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Three positive solutions of N-dimensional p-Laplacian with indefinite weight

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Abstract. This paper is concerned with the global behavior of components of positive radial solutions for the quasilinear elliptic problem with indefinite weight

$$\begin{aligned} &\operatorname{div}(\varphi_p(\nabla u)) + \lambda h(x) f(u) = 0, && \text{in } B, \\ &u = 0, && \text{on } \partial B, \end{aligned}$$

where $\varphi_p(s) = |s|^{p-2}s$, B is the unit open ball of $\mathbb{R}^N$ with $N \geq 1$, $1 < p < \infty$, $\lambda > 0$ is a parameter, $f \in C([0,\infty),[0,\infty))$ and $h \in C(\bar{B})$ is a sign-changing function. We manage to determine the intervals of λ in which the above problem has one, two or three positive radial solutions by using the directions of a bifurcation.

Keywords: positive solutions, p-Laplacian, indefinite weight, bifurcation.

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1 Introduction

In this paper, we investigate the existence of three positive radial solutions for the N-dimensional p-Laplacian problem

$$\begin{aligned} &\operatorname{div}(\varphi_p(\nabla u)) + \lambda h(x) f(u) = 0, && \text{in } B, \\ &u = 0, && \text{on } \partial B, \end{aligned} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, B is the unit open ball of $\mathbb{R}^N (N \geq 1)$, $1 < p < \infty$, $\lambda > 0$ is a parameter, $f \in C([0,\infty),[0,\infty))$, $f(0) = 0$, $f(s) > 0$ for $s > 0$, and h is a sign-changing function satisfying

$$H(B) = \{h \in C(\bar{B}) \text{ is radially symmetric} \mid h(x) > 0,\ x \in \Omega \text{ and } h(x) \leq 0,\ x \in \bar{B} \backslash \Omega\}$$

with the annular domain $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\} \subset B$ for some $0 < r_1 < r_2 < 1$.

A radial solution of (1.1) can be considered as a solution of the problem

$$\begin{aligned} &(r^{N-1}\varphi_p(u'))' + \lambda r^{N-1} h(r) f(u) = 0, \quad r \in I, \\ &u'(0) = 0 = u(1), \end{aligned} \tag{1.2}$$

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where $r = |x|$ with $x \in B$, $h \in H(I)$ with $I = (0,1)$, and

$$H(I) = \{h \in C(\bar{I}) \mid h(r) > 0,\ r \in (r_1, r_2) \text{ and } h(r) \le 0,\ r \in \bar{I} \backslash (r_1, r_2)\}.$$

It is known that the existence of three positive solutions for one-dimensional p-Laplacian problem with indefinite weight

$$\begin{aligned} &(\varphi_p(u'(x)))' + \lambda h(x) f(u(x)) = 0, \quad x \in I, \\ &u(0) = 0 = u(1) \end{aligned} \tag{1.3}$$

was mainly studied by three positive solutions theorem in Amann [1] based on the method of lower and upper solutions. Notice that even they established the basic three positive solutions theorem for the indefinite weight case, they could only apply it for a positive weight case. This implies that it is difficult to construct upper and lower solutions for the indefinite weight. Variational approach [3, 16] can also be applied to get three solutions, however, this method does not guarantee positivity or nontriviality of all solutions at most cases.

To overcome the difficulties mentioned above, very recently, Sim and Tanaka [17] employed a bifurcation technique to show the existence of three positive solutions for the one-dimensional p-Laplacian problem (1.3) with $h \in H(I)$, see [17, Theorem 1.1] for more details.

Up to our knowledge, the existence of three positive radial solutions have never been established for N-dimensional p-Laplacian problem (1.1) (or (1.2)) with indefinite weight h on the unit ball of $\mathbb{R}^N$. For example, Dai, Han and Ma [4]only showed the existence of one positive radial solution of (1.1) (or (1.2)) with indefinite weight. So it is the main purpose of this paper to obtain a similar result to Sim and Tanaka [17] for (1.1) (or (1.2)) with $h \in H(B)$. Indeed, problem with indefinite weight arises from the selection-migration model in population genetics. In this model, $h(r)$ changes sign corresponding to the fact that an allele A_1 holds an advantage over a rival allele A_2 at same points and is at a disadvantage at others; the parameter λ corresponds to the reciprocal of diffusion, for detail, see [8].

For other results on the study of positive solutions of N-dimensional p-Laplacian problem (1.1) or (1.2) we refer the reader to [4–7, 9–11]. It is worth noting that Dai, Han and Ma [4] studied the unilateral global bifurcation phenomena for (1.2) with indefinite weight and constructed the eigenvalue theory of the following problem with indefinite weight

$$\begin{aligned} &(r^{N-1}\varphi_p(u'))' + \lambda r^{N-1} h(r) \varphi_p(u) = 0, \quad r \in I, \\ &u'(0) = 0 = u(1). \end{aligned} \tag{1.4}$$

Let μ_1 be the first positive eigenvalue of (1.4). Then from the variational characterization of μ_1, it follows that

$$\mu_1 = \sup\left\{\mu > 0 \mid \mu \int_B h(x)|\phi(x)|^p dx \le \int_B |\nabla\phi|^p dx,\ \text{for } \phi \in C^\infty_{r,c}(B) \text{ and } \int_B h(x)|\phi(x)|^p dx > 0\right\},$$

where $C^\infty_{r,c}(B) = \{\phi \in C^\infty_c(B) \mid \phi \text{ is radially symmetric}\}$. For the spectrum of the p-Laplacian operator with indefinite weight we refer the reader to [2, 14].

We turn now to a more detailed statement of our assumptions and main conclusions. Throughout the paper we shall assume, without further comment, the following hypotheses concerning the function f:

(H1) $f : [0, \infty) \to [0, \infty)$ is continuous, $f(0) = 0$, $f(s) > 0$ for all $s > 0$;

(H2) there exist $\alpha > 0$, $f_0 > 0$ and $f_1 > 0$ such that $\lim_{s\to 0^+} \frac{f(s)-f_0 s^{p-1}}{\varphi_{p+\alpha}(s)} = -f_1$;

(H3) $f_\infty := \lim_{s\to\infty} \frac{f(s)}{\varphi_p(s)} = 0$;

(H4) there exists $s_0 > 0$ such that

$$\min_{s\in[s_0,2s_0]} \frac{f(s)}{\varphi_p(s)} \geq \frac{f_0}{\mu_1 h_0}\left(\frac{2\big(\nu_2(p)-\nu_1(p)\big)}{r_2-r_1}\right)^p,$$

where $h_0 = \min_{r\in[\frac{3r_1+r_2}{4},\frac{r_1+3r_2}{4}]} h(r)$, $\nu_1(p)$ and $\nu_2(p)$ are the first two zeros of the initial value problem

$$\begin{aligned} &(r^{N-1}\varphi_p(u'))' + r^{N-1}\varphi_p(u) = 0, \quad r > 0, \\ &u(0) = 1, \quad u'(0) = 0. \end{aligned} \tag{1.5}$$

It is well known [15] that (1.5) has a unique solution Φ defined on $[0,\infty)$, which is oscillatory. Let

$$0 < \nu_1(p) < \nu_2(p) < \cdots < \nu_n(p) < \cdots$$

be the zeros of Φ. These zeros are simple and $\nu_n(p) \to +\infty$ as $n \to +\infty$.

