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On a class of superlinear nonlocal fractional problems without Ambrosetti–Rabinowitz type conditions

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Abstract. In this note, we deal with the existence of infinitely many solutions for a problem driven by nonlocal integro-differential operators with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\mathcal{L}_K u = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ and the nonlinear term f satisfies superlinear at infinity but does not satisfy the the Ambrosetti–Rabinowitz type condition. The aim is to determine the precise positive interval of λ for which the problem admits at least two nontrivial solutions by using abstract critical point results for an energy functional satisfying the Cerami condition.

Keywords: integrodifferential operators, variational method, weak solutions, sign-changing potential.

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1 Introduction and main results

Recently, the fractional and non-local operators of elliptic type have been widely investigated. The interest in studying this type of operators of elliptic type re- lies not only on pure mathematical research but also on the significant applications to many areas, such as quasi-geostrophic flows, anomalous diffusion, continuum mechanics, crystal dislocation, soft thin films, semipermeable membranes, flame propagation turbulence, water waves and probability and finance, see [2,3,5,6,9,10] and the references therein.

The present study is concerned with the multiplicity of nontrivial weak solutions for the nonlocal fractional equations, namely,

$$\begin{cases} -\mathcal{L}_K u = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

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where λ is a real parameter, Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, $\mathcal{L}_K$ is the non-local operator defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \ x \in \mathbb{R}^n.$$

Here $K : \mathbb{R}^n \backslash 0 \to (0, +\infty)$ is a kernel function having the following properties

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^n) \text{ where } \gamma(x) = \min\{|x|^2, 1\}, \\ \text{there exists } k_0 > 0 \text{ such that } K(x) \geq k_0|x|^{-(n+2s)}, \ \forall x \in \mathbb{R}^n\backslash\{0\}, \\ K(-x) = K(x), \ \forall x \in \mathbb{R}^n\backslash\{0\}. \end{cases} \tag{1.2}$$

A typical example for K is given by $K(x) = |x|^{-(n+2s)}$. In this case, $\mathcal{L}_K$ is the fractional Laplacian operator $-(-\triangle)^s$ which (up to normalization factors) is defined as

$$-(-\triangle)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \qquad x \in \mathbb{R}^n.$$

If $\lambda = 1$, then problem (1.1) reduces to the following nonlocal elliptic equation

$$\begin{cases} -\mathcal{L}_K u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{1.3}$$

which has been studied by Servadei and Valdinoci [14] by using the fountain theorem. The author proved the existence of solutions under the following assumptions.

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exist $a_1, a_2 > 0$ and $q \in (2, 2_s^*)$ such that

$$|f(x,t)| \leq a_1 + a_2|t|^{q-1} \quad \text{for a.e. } x \in \Omega, \ t \in \mathbb{R},$$

where 2_s^* is the fractional Sobolev critical exponent defined by $2_s^* = \frac{2n}{n-2s}$.

(f_2) $\lim_{|t|\to 0} \frac{f(x,t)}{|t|} = 0$ uniformly for a.e. $x \in \overline{\Omega}$.

(f_3) There exist $\mu > 2$ and $r > 0$ such that for a.e. $x \in \Omega$, $t \in \mathbb{R}$, $|t| \geq r$

$$0 < \mu F(x,t) \leq t f(x,t),$$

where $F(x,t) := \mu \int_0^t f(x,\tau)d\tau$.

Moreover, there have been a large number of papers that study the existence of the solutions to (1.3), we refer the reader to [8, 14, 17, 18] and the references therein. For example, using Symmetric version of mountain pass lemma, Zhang, Molica Bisci and Servadei [17] studied the existence of infinitely many solutions of problem (1.3) when $f \in C(\overline{\Omega} \times \mathbb{R}^n)$, (f_1), (f_3) and the following symmetry condition:

(f_4) $f(x,-t) = -f(x,t)$ for a.e. $x \in \Omega$, $t \in \mathbb{R}$.

