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Multiple solutions of nonlinear partial functional differential equations and systems

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Abstract. We shall consider weak solutions of initial-boundary value problems for semilinear and nonlinear parabolic differential equations with certain nonlocal terms, further, systems of elliptic functional differential equations. We shall prove theorems on the number of solutions and find multiple solutions.

These statements are based on arguments for fixed points of some real functions and operators, respectively, and existence-uniqueness theorems on partial differential equations (without functional terms).

Keywords: partial functional differential equations, multiple solutions.

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1 Introduction

It is well known that mathematical models of several applications are functional differential equations of one variable (e.g. delay equations). In the monograph by Jianhong Wu [8] semilinear evolutionary partial functional differential equations and applications are considered, where the book is based on the theory of semigroups and generators. In the monograph by A. L. Skubachevskii [7] linear elliptic functional differential equations (equations with nonlocal terms and nonlocal boundary conditions) and applications are considered. A nonlocal boundary value problem, arising in plasma theory, was considered by A. V. Bitsadze and A. A. Samarskii in [1].

It turned out that the theory of pseudomonotone operators is useful to study nonlinear (quasilinear) partial functional differential equations (both stationary and evolutionary equations) and to prove existence of weak solutions (see [2,4,5]).

In [6] we considered some nonlinear elliptic functional differential equations where we proved theorems on the number of weak solutions of boundary value problems for such equations and showed existence of multiple solutions.

In the present work we shall consider nonlinear parabolic functional equations and systems of elliptic functional equations. By using ideas of [6]: arguments for fixed points of

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certain real functions and operators, respectively, we shall prove theorems on the number of solutions of such problems and show existence of multiple solutions.

First we recall the definition of weak solutions of boundary value problems for the nonlinear (quasilinear) elliptic equation

$$-\sum_{j=1}^{n} D_j[a_j(x,u,Du)] + a_0(x,u,Du) = F, \qquad x \in \Omega \tag{1.1}$$

with (zero) Dirichlet boundary condition $u(x) = 0$ on $\partial\Omega$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary (e.g. $\partial\Omega \in C^1$), $1 < p < \infty$. Denote by $W^{1,p}(\Omega)$ the usual Sobolev space of real valued functions with the norm

$$\| u \| = \left[\int_\Omega (|Du|^p + |u|^p) \right]^{1/p}.$$

Further, let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace containing $C_0^\infty(\Omega)$, $V^\star$ the dual space of V, the duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$.

Weak solutions of (1.1) are defined as functions $u \in V$ satisfying

$$\int_\Omega \left[\sum_{j=1}^{n} a_j(x,u,Du) D_j v + a_0(x,u,Du) v \right] dx = \langle F, v \rangle$$

for all $v \in V$ where $F \in V^\star$ is a given element and $V = W_0^{1,p}(\Omega)$ (the closure of $C_0^1(\Omega)$ in $W^{1,p}(\Omega)$) in the case of homogeneous Dirichlet boundary condition and $V = W^{1,p}(\Omega)$ in the case of homogeneous Neumann boundary condition

$$\partial_\nu^\star u = \sum_{j=1}^{n} a_j(x,u,Du)\nu_j = 0 \quad \text{for } x \in \partial\Omega$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal on the boundary $\partial\Omega$.

By using the theory of monotone operators, one can prove existence and uniqueness theorems on weak solutions of the above boundary value problems. Namely, consider the (nonlinear) operator $A : V \to V^\star$, defined by

$$\langle A(u), v \rangle := \int_\Omega \left[\sum_{j=1}^{n} a_j(x,u,Du) D_j v + a_0(x,u,Du) v \right] dx, \qquad v \in V. \tag{1.2}$$

One can formulate sufficient conditions on functions a_j which imply that the operator $A : V \to V^\star$ is bijection, i.e. for arbitrary $F \in V^\star$ there exists a unique solution $u \in V$ of the equation $A(u) = F$. (See [3,9].)

Namely, these sufficient conditions are:

$(A1)$ the functions $a_j : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy the Carathéodory conditions;

$(A2)$ there exist a constant c_1 and a function $k_1 \in L^q(\Omega)$ $(1/p + 1/q = 1)$ such that

$$|a_j(x,\xi)| \le c_1|\xi|^{p-1} + k_1(x);$$

$(A3)$ there exists a constant $c_2 > 0$ such that the inequality

$$\sum_{j=0}^{n} [a_j(x,\xi) - a_j(x,\xi^\star)](\xi_j - \xi_j^\star) \ge c_2 \sum_{j=0}^{n} |\xi_j - \xi_j^\star|.$$

holds.

A typical example, having this property, is operator A defined by p-Laplacian $\triangle_p$:

$$A = -\triangle_p u + c_0 u|u|^{p-2} = -\sum_{j=1}^{n} D_j[(D_j u)|Du|^{p-2}] + c_0 u|u|^{p-2}$$

with $p \geq 2$ and constant $c_0 > 0$. (Clearly, $\triangle_2 u = \triangle u$.)

Now we remind the definition of weak solutions of initial-boundary value value problems for nonlinear parabolic differential equations

$$D_t u - \sum_{j=1}^{n} D_j[a_j(t,x,u,Du)] + a_0(t,x,u,Du) = F \tag{1.3}$$

(for simplicity) with homogeneous initial and boundary condition.

Denote by $L^p(0,T;V)$ the Banach space of functions $u : (0,T) \to V$ ($V \subset W^{1,p}(\Omega)$ is a closed linear subspace) with the norm

$$\| u \| = \left[\int_0^T \| u(t) \|_V^p \, dt \right]^{1/p} \qquad (1 < p < \infty).$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p + 1/q = 1$. Weak solutions of (1.3) with zero initial and boundary condition is a function $u \in L^p(0,T;V)$ satisfying $D_t u \in L^q(0,T;V^\star)$ and

$$D_t u + A(u) = F, \qquad u(0) = 0$$

where $F \in L^q(0,T;V^\star)$ is a given function,

$$\langle [A(u)](t), v \rangle = \int_\Omega \left[\sum_{j=1}^{n} a_j(t,x,u,Du) D_j v + a_0(t,x,u,Du) v \right] dx \tag{1.4}$$

for all $v \in V$, almost all $t \in [0,T]$. (For $p \geq 2$, $u \in L^p(0,T;V)$ and $D_t u \in L^q(0,T;V^\star)$ imply $u \in C([0,T];L^2(\Omega))$ thus the initial condition $u(0) = 0$ makes sense.)

