, 2, \ldots, N,$$

and

$$\int_\Omega \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx \geq 1, \quad \text{for all } i = 1, 2, \ldots, N.$$

Hence, using this inequality, (4.4) and (4.5) with the Sobolev Embedding Theorem, we find positive constants, such that

$$\begin{aligned} \widehat{\Phi}(u, v) \geq & K\|(u, v)\| - c_3\|a_1\|_{L^\infty(\Omega)}\|\underline{u}\|_{L^\infty(\Omega)}\|(u, v)\| - c_4\|a_1\|_{L^\infty(\Omega)}\|(u, v)\| \\ & - c_5\|a_1\|_{L^\infty(\Omega)}\|(u, v)\|^r - c_6\|a_2\|_{L^\infty(\Omega)}\|\underline{v}\|_{L^\infty(\Omega)}\|(u, v)\| \\ & - c_7\|a_2\|_{L^\infty(\Omega)}\|(u, v)\| - c_8\|a_2\|_{L^\infty(\Omega)}\|(u, v)\|^r - c_9\|a_1\|_{L^\infty(\Omega)}\|(u, v)\|^r \\ & - c_{10}\|a_1\|_{L^\infty(\Omega)}\|(u, v)\|^r - c_{11}\|a_2\|_{L^\infty(\Omega)}\|(u, v)\|^r \\ & - c_{12}\|a_2\|_{L^\infty(\Omega)}\|(u, v)\|^r, \quad \text{for all } \|(u, v)\| = R, \end{aligned} \tag{4.10}$$

where $K = \min\{\frac{k_1}{p_N}, \frac{k_2}{p_N}\}$. Note that, if $(u,v) \in \partial B_R(0)$ with $R > 1$ and for $\|a_j\|_{L^\infty(\Omega)}$ sufficiently small, with $j = 1,2$, there exists $\beta \in \mathbb{R}$ such that $\widehat{\Phi}(u,v) \geq \beta$, for all $(u,v) \in \partial B_R(0)$. Hence, the choices of β, R and $\|a_j\|_{L^\infty(\Omega)}$ combined with inequalities (4.9) and (4.10) result in

$$\widehat{\Phi}(\underline{u},\underline{v}) < 0 < \beta \leq \inf_{(u,v)\in\partial B_R(0)} \widehat{\Phi}(u,v),$$

which shows the condition *i)*.

Now, by definition of $\widehat{G}_s$ we have

$$\widehat{G}_{s\underline{u}}(x, s\underline{u}, 0) \geq F(x, s\underline{u}, 0) \quad \text{for all } s \geq 1, \text{ a.e. in } \Omega.$$

We invoke (H_1) and (4.3) to obtain

$$\widehat{\Phi}(s\underline{u}, 0) \leq \sum_{i=1}^{N} \frac{s^{p_N}}{p_1} \int_\Omega \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_N} - \int_\Omega F(x, s\underline{u}, 0)\, dx.$$

Using (H_3), there exists $\widetilde{K}_1 > 0$ such that

$$F(x,s,0) \geq \widetilde{K}_1 s^{\frac{1}{\theta_s}}, \quad \text{for all } s \geq \max\{1, s_0\},$$

where s_0 are the constants that appear in (H_3). Then,

$$\widehat{\Phi}(s\underline{u}, 0) \leq s^{p_N} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_1} \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i} - \widetilde{K}_1 s^{\frac{1}{\theta_s}} \int_\Omega |\underline{u}|^{\frac{1}{\theta_s}}\, dx.$$

Since $\frac{1}{p^*} < \theta_s < \frac{1}{p_N}$, we conclude that $\widehat{\Phi}(s\underline{u}, 0) \to -\infty$ as $s \to +\infty$. So, we may find $e = s_0(\underline{u}, 0) \in E$ such that $\|e\| > R$ and $\widehat{\Phi}(e) < \beta$, which satisfies the condition *ii)*. □

Proof of Theorem 1.2. Let $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ be the subsolution and the supersolution of (1.1) given in Lemma (3.1) and (u_1, v_1) the solution of (1.1) obtained in Theorem 1.1.