For the case that $f(x,t)$ satisfies asymptotically linear at infinity with respect to t, Luo, Tang and Gao [8] obtained the existence of sign-changing solutions of (1.3) by combining constraint variational method with the quantitative deformation lemma. In references [4, 11, 12, 17, 18], some new superlinear growth conditions are established instead of (f_3), Among them, a few are weaker than (f_3), but most complement with it, for example,

(f_5) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^2} = +\infty$ uniformly for a.e. $x \in \overline{\Omega}$ and there exists $\gamma \geq 1$ such that for a.e. $x \in \Omega$, $\mathcal{F}(x,t') \leq \gamma \mathcal{F}(x,t)$ for any t, $t' \in \mathbb{R}$ with $0 < t' \leq t$, where $\mathcal{F}(x,t) = \frac{1}{2} t f(x,t) - F(x,t)$;

(f_6) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^2} = +\infty$ uniformly for a.e. $x \in \overline{\Omega}$ and there exists $\bar{t} > 0$ such that for a.e. $x \in \Omega$, the function $t \mapsto \frac{f(x,t)}{t}$ is increasing in $t \geq \bar{t}$ and decreasing in $t \leq -\bar{t}$;

$(f_7)_1$ there exists a positive constant $r_0 > 0$ such that $F(x,t) \geq 0$, $(x,t) \in \Omega \times \mathbb{R}$ and $|t| \geq r_0$, and

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^2} = +\infty, \quad \text{a.e. } x \in \overline{\Omega};$$

$(f_7)_2$ there exist a constant $C_1 > 0$ such that

$$\mathcal{F}(x,t) \geq C_1(|t|^2 - 1), \qquad (x,t) \in \Omega \times \mathbb{R};$$

(f_8) there exist constants $C_2 > 0$ and $\kappa > \frac{N}{2s}$ such that

$$|F(x,t)|^\kappa \leq C_2 |u|^{2\kappa} \mathcal{F}(x,t), \qquad (x,t) \in \Omega \times \mathbb{R} \text{ and } |t| \geq r_0;$$

(f_9) there exist constants $\mu > 2$, $2 < \alpha < 2_s^*$ and $C_3 > 0$ such that

$$f(x,t)t - \mu F(x,t) \geq C_3(|t|^\alpha - 1), \qquad (x,t) \in \Omega \times \mathbb{R};$$

(f_{10}) there exist constants $\mu > 2$ and $C_4 > 0$ such that

$$\mu F(x,t) \leq t f(x,t) + C_4 |t|^2, \qquad (x,t) \in \Omega \times \mathbb{R}.$$

Specifically, Zhang–Molica Bisci–Servadei [17] and Zhang–Tang–Chen [18] obtained the existence of infinitely many nontrivial solutions of (1.3) under the assumptions $f \in C(\overline{\Omega} \times \mathbb{R}^n)$, (f_1), (f_3)–(f_5), or $f \in C(\overline{\Omega} \times \mathbb{R}^n)$, (f_1), (f_3), (f_4) and (f_6), or $f \in C(\overline{\Omega} \times \mathbb{R}^n)$, (f_1), (f_3), (f_4), $(f_7)_{1,2}$ and (f_8), or $f \in C(\overline{\Omega} \times \mathbb{R}^n)$, (f_1), (f_3), (f_4), $(f_7)_{1,2}$ and (f_9), respectively.

However, regarding the existence of two distinct nontrivial weak solutions for (1.1) or (1.3), to the best of our knowledge, there are no results in the literature. Motivated by the above works, we shall further study the two nontrivial solutions of (1.1) with sign-changing potential and subcritical 2-superlinear nonlinearity. The aim of this study, as in [1], is to determine the precise positive interval of for which problem (1.1) admits at least two nontrivial solutions using abstract critical point theorems. Now, we are ready to state the main results of this article.