There are well-known conditions on functions a_j which imply that the operator $A : L^p(0,T;V) \to L^q(0,T;V^\star)$ (defined in (1.4)) is bijection, so for arbitrary $F \in L^q(0,T;V^\star)$ there exists a unique weak solution $u \in L^p(0,T;V)$ of the problem

$$D_t u + A(u) = F, \qquad u(0) = 0.$$

(See [3,9].) A simple example for A is

$$A(u) = -\triangle_p u + c_0 u|u|^{p-2}$$

with a positive constant c_0 (here A is not depending on t).

2 Parabolic equations with real valued functionals, applied to the solution

First consider a semilinear parabolic functional equation of the form

$$D_t u + Au = D_t u - \sum_{j,k=1}^{n} D_j[a_{jk}(x) D_k u] + a_0(x) u = k(Mu) F_1 + F_2 \tag{2.1}$$

(i.e. the elliptic operator A in (1.4) is linear), where $M : L^2(0,T;V) \to \mathbb{R}$ is a given linear continuous functional, $V \subset W^{1,2}(\Omega)$, $k : \mathbb{R} \to \mathbb{R}$ is a given continuous function, $F_1, F_2 \in L^2(0,T;V^\star)$. Further, $a_{jk}, a_0 \in L^\infty(\Omega)$, $a_{jk} = a_{kj}$ and the functions a_{jk} satisfy the uniform ellipticity condition

$$c_1|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k + a_0(x)\xi_0^2 \leq c_2|\xi|^2$$

for all $\xi = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, $x \in \Omega$ with some positive constants c_1, c_2. It is well known that in this case for all $F \in L^2(0,T;V^\star)$ there exists a unique weak solution u of

$$D_t u - \sum_{j,k=1}^{n} D_j[a_{jk}(x)D_k u] + a_0(x)u = F, \tag{2.2}$$

denoted by $u = B^{-1}F$ where $B^{-1} : L^2(0,T;V^\star) \to L^2(0,T;V)$ is a linear continuous operator. Consequently, $u \in L^2(0,T;V)$ is a weak solution of (2.1) if and only if

$$u = k(Mu)B^{-1}F_1 + B^{-1}F_2. \tag{2.3}$$

This equality implies that

$$Mu = k(Mu)M(B^{-1}F_1) + M(B^{-1}F_2). \tag{2.4}$$

Theorem 2.1. *A function $u \in L^2(0,T;V)$ is a weak solution of (2.1) if and only if $\lambda = Mu$ satisfies the equation*

$$\lambda = k(\lambda)M(B^{-1}F_1) + M(B^{-1}F_2), \tag{2.5}$$

and

$$u = k(\lambda)B^{-1}F_1 + B^{-1}F_2. \tag{2.6}$$

Proof. If u satisfies (2.1) then by (2.4) $\lambda = Mu$ satisfies the equation (2.5) and by (2.3) u satisfies (2.6). Conversely, if λ is a solution of (2.5) then for u, defined by (2.6) we have

$$Mu = k(\lambda)M(B^{-1}F_1) + M(B^{-1}F_2) = \lambda$$

and

$$\begin{aligned} D_t u + Au &= k(\lambda)[D_t(B^{-1}F_1) + A(B^{-1}F_1)] + [D_t(B^{-1}F_2) + A(B^{-1}F_2)] \\ &= k(\lambda)F_1 + F_2 = k(Mu)F_1 + F_2. \end{aligned}$$

□

Corollary 2.2. *The number of weak solutions u of (2.1) (with homogeneous initial-boundary condition) equals the number of solutions λ of equation (2.5).*

E.g. assume that $k \in C^1(\mathbb{R})$ and the function h defined by

$$h(\lambda) = \lambda - k(\lambda)M(B^{-1}F_1)$$

has the property $\inf_{\lambda\in\mathbb{R}} h'(\lambda) > 0$ or $\sup_{\lambda\in\mathbb{R}} h'(\lambda) < 0$. Then for any $F_2 \in L^2(0,T;V^\star)$ the problem (2.1) has exactly one solution u. In this case the mapping $L^2(0,T;V^\star) \to L^2(0,T;V)$ which maps F_2 to u is continuous since $h^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.

Further, assuming $M(B^{-1}F_1) \neq 0$, for arbitrary $N = 0, 1, \dots, \infty$ we can construct continuous functions $k : \mathbb{R} \to \mathbb{R}$ such that the initial-boundary value problem (2.1) has exactly N weak solutions, as follows. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function having N zeros and define function k by the formula

$$k(\lambda) = \frac{g(\lambda) + \lambda - M(B^{-1}F_2)}{M(B^{-1}F_1)}. \tag{2.7}$$

Then, clearly, equation (2.5) has N solutions.

Corollary 2.3. *The number of solutions of* (2.1)*, with fixed function k depends on F_2 (on the value of $M(B^{-1}F_2)$).*

This statement can be illustrated as follows. Let $F_2 \in L^2(0, T; V^\star)$ be fixed and consider μF_2 instead of F_2 with some parameter $\mu \in \mathbb{R}$. Then equation (2.5) has the form

$$\lambda = k(\lambda) M(B^{-1}F_1) + \mu M(B^{-1}F_2),$$

thus in this case

$$g(\lambda) = g_\mu(\lambda) = k(\lambda) M(B^{-1}F_1) + \mu M(B^{-1}F_2) - \lambda. \tag{2.8}$$

Assume that $M(B^{-1}F_1) \neq 0$, $M(B^{-1}F_2) \neq 0$. It is not difficult to construct function k such that for $\mu > \mu_0$ the function g_μ has no zeros and for $\mu = \mu_0$ has infinitely many zeros.

