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Lipschitz stability of generalized ordinary differential equations and impulsive retarded differential equations

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Abstract. We consider a class of retarded functional differential equations with preassigned moments of impulsive effect and we study the Lipschitz stability of solutions of these equations using the theory of generalized ordinary differential equations and Lyapunov functionals. We introduce the concept of variational Lipschitz stability and Lipschitz stability for generalized ordinary differential equations and we develop the theory in this direction by establishing conditions for the trivial solutions of generalized ordinary differential equations to be variationally Lipschitz stable. Thereby, we apply the results to get the corresponding ones for impulsive functional differential equations.

Keywords: generalized ODEs, impulsive RFDEs, variational Lipschitz stability, Lipschitz stability.

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1 Introduction

We denote by $G^{-}([a,b],\mathbb{R}^n)$ the Banach space, equipped with the usual supremum norm, of all functions from $[a,b] \subset \mathbb{R}$ to $\mathbb{R}^n$ which are regulated and continuous from the left. We denote by $G^{-}([a,+\infty),\mathbb{R}^n)$ the space of all bounded functions $f : [a,+\infty) \to \mathbb{R}^n$ such that for every real number $b > a$, the restriction $f|_{[a,b]}$ belongs to $G^{-}([a,b],\mathbb{R}^n)$.

Let $r > 0$ and $t_0 \geq 0$. Given a function $y \in G^{-}([t_0 - r,+\infty),\mathbb{R}^n)$ and $t \in [t_0,+\infty)$, we define $y_t \in G^{-}([-r,0],\mathbb{R}^n)$ as usual, by

$$y_t(\theta) = y(t+\theta), \qquad \theta \in [-r,0].$$

In this work, we consider the following retarded functional differential equation with preassigned moments of impulsive effect (impulsive RFDE):

$$\begin{cases} \dot{y}(t) = f(y_t,t), & t \neq t_k,\ t \geq t_0, \\ \Delta y(t) = I_k(y(t)), & t = t_k,\ k \in \mathbb{N}, \end{cases} \tag{1.1}$$

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subject to the initial condition

$$y_{t_0} = \phi, \tag{1.2}$$

where $f : G^-([-r,0],\mathbb{R}^n) \times [t_0,+\infty) \to \mathbb{R}^n$ and $\phi \in G^-([-r,0],\mathbb{R}^n)$. We also consider that, for each $y \in G^-([t_0 - r,+\infty),\mathbb{R}^n)$, the application $t \mapsto f(y_t,t)$ is Lebesgue integrable over $[t_0 - r,+\infty)$. The jumps occur at preassigned times t_k, $k \in \mathbb{N}$, such that $t_0 < t_k$, $t_k < t_{k+1}$ for $k \in \mathbb{N}$ and $t_k \to +\infty$ as $k \to +\infty$, and their action is described by the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k \in \mathbb{N}$. We assume that I_k, $k \in \mathbb{N}$, are bounded and Lipschitz continuous functions and

$$\Delta y(t) = y(t+) - y(t-) = y(t+) - y(t) = I_k(y(t)), \qquad k \in \mathbb{N},$$

for every $y \in G^-([t_0 - r,+\infty),\mathbb{R}^n)$ and $t \geq t_0$.

In [3], the authors defined the notion of Lipschitz stability of solutions for a certain system of impulsive functional-differential equations. However, this notion has been introduced by Dannan and Elaydi [4] for ordinary differential equations without impulses.

Bainov and Stamova [3] established sufficient conditions to guarantee the Lipschitz stability of the zero solution for the following impulsive functional differential equation:

$$\begin{cases} \dot{y}\,(t) = f\,(y_t, y(t), t)\,, & t \neq t_k,\ t \geq t_0, \\ \Delta y\,(t) = I_k\,(y\,(t))\,, & t = t_k,\ k \in \mathbb{N}, \end{cases} \tag{1.3}$$

with $f : G^-([-r,0],\mathbb{R}^n) \times \mathbb{R}^n \times [t_0,+\infty) \to \mathbb{R}^n$.

Here, our main goal is to establish results on Lipschitz stability for the zero solution of equation (1.2). In [3], the results were obtained by virtue of a comparison equation and differential inequalities for piecewise continuous functions. In this work, by considering a Lyapunov function $U : [t_0,\, +\infty) \times \overline{E}_\rho \to \mathbb{R}$ satisfying some conditions, where $\overline{E}_\rho$ is the closure of $E_\rho = \{\psi \in G^-([-r,0],\mathbb{R}^n) : \|\psi\| < \rho\}$, $\rho > 0$, we use the correspondence theorem between impulsive RFDEs and generalized ordinary differential equations (generalized ODEs) to show that the trivial solution $y \equiv 0$ of (1.1)–(1.2) is uniformly Lipschitz stable and globally uniformly Lipschitz stable. Then, the main tools used to obtain our results are Lyapunov's functions and the correspondence between impulsive RFDEs and generalized ODEs. One of the advantages of treating impulsive RFDEs impulses by means of the theory of generalized ODEs is that the theory of generalized ODEs is developed to a great extent. The assumptions usually concern the indefinite integral (in some sense) of the functions involved in the equations instead of the functions themselves. Furthermore, because impulsive RFDEs can be regarded as generalized ODEs, it is possible to obtain quite good results with short proofs just by transferring the results from one space to the other through the relation between the solutions [5]. Note that, in contrast to (1.1), Bainov and Stamova [3] have considered in system (1.3) that the right hand side of the functional differential equation depends also on $y(t)$. For the sake of simplicity, we have chosen to study equation (1.2), but according to [2], our results can be accomplished to equation (1.3), as one can verify in Section 5, [2]. However, our mathematical proposal here is not to introduce more general results than those present in [3], but to show another strategy and different conditions to obtain similar results. Such strategy was also used in [1], [2], and [6].

2 A brief exposition on generalized ODEs

A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $\{(\tau_i, [s_{i-1}, s_i]) : i = 1,2,\ldots,k\}$, where $a = s_0 \leq s_1 \leq \cdots \leq s_k = b$ is a division of $[a,b]$ and

$\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$.

A *gauge* on $[a, b]$ is any function $\delta : [a, b] \to (0, +\infty)$. Given a gauge δ on $[a, b]$, a tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$ is *δ-fine* if, for every i,

$$[s_{i-1}, s_i] \subset \{t \in [a, b] : |t - \tau_i| < \delta(\tau_i)\}.$$

Let X be a Banach space. The next type of integration is due to Jaroslav Kurzweil. See [8] and also [9].

Definition 2.1. A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is *Kurzweil integrable* over $[a, b]$, if there is a unique element $I \in X$ such that given $\varepsilon > 0$, there is a gauge δ of $[a, b]$ such that for every δ-fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$, we have $\|S(U, d) - I\| < \varepsilon$, where $S(U, d) = \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$. In this case, we write $I = \int_a^b DU(\tau, t)$ and use the convention $\int_a^b DU(\tau, t) = -\int_b^a DU(\tau, t)$.

The Kurzweil integral is linear, additive with respect to adjacent intervals and it encompasses the known Lebesgue integral. For more properties of this kind of integration, the reader may consult [9].

Let X be a Banach space and consider the set $\Omega = O \times [t_0, +\infty)$, where $O \subset X$ is an open set. Assume that $G : \Omega \to X$ is a given X-valued function defined for all $(x, t) \in \Omega$.

Definition 2.2. A function $x : [\alpha, \beta] \to X$ is called *a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x, t) \tag{2.1}$$

in the interval $[\alpha, \beta] \subset [t_0, +\infty)$ if $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - x(\gamma) = \int_\gamma^v DG(x(\tau), t) \tag{2.2}$$

holds for every γ, $v \in [\alpha, \beta]$.