Note that (H2) implies

$$\lim_{s\to 0^+} \frac{f(s)}{\varphi_p(s)} = f_0. \tag{1.6}$$

Combining this with (H3), we can deduce that there exists $f^* > 0$ satisfying

$$f(s) \leq f^* s^{p-1}, \qquad s \geq 0. \tag{1.7}$$

Let $Y = C[0,1]$ with the norm $\|u\|_\infty = \max_{r\in[0,1]} u(r)$ and $X = \{u \in C^1[0,1] : u'(0) = 0 = u(1)\}$ be a Banach space under the norm $\|u\| = \max\{\|u\|_\infty, \ \|u'\|_\infty\}$. Let $P := \{u \in X : \ u > 0 \text{ on } [0,1)\}$ and $\mathbb{R}^+ = [0,+\infty)$.

To wit, our principal result can now be stated.

Theorem 1.1. *Assume (H1)–(H4) hold. Let $h \in H(I)$. Then the pair $(\frac{\mu_1}{f_0},0)$ is a bifurcation point of problem (1.2), and there is an unbounded continuum $\mathcal{C}$ of the set of positive solutions of problem (1.2) in $\mathbb{R}\times X$ bifurcating from $(\frac{\mu_1}{f_0},0)$ such that $\mathcal{C} \subseteq \big((\mathbb{R}^+\times P)\cup\{(\frac{\mu_1}{f_0},0)\}\big)$ and $\lim_{\lambda\to+\infty}\|u_\lambda\| = +\infty$ for $(\lambda,u_\lambda) \in \mathcal{C}\setminus\{(\frac{\mu_1}{f_0},0)\}$. Moreover, there exist $(\lambda_*,u_{\lambda_*})$ and $(\lambda^*,u_{\lambda^*}) \in \mathcal{C}$ which satisfy $0 < \lambda_* < \frac{\mu_1}{f_0} < \lambda^*$ and $\|u_{\lambda^*}\| < \|u_{\lambda_*}\|$, such that the continuum $\mathcal{C}$ grows to the right from the bifurcation point $(\frac{\mu_1}{f_0},0)$, to the left at $(\lambda^*,u_{\lambda^*})$ and to the right at $(\lambda_*,u_{\lambda_*})$.*

From Theorem 1.1, we can easily derive the following corollary, which gives the ranges of parameter guaranteeing problem (1.2) has one, two or three positive solutions (see Figure 1).

Corollary 1.2. *Assume (H1)–(H4) hold. Let $h \in H(I)$. Then there exist $\lambda_* \in (0,\frac{\mu_1}{f_0})$ and $\lambda^* > \frac{\mu_1}{f_0}$ such that*

(i) (1.2) *has at least one positive solution if $\lambda = \lambda_*$;*

(ii) (1.2) *has at least two positive solutions if $\lambda_* < \lambda \leq \frac{\mu_1}{f_0}$;*

(iii) (1.2) *has at least three positive solutions if* $\frac{\mu_1}{f_0} < \lambda < \lambda^*$*;*

(iv) (1.2) *has at least two positive solutions if* $\lambda = \lambda^*$*;*

(v) (1.2) *has at least one positive solution if* $\lambda > \lambda^*$*.*

Remark 1.3. Note that (H2) has been used in [17] studying the one-dimensional p-Laplacian with a sign-changing weight. Indeed, under (H2) there is an unbounded continuum $\mathcal{C}$ which is bifurcating from $\frac{\mu_1}{f_0}$. Conditions (H1)–(H3) and $h \in H(I)$ push the direction of continuum $\mathcal{C}$ to the right near $u = 0$. Moreover, it follows from (H3) and (H4) that the nonlinearity is superlinear at some point and is sublinear near ∞, which make continuum $\mathcal{C}$ turn to the left at some point and to the right near $\lambda = \infty$. Nevertheless, assumptions (H2) and (H4) are technical and need to be further improved.

Remark 1.4. Let us consider the nonlinear function

$$f(s) = s^{p-1}[s^2 - 4s + 5]a^{-\frac{s}{m}}, \qquad s \geq 0,$$

where $a > 1, m > \ln a$. It is not difficult to prove that f satisfies (H1), (H2) and (H3) with

$$\alpha = 1, \qquad f_0 = 5, \qquad f_1 = 4 + \frac{5 \ln a}{m}.$$

Let $g(s) := \frac{f(s)}{s^{p-1}}$. We can easily verify that g is increasing on

$$\left(\frac{2\ln a + m - \sqrt{m^2 - (\ln a)^2}}{\ln a}, \ \frac{2\ln a + m + \sqrt{m^2 - (\ln a)^2}}{\ln a} \right),$$

and is decreasing on $\left(\frac{2\ln a + m + \sqrt{m^2 - (\ln a)^2}}{\ln a}, \infty\right)$. Consequently,

$$\min_{s \in [2 + \frac{m}{\ln a}, \ 2(2 + \frac{m}{\ln a})]} \frac{f(s)}{s^{p-1}} = \min \left\{ g\left(2 + \frac{m}{\ln a}\right), \ g\left(4 + \frac{2m}{\ln a}\right) \right\} \to \infty, \qquad m \to \infty$$

and so (H4) is satisfied. Then f satisfies all of the conditions in Theorem 1.1.

The contents of this paper have been distributed as follows. In Section 2, we establish a global bifurcation phenomena from the trivial branch with the rightward direction. In Section 3, we show that the bifurcation curve grows to the left at some point under (H4) condition. In Section 4, we get the second turn of the bifurcation curve which grows to right near $\lambda = \infty$. Moreover, we give the proof of Theorem 1.1.