Theorem 1.1. Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with continuous boundary. Let $K : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$ be a function satisfying (1.1) and (f_1), $(f_7)_1$ and (f_8) hold. Then there exists a positive constant λ_0 such that the problem (1.1) admits at least two distinct weak solutions for each $\lambda \in (0, \lambda_0)$.

Theorem 1.2. Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded set of $\mathbb{R}^n$ with continuous boundary. Let $K : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$ be a function satisfying (1.1) and (f_1), $(f_7)_1$ and (f_{10}) hold. Then there exists a positive constant λ_0 such that the problem (1.1) admits at least two distinct weak solutions for each $\lambda \in (0, \lambda_0)$.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on fractional Lebesgue–Sobolev spaces. In Section 3, several existence results about at least two distinct nontrivial weak solutions for problem (1.1) are obtained by abstract critical point theory and the compactness result of the Palais–Smale type.

2 Preliminaries

In order to discuss problem (1.1), we need some facts on space X_0 which are called fractional Sobolev space. For this reason, we will recall some properties involving the fractional Sobolev space, which can be found in [14–16] and references therein.

Let X denote the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$((x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)}) \in L^2(\Omega\times\Omega, dxdy).$$

The function space X is equipped with the following norm

$$\|u\|_X = \Big(\|u\|_{L^2(\Omega)} + \int_{\Omega\times\Omega}(|u(x)-u(y)|^2)K(x-y)dxdy\Big)^{1/2}. \tag{2.1}$$

The function space X_0 is defined by

$$X_0 := \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n\backslash\Omega\} \tag{2.2}$$

endowed with the Luxemburg norm

$$\|u\|_{X_0} := \Big(\int_{\Omega\times\Omega}(|u(x)-u(y)|^2)K(x-y)dxdy\Big)^{1/2}$$

and $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (for this see [14, Lemma 7]), with scalar product

$$\langle u,v\rangle = \int_{\Omega\times\Omega}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy.$$

By Lemma 6 in [14], Servadei and Valdinoci proved a sort of Poincaré–Sobolev inequality for the functions in X_0 as follows.

Lemma 2.1. let $K : \mathbb{R}^n\backslash 0 \to (0,+\infty)$ be a function satisfying assumption (1.2). Then there exists a constant $c > 1$, depending only on N, s, λ and Ω, such that for any $u \in X_0$

$$\|u\|_{X_0} \le \|u\|_X \le c\|u\|_{X_0}.$$

By the above lemma, $\|u\|_{X_0}$ is an equivalent norm in X_0. We will use the equivalent norm in the following discussion and write $\|u\| = \|u\|_{X_0}$ for simplicity. The following embedding theorem will play a crucial role in our subsequent arguments.

Lemma 2.2. let $K : \mathbb{R}^n\backslash 0 \to (0,+\infty)$ be a function satisfying assumption (1.2). Then the embedding $X_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [2, 2_s^*]$, i.e., there exists $c_r > 0$ such that $|u|_r \le c_r\|u\|$, $u \in X_0$. Moreover, X_0 is compactly embedded into $L^r(\Omega)$ only for $r \in [2, 2_s^*)$, where $L^r(\Omega)$ denotes Lebesgue space with the standard norm $|u|_r$.

In order to prove our main result, we define the energy functional φ_λ on X_0 by

$$\varphi_\lambda(u) = J(u) - \lambda\Psi(u), \tag{2.3}$$

where $J(u) = \frac{1}{2}\int_{\Omega\times\Omega}|u(x)-u(y)|^2K(x-y)dxdy$ and $\Psi(u) = \int_\Omega F(x,u)dx$, F is the function defined in (f_3). By [13], the energy functional $\varphi_\lambda : X_0 \to \mathbb{R}$ is well defined and of class $C^1(X_0, \mathbb{R})$. Moreover, the derivative of φ_λ is

$$\langle\varphi'_\lambda(u), v\rangle = \int_{\Omega\times\Omega}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy - \lambda\int_\Omega f(x,u)vdx,$$

for all $u, v \in X_0$. Obviously, solutions for problem (1.1) are corresponding to critical points of the energy functional φ_λ.