The integral on the right-hand side of (2.2) is in the sense of Definition 2.1.

Given an initial condition $(\tilde{x}, t^*) \in \Omega$, a solution of the initial value problem for equation (2.1) is given as follows.

Definition 2.3. A function $x : [\alpha, \beta] \to X$ is a *solution of the generalized ordinary differential equation* (2.1) *with the initial condition* $x(t^*) = \tilde{x}$*, in the interval* $[\alpha, \beta] \subset [t_0, +\infty)$, if $t^* \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and the equality

$$x(v) - \tilde{x} = \int_{t^*}^v DG(x(\tau), t) \tag{2.3}$$

holds for every $v \in [\alpha, \beta]$.

In the sequel, we define a special class of functions $G : \Omega \to X$ for which we can obtain interesting properties of the solutions of (2.1).

Definition 2.4. Let $h : [t_0, +\infty) \to \mathbb{R}$ be a nondecreasing function. We say that a function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$ if

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{2.4}$$

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)| \tag{2.5}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

Assume that $G : \Omega \to X$ satisfies condition (2.4). Let us denote by $\operatorname{var}_\alpha^\beta(x)$ the variation of a function $x : [t_0, +\infty) \to X$ on a compact interval $[\alpha, \beta] \subset [t_0, +\infty)$. If $x : [\alpha, \beta] \to X$ is a local solution of (2.1), then

$$\|x(s_1) - x(s_2)\| \leq |h(s_2) - h(s_1)| \tag{2.6}$$

for all $s_1, s_2 \in [\alpha, \beta]$, and hence x is of bounded variation on $[\alpha, \beta]$ with

$$\operatorname{var}_\alpha^\beta x \leq h(\beta) - h(\alpha) < +\infty. \tag{2.7}$$

Furthermore, every point in $[\alpha, \beta]$ at which the function h is continuous is a continuity point of the solution $x : [\alpha, \beta] \to X$ and one has

$$x(\sigma+) - x(\sigma) = G(x(\sigma), \sigma+) - G(x(\sigma), \sigma), \quad \text{for } \sigma \in [\alpha, \beta) \tag{2.8}$$

and

$$x(\sigma) - x(\sigma-) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-), \quad \text{for } \sigma \in (\alpha, \beta], \tag{2.9}$$

where

$$G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s), \quad \text{for } \sigma \in [\alpha, \beta)$$

and

$$G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s), \quad \text{for } \sigma \in (\alpha, \beta].$$

To verify the proofs of the above assertions, see [9, Lemmas 3.10 and 3.12].

Now we present a result on the existence of the integral involved in the definition of the solution of the generalized ODE (2.1).

Lemma 2.5. *Let $G \in \mathcal{F}(\Omega, h)$. Suppose $[\alpha, \beta] \subset [t_0, +\infty)$, $x : [\alpha, \beta] \to X$ is of bounded variation on $[\alpha, \beta]$ and $(x(s), s) \in \Omega$ for every $s \in [\alpha, \beta]$. Then the integral $\int_\alpha^\beta DG(x(\tau), t)$ exists and the function $s \mapsto \int_\alpha^s DG(x(\tau), t) \in X$, $s \in [\alpha, \beta]$, is of bounded variation.*

The next result concerns local existence and uniqueness of a solution of a generalized ODEs of type (2.1) with right-hand side in $\mathcal{F}(\Omega, h)$. A proof of such result can be found in [5] (see Theorem 2.15 there).

Theorem 2.6 (Local existence and uniqueness). *Let $G \in \mathcal{F}(\Omega, h)$, where the function h is nondecreasing and left continuous. If for every $(\tilde{x}, t^*) \in \Omega$ such that for $\tilde{x}_+ = \tilde{x} + G(\tilde{x}, t^*+) - G(\tilde{x}, t^*)$, one has $(\tilde{x}_+, t^*) \in \Omega$, then there exists $\Delta > 0$ such that there exists a unique solution $x : [t^*, t^* + \Delta] \to X$ of the generalized ordinary differential equation (2.1) for which $x(t^*) = \tilde{x}$.*

3 Lipschitz stability for generalized ODEs

In this section, $(X, \|\cdot\|)$ is a Banach space and we set $\Omega = B_c \times [t_0, +\infty)$, where $B_c = \{y \in X : \|y\| < c\}$, with $c > 0$ and $t_0 \geq 0$. We also assume that $G \in \mathcal{F}(\Omega, h)$, where $h : [t_0, +\infty) \to \mathbb{R}$ is a left continuous nondecreasing function, and $G(0, t) - G(0, s) = 0$ for all $t, s \in [t_0, +\infty)$. Then for every $[\gamma, v] \subset [t_0, +\infty)$, we have

$$\int_\gamma^v DG(0, t) = G(0, v) - G(0, \gamma) = 0.$$

Thus $x \equiv 0$ is a solution of the generalized ODE (2.1) on $[t_0, +\infty)$. Note also that, by (2.6), every solution of (2.1) is continuous from the left, whenever h from the definition of $\mathcal{F}(\Omega, h)$ is left continuous.

In the sequel, we introduce the concept of variational Lipschitz stability for generalized ODEs. This concept was inspired by the concept of variational stability introduced by Š. Schwabik in [10].

Definition 3.1. The trivial solution $x \equiv 0$ of the generalized ODE (2.1) is said to be *variationally uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that if $\bar{x} : [\alpha, \beta] \to B_c$, $t_0 \le \alpha < \beta < +\infty$, is a function of bounded variation on $[\alpha, \beta]$ and it is left continuous on $(\alpha, \beta]$ such that

$$\|\bar{x}(\alpha)\| < \delta$$

and

$$\operatorname{var}_\alpha^\beta \left(\bar{x}(s) - \int_\alpha^s DG(\bar{x}(\tau), t) \right) < \delta,$$

then

$$\|\bar{x}(t)\| \le M \|\bar{x}(\alpha)\| \quad \text{for all } t \in [\alpha, \beta].$$

In the next lines, we introduce the concept of Lipschitz stability for generalized ODEs.

Definition 3.2. The trivial solution $x \equiv 0$ of the generalized ODE (2.1) is said to be:

i) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that, if $\bar{x} : [\alpha, \beta] \to B_c$, $t_0 \le \alpha < \beta < +\infty$, is a solution of the generalized ODE (2.1) on $[\alpha, \beta]$ such that

$$\|\bar{x}(\alpha)\| < \delta$$

then

$$\|\bar{x}(t)\| \le M \|\bar{x}(\alpha)\| \quad \text{for all } t \in [\alpha, \beta].$$

ii) *uniformly globally Lipschitz stable* if there exists $M > 0$ such that, if $\bar{x} : [\alpha, \beta] \to B_c$, $t_0 \le \alpha < \beta < +\infty$, is a solution of the generalized ODE (2.1) on $[\alpha, \beta]$, then

$$\|\bar{x}(t)\| \le M \|\bar{x}(\alpha)\| \quad \text{for all } t \in [\alpha, \beta].$$

Note that, if the trivial solution $x \equiv 0$ of the generalized ODE (2.1) is variationally uniformly Lipschitz stable, then it is uniformly Lipschitz stable.

In what follows, we establish conditions under which the trivial solution of (2.1) is variationally uniformly Lipschitz stable. We shall need the following auxiliary result whose proof can be found in [10], pages 395–400.