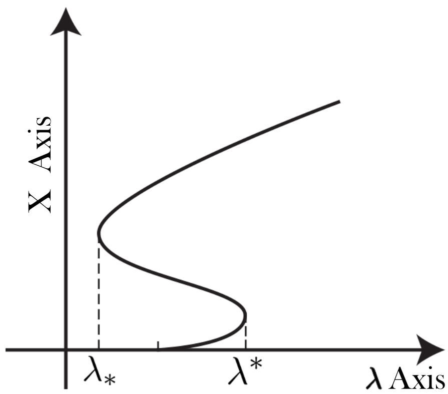

Figure 1.1: Bifurcation diagram of Theorem 1.1.

2 Global bifurcation phenomena with the rightward direction

We firstly introduce the following important result, which is proved in [4, Theorem 3.1], see also [5, Theorem 2.1] or [13] studying the semilinear problem.

Lemma 2.1 (See [4]). *Let $h \in H(I)$. Assume $g : I \times \mathbb{R} \times \mathbb{R}$ satisfies Carathéodory condition in the first two variable, $g(r, s, 0) \equiv 0$ for $(r, s) \in I \times \mathbb{R}$ and*

$$\lim_{s \to 0} \frac{g(r, s, \lambda)}{\varphi_p(s)} = 0 \tag{2.1}$$

uniformly for $r \in I$ and λ on bounded sets. Then from each $(\lambda_k^\nu, 0)$ bifurcates an unbounded continuum $\mathcal{C}_k^\nu$ of solutions to problems

$$\begin{aligned} &(r^{N-1}\varphi_p(u'(r)))' + \lambda r^{N-1}h(r)\varphi_p(u(r)) + r^{N-1}g(r, u, \lambda) = 0, \quad r \in I, \\ &u'(0) = 0 = u(1), \end{aligned} \tag{2.2}$$

with exactly $k-1$ simple zeros, where λ_k^ν is the eigenvalue of (1.4) and $\nu \in \{-1, 1\}$. □

Now, we are ready to show the unbounded continuum $\mathcal{C}$ of positive solutions to problem (1.2). Thanks to (1.6), there exists $\delta > 0$ such that $f(s) = f_0\varphi_p(s) + \xi(s), s \in (0, \delta)$, here $\xi \in C[0, \infty)$ and

$$\lim_{s \to 0^+} \frac{\xi(s)}{\varphi_p(s)} = 0. \tag{2.3}$$

Let us consider the auxiliary problem

$$\begin{aligned} &(r^{N-1}\varphi_p(u'(r)))' + \lambda f_0 r^{N-1}h(r)\varphi_p(u(r)) + \lambda r^{N-1}h(r)\xi(u(r)) = 0, \quad r \in I, \\ &u'(0) = 0 = u(1) \end{aligned} \tag{2.4}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

From Lemma 2.1, we can easily obtain the following result.

Lemma 2.2. *Assume (H1)–(H3) hold. Let $h \in H(I)$. Then from $(\frac{\mu_1}{f_0}, 0)$ there emanates an unbounded continuum $\mathcal{C}$ of positive solutions to problem (2.4) (i.e. (1.2)) in $\mathbb{R}^+ \times X$.*

Remark 2.3. Let $g(r, u, \lambda) = \lambda h(r)\xi(u)$ and $\lambda_1^+ = \mu_1$ is the first positive eigenvalue of (1.4). Then Lemma 2.2 is an immediate consequence of Lemma 2.1. Moreover, by virtue of the relationship between function limit and infinitesimal quantity, it follows from (1.6) that there exists $\delta > 0$ such that

$$\frac{f(s)}{\varphi_p(s)} = f_0 + \alpha(s) \quad \text{for } 0 < s < \delta, \tag{2.5}$$

where $\lim_{s \to 0^+} \alpha(s) = 0$, i.e. $\alpha(s)$ is the infinitesimal quantity when $s \to 0^+$. Consequently, (2.5) implies that $f(s) = f_0\varphi_p(s) + \xi(s)$ for $s \in (0, \delta)$, where $\xi(s) = \alpha(s)\varphi_p(s)$ and satisfies (2.3). For example, let us consider the nonlinear function $f(s) = 2s^{p-1} + s^p$, obviously, $f_0 := 2 = \lim_{s \to 0^+} \frac{f(s)}{\varphi_p(s)}$ and $\xi(s) = s^p$.

Lemma 2.4. *Let the hypotheses of Lemma 2.2 hold. Suppose $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ is a sequence of positive solutions to (1.2) which satisfies*

$$\|u_n\| \to 0 \quad \textit{and} \quad \lambda_n \to \frac{\mu_1}{f_0} \quad \textit{as } n \to \infty.$$

Then there exists a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|}$ converges uniformly to ϕ_1 on $[0, 1]$. Here ϕ_1 is the eigenfunction corresponding to μ_1 satisfying $\|\phi_1\| = 1$.

Proof. Put $v_n := \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$, and hence $\|v_n'\|_\infty$ and $\|v_n\|_\infty$ are bounded. Applying the Arzelà–Ascoli theorem, a subsequence of $\{v_n\}$ uniformly converges to a limit v. We again denote by $\{v_n\}$ the subsequence. Observe that $v'(0) = 0 = v(1)$ and $\|v\| = 1$. Now, from the equation of (1.2) with $\lambda = \lambda_n$ and $u = u_n$, we obtain

$$\varphi_p(u_n') = -\lambda_n \int_0^r h(t)\left(\frac{t}{r}\right)^{N-1} f(u_n)dt. \tag{2.6}$$

Dividing the both sides of (2.6) by $\|u_n\|^{p-1}$, we get

$$\varphi_p(v_n') = -\lambda_n \int_0^r h(t)\left(\frac{t}{r}\right)^{N-1} \frac{f(u_n(t))}{\varphi_p(u_n(t))}\varphi_p(v_n(t))dt =: w_n(r), \tag{2.7}$$

whence also

$$v_n(r) = -\int_r^1 \varphi_p^{-1}(w_n(t))dt. \tag{2.8}$$

On the other hand, it can be easily seen that $\frac{f(u_n(r))}{\varphi_p(u_n(r))} \to f_0$ (recall $u_n(r) \to 0$ for all $r \in [0,1]$) as $n \to \infty$. Then, by virtue of (1.7), it follows from Lebesgue's dominated convergence theorem that $w_n(r)$ tends to $w(r)$,

$$w(r) := -\mu_1 \int_0^r \left(\frac{t}{r}\right)^{N-1} h(t)\varphi_p(v(t))dt, \quad \text{for all } r \in [0,1].$$