E.g. let

$$k(\lambda) = \frac{\lambda}{M(B^{-1}F_1)} + \sin\lambda,$$

then

$$g_\mu(\lambda) = M(B^{-1}F_1)\left[\sin\lambda + \mu\frac{M(B^{-1}F_2)}{M(B^{-1}F_1)}\right].$$

Consequently, for

$$\mu = \mu_0 = \frac{M(B^{-1}F_1)}{M(B^{-1}F_2)}$$

g_μ has infinitely many zeros and for $\mu > \mu_0$ it has no zeros.

It is not difficult to show that if

$$k(\lambda) = \frac{\lambda}{M(B^{-1}F_1)} + \sin\lambda + \frac{1}{\lambda}$$

then for $\mu = \mu_0$ the function g_μ defined by (2.8) has no positive zeros but for $0 < \mu/\mu_0 < 1$ the function g_μ has infinitely many zeros.

Further, by using (2.5), if $M(B^{-1}F_1) \neq 0$, it is not difficult to construct continuous functions k such that for arbitrary $F_2 \in L^2(0, T; V^\star)$ the problem (2.1) has 3 solutions.

Remark 2.4. Assume that $k \in C^1(\mathbb{R})$, for some $F_2 \in L^2(0, T; V^\star)$ problem (2.1) has N zeros: $u_1, u_2, \ldots, u_N$ and for the function

$$g(\lambda) = k(\lambda) M(B^{-1}F_1) + M(B^{-1}F_2) - \lambda,$$
$$g'(\lambda_j) = k'(\lambda_j) M(B^{-1}F_1) - 1 \neq 0 \quad \text{for } j = 1, \ldots, N, \text{ where } \lambda_j = Mu_j.$$

Then there exist $\varepsilon > 0$, $\delta > 0$ (they are independent) such that

$$\| \tilde{F}_2 - F_2 \|_{L^2(0,t;V^\star)} < \delta$$

implies: for every j there exists a unique $\tilde{u}_j \in L^2(0, T; V)$ weak solution of (2.1) with the property

$$\| \tilde{u}_j - u_j \|_{L^2(0,t;V)} < \varepsilon,$$

where on the right hand side of (2.1) $\tilde{F}_2$ is instead of F_2. Further, $\tilde{u}_j$ depends continuously on $\tilde{F}_2$, belonging to δ neighborhood of F_2.

Proof. Consider the function h defined by

$$h(\lambda, c) = k(\lambda)M(B^{-1}F_1) - \lambda + c$$

and apply the implicit function theorem to this function. Since

$$h(\lambda_j, M(B^{-1}F_2)) = 0, \qquad \partial_1 h(\lambda_j, M(B^{-1}F_2)) \neq 0 \qquad (j = 1, \dots, N),$$

there exist $\varepsilon_0, \delta_0 > 0$ such that for every fixed j, $|\tilde{c} - c| < \delta_0$ implies that there exists a unique $\tilde{\lambda}_j$ satisfying

$$h(\tilde{\lambda}_j, \tilde{c}) = 0, \qquad |\tilde{\lambda}_j - \lambda_j| < \varepsilon_0$$

and $\tilde{\lambda}_j$ depends continuously on $\tilde{c}$ (in the δ_0 neighborhood of $c = M(B^{-1}F_2)$). Hence we obtain the statement of Remark 2.4. □

In this case the problem (2.1) with the right hand side $\tilde{F}_2$ may have other solutions, too. (See the first example in Corollary 2.3.)

Remark 2.5. The linear continuous functional $M : L^2(0, T; V) \to \mathbb{R}$ may have the form

$$Mu = \int_0^T \int_\Omega \left[K_0(t, x)u(t, x) + \sum_{j=1}^n K_j(t, x)D_j u(t, x) \right] dtdx, \tag{2.9}$$

where $K \in L^2((0, T) \times \Omega)$. In this case the value of solutions of the initial-boundary problem for (2.1) in some time t are connected with the values of u in all $t \in [0, T]$.

Now consider *nonlinear* parabolic functional equations of the form

$$D_t u + [l(Mu)]^\gamma A(u) = [l(Mu)]^\beta F, \tag{2.10}$$

where the nonlinear operator A has the form (1.4) and has the property

$$A(\mu u) = \mu^{p-1} A(u), \quad \text{for all } \mu > 0 \quad (p > 1) \tag{2.11}$$

(e.g. $A(u) = -\triangle_p u + c_0 u|u|^{p-1}$ with $c_0 > 0$ has this property), further, $M : V \to \mathbb{R}$ is (homogeneous) functional with the property

$$M(\mu u) = \mu^\sigma M(u) \quad \text{for all } \mu > 0 \text{ with some } \sigma > 0; \tag{2.12}$$

l is a given positive continuous function and the numbers β, γ satisfy

$$\gamma = \beta(2 - p), \qquad \beta > 0.$$

Theorem 2.6. *A function $u \in V$ is a weak solution of* (2.10) *with zero initial and boundary condition if and only if $\lambda = M(u)$ satisfies the equation*

$$\lambda = [l(\lambda)]^{\beta\sigma} M[B^{-1}(F)] \quad \textit{and} \quad u = [l(\lambda)]^\beta B^{-1} F, \tag{2.13}$$

where B is defined by $B(u) = D_t u + A(u)$, i.e. $B^{-1}(u)$ is the unique weak solution of (1.3) *(with zero initial and boundary condition).*

Proof. Define u_μ by $\mu^{-\beta}u$ with some positive number μ. Then

$$u = \mu^\beta u_\mu, \qquad D_t u = \mu^\beta D_t u_\mu, \qquad A(u) = \mu^{\beta(p-1)} A(u_\mu),$$

thus a function u satisfies the equation

$$D_t u + \mu^\gamma A(u) = \mu^\beta F \tag{2.14}$$

if and only if

$$D_t u_\mu + A(u_\mu) = F. \tag{2.15}$$

Consequently, a function $u \in V$ satisfies (2.10) in weak sense (i.e. (2.14) with $\mu = l[M(u)]$) if and only if

$$\tilde{u} = [l(M(u))]^{-\beta} u \quad \text{satisfies} \quad D_t \tilde{u} + A(\tilde{u}) = F. \tag{2.16}$$

The solution of (2.16) is $\tilde{u} = B^{-1}(F)$, therefore the weak solution of (2.10) is

$$u = [l(M(u))]^\beta \tilde{u} = [l(M(u))]^\beta B^{-1} F,$$

hence

$$M(u) = [l(M(u))]^{\beta\sigma} M(\tilde{u}) = [l(M(u))]^{\beta\sigma} M[B^{-1}F]. \tag{2.17}$$

Thus, if u satisfies (2.10) then $\lambda = M(u)$ and u satisfy (2.13).