Lemma 3.3 ([10, Lemma 1]). *Let $G \in \mathcal{F}(\Omega, h)$. Suppose $V : [t_0, +\infty) \times X \to \mathbb{R}_+$ is such that $V(\cdot, x) : [t_0, +\infty) \to \mathbb{R}_+$ is left continuous on $(t_0, +\infty)$ for $x \in X$ and satisfies*

$$|V(t, z) - V(t, y)| \le K \|z - y\|, \qquad z, y \in X,\ t \in [t_0, +\infty),$$

where $K > 0$. Suppose, in addition, that there is a function $\Phi : X \to \mathbb{R}$ such that, for every solution $x : [a, b] \to X$ of (2.1) with $[a, b] \subset [t_0, +\infty)$, one has

$$D^+V(t, x(t)) = \limsup_{\eta \to 0^+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \le \Phi(x(t)), \qquad t \in [a, b].$$

If $\bar{x} : [\alpha, \beta] \to X$, $t_0 \le \alpha < \beta < +\infty$, is left continuous on $(\alpha, \beta]$ and of bounded variation on $[\alpha, \beta]$, then

$$V(\beta, \bar{x}(\beta)) - V(\alpha, \bar{x}(\alpha)) \le K \operatorname{var}_\alpha^\beta \left(\bar{x}(s) - \int_\alpha^s DG(\bar{x}(\tau), t) \right) + M(\beta - \alpha),$$

where $M = \sup_{t \in [\alpha, \beta]} \Phi(\bar{x}(t))$.

Theorem 3.4. *Let $V : [t_0, +\infty) \times \overline{B_\rho} \to \mathbb{R}_+$ be a function, where $\overline{B_\rho} = \{y \in X : \|y\| \leq \rho\}$, $0 < \rho < c$. Suppose V satisfies the following conditions:*

(i) $V(t, 0) = 0$ *for all* $t \in [t_0, +\infty)$;

(ii) $V(\cdot, x) : [t_0, +\infty) \to \mathbb{R}_+$ *is continuous from the left on* $(t_0, +\infty)$ *for all* $x \in \overline{B_\rho}$;

(iii) There is a positive constant K such that

$$|V(t,z) - V(t,y)| \leq K\|z - y\|, \qquad t \in [t_0, +\infty),\ z, y \in \overline{B_\rho};$$

(iv) There is a monotone increasing function $b : [0, +\infty) \to [0, +\infty)$ satisfying $b(0) = 0$ such that $V(t, x) \geq b(\|x\|)$ for all $t \in [t_0, +\infty)$ and $x \in \overline{B_\rho}$;

(v) For all solution $x : [\alpha, \beta] \to \overline{B}_\rho$ of (2.1), one has

$$D^+V(t, x(t)) = \limsup_{\eta \to 0^+} \frac{V(t + \eta, x(t + \eta)) - V(t, x(t))}{\eta} \leq 0, \qquad t \in [\alpha, \beta].$$

Then the trivial solution $x \equiv 0$ of (2.1) is variationally uniformly Lipschitz stable.

Proof. Let $\overline{x} : [\alpha, \beta] \to \overline{B_\rho}$ be a function of bounded variation on $[\alpha, \beta]$ and continuous from the left on $(\alpha, \beta]$, with $[\alpha, \beta] \subset [t_0, +\infty)$. By Lemma 3.3, we have

$$V(t, \overline{x}(t)) - V(\alpha, \overline{x}(\alpha)) \leq K \operatorname{var}_\alpha^\beta \left(\overline{x}(s) - \int_\alpha^s DG(\overline{x}(\tau), t) \right) \quad \text{for all } t \in [\alpha, \beta], \tag{3.1}$$

and properties (i) and (iii) yield

$$V(\alpha, \overline{x}(\alpha)) \leq K\|\overline{x}(\alpha)\|. \tag{3.2}$$

Choose $\varepsilon > 0$ in such a way that the inequality $0 < b(\varepsilon) \leq b(\|x(\alpha)\| + \varepsilon)$ holds. Let $\delta > 0$ be such that $2K\delta < b(\varepsilon)$. If

$$\|\overline{x}(\alpha)\| < \delta \quad \text{and} \quad \operatorname{var}_\alpha^\beta \left(\overline{x}(s) - \int_\alpha^s DG(\overline{x}(\tau), t) \right) < \delta,$$

then (3.1) and (3.2) imply

$$V(t, \overline{x}(t)) \leq 2K\delta < b(\varepsilon) \leq b(\|\overline{x}(\alpha)\| + \varepsilon), \qquad t \in [\alpha, \beta]. \tag{3.3}$$

On the other hand, by (iv), we have

$$V(t, \overline{x}(t)) \geq b(\|\overline{x}(t)\|), \qquad t \in [\alpha, \beta]. \tag{3.4}$$

Since b is an increasing function, we conclude that

$$\|\overline{x}(t)\| < \|\overline{x}(\alpha)\| + \varepsilon, \qquad t \in [\alpha, \beta],$$

by (3.3) and (3.4).

By taking $M = 1$ in Definition 3.1, we get the result, since $\varepsilon > 0$ can be as small as we want it to be. □

The following result presents conditions under which the trivial solution of (2.1) is uniformly globally Lipschitz stable.

Theorem 3.5. *Let $V : [t_0, +\infty) \times \overline{B_\rho} \to \mathbb{R}_+$ be a function satisfying conditions (i), (ii), (iii) and (v) of Theorem (3.4) and such that:*

(iv') $V(t, x) \geq \|x\|$ for all $(t, x) \in [t_0, +\infty) \times \overline{B_\rho}$;

Then, the trivial solution $x \equiv 0$ of (2.1) is uniformly globally Lipschitz stable.

Proof. Let $\overline{x} : [\alpha, \beta] \to \overline{B_\rho}$ be a solution of the generalized ODE (2.1) on $[\alpha, \beta]$.

From Lemma 3.3 and (iv'), it follows that, for each $t \in [\alpha, \beta]$,

$$\|\overline{x}(t)\| \leq V(t, \overline{x}(t)) \leq V(\alpha, \overline{x}(\alpha)) + K\mathrm{var}_\alpha^\beta \left(\overline{x}(s) - \int_\alpha^s DG(\overline{x}(\tau), t) \right) \leq K\|\overline{x}(\alpha)\|,$$

since $\mathrm{var}_\alpha^\beta \left(\overline{x}(s) - \int_\alpha^s DG(\overline{x}(\tau), t) \right) = 0$ and $V(\alpha, \overline{x}(\alpha)) \leq K\|\overline{x}(\alpha)\|$ (see (3.2)). This completes the proof. □

4 The correspondence between impulsive RFDEs and generalized ODEs

In this section, we describe a certain class of retarded functional differential equations with pre-fixed moments of impulsive effect. Then, we show that any solution of this class of RFDEs with pre-fixed impulses admits a one-to-one correspondence with a solution of a certain class of generalized ODEs. This correspondence is crucial to the proof of our main results.

4.1 A class of impulsive RFDEs

Consider the impulsive RFDE

$$\begin{cases} \dot{y}(t) = f(y_t, t), & t \neq t_k,\ t \geq t_0, \\ \Delta y(t) = I_k(y(t)), & t = t_k,\ k \in \mathbb{N}, \end{cases} \tag{4.1}$$

subject to initial condition

$$y_{t_0} = \phi, \tag{4.2}$$

where $\phi \in G^-([-r, 0], \mathbb{R}^n)$ and $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$. The impulse operators $I_k(x)$, $k \in \mathbb{N}$, are bounded continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and

$$\Delta y(t) = y(t+) - y(t-) = y(t+) - y(t)$$

for all $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$ and all $t \geq t_0$.

With respect to the moments of impulsive effect t_k, $k \in \mathbb{N}$, we consider the following conditions:

(C1) $t_0 < t_k$ and $t_k < t_{k+1}$ for $k \in \mathbb{N}$;

(C2) $t_k \to +\infty$ as $k \to +\infty$.