Consequently, combining (2.8) and Lebesgue's dominated convergence theorem, we can deduce

$$v(r) = -\int_r^1 \varphi_p^{-1}(w(t))dt = \int_r^1 \varphi_p^{-1}\Big(\mu_1 \int_0^s \left(\frac{t}{s}\right)^{N-1} h(t)\varphi_p(v(t))dt\Big)ds,$$

which is equivalent to (1.4) with $\lambda = \mu_1$, and hence $v \equiv \phi_1$. □

Lemma 2.5. *Suppose $\alpha > 0$ and $h \in H(I)$. Let ϕ_1 be the eigenfunction corresponding to μ_1. Then*

$$\int_B h(x)[\phi_1(x)]^{p+\alpha}dx > 0.$$

Proof. Multiplying $-\operatorname{div}(\varphi_p(\nabla u)) = \mu_1 h(x)\varphi_p(u)$ by $\phi_1^{1+\alpha}$ and integrating it over B, we obtain

$$\begin{aligned}
&\mu_1 \int_B h(x)[\phi_1]^{p+\alpha}(x)dx \\
&\quad = -\int_B \operatorname{div}(|\nabla\phi_1|^{p-2}\nabla\phi_1)\phi_1^{\alpha+1}(x)dx \\
&\quad = -\int_{\partial B} \phi_1^{\alpha+1}(x)|\nabla\phi_1|^{p-2}\nabla\phi_1 \cdot \nu dx + (\alpha+1)\int_B \phi_1^{\alpha}(x)\nabla\phi_1|\nabla\phi_1|^{p-2}\nabla\phi_1 dx \\
&\quad = (\alpha+1)\int_B \phi_1^{\alpha}(x)|\nabla\phi_1|^p dx > 0.
\end{aligned}$$

□

The next result establishes that the continuum $\mathcal{C}$ grows to the right from $(\frac{\mu_1}{f_0}, 0)$.

Lemma 2.6. *Let the hypotheses of Lemma 2.2 hold. Then there exists $\delta > 0$ such that $(\lambda, u) \in \mathcal{C}$ and $|\lambda - \frac{\mu_1}{f_0}| + \|u\| \le \delta$ imply $\lambda > \frac{\mu_1}{f_0}$.*

Proof. For contradiction we assume that there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ satisfying

$$\|u_n\| \to 0, \qquad \lambda_n \to \frac{\mu_1}{f_0} \quad \text{and} \quad \lambda_n \le \frac{\mu_1}{f_0}. \tag{2.9}$$

From Lemma 2.4, there exists a subsequence of $\{u_n\}$, we again denote it by $\{u_n\}$, such that $\frac{u_n}{\|u_n\|}$ converges uniformly to ϕ_1 on $[0,1]$, here $\phi_1 > 0$ is the eigenfunction corresponding to μ_1 satisfying $\|\phi_1\| = 1$. Multiplying the equation of (1.1) applied to (λ_n, u_n) by u_n and integrating over B, we see that

$$\lambda_n \int_B h(x) f(u_n) u_n dx = \int_B |\nabla u_n|^p dx.$$

It follows from the definition of μ_1 that

$$\lambda_n \int_B h(x) f(u_n) u_n dx \ge \mu_1 \int_B h(x) |u_n|^p dx,$$

whence also

$$\int_B h(x) \frac{f(u_n) - f_0 (u_n)^{p-1}}{(u_n)^{p-1+\alpha}} \left| \frac{u_n}{\|u_n\|} \right|^{p+\alpha} dx \ge \frac{\mu_1 - f_0 \lambda_n}{\lambda_n \|u_n\|^\alpha} \int_B h(x) \left| \frac{u_n}{\|u_n\|} \right|^p dx.$$

Together with (H2), Lemma 2.5 and Lebesgue's dominated convergence theorem, then gives

$$\int_B h(x) \frac{f(u_n) - f_0 (u_n)^{p-1}}{(u_n)^{p-1+\alpha}} \left| \frac{u_n}{\|u_n\|} \right|^{p+\alpha} dx \to -f_1 \int_B h(x) |\phi_1|^{p+\alpha} dx < 0,$$

but

$$\int_B h(x) \left| \frac{u_n}{\|u_n\|} \right|^p dx \to \int_B h(x) |\phi_1|^p dx > 0.$$

Consequently, $\lambda_n > \frac{\mu_1}{f_0}$, which contradicts (2.9). $\square$

Remark 2.7. Lemma 2.6 implies that the bifurcation continuum $\mathcal{C}$ has the rightward direction from the bifurcation point $(\frac{\mu_1}{f_0}, 0)$.

3 Direction turn of bifurcation

In this section, in view of the condition (H4), we show that the continuum $\mathcal{C}$ grows to the left at some point.

Lemma 3.1. *Let $h \in H(I)$. Suppose u is a positive solution of* (1.2). *Then*

$$\frac{\|u\|_\infty}{2} \le u(r) \le \|u\|_\infty, \qquad r \in \left[\frac{3r_1 + r_2}{4}, \frac{r_1 + 3r_2}{4} \right]. \tag{3.1}$$

Proof. It readily follows from the equation of (1.2) that $u'(r)$ is nondecreasing on $[0,1] \backslash (r_1, r_2)$ and $u'(r)$ is decreasing on (r_1, r_2) because $h \in H(I)$. On the other hand, according to $u(1) = 0 = u'(0)$ and $u(r) > 0$ for all $(0,1)$, it becomes apparent that $u'(1) \le 0$. Therefore, u is convex on $[0,1] \backslash (r_1, r_2)$ and concave on (r_1, r_2). If $r_0 \in [r_1, \frac{r_1+r_2}{2}]$ is a point of a maximum of u, then, for all $r_1 \le r \le r_0$, we have

$$\frac{u(r) - u(r_1)}{r - r_1} \ge \frac{\|u\|_\infty - u(r_1)}{r_0 - r_1},$$

whence also

$$\frac{u(r)}{r-r_1} \geq \frac{\|u\|_\infty}{r_0-r_1} + \frac{u(r_1)}{r-r_1} - \frac{u(r_1)}{r_0-r_1} \geq \frac{\|u\|_\infty}{r_0-r_1} \geq \frac{\|u\|_\infty}{\frac{r_1+r_2}{2}-r_1}.$$

Thus

$$u(r) \geq \frac{\|u\|_\infty}{\frac{r_1+r_2}{2}-r_1}(r-r_1).$$

Observe that $\frac{r-r_1}{\frac{r_1+r_2}{2}-r_1} \geq \frac{1}{2}$ is equivalent to $r \geq \frac{3r_1+r_2}{4}$.