A sequence $\{u_n\} \subset X_0$ is said to be a $(C)_c$-sequence if $\varphi_\lambda(u_n) \to c$ and $\|\varphi'_\lambda(u_n)\|(1+\|u_n\|) \to 0$. φ_λ is said to satisfy the $(C)_c$-condition if any $(C)_c$-sequence has a convergent subsequence. If this condition is satisfied at every level $c \in \mathbb{R}$, then we say that φ_λ satisfies (C)-condition.

In order to prove our main result, we state the following lemma which will play a crucial role in the proof of main theorems.

Lemma 2.3 ([7, Theorem 2.6])**.** Let E be a real Banach space, $G, H : E \to \mathbb{R}$ be two continuous Gâteaux differentiable functionals such that G is bounded from below and $G(0) = H(0) = 0$. Fix $\nu > 0$ and assume that, for each

$$\lambda \in \left(0, \frac{\nu}{\sup_{G(u) \le \nu} H(u)}\right),$$

the functional $I_\lambda := G - \lambda H$ satisfies the (C)-condition and it is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{\nu}{\sup_{G(u) \le \nu} H(u)}\right),$$

the functional I_λ admits two distinct critical points.

3 Proof of the main results

In this section, we prove our main result. As we will see, in order to obtain the existence of at least two weak solutions for problem (1.1) we use variational methods. Firstly, we are ready to prove the following result about the compactness of the functional φ_λ.

Lemma 3.1. Assume that (f_1), $(f_7)_1$ and (f_8) hold. Then for all $\lambda > 0$, any $(C)_c$ sequence is bounded in X_0.

Proof. Let $\{u_n\} \subset X_0$ be a $(C)_c$ sequence, that is,

$$\varphi_\lambda(u_n) \to c \quad \text{and} \quad |\varphi'_\lambda(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.1}$$

To complete our goals, arguing by contradiction, suppose that $\|u_n\| \to \infty$, as $n \to \infty$. Observe that for n large,

$$\begin{aligned} c + 1 &\ge \varphi_\lambda(u_n) - \frac{1}{2}\langle \varphi'_\lambda(u_n), u_n \rangle \\ &= \lambda \int_\Omega \mathcal{F}(x, u_n) dx. \end{aligned} \tag{3.2}$$

Since $\|u_n\| > 1$ for n large, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\varphi_\lambda(u_n)}{\|u_n\|^2} \\ &= \frac{1}{2} - \lambda \lim_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^2} dx, \end{aligned}$$

which implies that

$$\frac{1}{2\lambda} \le \limsup_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^2} dx. \tag{3.3}$$

For $0 \leq \alpha < \beta$, let

$$\Omega_n(\alpha, \beta) = \{x \in \Omega : \alpha \leq |u_n(x)| < \beta\}.$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $|v_n|_q \leq c_q\|v_n\| = c_q$ for $q \in [1, 2_s^*)$. Since X_0 is a reflexive space (see [15, Lemma 7]), going if necessary to a subsequence, we may assume that

$$\begin{aligned} & v_n \rightharpoonup v \quad \text{in } X_0; \\ & v_n \to v \quad \text{in } L^q(\Omega), \quad 1 \leq q < 2_s^*; \\ & v_n(x) \to v(x) \quad \text{a.e. on } \Omega. \end{aligned} \tag{3.4}$$

Now, we consider two possible cases: $v = 0$ or $v \neq 0$.