Conversely, if $\lambda \in \mathbb{R}$ is a solution of (2.13) then

$$u = [l(\lambda)]^\beta B^{-1}(F)$$

is a solution of (2.10) because

$$M(u) = [l(\lambda)]^{\beta\sigma} M[B^{-1}(F)] = \lambda,$$

so $\lambda = M(u)$, further, u satisfies (2.10) in weak sense, since (2.10) holds if and only if

$$D_t \tilde{u} + A(\tilde{u}) = F, \quad \text{where} \quad \tilde{u} = [l(M(u))]^{-\beta} u. \qquad \square$$

Corollary 2.7. *The number of weak solutions of* (2.10) *equals the number of roots of* (2.13).

Further, assuming $M[B^{-1}(F)] > 0$, for arbitrary $N = 1, 2, \ldots, \infty$ one can construct a continuous positive function l such that (2.10) *has exactly N solutions, in the following way. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(\lambda) + \lambda > 0$ for all $\lambda \in \mathbb{R}$ and g has N real roots. Then for*

$$l(\lambda) = \left[\frac{g(\lambda) + \lambda}{M(B^{-1}(F))} \right]^{1/(\beta\sigma)}$$

(2.10) *has N weak solutions.*

Remark 2.8. Let the functional l be fixed. Then the number of solutions of (2.10) depends on F. Similar examples can be constructed as in Corollary 2.3.

Remark 2.9. An example for functional M with property (2.12) is integral operator of the form

$$M(u) = \int_0^T \int_\Omega K(t,x) |u(t,x)|^\sigma \, dt dx.$$

3 Parabolic equations with nonlocal operators

Now consider partial functional equations of the form

$$Bu = D_t u + Au = C(u), \tag{3.1}$$

where A is a uniformly elliptic linear differential operator (see (2.1) or (2.2)) and $C : L^2(0,T;V) \to L^2(0,T;V^\star)$ is a given (possibly nonlinear) operator. Clearly, $u \in V$ satisfies (3.1) if and only if

$$u = B^{-1}[C(u)] =: G(u), \tag{3.2}$$

where $G : L^2(0,T;V) \to L^2(0,T;V$ is a given (possibly nonlinear) operator, i.e. u is a fixed point of G. Then

$$C(u) = B[G(u)]. \tag{3.3}$$

Now we consider three particular cases for G.

1. The operator G is defined by

$$[G(u)](t,x) = (Lu)(t,x) + F(t,x) = \int_0^T \int_\Omega K(t,\tau,x,y)u(\tau,y)d\tau dy + F(t,x), \tag{3.4}$$

where $K \in L^2([0,T] \times [0,T] \times \Omega \times \Omega)$, $u \in L^2((0,T) \times \Omega)$.

Theorem 3.1. *If K and F are sufficiently smooth and "good" then the solution $u \in L^2((0,T) \times \Omega)$ of (3.2) with the operator (3.4) belongs to $L^2(0,T;V)$, $D_t u$ belongs to $L^2(0,T;V^\star)$, $u(0) = 0$,*

$$(Cu)(t,x) = \int_0^T \int_\Omega [D_t K(t,\tau,x,y) + A_x K(t,\tau,x,y)]u(\tau,y)d\tau dy + D_t F(t,x) + A_x F(t,x)$$

and the equation (3.1) has the form

$$(Bu)(t,x) = \int_0^T \int_\Omega [D_t K(t,\tau,x,y) + A_x K(t,\tau,x,y)]u(\tau,y)d\tau dy + D_t F(t,x) + A_x F(t,x). \tag{3.5}$$

($A_x K(t,\tau,x,y)$ denotes the differential operator applied to $x \to K(t,\tau,x,y)$.)

Further, if 1 is an eigenvalue of the linear integral operator L with multiplicity N then (3.5) may have N linearly independent solutions.

Proof. Equation (3.5) is equivalent with

$$u(t,x) = (Gu)(t,x) = \int_0^T \int_\Omega K(t,\tau,x,y)u(\tau,y)d\tau dy + F(t,x)$$

which implies Theorem 3.1 since for a solution $u \in L^2((0,T) \times \Omega)$ of the last equation we have $u \in L^2(0,T;V)$, $D_t u \in L^2(0,T;V^\star)$ by the assumption of the theorem. □

Remark 3.2. Similarly to the problems in the previous section, the value of solutions u of (3.5) in some time t, are connected with the values of u for $t \in [0,T]$.

2. Now consider the case

$$(Gu)(t,x) = \int_\Omega K(x,y)u(t,y)dy, \tag{3.6}$$

where $K \in L^2(\Omega \times \Omega)$, $u \in L^2((0,T) \times \Omega)$.

Theorem 3.3. *Assume that K is sufficiently smooth such that the operator G, defined by* (3.6)*, i.e.*

$$(Gv)(x) = \int_\Omega K(x,y)v(y)dy,$$

maps V into V. Then the equation (3.1) *has the form*

$$(Bu)(t,x) = \int_\Omega \left[K(x,y)D_t u(t,y) + A_x[K(x,y)]u(t,y)\right] dy. \tag{3.7}$$

Further, if 1 is an eigenvalue of the integral operator (3.6)*, applied to $v \in L^2(\Omega)$, then the equation* (3.7) *with zero initial and boundary condition has infinitely many linearly independent solutions.*

Proof. The equality (3.7) follows directly from (3.1) and (3.3).