Analogously, if $r_0 \in [\frac{r_1+r_2}{2}, r_2]$ is a point of a maximum of u. Then, for all $r_0 \leq r \leq r_2$, it follows that

$$u(r) \geq \frac{\|u\|_\infty}{r_2-\frac{r_1+r_2}{2}}(r_2-r),$$

and $\frac{r_2-r}{r_2-\frac{r_1+r_2}{2}} \geq \frac{1}{2}$ is equivalent to $r \leq \frac{r_1+3r_2}{4}$. □

Lemma 3.2. *Assume (H1) and (H4) hold. Let $h \in H(I)$. Suppose u is a positive solution of* (1.2) *with $\|u\|_\infty = 2s_0$. Then $\lambda < \mu_1/f_0$.*

Proof. It can be easily seen from Lemma 3.1 that

$$s_0 \leq u(r) \leq 2s_0, \qquad r \in J := \left[\frac{3r_1+r_2}{4}, \frac{r_1+3r_2}{4}\right].$$

For contradiction we assume $\lambda \geq \mu_1/f_0$. Then it follows from (H4) that, for all $r \in J$,

$$\lambda h(r)\frac{f(u(r))}{\varphi_p(u(r))} \geq \frac{\mu_1}{f_0}h_0\frac{f_0}{\mu_1 h_0}\left(\frac{2\,(\nu_2(p)-\nu_1(p))}{r_2-r_1}\right)^p \geq \left(\frac{2\,(\nu_2(p)-\nu_1(p))}{r_2-r_1}\right)^p.$$

Put

$$v(r) = \Phi\left[\frac{2\,(\nu_2(p)-\nu_1(p))}{r_2-r_1}r + \frac{(r_1+3r_2)\nu_1(p)-(r_2+3r_1)\nu_2(p)}{2(r_2-r_1)}\right].$$

Recall that Φ is a unique solution of (1.5). Then v is a solution of

$$\begin{aligned} &(r^{N-1}\varphi_p(v'))' + \left(\frac{2\,(\nu_2(p)-\nu_1(p))}{r_2-r_1}\right)^p r^{N-1}\varphi_p(v) = 0, \qquad r \in J,\\ &v\left(\frac{3r_1+r_2}{4}\right) = 0, \qquad v\left(\frac{r_1+3r_2}{4}\right) = 0. \end{aligned} \tag{3.2}$$

On the other hand, u is a solution of

$$(r^{N-1}\varphi_p(u'))' + \lambda h(r)\frac{f(u)}{\varphi_p(u)}r^{N-1}\varphi_p(u) = 0, \qquad r \in J.$$

Applying the Sturm comparison Theorem [15, Lemma 4.1], we can deduce that u has at least one zero on J, an obvious contradiction. □

Remark 3.3. It follows from Lemma 3.2 that there exists a direction turn of the bifurcation continuum $\mathcal{C}$ which grows to the left at some point $(\lambda^*, u_{\lambda^*}) \in \mathcal{C}$.

4 Second turn and proof of main result

In this section, we prove that there is the second direction turn of bifurcation and complete the proof of Theorem 1.1.

Lemma 4.1. *Suppose u is a positive solution of* (1.2) *under the hypotheses (H1)–(H3) and $h \in H(I)$. Then there exists a constant $C > 0$ independent of u such that*

$$|u'(r)| \leq \lambda^{\frac{1}{p-1}} C \|u\|_\infty, \qquad r \in [0,1]. \tag{4.1}$$

Proof. An easy integration for (1.2) now yields

$$-\varphi_p(u'(r)) = \lambda \int_0^r h(t) \left(\frac{t}{r}\right)^{N-1} f(u) dt, \qquad r \in [0,1].$$

Together this with (1.7), we are lead to

$$|u'|^{p-1} = \lambda \left| \int_0^r h(t) \left(\frac{t}{r}\right)^{N-1} f(u) dt \right| \leq \lambda f^* \|u\|_\infty^{p-1} \int_0^1 |h(t)| dt,$$

the result follows at once. □

The following result provides us with a lower bound for the parameter.

Lemma 4.2. *Let the hypotheses of Lemma 4.1 hold. If u is a positive solution of* (1.2)*, then there exists $\lambda_* > 0$ such that $\lambda \geq \lambda_*$.*

Proof. Let r_0 be a point of a maximum of u. According to (4.1), we obtain

$$\|u\|_\infty = u(r_0) = \int_1^{r_0} u'(r) dr \leq \int_{r_0}^1 |u'(r)| dr \leq \lambda^{\frac{1}{p-1}} C \|u\|_\infty \int_{r_0}^1 dr \leq \lambda^{\frac{1}{p-1}} C \|u\|_\infty,$$

and hence, $\lambda \geq C^{-(p-1)}$, where C is a constant defined in Lemma 4.1. □

Remark 4.3. Lemma 4.2 implies that the bifurcation continuum $\mathcal{C}$ can not intersect with the X axis.

The next result shows that there is an upper estimate of the C^1-norm of positive solutions of (1.2).

Lemma 4.4. *Let the hypotheses of Lemma 4.1 hold. Suppose $J \subset (0, \infty)$ is a compact interval. Then there exists $M_J > 0$ such that all possible positive solutions u of* (1.2) *with $\lambda \in J$ satisfy*

$$\|u\| \leq M_J.$$

Proof. Now we proceed as in [12], repeating the arguments for completeness. Put $J := [a, b]$. For contradiction, we suppose that there exists a sequence $\{u_n\}$ of positive solutions of (1.2) with $\lambda_n \in J$, $\|u_n\| \to \infty$ as $n \to \infty$, which implies that $\|u_n\|_\infty \to \infty$ (as $n \to \infty$) by Lemma 4.1. Taking

$$\alpha \in \left(0, \frac{1}{b\varphi_p(\gamma_p Q)}\right), \quad \text{where } \gamma_p = \max\left\{1, 2^{\frac{2-p}{p-1}}\right\}, \ Q = \varphi_p^{-1}\left(\int_0^1 |h(s)| ds\right).$$

Thanks to (H3), there exists $u_\alpha > 0$ such that

$$f(u) < \alpha u^{p-1} \quad \text{for all } u > u_\alpha.$$

Define

$$m_\alpha = \max_{s\in[0,u_\alpha]} f(s), \quad A_n = \{r : u_n(r) \le u_\alpha \text{ for } r \in [0,1]\}, \quad B_n = \{r : u_n(r) > u_\alpha \text{ for } r \in [0,1]\}.$$