Using the Lemma 4.2, we conclude, with the Mountain Pass Theorem (see [3]), that

$$\widehat{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widehat{\Phi}(\gamma(t)), \quad \text{where } \Gamma = \{\gamma \in C\big([0,1], W_0^{1,\overrightarrow{p}}(\Omega)\big) : \gamma(0) = (\underline{u},\underline{v}),\ \gamma(1) = e\},$$

is critical value of $\widehat{\Phi}$.

By (3.5),(3.6),(4.1) and (4.2), $G_s(x,s,t) = \widehat{G}_s(x,s,t)$ for $s \in [0, \overline{u}]$ and $G_t(x,s,t) = \widehat{G}_t(x,s,t)$ for $t \in [0, \overline{v}]$, thus $\Phi(u,v) = \widehat{\Phi}(u,v)$ with $u \in [0, \overline{u}]$ and $v \in [0, \overline{v}]$, where Φ and $\widehat{\Phi}$ are given in (3.8) and (4.3), respectively. Then,

$$\widehat{\Phi}(u_1, v_1) = \inf_{\mathcal{M}} \Phi(u,v),$$

where

$$\mathcal{M} = \big\{(u,v) \in E : \underline{u} \leq u \leq \overline{u} \text{ a.e. in } \Omega \text{ and } \underline{v} \leq v \leq \overline{v} \text{ a.e. in } \Omega\big\}.$$

was given in the proof of in Theorem 1.1.

Therefore, the problem (1.1) has two weak solutions $v_1, v_2 \in W_0^{1,\overrightarrow{p}}(\Omega)$, such that

$$\widehat{\Phi}(u_1, v_1) \leq \widehat{\Phi}(\underline{v}) < 0 < \beta \leq \widehat{c} = \widehat{\Phi}(u_2, v_2).$$

Recall that $\underline{u} \leq u_1 \leq \overline{u}$ a.e. in Ω and $\underline{v} \leq v_1 \leq \overline{v}$ a.e. in Ω, thus $(u_1, v_1) > 0$. Now, we will show that $(u_2, v_2) > 0$.

Taking $((\underline{u}, \underline{v}) - (u_2, v_2))^+$, as test function and defining $\{(u_2, v_2) < (\underline{u}, \underline{v})\} := \{x \in \Omega : u_2(x) < \underline{u}(x) \text{ and } v_2(x) < \underline{v}(x)\}$, we have

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_2}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_2}{\partial x_i} \frac{\partial (\underline{u} - u_2)^+}{\partial x_i} dx + \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial v_2}{\partial x_i} \right|^{p_i - 2} \frac{\partial v_2}{\partial x_i} \frac{\partial (\underline{v} - v_2)^+}{\partial x_i} dx$$
$$= \int_{\{u_2 < \underline{u}\}} (a_1 \underline{u} + F_s(x, \underline{u}, t))(\underline{u} - u_2)^+ \, dx + \int_{\{v_2 < \underline{v}\}} (a_2 \underline{v} + F_t(x, s, \underline{v}))(\underline{v} - v_2)^+ \, dx. \quad (4.11)$$

Since $(\underline{u}, \underline{v})$ is a subsolution of (1.1), using (4.11) we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \underline{u}}{\partial x_i} \frac{\partial (\underline{u} - u_2)^+}{\partial x_i} dx - \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u_2}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_2}{\partial x_i} \frac{\partial (\underline{u} - u_2)^+}{\partial x_i} dx \leq 0$$

and

$$\sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial \underline{v}}{\partial x_i} \right|^{p_i - 2} \frac{\partial \underline{v}}{\partial x_i} \frac{\partial (\underline{v} - v_2)^+}{\partial x_i} dx - \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial v_2}{\partial x_i} \right|^{p_i - 2} \frac{\partial v_2}{\partial x_i} \frac{\partial (\underline{v} - v_2)^+}{\partial x_i} dx \leq 0.$$

From inequality (2.2), we conclude that $\|(\underline{u} - u_2)^+\|_{1,\overrightarrow{p}} \leq 0$ and $\|(\underline{v} - v_2)^+\|_{1,\overrightarrow{p}} \leq 0$, this implies $0 < \underline{u} \leq u_2$ a.e. in Ω and $0 < \underline{v} \leq v_2$ a.e. in Ω. We concluded that $(u_2, v_2) > 0$. □

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