(1) If $v = 0$, then we have that $v_n \to 0$ in $L^q(\Omega)$ for all $q \in [1, 2_s^*)$, and $v_n(x) \to 0$ a.e. on Ω. Hence, it follows from (f_1) that

$$\int_{\Omega_n(0,r_0)} \frac{|F(x, u_n)|}{\|u_n\|^2} dx \leq \frac{(a_1 r_0 + \frac{a_2}{q} r_0^q)\text{meas}(\Omega)}{\|u_n\|^2} \to 0 \quad \text{as } n \to +\infty, \tag{3.5}$$

where meas$(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Set $\kappa' = \frac{\kappa}{\kappa-1}$. Since $\kappa > \frac{N}{2s}$ one sees that $2\kappa' \in (1, 2_s^*)$. Hence, we deduce from (f_8), (3.2) and (3.4) that

$$\begin{aligned} & \int_{\Omega_n(r_0,+\infty)} \frac{F(x, u_n)}{u_n^2} v_n^2 dx \\ & \leq \left(\int_{\Omega_n(r_0,+\infty)} \frac{F(x, u_n)^\kappa}{u_n^{2\kappa}} dx \right)^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,+\infty)} v_n^{2\kappa'} dx \right)^{\frac{1}{\kappa'}} \\ & \leq \left(\int_{\Omega_n(r_0,+\infty)} \frac{F(x, u_n)^\kappa}{u_n^{2\kappa}} dx \right)^{\frac{1}{\kappa}} \left(\int_\Omega v_n^{2\kappa'} dx \right)^{\frac{1}{\kappa'}} \\ & \leq C_2^{\frac{1}{\kappa}} \left(\int_{\Omega_n(r_0,+\infty)} \mathcal{F}(x, u_n) dx \right)^{\frac{1}{\kappa}} \left(\int_\Omega v_n^{2\kappa'} dx \right)^{\frac{1}{\kappa'}} \\ & \leq [C_2(\frac{c+1}{\lambda})]^{\frac{1}{\kappa}} \left(\int_\Omega v_n^{2\kappa'} dx \right)^{\frac{1}{\kappa'}} \\ & \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.6}$$

Combining (3.5) with (3.6), we get

$$\begin{aligned} & \int_\Omega \frac{|F(x, u_n)|}{\|u_n\|^2} dx \\ & = \int_{\Omega_n(0,r_0)} \frac{|F(x, u_n)|}{\|u_n\|^2} dx + \int_{\Omega_n(r_0,+\infty)} \frac{F(x, u_n)}{\|u_n\|^2} dx \\ & = \int_{\Omega_n(0,r_0)} \frac{|F(x, u_n)|}{\|u_n\|^2} dx + \int_{\Omega_n(r_0,+\infty)} \frac{F(x, u_n)}{u_n^2} v_n^2 dx \\ & \to 0, \quad \text{as } n \to \infty, \end{aligned} \tag{3.7}$$

which contradicts (3.3).

(2) If $v \neq 0$, set

$$\Omega_{\neq} := \{x \in \Omega : v(x) \neq 0\},$$

then $\mathrm{meas}(\Omega_{\neq}) > 0$. For a.e. $x \in \Omega_{\neq}$, we have

$$\lim_{n\to\infty} |u_n(x)| = +\infty.$$

Hence,

$$\Omega_{\neq} \subset \Omega_n(r_0, \infty) \quad \text{for large } n \in N.$$

As the proof of (3.5), we also obtain that

$$\int_{\Omega_n(0,r_0)} \frac{|F(x,u_n)|}{\|u_n\|^2} dx \le \frac{(a_1 r_0 + \frac{a_2}{q} r_0^q)\mathrm{meas}(\Omega)}{\|u_n\|^2} \to 0 \quad \text{as } n \to +\infty. \tag{3.8}$$