Let 1 be an eigenvalue and $v \in L^2(\Omega)$ an eigenfunction of G then by assumption $v \in V$. Further, let $\tau \in C^1[0,T]$ with the property $\tau(0) = 0$ then functions u, defined by $u(t,x) = \tau(t)v(x)$ are weak solutions of (3.7) with 0 initial condition. □

Remark 3.4. In the case of equations (3.7) the value of solutions u in some t are connected with the values of u only in t. (Compare to Remarks 2.5 and 3.2.)

3. Now consider operators G of the form

$$G(u) = Lu + h(Pu)F + H, \tag{3.8}$$

where operator L is defined in (3.4) and its kernel has the same smoothness property, $P : L^2(0,T;V) \to \mathbb{R}$ is a linear continuous functional, $h : \mathbb{R} \to \mathbb{R}$ is a given continuous function and $F, H \in L^2(0,T;V)$, $D_tF, D_tH \in L^2(0,T;V)$. Here assume that 1 is not an eigenvalue of the integral operator $L : L^2((0,T)\times\Omega) \to L^2((0,T)\times\Omega)$.

Theorem 3.5. *In this case equation* (3.1) *has the form*

$$Bu = \int_0^T \int_\Omega [D_t K(t,\tau,x,y) + A_x K(t,\tau,x,y)]u(\tau,y)d\tau dy + h(Pu)BF + BH. \tag{3.9}$$

Further, u is a weak solution of (3.9) *if and only if $u = h(\lambda)P[(I-L)^{-1}F] + (I-L)^{-1}H$ where λ is a root of the equation*

$$\lambda = h(\lambda)P[(I-L)^{-1}F] + P[(I-L)^{-1}H]. \tag{3.10}$$

Thus the number of solutions of (3.9) *equals the number of the roots of* (3.10).

Proof. Equation (3.9) is fulfilled if and only if u is a solution, belonging to $L^2((0,T)\times\Omega)$ of

$$u(t,x) = \int_0^T \int_\Omega K(t,\tau,x,y)u(\tau,y)d\tau dy + h(Pu)F(t,x) + H(t,x),$$

since by the properties of F, H and L, for such a solution $u \in L^2(0,T;V)$, and $D_tu \in L^2(0,T;V)$ hold. Thus (3.9) is equivalent with $u \in L^2((0,T)\times\Omega)$ and

$$(I-L)u = h(Pu)F + H, \quad u = h(Pu)(I-L)^{-1}F + (I-L)^{-1}H. \tag{3.11}$$

Let $u_\lambda = h(\lambda)(I-L)^{-1}F + (I-L)^{-1}H$ then

$$P(u_\lambda) = h(\lambda)P[(I-L)^{-1}F] + P[(I-L)^{-1}H].$$

Consequently, (3.11) (and so (3.9)) is satisfied if and only if $\lambda = Pu$ satisfies (3.10). □

Corollary 3.6. *If $P[(I-L)^{-1}F] \neq 0$ then for arbitrary N $(=0,1,\dots,\infty)$ we can construct h such that (3.9) has N solutions, in the following way. Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous functions having N zeros. Then (3.9) has N solutions if*

$$h(\lambda) = \frac{g(\lambda)+\lambda - P[(I-L)^{-1}H]}{P[(I-L)^{-1}F]}.$$

Remark 3.7. The linear functional $P : L^2(0,T;V) \to \mathbb{R}$ may have the form (2.9).

Remark 3.8. For fixed functions h, F the number of solutions of (3.9) depends on H by (3.10). It may happen that the number of solutions of the problem with μF (where μ is a real parameter) is 0 for $\mu > \mu_0$ and is some N $(=1,2,\dots,\infty)$ for $\mu=\mu_0$. (See Corollary 2.3.)

Further, assuming that for the function φ defined by

$$\varphi(\lambda) = \lambda - h(\lambda)P[(I-L)^{-1}F]$$

we have

$$\inf_{\lambda\in\mathbb{R}} \varphi'(\lambda) > 0 \quad \text{or} \quad \sup_{\lambda\in\mathbb{R}} \varphi'(\lambda) < 0$$

then for any (sufficiently smooth) H the equation (3.9) has exactly one solution.

4 Systems of elliptic functional equations

First consider systems of semilinear elliptic functional differential equations of the form

$$A_1 u = l_1(Mu)F_1 + k_1(Nv)G_1 + H_1, \tag{4.1}$$

$$A_2 v = l_2(Mu)F_2 + k_2(Nv)G_2 + H_2, \tag{4.2}$$

where $A_j : V \to V^\star$ are uniformly elliptic linear differential operators ($V \subset W^{1,2}(\Omega)$), $F_j, G_j, H_j \in V^\star$; $M, N : V \to \mathbb{R}$ are linear continuous functionals and $l_j, k_j : \mathbb{R}\to\mathbb{R}$ are continuous functions.