Let $\delta_n \in (0,1)$ satisfy $u_n(\delta_n) = \max_{r\in[0,1]} u_n(r)$. Then, for all $r \in [\delta_n, 1]$, it follows

$$\begin{aligned}
u_n(\delta_n) &= \int_{\delta_n}^1 \varphi_p^{-1}\left(\frac{1}{r^{N-1}}\lambda_n \int_{\delta_n}^r \tau^{N-1}h(\tau)f(u_n(\tau))d\tau\right) dr \\
&\le \int_{\delta_n}^1 \varphi_p^{-1}\left(\lambda_n \int_{\delta_n}^1 |h(\tau)|f(u_n(\tau))d\tau\right) dr \\
&\le \varphi_p^{-1}(\lambda_n) \int_{\delta_n}^1 \varphi_p^{-1}\left(\int_{A_n} |h(\tau)|f(u_n(\tau))d\tau + \int_{B_n} |h(\tau)|f(u_n(\tau))d\tau\right) dr \\
&\le \varphi_p^{-1}(\lambda_n) \int_{\delta_n}^1 \varphi_p^{-1}\left(m_\alpha \int_{A_n} |h(\tau)|d\tau + \int_{B_n} |h(\tau)|f(u_n(\tau))d\tau\right) dr.
\end{aligned}$$

Then

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \le \gamma_p \int_{\delta_n}^1 \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{\|u_n\|_\infty} + \varphi_p^{-1}\left(\int_{B_n} \frac{|h(\tau)|f(u_n(\tau))}{\|u_n\|_\infty^{p-1}}d\tau\right)\right] dr.$$

Combining this with $\frac{f(u_n(\tau))}{\|u_n\|_\infty^{p-1}} \le \frac{f(u_n(\tau))}{u_n^{p-1}(\tau)} \le \alpha$ since $u_n(\tau) > u_\alpha$ for $\tau \in B_n$, we obtain

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \le \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{\|u_n\|_\infty} + \varphi_p^{-1}(\alpha)Q\right].$$

It follows from $\lambda_n \in J$ that $\frac{1}{\varphi_p^{-1}(\lambda_n)} \ge \frac{1}{\varphi_p^{-1}(b)}$ for all n, and hence

$$\frac{1}{\varphi_p^{-1}(b)} \le \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{\|u_n\|_\infty} + \varphi_p^{-1}(\alpha)Q\right].$$

According to $\|u_n\|_\infty \to \infty$ as $n \to \infty$, we must have

$$\frac{1}{\varphi_p^{-1}(b)} \le \gamma_p\varphi_p^{-1}(\alpha)Q < \gamma_p\varphi_p^{-1}\left(\frac{1}{b\varphi_p(\gamma_p Q)}\right) Q = \frac{1}{\varphi_p^{-1}(b)},$$

an obvious contradiction. □

Now put

$$\underline{f}(s) = \min_{s/2\le t\le s} \frac{f(t)}{t^{p-1}}.$$

Lemma 4.5. *Suppose (H1)–(H4) are satisfied and $h \in H(I)$. Let u be a positive solution of (1.2). Then there exists a constant $C > 0$ independent of u such that*

$$\lambda \underline{f}(\|u\|) \le C. \tag{4.2}$$

Proof. Let r_0 be a point of a maximum of u. As in the proof of Lemma 3.1, it readily follows that

$$u''(r) > 0, \quad r \in [0,1] \setminus (r_1, r_2), \qquad u'' < 0, \quad r \in (r_1, r_2).$$

Therefore, $r_0 \in (r_1, r_2)$, which is divided into four cases.

Case 1. Let $\frac{r_1+r_2}{2} \le r_0 \le \frac{r_1+3r_2}{4}$. Owing to $\frac{r_1+r_2}{2} > \frac{3r_1+r_2}{4}$. Integrating the equation of (1.2) from r_0 to r, we are lead to

$$r^{N-1}\varphi_p(u') = \int_{r_0}^{r} (t^{N-1}\varphi_p(u'))'dt = -\int_{r_0}^{r} \lambda t^{N-1}h(t)f(u(t))dt = \int_{r}^{r_0} \lambda t^{N-1}h(t)f(u(t))dt,$$

and then integrating it from $\frac{3r_1+r_2}{4}$ to r_0,

$$u(r_0) - u\left(\frac{3r_1+r_2}{4}\right) = \int_{\frac{3r_1+r_2}{4}}^{r_0} \varphi_p^{-1}\left(\frac{1}{r^{N-1}}\int_r^{r_0} \lambda t^{N-1}h(t)f(u(t))dt\right)dr.$$

It follows from Lemma 3.1 that

$$\begin{aligned}\|u\|_\infty = u(r_0) &\ge \int_{\frac{3r_1+r_2}{4}}^{r_0} \varphi_p^{-1}\left(\frac{1}{r^{N-1}}\int_r^{r_0} \lambda t^{N-1}h(t)\frac{f(u(t))}{[u(t)]^{p-1}}[u(t)]^{p-1}dt\right)dr\\ &\ge \varphi_p^{-1}(\lambda \underline{f}(\|u\|_\infty))\frac{\|u\|_\infty}{2}\int_{\frac{3r_1+r_2}{4}}^{\frac{r_1+r_2}{2}} \varphi_p^{-1}\left(\int_r^{\frac{r_1+r_2}{2}} h(t)dt\right)dr.\end{aligned}$$

Now put

$$M_1 := \int_{\frac{3r_1+r_2}{4}}^{\frac{r_1+r_2}{2}} \varphi_p^{-1}\left(\int_r^{\frac{r_1+r_2}{2}} h(t)dt\right)dr > 0.$$

Necessarily,

$$\lambda \underline{f}(\|u\|) \le \lambda \underline{f}(\|u\|_\infty) \le \frac{2^{p-1}}{M_1^{p-1}}. \tag{4.3}$$