Remark 4.1. In this work, we do not consider the initial time t_0 as a moment of impulsive effect.

Let $G_1 \subset G^-([t_0 - r, +\infty), \mathbb{R}^n)$ be an open set with the prolongation property that is, if y is an element of G_1 and $\bar{t} \in [t_0, +\infty)$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & t_0 - r \leq t \leq \bar{t}, \\ y(\bar{t}), & \bar{t} < t < +\infty, \end{cases}$$

is also an element of G_1. In particular, any open ball in $G^-([t_0 - r, +\infty), \mathbb{R}^n)$ has this property.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $|\cdot|$. We assume that $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$ is such that, for every $y \in G_1$, $t \mapsto f(y_t, t)$ is Lebesgue integrable on $[t_0, +\infty)$ and moreover:

(A) There is a Lebesgue integrable function $M : [t_0, +\infty) \to \mathbb{R}$ such that for all $x \in G_1$ and all $u_1, u_2 \in [t_0, +\infty)$

$$\left| \int_{u_1}^{u_2} f(x_s, s)\, ds \right| \leq \int_{u_1}^{u_2} M(s)\, ds;$$

(B) There is a Lebesgue integrable function $L : [t_0, +\infty) \to \mathbb{R}$ such that for all $x, y \in G_1$ and all $u_1, u_2 \in [t_0, +\infty)$

$$\left| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \right| \leq \int_{u_1}^{u_2} L(s) \|x_s - y_s\|\, ds.$$

For the impulse operators $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k \in \mathbb{N}$, we assume:

(A′) There is a constant $K_1 > 0$ such that for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$ we have

$$|I_k(x)| \leq K_1;$$

(B′) There is a constant $K_2 > 0$ such that for all $k \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$ we have

$$|I_k(x) - I_k(y)| \leq K_2 |x - y|.$$

Consider the following definition of a solution of the initial value problem (4.1)–(4.2) on some interval $[t_0 - r, t_0 + \sigma]$ for $\sigma > 0$.

Definition 4.2. Let $\sigma > 0$ and consider the impulsive RFDE (4.1)–(4.2), where $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$, and for every $y \in G_1$, $t \mapsto f(y_t, t)$ is Lebesgue integrable over $t \in [t_0, t_0 + \sigma]$. If there is a function $y \in G_1$ such that

(i) $\dot{y}(t) = f(y_t, t)$, for almost every $t \in [t_0, t_0 + \sigma] \setminus \{s \in [t_0, t_0 + \sigma] : s = t_k, k \in \mathbb{N}\}$;

(ii) $y(t+) = y(t) + I_k(y(t))$, $t = t_k \in [t_0, t_0 + \sigma]$, $k \in \mathbb{N}$;

(iii) for each $k \in \mathbb{N}$, the function y_k given by

$$y_k(t) = \begin{cases} y_k(t_k) = y(t_k+) \\ y(t), & \text{if } t \in (t_k, t_{k+1}), \\ y_k(t_{k+1}) = y_k(t_{k+1}^-) \end{cases}$$

is absolutely continuous on $[t_k, t_{k+1}]$.

(iv) $y_{t_0} = \phi$,

then y is called a *solution* of (4.1) on $[t_0 - r, t_0 + \sigma]$ with initial condition (ϕ, t_0).

Let us consider the initial value problem (4.1)–(4.2) and assume that the function $f : G^{-}([-r,0],\mathbb{R}^n) \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ is such that, for every $y \in G_1$, $t \mapsto f(y_t, t)$ is Lebesgue integrable over $[t_0, t_0+\sigma]$. Then it is not difficult to see that $y \in G_1$ is a solution of (4.1)–(4.2) on $[t_0 - r, t_0+\sigma]$ if, and only if,

$$y(t) = \begin{cases} \phi(t-t_0), & t \in [t_0 - r, t_0], \\ \phi(0) + \int_{t_0}^{t} f(y_s, s)\, ds + \sum_{k=1}^{m} I_k(y(t_k)), & t \in [t_0, t_0+\sigma], \end{cases}$$

where m is such that $t_0 < t_1 < t_2 < \cdots < t_m \leq t_0 + \sigma$.

Remark 4.3. The sum $\sum_{k=1}^{m} I_k(y(t_k))$ can be rewritten as

$$\sum_{k=1}^{m} H_{t_k}(t) H_{t_k}(\vartheta) I_k(y(t_k)),$$

where m is such that $t_0 < t_1 < t_2 < \cdots < t_m \leq t_0 + \sigma$ and H_{t_k} denotes the left continuous Heaviside function concentrated at t_k, that is, the function given by

$$H_k(t) = \begin{cases} 0, & \text{for } t_0 \leq t \leq t_k, \\ 1, & \text{for } t > t_k. \end{cases}$$

4.2 Impulsive RFDEs and generalized ODEs

The results of this section can be found in [5] and they are only exposed here to provide a good understanding of the whole work to the reader.

Let $t_0 \geq 0$, $\sigma > 0$ and $r > 0$. Suppose conditions (A) and (B) of the previous subsection are satisfied for the application $f(\phi, t) : G^{-}([-r,0],\mathbb{R}^n) \times [t_0, t_0+\sigma] \to \mathbb{R}^n$ on the right-hand side of (4.1). Then, for $y \in G_1$ and $t \in [t_0, t_0+\sigma]$, define

$$F(y,t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0 \text{ or } t_0 - r \leq t \leq t_0, \\ \int_{t_0}^{\vartheta} f(y_s, s)\, ds, & t_0 \leq \vartheta \leq t \leq t_0 + \sigma, \\ \int_{t_0}^{t} f(y_s, s)\, ds, & t_0 \leq t \leq \vartheta \leq t_0 + \sigma. \end{cases} \tag{4.3}$$

Thus, for each pair $(y,t) \in G_1 \times [t_0, t_0+\sigma]$, equation (4.3) defines an element $F(y,t)$ of the space $C([t_0 - r, t_0+\sigma],\mathbb{R}^n)$ of continuous functions from $[t_0 - r, t_0+\sigma]$ to $\mathbb{R}^n$, that is,

$$F : G_1 \times [t_0, t_0+\sigma] \to C([t_0 - r, t_0+\sigma],\mathbb{R}^n)$$

and $F(y,t)(\tau) \in \mathbb{R}^n$ is the value that $F(y,t)$ assumes at a point $\tau \in [t_0 - r, t_0+\sigma]$.

As proved in [5], $F \in \mathcal{F}(\Omega, h_1)$, with $h_1 : [t_0, t_0+\sigma] \to \mathbb{R}$ given by

$$h_1(t) = \int_{t_0}^{t} [M(s) + L(s)] ds, \qquad t \in [t_0, t_0+\sigma]. \tag{4.4}$$

The function h_1 is absolutely continuous and nondecreasing, since $M, L : [t_0, t_0+\sigma] \to \mathbb{R}$ are non-negative and Lebesgue integrable functions.

Now, let us assume that conditions $(C1)$ and $(C2)$ are satisfied. For $y \in G_1$ and $t \in [t_0, t_0 + \sigma]$, define

$$J(y,t)(\vartheta) = \sum_{k=1}^{m} H_{t_k}(t) H_{t_k}(\vartheta) I_k(y(t_k)), \tag{4.5}$$

where $\vartheta \in [t_0 - r, t_0 + \sigma]$.

As in [5], one can show that $J \in \mathcal{F}(\Omega, h_2)$, where $h_2 : [t_0, t_0 + \sigma] \to \mathbb{R}$ is given by

$$h_2(t) = \max(K_1, K_2) \sum_{k=1}^{m} H_{t_k}(t). \tag{4.6}$$

It is easy to verify that h_2 is a nondecreasing real function which is continuous from the left at every point.