Clearly, u, v are weak solutions of (4.1), (4.2) with homogeneous boundary conditions if and only if

$$u = l_1(Mu)A_1^{-1}F_1 + k_1(Nv)A_1^{-1}G_1 + A_1^{-1}H_1, \tag{4.3}$$

$$v = l_2(Mu)A_2^{-1}F_2 + k_2(Nv)A_2^{-1}G_2 + A_2^{-1}H_2. \tag{4.4}$$

Remark 4.1. Functionals M, N may have the form

$$Mu = \int_\Omega \left[a(x)u(x) + \sum_{j=1}^n b_j(x)D_j u(x)\right] dx, \quad \text{where } a, b_j \in L^2(\Omega).$$

Theorem 4.2. *Functions $u, v \in V$ satisfy (4.1), (4.2) if and only if*

$$u = l_1(\lambda_1)A_1^{-1}F_1 + k_1(\lambda_2)A_1^{-1}G_1 + A_1^{-1}H_1, \tag{4.5}$$

$$v = l_2(\lambda_1)A_2^{-1}F_2 + k_2(\lambda_2)A_2^{-1}G_2 + A_2^{-1}H_2, \tag{4.6}$$

where $\lambda_1 = Mu$ and $\lambda_2 = Nv$ are roots of the algebraic system

$$\lambda_1 = l_1(\lambda_1)M(A_1^{-1}F_1) + k_1(\lambda_2)M(A_1^{-1}G_1) + M(A_1^{-1}H_1), \tag{4.7}$$

$$\lambda_2 = l_2(\lambda_1)N(A_2^{-1}F_2) + k_2(\lambda_2)N(A_2^{-1}G_2) + N(A_2^{-1}H_2). \tag{4.8}$$

Proof. If u, v are solutions of (4.1), (4.2) then by (4.3), (4.4)

$$Mu = l_1(Mu)M(A_1^{-1}F_1) + k_1(Nv)M(A_1^{-1}G_1) + M(A_1^{-1}H_1), \tag{4.9}$$
$$Nv = l_2(Mu)N(A_2^{-1}F_2) + k_2(Nv)N(A_2^{-1}G_2) + N(A_2^{-1}H_2). \tag{4.10}$$

Thus $\lambda_1 = Mu$ and $\lambda_2 = Nv$ satisfy (4.7), (4.8).

Conversely, if λ_1, λ_2 are roots of (4.7), (4.8) then for the functions u, v defined by (4.5), (4.6) we have

$$Mu = l_1(\lambda_1)M(A_1^{-1}F_1) + k_1(\lambda_2)M(A_1^{-1}G_1) + M(A_1^{-1}H_1) = \lambda_1,$$
$$Nv = l_2(\lambda_1)N(A_2^{-1}F_2) + k_2(\lambda_2)N(A_2^{-1}G_2) + N(A_2^{-1}H_2) = \lambda_2$$

and

$$A_1u = l_1(\lambda_1)F_1 + k_1(\lambda_2)G_1 + H_1 = l_1(Mu)F_1 + k_1(Nv)G_1 + H_1,$$
$$A_2v = l_2(\lambda_1)F_2 + k_2(\lambda_2)G_2 + H_2 = l_2(Mu)F_2 + k_2(Nv)G_2 + H_2,$$

i.e. u and v satisfy the system (4.1), (4.2). □

Corollary 4.3. *The number of weak solutions of* (4.1), (4.2) *equals the number of roots of the algebraic system* (4.7), (4.8).

Theorem 4.4. *Assume that the function χ defined by*

$$\chi(\lambda_1) = \lambda_1 - l_1(\lambda_1)M(A_1^{-1}F_1) \tag{4.11}$$

is strictly monotone and its range is $\mathbb{R}$. Then λ_1, λ_2 are solutions of the system (4.7), (4.8) *if and only if λ_2 is the root of the equation*

$$\lambda_2 = g(\lambda_2) := l_2\{\chi^{-1}[k_1(\lambda_2)M(A_1^{-1}G_1) + M(A_1^{-1}H_1)]\}N(A_2^{-1}F_2)$$
$$+ k_2(\lambda_2)N(A_2^{-1}G_2) + N(A_2^{-1}H_2) \tag{4.12}$$

and

$$\lambda_1 = \chi^{-1}[k_1(\lambda_2)M(A_1^{-1}G_1) + M(A_1^{-1}H_1)]. \tag{4.13}$$

Consequently, the number of solutions of the system (4.1), (4.2) *equals the number of $\lambda_2 \in \mathbb{R}$ roots of* (4.12).

Further, if $N(A_2^{-1}G_2) \neq 0$ then for arbitrary continuous functions k_1, l_2 one can construct continuous functions k_2 such that (4.1), (4.2) *has N $(= 0, 1, \dots, \infty)$ solutions as follows. Let g be any continuous function for which $\lambda_2 = g(\lambda_2)$ has N λ_2 roots. If*

$$k_2(\lambda_2) = \frac{g(\lambda_2)}{N(A_2^{-1}G_2)}$$
$$+ \frac{-l_2\{\chi^{-1}[k_1(\lambda_2)M(A_1^{-1}G_1) + M(A_1^{-1}H_1)]\}N(A_1^{-1}F_2) - N(A_2^{-1}H_2)}{N(A_2^{-1}G_2)} \tag{4.14}$$

Then (4.1), (4.2) *has N solutions.*

Proof. By the assumption of our theorem, χ is a continuous bijection between $\mathbb{R}$ and $\mathbb{R}$, thus equation (4.7) is equivalent with (4.13), hence (4.8) is equivalent with (4.12)

Further, if $N(A_2^{-1}G_2) \neq 0$ then (4.12) is equivalent with (4.14). □

Remark 4.5. From equation (4.12) one can see that with fixed functions k_1, k_2, l_1, l_2 and fixed F_1, F_2, G_1, H_1 the number of solutions may depend on G_2 and H_2. One can construct examples (e.g. by choosing appropriate function k_2) such that the number of solutions with μG_2 instead of G_2 (or with μH_2 instead of H_2), where μ is a real parameter, is 0 for $\mu > \mu_0$ and is N $(= 1, 2, \ldots, \infty)$ for $\mu = \mu_0$. (See Corollary 2.3.)

Further, if for any $H_1 \in V^\star$, the function ψ defined by $\psi(\lambda_2) = \lambda_2 - g(\lambda_2)$ (g is defined by (4.12)), is strictly monotone and its range is $\mathbb{R}$ then for any $H_1, H_2 \in V^\star$ there is a unique solution of (4.1), (4.2) with zero initial and boundary condition.