Case 2. Let $\frac{r_1+3r_2}{4} < r_0 < r_2$. It should be noted that $u'(\frac{r_1+3r_2}{4}) > 0$. Then, integrating the equation of (1.2) in $(\frac{3r_1+r_2}{4}, r)$ shows that

$$\begin{aligned}u(r) - u\left(\frac{3r_1+r_2}{4}\right) &= \int_{\frac{3r_1+r_2}{4}}^{r} \varphi_p^{-1}\left(\frac{1}{s^{N-1}}\int_s^{r_0} \lambda t^{N-1}h(t)\frac{f(u(t))}{[u(t)]^{p-1}}[u(t)]^{p-1}dt\right)ds\\ &= \int_{\frac{3r_1+r_2}{4}}^{r} \varphi_p^{-1}\left(\frac{1}{s^{N-1}}\int_s^{\frac{r_1+3r_2}{4}} \lambda t^{N-1}h(t)\frac{f(u(t))}{[u(t)]^{p-1}}[u(t)]^{p-1}dt\right)ds,\end{aligned}$$

and hence

$$\|u\|_\infty \ge u\left(\frac{r_1+3r_2}{4}\right) \ge \int_{\frac{3r_1+r_2}{4}}^{\frac{r_1+3r_2}{4}} \varphi_p^{-1}\left(\frac{1}{s^{N-1}}\int_s^{\frac{r_1+3r_2}{4}} \lambda t^{N-1}h(t)\frac{f(u(t))}{[u(t)]^{p-1}}[u(t)]^{p-1}dt\right)ds.$$

Owing to Lemma 3.1, one gets

$$\begin{aligned}\|u\|_\infty &\ge \varphi_p^{-1}(\lambda \underline{f}(\|u\|_\infty))\frac{\|u\|_\infty}{2}\int_{\frac{3r_1+r_2}{4}}^{\frac{r_1+3r_2}{4}} \varphi_p^{-1}\left(\int_s^{\frac{r_1+3r_2}{4}} h(t)dt\right)ds\\ &\ge \varphi_p^{-1}(\lambda \underline{f}(\|u\|_\infty))\frac{\|u\|_\infty}{2}\int_{\frac{3r_1+r_2}{4}}^{\frac{r_1+r_2}{2}} \varphi_p^{-1}\left(\int_s^{\frac{r_1+r_2}{2}} h(t)dt\right)ds\\ &= \varphi_p^{-1}(\lambda \underline{f}(\|u\|_\infty))\frac{\|u\|_\infty}{2}M_1,\end{aligned}$$

and (4.3) is satisfied.

Case 3. Let $\frac{3r_1+r_2}{4} < r_0 < \frac{r_1+r_2}{2}$. Analogously, integrating the equation of (1.2) in (r_0, r) and then integrating it over (r_0, r), we find that

$$u(r_0) - u(r) = \int_{r_0}^{r} \varphi_p^{-1}\left(\frac{1}{s^{N-1}}\int_{r_0}^{s} \lambda t^{N-1} h(t)\frac{f(u(t))}{[u(t)]^{p-1}}[u(t)]^{p-1}dt\right) ds,$$

whence also

$$\begin{aligned} u(r_0) = \|u\|_\infty &\geq \int_{r_0}^{\frac{r_1+3r_2}{4}} \varphi_p^{-1}\left(\frac{1}{s^{N-1}}\int_{r_0}^{s} \lambda t^{N-1} h(t)\frac{f(u(t))}{[u(t)]^{p-1}}[u(t)]^{p-1}dt\right) ds \\ &\geq \varphi_p^{-1}(\lambda \underline{f}(\|u\|_\infty))\frac{\|u\|_\infty}{2}\int_{r_0}^{\frac{r_1+3r_2}{4}} \varphi_p^{-1}\left(\frac{1}{s^{N-1}}\int_{r_0}^{s} t^{N-1}h(t)dt\right) ds \\ &\geq \varphi_p^{-1}(\lambda \underline{f}(\|u\|_\infty))\frac{\|u\|_\infty}{2}\varphi_p^{-1}\left[\left(\frac{2(r_1+r_2)}{r_1+3r_2}\right)^{N-1}\right]\int_{\frac{r_1+r_2}{2}}^{\frac{r_1+3r_2}{4}} \varphi_p^{-1}\left(\int_{\frac{r_1+r_2}{2}}^{s} h(t)dt\right) ds. \end{aligned}$$

Put

$$M_2 := \int_{\frac{r_1+r_2}{2}}^{\frac{r_1+3r_2}{4}} \varphi_p^{-1}\left(\int_{\frac{r_1+r_2}{2}}^{s} h(t)dt\right) ds > 0.$$

Consequently,

$$\lambda \underline{f}(\|u\|) \leq \lambda \underline{f}(\|u\|_\infty) \leq \frac{2^{p-1}}{M_2^{p-1}}\left(\frac{r_1+3r_2}{2(r_1+r_2)}\right)^{N-1}. \tag{4.4}$$

Case 4. Let $r_1 < r_0 < \frac{3r_1+r_2}{4}$. Arguing as above, we can also prove (4.4).

Consequently, $\lambda \underline{f}(\|u\|) \leq C$, where

$$C = \frac{2^{p-1}}{\min\{M_1^{p-1}, M_2^{p-1}\}}\left(\frac{r_1+3r_2}{2(r_1+r_2)}\right)^{N-1}. \qquad \square$$

The next result establishes that the continuum $\mathcal{C}$ grows to (∞, ∞) in $[0, \infty) \times X$.

Lemma 4.6. *Let the hypotheses of Lemma 4.5 hold. Then $\mathcal{C}$ joins $(\frac{\mu_1}{f_0}, 0)$ to (∞, ∞) in $[0, \infty) \times X$.*

Proof. From Lemma 2.2, it follows that $\mathcal{C}$ is unbounded, and hence, there exists $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that

$$|\lambda_n| + \|u_n\| \to \infty. \tag{4.5}$$

Clearly, by virtue of Lemma 4.2, $\lambda_n > 0$. We first claim that $\{\lambda_n\}$ is unbounded. Suppose for contradiction that there exists a bounded subsequence $\{\lambda_{n_k}\}$. Then it follows from Lemma 4.4 that $\|u_{n_k}\|$ is bounded, which contradicts (4.5). Thus, claim is valid.

On the other hand, owing to (4.2) (in Lemma 4.5), we must have $\underline{f}(\|u_n\|) \to 0$. It can be easily seen that $\|u_n\| \to \infty$ according to (1.6). $\square$

Remark 4.7. Lemma 4.6 means that there is the second direction turn of the unbounded continuum $\mathcal{C}$, i.e. it grows to the right at $(\lambda_*, u_{\lambda_*})$.

Now, we are ready to establish the main result (Theorem 1.1) in this paper.