Taking $F(y,t)$ given by (4.3) and $J(y,t)$ given by (4.5), define for $y \in G_1$ and $t \in [t_0, t_0 + \sigma]$,

$$G(y,t)(\vartheta) = F(y,t)(\vartheta) + J(y,t)(\vartheta), \tag{4.7}$$

where $\vartheta \in [t_0 - r, t_0 + \sigma]$. Then the value of $G(y,t)$ belongs to $G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$, that is, we have

$$G : G_1 \times [t_0, t_0 + \sigma] \to G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n).$$

Besides, the function G given by (4.7) belongs to the class $\mathcal{F}(\Omega, h)$, where $\Omega = G_1 \times [t_0, t_0 + \sigma]$ and $h = h_1 + h_2$.

Assume that the functions f and I_k in equation (4.1) satisfy conditions (A), (B), (A') and (B') of the previous subsection and that the impulsive moments t_k satisfy conditions $(C1)$ and $(C2)$. Consider the generalized ODE

$$\frac{dx}{d\tau} = DG(x,t), \tag{4.8}$$

where the function G is given by (4.7).

Now, we proceed to establish the correspondence between the impulsive RFDEs and a class of generalized ODEs on each compact interval $[t_0 - r, t_0 + \sigma]$, $\sigma > 0$.

Lemma 4.4. *Let x be a solution of (4.8) on the interval $[t_0, t_0 + \sigma]$, with G given by (4.7) and with initial condition $x(t_0) \in G_1$ given by $x(t_0)(\vartheta) = \phi(\vartheta)$ for $\vartheta \in [t_0 - r, t_0]$ and $x(t_0)(\vartheta) = x(t_0)(t_0)$ for $\vartheta \in [t_0, t_0 + \sigma]$. Then if $v \in [t_0, t_0 + \sigma]$, we have*

$$x(v)(\vartheta) = x(v)(v), \qquad \vartheta \geq v, \ \vartheta \in [t_0, t_0 + \sigma] \tag{4.9}$$

and

$$x(v)(\vartheta) = x(\vartheta)(\vartheta), \qquad v \geq \vartheta, \ \vartheta \in [t_0, t_0 + \sigma]. \tag{4.10}$$

The next results give a one-to-one relation between the solution of the impulsive system (4.1)–(4.2) and the solution of the generalized ODE (4.8) with initial condition described in terms of the initial condition of (4.1)–(4.2).

Theorem 4.5 (Theorem 3.4, [5])**.** *Consider system (4.1)–(4.2), where for every $y \in G_1$, the function $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is such that $t \mapsto f(y_t, t)$ is Lebesgue integrable over $[t_0, t_0 + \sigma]$ and conditions $(C1)$ to $(C5)$ and (A), (B), (A'), (B') are fulfilled. Let y be a solution of system (4.1)–(4.2) on the interval $[t_0 - r, t_0 + \sigma]$. Given $t \in [t_0, t_0 + \sigma]$, let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma]. \end{cases} \tag{4.11}$$

Then $x(t) \in G^{-}([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *and* x *is a solution of* (4.8) *on* $[t_0, t_0 + \sigma]$ *with initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta), & \vartheta \in [t_0 - r, t_0], \\ y(t), & \vartheta \in [t_0, t_0 + \sigma]. \end{cases}$$

Theorem 4.6 ([5, Theorem 3.5])**.** *Let* G *be given by* (4.7) *and let* x *be a solution of* (4.8) *on the interval* $[t_0, t_0 + \sigma]$ *satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \le \vartheta \le t_0, \\ x(t_0)(t_0), & t_0 \le \vartheta \le t_0 + \sigma, \end{cases}$$

where $\phi \in G^{-}([-r, 0], \mathbb{R}^n)$*. For every* $\vartheta \in [t_0 - r, t_0 + \sigma]$*, let*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \le \vartheta \le t_0, \\ x(\vartheta)(\vartheta), & t_0 \le \vartheta \le t_0 + \sigma. \end{cases} \tag{4.12}$$

Then y *is a solution of system* (4.1)–(4.2) *on* $[t_0 - r, t_0 + \sigma]$.

Now, consider the generalized ODE (4.8). Let $t^* \in [t_0, +\infty)$ and $\phi \in G^{-}([-r, 0], \mathbb{R}^n)$ be given and define a function $\widetilde{x} \in G_1$ by

$$\widetilde{x}(\vartheta) = \begin{cases} \phi(\vartheta - t^*), & \text{for } \vartheta \in [t^* - r, t^*], \\ \phi(0), & \text{for } \vartheta \in [t^*, +\infty). \end{cases} \tag{4.13}$$

Looking at the initial value problem (4.8)–(4.13), Theorem 2.6 together with Theorems 4.5 and 4.6, can be used to obtain the following local existence and uniqueness result.

Theorem 4.7. *If conditions* (C1)*,* (C2)*,* (A)*,* (B)*,* (A′) *and* (B′) *are fulfilled and if* $\widetilde{x} \in G_1$ *given by* (4.13) *is such that*

$$\widetilde{x}(\vartheta) + H_{t_k}(\vartheta) I_k(\widetilde{x}(t^*)) \in G_1, \tag{4.14}$$

when $t^* = t_k$ *for some* $k \in \mathbb{N}$*, then there is a* $\Delta > 0$ *such that, in the interval* $[t^*, t^* + \Delta]$*, there is a unique solution* $y : [t^*, t^* + \Delta] \to \mathbb{R}^n$ *of the impulsive RFDE* (4.1) *for which* $y_{t^*} = \phi$.

By Theorem 2.6, for $\widetilde{x} \in G_1$, the relation

$$\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t^*+) - G(\widetilde{x}, t^*) \in G_1$$

is needed. This condition assures that the solution of the initial value problem (4.8)–(4.13) does not jump out of the set G_1 immediately at the moment t^*. Notice that, in our situation, where the function G is given by (4.8), we have

$$G(\widetilde{x}, t^*+) - G(\widetilde{x}, t^*) = 0, \quad \text{if } t^* \neq t_k,\ k = 1, 2, \ldots, m$$

and

$$[G(\widetilde{x}, t^*+) - G(\widetilde{x}, t^*)](\vartheta) = H_{t_k}(\vartheta) I_k(\widetilde{x}(t^*)), \quad \text{if } t^* = t_k,\ k = 1, 2, \ldots, m \tag{4.15}$$

and (4.15) implies condition (4.14) in Theorem 4.7.

5 Lipschitz stability for impulsive RFDEs

We consider the RFDE with impulse action (4.1)–(4.2) and we assume that conditions $(C1)$, $(C2)$, (A), (B), (A') and (B') of the previous section are fulfilled.

In the sequel, we consider

$$f(0,t) = 0 \text{ for every } t \text{ and } I_k(0) = 0,\ k \in \mathbb{N},$$

so that the function $y \equiv 0$ is a solution of the impulsive RFDE (4.1) in any interval contained in $[t_0, +\infty)$.

Our goal in this section is to present Lipschitz stability results for the trivial solution of the impulsive RFDE (4.1) through the theory of generalized ODEs using Lyapunov functionals.

Given $c > 0$, let ρ be such that $0 < \rho < c$. Consider the sets $E_c = \{\psi \in G^-([-r,0],\mathbb{R}^n) : \|\psi\| < c\}$, $\overline{E}_\rho = \{\psi \in G^-([-r,0],\mathbb{R}^n) : \|\psi\| \leq \rho\}$ and $\overline{B}_\rho = \{\psi \in G_1 : \|\psi\| \leq \rho\}$.