Now consider the following system of nonlinear elliptic functional differential equations:

$$A_1 u = l_1(M(u))k_1(N(v))F_1, \tag{4.15}$$

$$A_2 v = l_2(M(u))k_2(N(v))F_2, \tag{4.16}$$

where the nonlinear elliptic differential operators $A_j : V \to V^\star$ of the form (1.2) are bijections ($V \subset W^{1,p}(\Omega)$) and have the property (2.11), i.e.

$$A_j(\mu u) = \mu^{p-1} A_j(u) \qquad (\mu > 0,\ p > 1) \tag{4.17}$$

(e.g. A_j may have the form $A_j u = -\triangle_p u + c_j u|u|^{p-1}$ with constants $c_j > 0$); k_j, l_j are given continuous functions, $M, N : V \to \mathbb{R}$ are nonnegative continuous functionals with the property

$$M(\mu u) = \mu^\sigma M(u), \qquad N(\mu v) = \mu^\sigma N(v) \qquad (\mu > 0, \sigma > 0) \tag{4.18}$$

and $F_j \in V^\star$.

Remark 4.6. Functionals M (and N) may have e.g. the form

$$M(u) = \int_\Omega |f||u|^\sigma.$$

Clearly, $u, v \in V$ are solutions of (4.15), (4.16) if and only if

$$u = [l_1(M(u))]^{1/(p-1)}[k_1(N(v))]^{1/(p-1)} A_1^{-1}(F_1), \tag{4.19}$$

$$v = [l_2(M(u))]^{1/(p-1)}[k_2(N(v))]^{1/(p-1)} A_2^{-1}(F_2). \tag{4.20}$$

Theorem 4.7. *Functions u, v satisfy (4.15), (4.16) if and only if*

$$u = [l_1(\lambda_1)]^{1/(p-1)}[k_1(\lambda_2)]^{1/(p-1)} A_1^{-1}(F_1), \tag{4.21}$$

$$v = [l_2(\lambda_1)]^{1/(p-1)}[k_2(\lambda_2)]^{1/(p-1)} A_2^{-1}(F_2). \tag{4.22}$$

where λ_1, λ_2 are roots of the algebraic system

$$\lambda_1 = [l_1(\lambda_1)]^{\frac{\sigma}{p-1}}[k_1(\lambda_2)]^{\frac{\sigma}{p-1}} M[A_1^{-1}(F_1)], \tag{4.23}$$

$$\lambda_2 = [l_2(\lambda_1)]^{\frac{\sigma}{p-1}}[k_2(\lambda_2)]^{\frac{\sigma}{p-1}} N[A_2^{-1}(F_2)]. \tag{4.24}$$

Proof. If u, v are solutions of (4.15), (4.16) then by (4.19), (4.20)

$$M(u) = [l_1(M(u))]^{\frac{\sigma}{p-1}}[k_1(N(v))]^{\frac{\sigma}{p-1}} M[A_1^{-1}(F_1)], \tag{4.25}$$

$$N(v) = [l_2(M(u))]^{\frac{\sigma}{p-1}}[k_2(N(v))]^{\frac{\sigma}{p-1}} M[A_2^{-1}(F_2)], \tag{4.26}$$

thus $\lambda_1 = M(u)$ and $\lambda_2 = N(v)$ satisfy (4.23), (4.24) and (4.19), (4.20) imply (4.21), (4.22).

Conversely, if λ_1, λ_2 are roots of (4.23), (4.24) then for the functions u, v defined by (4.21), (4.22) we have

$$M(u) = [l_1(\lambda_1)]^{\frac{\sigma}{p-1}}[k_1(\lambda_2)]^{\frac{\sigma}{p-1}} M[A_1^{-1}(F_1)] = \lambda_1,$$
$$N(v) = [l_2(\lambda_1)]^{\frac{\sigma}{p-1}}[k_2(\lambda_2)]^{\frac{\sigma}{p-1}} M[A_2^{-1}(F_2)] = \lambda_2$$

and

$$A_1(u) = [l_1(\lambda_1)][k_1(\lambda_2)]F_1 = [l_1(M(u))][k_1(N(v))]F_1,$$
$$A_2(v) = [l_2(\lambda_1)][k_1(\lambda_2)]F_2 = [l_2(M(u))][k_2(N(v))]F_2,$$

i.e. u, v satisfy the system (4.15), (4.16). □

Corollary 4.8. *The number of weak solutions of* (4.15), (4.16) *equals the number of roots of the algebraic system* (4.23), (4.24).

Theorem 4.9. *Assume that the function χ defined by*

$$\chi(\lambda_1) = \frac{\lambda_1}{[l_1(\lambda_1)]^{\frac{\sigma}{p-1}}}$$

is strictly monotone and its range is $\mathbb{R}$. Then λ_1, λ_2 are solutions of (4.23), (4.24) *if and only if λ_2 is a root of the equation*

$$\lambda_2 = \left\{l_2\left[\chi^{-1}(k_1(\lambda_2))^{\frac{\sigma}{p-1}} M(A_1^{-1}(F_1))\right]\right\}^{\frac{\sigma}{p-1}} \times [k_2(\lambda_2)]^{\frac{\sigma}{p-1}} N[A_2^{-1}(F_2)] \tag{4.27}$$

and

$$\lambda_1 = \chi^{-1}\left\{[k_1(\lambda_2)]^{\frac{\sigma}{p-1}} M[A_1^{-1}(F_1)]\right\}. \tag{4.28}$$

Consequently, the number of roots of (4.27) *equals the number of solutions of system* (4.15), (4.16).