Proof. It follows from Lemma 2.2 and Lemma 2.6 that $\mathcal{C}$ is bifurcating from $(\frac{\mu_1}{f_0}, 0)$ and goes rightward. Moreover, by Lemma 4.6, there exist $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that $\lambda_n \to \infty$ and $\|u_n\| \to \infty$ as $n \to \infty$. Thus, there exists $(\lambda_0, u_0) \in \mathcal{C}$ such that $\|u_0\|_\infty = 2s_0$ (s_0 be defined by (H4)), and hence, $\lambda_0 < \frac{\mu_1}{f_0}$ by Lemma 3.2. Therefore, together with Lemmas 2.6, 3.2 and 4.4, it deduces that $\mathcal{C}$ passes through some points $(\frac{\mu_1}{f_0}, v_1)$ and $(\frac{\mu_1}{f_0}, v_2)$ with

$$\|v_1\|_\infty < 2s_0 < \|v_2\|_\infty,$$

and there exist $\underline{\lambda}$ and $\bar{\lambda}$ satisfying $0 < \underline{\lambda} < \frac{\mu_1}{f_0} < \bar{\lambda}$, as well as we have

(i) if $\lambda \in (\frac{\mu_1}{f_0}, \bar{\lambda}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in \mathcal{C}$ with $\|u\|_\infty < \|v\|_\infty < 2s_0$;

(ii) if $\lambda \in [\underline{\lambda}, \frac{\mu_1}{f_0}]$, then there exist u and v such that $(\lambda, u), (\lambda, v) \in \mathcal{C}$ with $\|u\|_\infty < 2s_0 < \|v\|_\infty$.

Now put

$$\lambda^* = \sup\{\bar{\lambda} : \bar{\lambda} \text{ satisfies (i)}\}, \qquad \lambda_* = \inf\{\underline{\lambda} : \underline{\lambda} \text{ satisfies (ii)}\}.$$

Then, by the standard argument, there exists at least one positive solution of (1.2) with $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively. Clearly, $\mathcal{C}$ turns to the left at $(\lambda^*, u_{\lambda^*})$ and to the right at $(\lambda_*, u_{\lambda_*})$. Consequently, Lemma 3.2 implies that, for any $\lambda > \frac{\mu_1}{f_0}$, there exists w such that $(\lambda, w) \in \mathcal{C}$ with $\|w\|_\infty > 2s_0$. This ends the proof of Theorem 1.1. $\square$

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References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18**(1976), 620–709. https://doi.org/10.1137/1019102; MR0415432; Zbl 0375.47026

[2] A. Anane, O. Chakrone, M. Monssa, Spectrum of one dimensional p-Laplacian with indefinite weight, *Electron. J. Qual. Theory Differ. Equ.* **2002**, No. 17, 1–11. https://doi.org/10.14232/ejqtde.2002.1.17; MR1942086; Zbl 1022.35033

[3] G. Bonanno, Existence of three solutions for a two point boundary value problem, *Appl. Math. Lett.* **13**(2000), No. 5, 53–57. https://doi.org/10.1016/S0893(00)00033-1; MR1760263; Zbl 1009.34019

[4] G. Dai, X. Han, R. Ma, Unilateral global bifurcation and nodal solutions for the p-Laplacian with sign-changing weight, *Complex Var. Elliptic Equ.* **59**(2014), No. 6, 847–862. https://doi.org/10.1080/17476933.2013.791686; MR3195915; Zbl 1295.35056

[5] G. Dai, R. Ma, Unilateral global bifurcation phenomena and nodal solutions for p-Laplacian, *J. Differential Equations* **252**(2012), No. 3, 2448–2468. https://doi.org/10.1016/j.jde.2011.09.026; MR2860625; Zbl 1242.34068

[6] G. Dai, R. Ma, Unilateral global bifurcation for p-Laplacian with non-$p-1$-linearization nonlinearity, *Discrete Contin. Dyn. Syst.* **35**(2015), No. 1, 99–116. https://doi.org/10.3934/dcds.2015.35.99; MR3286950; Zbl 1319.34034

[7] P. Drábek, Y. X. Huang, Bifurcation problems for the p-Laplacian in R^N, *Trans. Amer. Math. Soc.* **349**(1997), No. 1, 171–188. https://doi.org/10.1090/S0002-9947-97-01788-1; MR1390979; Zbl 0868.35010

[8] W. H. Fleming, A selection-migration model in population genetics, *J. Math. Biol.* **2**(1975), 219–233. https://doi.org/10.1007/BF00277151; MR0403720; Zbl 0325.92009

[9] P. Girg, P. Takáč, Bifurcations of positive and negative continua in quasilinear elliptic eigenvalue problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9**(2008), 275–327. https://doi.org/10.1007/s00023-008-0356-x; MR2399190; Zbl 1145.35020

[10] D. Hai, Nonexistence of positive solutions for a class of p-Laplacian boundary value problems, *Appl. Math. Lett.* **31**(2014), 12–15. https://doi.org/10.1016/j.aml.2013.12.013; MR3178421; Zbl 1317.35059

[11] L. Iturriaga, E. Massa, J. Sánchez, P. Ubilla, Positive solutions of the p-Laplacian involving a superlinear nonlinearity with zeros, *J. Differential Equations* **248**(2010), 309–327. https://doi.org/10.1016/j.jde.2009.08.008; MR2558168; Zbl 1181.35117

[12] Y.-H. Lee, I. Sim, Global bifurcation phenomena for singular one-dimensional p-Laplacian, *J. Differential Equations* **229**(2006), 229–256. https://doi.org/10.1016/j.jde.2006.03.021; MR2265626; Zbl 1113.34010

[13] R. Ma, T. Chen, Y. Lu, On the Bonheure–Noris–Weth conjecture in the case of linearly bounded nonlinearities, *Discrete Contin. Dyn. Syst. Ser. B* **21**(2016), 2649–2662. https://doi.org/10.3934/dcdsb.2016066; MR3555134; Zbl 1351.35057

[14] G. Meng, P. Yan, M. Zhang, Spectrum of one-dimensional p-Laplacian with an indefinite integrable weight, *Mediterr. J. Math.* **7**(2010), 225–248. https://doi.org/10.1007/s00009-010-0040-5; MR2645912; Zbl 1202.34147

[15] M. Del Pino, R. Manásevich, Global bifurcation from the eigenvalues of the p-Lapiacian, *J. Differential Equations* **92**(1991), 226–251. https://doi.org/10.1016/0022-0396(91)90048-E; MR1120904; Zbl 0781.35017

[16] B. Ricceri, A three critical points theorem revisited, *Nonlinear Anal.* **70**(2009), 3084–3089. https://doi.org/10.1016/j.na.2008.04.010; MR2536320; Zbl 1214.47079

[17] I. Sim, S. Tanaka, Three positive solutions for one-dimensional p-Laplacian problem with sign-changing weight, *Appl. Math. Lett.* **49**(2015), 42–50. https://doi.org/10.1016/j.aml.2015.04.007; MR3361694; Zbl 1342.35122