The following concept of Lipschitz stability of the trivial solution of (4.1) was investigated in [3] and [4], for instance.

Definition 5.1. The trivial solution $y \equiv 0$ of impulsive RFDE (4.1) is said to be:

(i) *Uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that, if $\phi \in E_c$ and $\overline{y} : [t_0 - r, +\infty) \to \mathbb{R}^n$ is a solution of (4.1) such that $\overline{y}_{t_0} = \phi$ and $\|\phi\| < \delta$, then

$$|\overline{y}(t, t_0, \phi)| \leq M\|\phi\| \quad \text{for all } t \geq t_0;$$

(ii) *Globally uniformly Lipschitz stable* if there exists $M > 0$ such that, if $\phi \in E_c$ and $\overline{y} : [t_0 - r, +\infty) \to \mathbb{R}^n$ is a solution of (4.1), with $\overline{y}_{t_0} = \phi$, then

$$|\overline{y}(t, t_0, \phi)| \leq M\|\phi\| \quad \text{for all } t \geq t_0.$$

Given $t \geq t_0$ and a function $\psi \in G^-([-r,0],\mathbb{R}^n)$, consider the impulsive RFDE (4.1) with initial condition $y_t = \psi$. We also consider the generalized ODE (4.8) subject to the initial condition $x(t) = \widetilde{x}$, where $\widetilde{x}(\tau) = \psi(\tau - t)$, for $t - r \leq \tau \leq t$, and $\widetilde{x}(\tau) = \psi(0)$, for $\tau \geq t$. Assume that $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t+) - G(\widetilde{x}, t) \in G_1$. By Theorem 4.7, there is a unique local solution $y : [t - r, v] \to \mathbb{R}^n$ of the impulsive RFDE (4.1) satisfying $y_t = \psi$. By Theorem 4.5, we can find a solution $x : [t, v] \to G^-([t, v], \mathbb{R}^n)$ of the generalized ODE (4.8), with initial condition $x(t) = \widetilde{x}$. Then $x(t)(t + \theta) = y(t + \theta)$ for any $\theta \in [-r, 0]$ and, therefore, $(x(t))_t = y_t$. In this case, we write $y_{t+\eta} = y_{t+\eta}(t, \psi)$ for every $\eta \geq 0$. Then, for $U : [t_0, +\infty) \times G^-([-r,0],\mathbb{R}^n) \to \mathbb{R}_+$, we define

$$D^+U(t,\psi) = \limsup_{\eta \to 0+} \frac{U(t+\eta, y_{t+\eta}(t,\psi)) - U(t, y_t(t,\psi))}{\eta}, \qquad t \geq t_0.$$

On the other hand, given $t \geq t_0$, if $\widetilde{x} \in G^-([t - r, +\infty), \mathbb{R}^n)$ is such that $\widetilde{x}(\tau) = \psi(\tau - t)$, $t - r \leq \tau \leq t$, and $\widetilde{x}(\tau) = \psi(0)$, $\tau \geq t$, then, by Theorem 2.6, there exists a unique solution $x : [t, \overline{v}] \to G^-([t, \overline{v}], \mathbb{R}^n)$ of the generalized ODE (4.8) such that $x(t) = \widetilde{x}$, with $[t, \overline{v}] \subset [t_0, +\infty)$. By Theorem 4.6, we can find a solution $y : [t - r, \overline{v}] \to \mathbb{R}^n$ of (4.1) which satisfies $y_t = \psi$ and is described in terms of x. In this case, we write $x_\psi(t)$ instead of $x(t)$ and we have $y_t(t, \psi) = (x_\psi(t))_t = \psi$. Consequently, $(t, x_\psi(t)) \mapsto (t, y_t(t, \psi))$ is a one-to-one mapping, and we can define a functional $V : [t_0, +\infty) \times G_1 \to \mathbb{R}_+$ by

$$V(t, x_\psi(t)) = U(t, y_t(t, \psi)). \tag{5.1}$$

Then we have

$$D^+U(t,\psi) = \limsup_{\eta\to 0^+} \frac{V(t+\eta, x_\psi(t+\eta)) - V(t, x_\psi(t))}{\eta}, \qquad t \geq t_0. \tag{5.2}$$

Remark 5.2. With the previous notations, given $t \geq t_0$, we have $\|y_t(t,\psi)\| = \|x_\psi(t)\|$, since

$$\begin{aligned}\|y_t(t,\psi)\| = \|y_t\| &= \sup_{-r\leq\theta\leq 0} |y(t+\theta)| = \sup_{t-r\leq\tau\leq t} |y(\tau)| = \sup_{t-r\leq\tau\leq t} |x_\psi(t)(\tau)| \\ &= \sup_{t-r\leq\tau<+\infty} |x_\psi(t)(\tau)| = \|x_\psi(t)\|,\end{aligned}$$

where we used Theorem 4.5 to obtain the fourth equality.

We proceed to show that the function V given by (5.1) satisfies some properties, provided the function U also satisfies some similar properties.

Lemma 5.3. *Let $U : [t_0,+\infty) \times \overline{E_\rho} \to \mathbb{R}_+$ and $V : [t_0,+\infty) \times \overline{B_\rho} \to \mathbb{R}_+$ be as in (5.1). The following statements are valid:*

(i) If $U(t,0) = 0$ for all $t \geq t_0$, then $V(t,0) = 0$ for all $t \geq t_0$;

(ii) If $U(\cdot,\psi) : [t_0,+\infty) \to \mathbb{R}_+$ is continuous from the left on $(t_0,+\infty)$ for all $\psi \in \overline{E_\rho}$, then $V(\cdot,z) : [t_0,+\infty) \to \mathbb{R}_+$ is continuous from the left on $(t_0,+\infty)$ for all $z \in \overline{B_\rho}$;

(iii) If there exist a constant $K > 0$ such that

$$|U(t,\psi) - U(t,\phi)| \leq K\|\psi - \phi\| \quad \text{for all } \psi,\phi \in \overline{E_\rho},$$

then

$$|V(t,z) - V(t,y)| \leq K\|z - y\| \quad \text{for all } z, y \in \overline{B_\rho}.$$

Proof. Item (ii) follows immediately from Definition 5.1.

Given $t \geq t_0$ and $\psi,\overline{\psi} \in \overline{E}_\rho$, let $y,\overline{y},\widehat{y} : [t-r,+\infty) \to \mathbb{R}^n$ be solutions of equation (4.1) such that $y_t = \psi$, $\overline{y}_t = \overline{\psi}$ and $\widehat{y}_t = 0$. Suppose $x,\overline{x},\widehat{x}$ are solutions on $[t,+\infty)$ of the generalized ODE (4.8) given by Theorem 4.5 with respect to y,$\overline{y}$ and $\widehat{y}$ respectively. Then $(x(t))_t = y_t = \psi$, $(\overline{x}(t))_t = \overline{y}_t = \overline{\psi}$ and $(\widehat{x}(t))_t = \widehat{y}_t = 0$. By Remark 5.2, $x_\psi(t), x_{\overline{\psi}}(t) \in \overline{B}_\rho$.

Let $V : [t_0,+\infty) \times G_1 \to \mathbb{R}_+$ be given by (5.1). We will prove $i)$ and $iii)$.

(i) $0 = U(t,0) = U(t,\widehat{y}_t(t,0)) = V(t,\widehat{x}(t)) = V(t,0)$, since $\widehat{x}(t)(\tau) = 0$ for all τ (see Theorem 4.5), that is, $\widehat{x}(t) \equiv 0$.