Further, if $N[A_2^{-1}(F_2)] > 0$ then for arbitrary continuous positive functions k_1, l_2 we can construct positive continuous functions k_2 such that the system has N $(= 0, 1, \ldots, \infty)$ solutions, in the following way. Let g be a continuous function having N zeros with the property $\lambda_2 + g(\lambda_2) > 0$ for all $\lambda_2 > 0$. Then (4.15), (4.16) *has N solutions if*

$$k_2(\lambda_2) = \frac{[\lambda_2 + g(\lambda_2)]^{\frac{p-1}{\sigma}}}{l_2\left[\chi^{-1}\left((k_1(\lambda_2))^{\frac{\sigma}{p-1}} M[A_1^{-1}(F_1)]\right)\right]} \times \frac{1}{\{N[A_2^{-1}(F_2)]\}^{\frac{p-1}{\sigma}}}.$$

Proof. By the assumption of the theorem, χ is a continuous bijection between $\mathbb{R}$ and $\mathbb{R}$, thus (4.23) is equivalent with (4.28), hence (4.24) is equivalent with (4.27). The further statements of the theorem can be proved similarly to the former theorems. □

Remark 4.10. If l_1 is identically 1 then $\chi(\lambda_1) = \lambda_1$ and (4.27), (4.28) have the form

$$\lambda_2 = \left\{l_2\left[(k_1(\lambda_2))^{\frac{\sigma}{p-1}} M(A_1^{-1}(F_1))\right]\right\}^{\frac{\sigma}{p-1}} \times [k_2(\lambda_2)]^{\frac{\sigma}{p-1}} N[A_2^{-1}(F_2)],$$
$$\lambda_1 = [k_1(\lambda_2)]^{\frac{\sigma}{p-1}} M[A_1^{-1}(F_1)].$$

Remark 4.11. Similarly can be considered systems of certain semilinear and nonlinear parabolic functional equations.

Finally, consider the system of semilinear elliptic functional differential equations

$$(Bu)(x) = \int_\Omega B_x \tilde{K}(x,y)u(y)dy + l(Pv)[B(F_1)](x), \tag{4.29}$$

$$(Cv)(x) = \int_\Omega C_x \tilde{L}(x,y)v(y)dy + k(Qu)[C(F_2)](x), \tag{4.30}$$

where B, C are uniformly elliptic linear differential operators, $\tilde{K}, \tilde{L} \in L^2(\Omega \times \Omega)$ are sufficiently smooth functions (in x), $P, Q : V \to \mathbb{R}$ are linear continuous functionals, $V \subset W^{1,2}(\Omega)$ is a closed linear subspace, $k, l : \mathbb{R} \to \mathbb{R}$ are continuous functions, $F_j \in V$.

Clearly, $u, v \in V$ satisfy (4.29), (4.30) if and only if

$$u(x) = \int_\Omega \tilde{K}(x,y)u(y)dy + l(Pv)F_1(x), \tag{4.31}$$

$$v(x) = \int_\Omega \tilde{L}(x,y)v(y)dy + k(Qu)F_2(x). \tag{4.32}$$

Theorem 4.12. *Assume that the operators K, L defined by*

$$(Ku)(x) = \int_\Omega \tilde{K}(x,y)u(y)dy, \qquad (Lv)(x) = \int_\Omega \tilde{L}(x,y)v(y)dy, \qquad u, v \in L^2(\Omega)$$

map $L^2(\Omega)$ into V and 1 is not eigenvalue of K and L. Then u, v are solutions of (4.29), (4.30) if and only if

$$u = l(\lambda_2)(I-K)^{-1}F_1, \tag{4.33}$$

$$v = k(\lambda_1)(I-K)^{-1}F_2, \tag{4.34}$$

where λ_1, λ_2 are roots of the system

$$\lambda_1 = l(\lambda_2)P[(I-K)^{-1}F_1], \tag{4.35}$$

$$\lambda_2 = k(\lambda_1)Q[(I-L)^{-1}F_2]. \tag{4.36}$$

Thus the number of solutions of (4.29), (4.30) equals the number of roots of (4.35), (4.36).

Proof. System (4.29), (4.30) is equivalent with (4.31), (4.32), which is equivalent with

$$(I-K)u = l(Pv)F_1, \tag{4.37}$$

$$(I-L)v = k(Qu)F_2. \tag{4.38}$$

(By $F_j \in V$ and the assumption on smoothness of $\tilde{K}$ and $\tilde{L}$, solutions $u, v \in L^2(\Omega)$ of (4.31), (4.32) should belong to V.) Let

$$u_{\lambda_2} = l(\lambda_2)(I-K)^{-1}F_1, \qquad v_{\lambda_1} = k(\lambda_1)(I-L)^{-1}F_2,$$

then

$$P(u_{\lambda_2}) = l(\lambda_2)P[(I-K)^{-1}F_1], \qquad Q(v_{\lambda_1}) = k(\lambda_1)Q[(I-L)^{-1}F_2].$$

Consequently, (4.29), (4.30) and so (4.37), (4.38) is satisfied if and only if $\lambda_1 = Qu$ and $\lambda_2 = Pv$ satisfy (4.35), (4.36). □

Theorem 4.13. *System (4.35), (4.36) is fulfilled if and only if λ_2 is a root of*

$$\lambda_2 = k[l(\lambda_2)P(I-K)^{-1}F_1]Q(I-L)^{-1}F_2 \tag{4.39}$$

and

$$\lambda_1 = l(\lambda_2)P(I-K)^{-1}F_1, \tag{4.40}$$

thus the number of solutions of (4.29), (4.30) equals the number of roots of (4.39).

If the function k is strictly monotone and its range is $\mathbb{R}$, further, $P(I-K)^{-1}F_1 \neq 0$, $Q(I-L)^{-1}F_2 \neq 0$ then for arbitrary N $(= 0, 1, \ldots, \infty)$ one can construct continuous functions l such that the system (4.29), (4.30) has N solutions, as follows. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function having N zeros. Then system (4.29), (4.30) has N solutions if

$$l(\lambda_2) = \frac{1}{P(I-k)^{-1}F_1} k^{-1} \left[\frac{\lambda_2 + g(\lambda_2)}{Q(I-L)^{-1}F_2} \right]. \tag{4.41}$$

Proof. Clearly, (4.35), (4.36) is fulfilled if and only if (4.39) and (4.40) are satisfied. Let

$$g(\lambda_2) = k[l(\lambda_2)P(I-K)^{-1}F_1]Q(I-L)^{-1}F_2 - \lambda_2, \tag{4.42}$$

this equality holds if and only if $l(\lambda_2)$ is defined by (4.41). Consequently, (4.39) has N roots if and only if the function g defined by (4.42) has N roots. □

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