(ii) From (5.1), it follows that

$$|V(t,x_\psi(t)) - V(t,\overline{x}_{\overline{\psi}}(t))| = |U(t,y_t(t,\psi)) - U(t,\overline{y}_t(t,\overline{\psi}))| = |U(t,\psi) - U(t,\overline{\psi})|.$$

Then by Remark 5.2, we obtain

$$|V(t,x_\psi(t)) - V(t,\overline{x}_{\overline{\psi}}(t))| \leq K\|\psi - \overline{\psi}\| = K\|x_\psi(t) - \overline{x}_{\overline{\psi}}(t)\|. \tag{5.3}$$

It is clear that, given $t \geq t_0$ and $z,\overline{z} \in \overline{B}_\rho$, there exist solutions x and $\overline{x}$ of the generalized ODE (4.8) with initial conditions ψ and $\overline{\psi}$ respectively, $\psi,\overline{\psi} \in G^-([-r,0],\mathbb{R}^n)$, such that $z = x_\psi(t)$, $(x_\psi(t))_t = y_t(t,\psi)$, $\overline{z} = \overline{x}_{\overline{\psi}}(t)$ and $(\overline{x}_{\overline{\psi}}(t))_t = \overline{y}_t(t,\overline{\psi})$, by Remark 5.2.

Since

$$\|\psi\| = \|y_t(t,\psi)\| = \|x_\psi(t)\| = \|z\| \leq \rho$$

and

$$\|\overline{\psi}\| = \|\overline{y}_t(t,\overline{\psi})\| = \|\overline{x}_{\overline{\psi}}(t)\| = \|\overline{z}\| \leq \rho,$$

by (5.3), we have that $|V(t,z) - V(t,\overline{z})| \leq K\|z - \overline{z}\|$ which completes the proof. □

Theorem 5.4. *Let $0 < \rho < c$ and $U : [t_0, +\infty) \times \overline{E_\rho} \to \mathbb{R}_+$ be such that:*

(i) $U(t, 0) = 0$ *for all* $t \geq t_0$*;*

(ii) $U(\cdot, \psi) : [t_0, \infty) \to \mathbb{R}_+$ *is continuous from the left on* $(t_0, +\infty)$ *for all* $\psi \in \overline{E_\rho}$*;*

(iii) there exists a positive constant K *such that* $|U(t, \psi) - U(t, \phi)| \leq K\|\psi - \phi\|$ *for all* $\psi, \phi \in \overline{E_\rho}$*;*

(iv) there exists a monotone increasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $b(0) = 0$ *and* $U(t, \psi) \geq b(\|\psi\|)$ *for all* $\psi \in \overline{E_\rho}$*;*

(v) for all solution $y : [\alpha - r, \beta] \to \overline{E_\rho}$ *of impulsive RFDE* (4.1)*, with* $t_0 \leq \alpha < \beta < +\infty$*, one has* $D^+U(t, y_t) \leq 0$ *for all* $t \in [\alpha, \beta]$*.*

Then the solution $y \equiv 0$ *of* (4.1) *is uniformly Lipschitz stable.*

Proof. Consider $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}_+$ given by (5.1). By Lemma 5.3, $V(\cdot, z) : [t_0, +\infty) \to \mathbb{R}_+$ is continuous from the left on $(t_0, +\infty)$ for all $z \in \overline{B_\rho}$,

$$V(t, 0) = 0, \quad \text{for } t \in [t_0, +\infty)$$

and

$$|V(t, z) - V(t, \overline{z})| \leq K\|z - \overline{z}\|, \quad \text{for } t \in [t_0, +\infty) \text{ and } z, \overline{z} \in \overline{B}_\rho.$$

Furthermore, by relation (5.1) and conditions (iv) and (v), the functional V satisfies conditions (iv) and (v) of Theorem 3.4. Then, all hypotheses of Theorem 3.4 are satisfied. Therefore, there exist $M > 0$ and $\delta > 0$ such that if $\overline{x} : [\alpha, \beta] \to \overline{B}_\rho$, $t_0 \leq \alpha < \beta < +\infty$, is a function of bounded variation on $[\alpha, \beta]$ and it is left continuous on $(\alpha, \beta]$ such that

$$\|\overline{x}(\alpha)\| < \delta$$

and

$$\mathrm{var}_\alpha^\beta \left(\overline{x}(s) - \int_\alpha^s DG(\overline{x}(\tau), t) \right) < \delta,$$

then

$$\|\overline{x}(t)\| \leq M\|\overline{x}(\alpha)\| \quad \text{for all } t \in [\alpha, \beta].$$

Let $\phi \in \overline{E}_\rho$ and $\overline{y} : [t_0 - r, +\infty) \to \mathbb{R}^n$ be a solution of (4.1) such that $\overline{y}_{t_0} = \phi$ and

$$\|\phi\| < \delta. \tag{5.4}$$

We need to prove that

$$|\overline{y}(t, t_0, \phi)| \leq M\|\phi\|, \qquad t \in [t_0, +\infty). \tag{5.5}$$

Let us denote $\overline{y}_t = \overline{y}_t(t_0, \phi)$ and define

$$\overline{x}(t)(\tau) = \begin{cases} \overline{y}(\tau), & t_0 - r \leq \tau \leq t, \\ \overline{y}(t), & \tau \geq t. \end{cases} \tag{5.6}$$

By Theorem 4.5, $\overline{x}(t)$ is a solution on $[t_0, +\infty)$ of the generalized ODE (4.8) satisfying the initial condition $\overline{x}(t_0) = \widetilde{x}$, where

$$\widetilde{x}(\tau) = \begin{cases} \phi(\tau - t_0), & t_0 - r \leq \tau \leq t_0, \\ \phi(0), & \tau \geq t_0. \end{cases} \tag{5.7}$$

Furthermore, $\overline{x}$ is of locally bounded variation on $[t_0, +\infty)$.

By (5.4) and (5.7), we have

$$\|\overline{x}(t_0)\| = \sup_{t_0 - r \le \tau < +\infty} |\widetilde{x}(\tau)| = \|\phi\| < \delta. \tag{5.8}$$

Also, for any $v \in [t_0, +\infty)$, we have

$$\mathrm{var}_{t_0}^{v} \left(\overline{x}(s) - \int_{t_0}^{s} DG(\overline{x}(\tau), t) \right) = 0 < \delta. \tag{5.9}$$

Hence, there exists $M > 0$ such that $\|\overline{x}(t)\| < M\|\overline{x}(t_0)\|$ for every $t \in [t_0, v]$. In particular,

$$\|\overline{x}(v)\| < M\|\overline{x}(t_0)\|.$$

Then (5.6) implies that

$$\begin{aligned} |\overline{y}(t)| \le \|\overline{y}_t\| &= \sup_{-r \le \theta \le 0} |\overline{y}(t+\theta)| \le \sup_{t_0 - r \le \tau \le v} |\overline{y}(\tau)| \\ &= \sup_{t_0 - r \le \tau \le v} |\overline{x}(v)(\tau)| = \sup_{t_0 - r \le \tau < +\infty} |\overline{x}(v)(\tau)| \\ &= \|\overline{x}(v)\| < M\|\overline{x}(t_0)\| = M\|\phi\|, \end{aligned} \tag{5.10}$$

for any $t \in [t_0, v]$. Since v is arbitrary, (5.5) holds and the proof is complete. □

Theorem 5.5. *Assume that* $U : [t_0, +\infty) \times \overline{E_\rho} \to \mathbb{R}_+$ *satisfies conditions (i), (ii), (iii) and (v) from Theorem 3.5 and, in addition,*

iv′) $U(t, \phi) \ge \|\phi\|$ *for all* $\phi \in \overline{E_\rho}$.

Then the trivial solution $y \equiv 0$ *of* (4.1) *is globally uniformly Lipschitz stable.*

Proof. It is simple to verify that the functional $V : [t_0, +\infty) \times \overline{B}_\rho \to \mathbb{R}_+$ given by (5.1) satisfies all condition of Theorem 3.5. Then, there exists $M > 0$ such that, if $\overline{x} : [\alpha, \beta] \to \overline{B}_\rho$, $t_0 \le \alpha < \beta < +\infty$, is a solution of (4.8) on $[\alpha, \beta]$, then

$$\|\overline{x}(t)\| \le M\|\overline{x}(\alpha)\| \quad \text{for all } t \in [\alpha, \beta].$$

Let $\phi \in \overline{E}_\rho$ and $\overline{y} : [t_0 - r, +\infty) \to \mathbb{R}^n$ be a solution of (4.1) such that $\overline{y}_{t_0} = \phi$. To prove that

$$|\overline{y}(t, t_0, \phi)| \le M\|\phi\|, \quad t \in [t_0, +\infty), \tag{5.11}$$

it is enough to follow the steps of the proof of the previous result. □

6 Example

Consider the following single species model exhibiting the so-called Allee effect in which the per-capita growth rate is a quadratic function of the density:

$$\begin{cases} y'(t) = y(t)[a + by(t-r) - cy^2(t-r)], & t \ne t_k,\ t \ge 0 \\ \Delta y(t) = I_k(y(t)), & t = t_k,\ k \in \mathbb{N}, \end{cases} \tag{6.1}$$

where $r > 0$ characterizes the maturity of the population, $a, c \in (0, +\infty)$, $b \in \mathbb{R}$ are constants, and the impulse operators I_k, $k \in \mathbb{N}$, characterize the value of the increase or decrease of the population under the action of external perturbations (for example, human action) at the moments $0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. This model was proposed by Gopalsamy and Ladas in [7], and it was also studied by Bainov and Stamova in [3].

Let $\phi : [-r, 0] \to \mathbb{R}_+$ be a continuous function. The initial conditions for (6.1) are assumed to be as follows:

$$y(s) = \phi(s) \geq 0 \quad \text{for} \ -r \leq s < 0, \qquad y(0) > 0.$$

Let $\rho > 0$ be such that $|y(t)| < \rho$ and $|y(t-r)| < \rho$ for $t \geq 0$.

We will assume that the impulse operators I_k, $k \in \mathbb{N}$, satisfy the following conditions:

(B1) $I_k(0) = 0$ for all $k \in \mathbb{N}$;

(B2) there exist a constant $K_1 > 0$ such that for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$|I_k(x)| \leq K_1;$$

(B3) there exist a constant $K_2 > 0$ such that for all $k \in \mathbb{N}$ and $x, y \in \mathbb{R}$, we have

$$|I_k(x) - I_k(y)| \leq K_2 |x - y|;$$

(B4) $x I_k(x) < 0$ for all $x \neq 0$.

For each $t \geq 0$, let $f(y_t, t) = y(t)[a + by(t-r) - cy^2(t-r)]$. It is not difficult to verify that f satisfies conditions (A) and (B) since $|y(t)| < \rho$ and $|y(t-r)| < \rho$ for $t \geq 0$.

We will show that the trivial solution of system (6.1) is uniformly Lipschitz stable if $b\rho \leq -a$. In order to do this, we define an auxiliary function $W : \mathbb{R} \to \mathbb{R}$ by $W(s) = |s|$. We claim that for each solution y of (6.1) on a compact interval $I \subset [0, +\infty)$, we have $D^+W(y(t)) \leq 0$ for all $t \geq 0$. Indeed, we will consider two cases: when $t = t_k$ for some $k \in \mathbb{N}$ and when $t \neq t_k$ for any $k \in \mathbb{N}$.

Case 1: If $t \neq t_k$ for $k \in \mathbb{N}$, then:

(i) if $y(t) \geq 0$, we obtain

$$\begin{aligned} D^+W(y(t)) &= y'(t) = y(t)[a + by(t-r) - cy^2(t-r)] \\ &\leq y(t)[a + by(t-r)] \leq \rho a + b\rho^2 \leq \rho a - a\rho = 0, \end{aligned}$$

since $c > 0$, $y^2(t-r) \geq 0$, and $b\rho \leq -a$.

(ii) If $y(t) < 0$, we have

$$D^+W(y(t)) = -y'(t) = -y(t)[a + by(t-r) - cy^2(t-r)] \leq -y(t)[a + by(t-r)] < 0,$$

since $y(t)cy^2(t-r) \leq 0$, $-y(t) > 0$, and $a + by(t-r) < a + b\rho \leq 0$.

Case 2: On the other hand, if $t = t_k$ for some $k \in \mathbb{N}$, then:

(i') if $y(t_k) > 0$, we have

$$0 < (1 - K_2) y(t_k) \leq I_k(y(t_k)) + y(t_k) \leq y(t_k),$$

by (B_1), (B_3), and (B_4). Thus, $W(I_k(y(t_k)) + y(t_k)) \leq W(y(t_k))$.

(ii′) If $y(t_k) < 0$, we get

$$y(t_k) < I_k(y(t_k)) + y(t_k) \le (1 - K_2)y(t_k) < 0,$$

that is, $W(I_k(y(t_k)) + y(t_k)) \le W(y(t_k))$.
Therefore, we can conclude in both cases that

$$W(y(t_k^+)) \le W(y(t_k)).$$

From the continuity of W, it follows that

$$W(y(t_k^+)) = \lim_{\sigma \to t_k^+} W(y(\sigma)) \le W(y(t_k)).$$

Then, there exists a right neighborhood of t_k, say $(t_k, t_k + \delta)$ for some $\delta > 0$, such that

$$W(y(\sigma)) \le W(y(t_k)), \qquad \text{for all } \sigma \in (t_k, t_k + \delta).$$

Let $\eta > 0$ be sufficiently small such that $t_k + \eta \in (t_k, t_k + \delta)$. Thus,

$$W(y(t_k + \eta)) \le W(y(t_k))$$

and

$$D^+W(y(t_k)) = \lim_{\eta \to 0^+} \frac{W(y(t_k + \eta)) - W(y(t_k))}{\eta} \le 0,$$

whence we can conclude that

$$D^+W(y(t)) \le 0 \quad \text{for all } t \ge 0. \tag{6.2}$$

Define a functional $U : [0, +\infty) \times G^-([-r, 0], \mathbb{R}) \to \mathbb{R}$ by

$$U(t, \phi) = \sup_{-r \le \theta \le 0} W(\phi(\theta)) = \sup_{-r \le \theta \le 0} |\phi(\theta)| = \|\phi\|.$$

It is easy to verify that U satisfies conditions (i), (ii), (iii) and (iv) from Theorem 5.4.
We will prove that U satisfies condition (v) from Theorem 5.4, that is, $D^+U(t, y(t)) \le 0$ for all $t \ge 0$. If $\theta_0 = 0$, it follows from (6.2) that $D^+U(t, y_t) = D^+W(y(t)) \le 0$.
Now, let $-r \le \theta_0 < 0$. For $\eta > 0$ sufficiently small, we have

$$\sup_{-r \le \theta \le 0} W(y_{t+\eta}(\theta)) = \sup_{-r \le \theta \le 0} W(y_t(\theta))$$

and, hence,

$$D^+U(t, y_t) = \limsup_{\eta \to 0^+} \frac{\sup\limits_{-r \le \theta \le 0} W(y_{t+\eta}(\theta)) - \sup\limits_{-r \le \theta \le 0} W(y_t(\theta))}{\eta} = 0.$$

Therefore, Theorem 5.4 implies the solution $y \equiv 0$ of (6.1) is uniformly Lipschitz stable.

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