By $(f_7)_1$, (3.8) and Fatou's lemma, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\varphi_\lambda(u_n)}{\|u_n\|^2} \\
&= \frac{1}{2} - \lambda \lim_{n\to\infty} \int_\Omega \frac{F(x,u_n)}{\|u_n\|^2} dx \\
&= \frac{1}{2} - \lambda \lim_{n\to\infty} \left[\int_{\Omega_n(0,r_0)} \frac{F(x,u_n)}{\|u_n\|^2} dx + \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^2} dx \right] \\
&= \frac{1}{2} - \lambda \lim_{n\to\infty} \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^2} dx \\
&\le \frac{1}{2} - \lambda \liminf_{n\to\infty} \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{\|u_n\|^2} dx \\
&= \frac{1}{2} - \lambda \liminf_{n\to\infty} \int_{\Omega_n(r_0,+\infty)} \frac{F(x,u_n)}{|u_n|^2} |v_n|^2 dx \\
&= \frac{1}{2} - \lambda \liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n)}{|u_n|^2} \chi_{\Omega_n(r_0,+\infty)}(x) |v_n|^2 dx \\
&\le \frac{1}{2} - \lambda \int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^2} \chi_{\Omega_n(r_0,+\infty)}(x) |v_n|^2 dx \\
&\to -\infty,
\end{aligned} \tag{3.9}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in X_0. The proof is accomplished. □

Lemma 3.2. Suppose that (f_1), $(f_7)_1$ and (f_8) hold. Then for all $\lambda > 0$, any $(C)_c$-sequence of φ_λ has a convergent subsequence in E.

Proof. Let $\{u_n\} \subset X_0$ be a $(C)_c$ sequence. In view of the Lemma 3.1, the sequence $\{u_n\}$ is bounded in X_0. Then, up to a subsequence we have $u_n \rightharpoonup u$ in X_0. According to Lemma 2.2, $u_n \to u$ in $L^q(\Omega)$ for $1 \le q < 2_s^*$.

It is easy to compute directly that

$$\begin{aligned}
&\int_\Omega |f(x,u_n)-f(x,u)||u_n-u|dx \\
&\le \int_\Omega (|f(x,u_n)|+|f(x,u)|)|u_n-u|dx \\
&\le \int_\Omega [(a_1+a_2|u_n|^{q-1})+(a_1+a_2|u|^{q-1})]|u_n-u|dx \\
&\le 2a_1\int_\Omega |u_n-u|dx + a_2\int_\Omega |u_n|^{q-1}|u_n-u|dx + a_2\int_\Omega |u|^{q-1}|u_n-u|dx \\
&\le 2a_1\int_\Omega |u_n-u|dx + a_2\left(\int_\Omega |u_n|^{(q-1)q'}dx\right)^{\frac{1}{q'}}\left(\int_\Omega |u_n-u|^q dx\right)^{\frac{1}{r}} \\
&\quad + a_2\left(\int_\Omega |u|^{(q-1)q'}dx\right)^{\frac{1}{q'}}\left(\int_\Omega |u_n-u|^q dx\right)^{\frac{1}{q}} \\
&= 2a_1\int_\Omega |u_n-u|dx + a_2\left(\int_\Omega |u_n|^q dx\right)^{\frac{q-1}{q}}\left(\int_\Omega |u_n-u|^q dx\right)^{\frac{1}{q}} \\
&\quad + a_2\left(\int_\Omega |u|^q dx\right)^{\frac{q-1}{q}}\left(\int_\Omega |u_n-u|^q dx\right)^{\frac{1}{q}} \\
&= 2a_1|u_n-u|_1 + a_2|u_n|_q^{q-1}|u_n-u|_q + a_2|u|_q^{q-1}|u_n-u|_q \\
&\to 0, \quad \text{as } n\to\infty,
\end{aligned} \tag{3.10}$$

where $\frac{1}{q}+\frac{1}{q'}=1$.

Noting that

$$\begin{aligned}
\|u_n-u\|^2 &= \langle u_n-u, u_n-u\rangle \\
&= \langle \varphi'_\lambda(u_n)-\varphi'_\lambda(u), u_n-u\rangle + \lambda\int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx.
\end{aligned} \tag{3.11}$$

Moreover, by (3.1), one yields

$$\lim_{n\to\infty}\langle \varphi'_\lambda(u_n)-\varphi'_\lambda(u), u_n-u\rangle = 0. \tag{3.12}$$

Finally, the combination of (3.10)–(3.12) implies

$$\|u_n-u\| \to 0, \quad \text{as } n\to+\infty. \tag{3.13}$$

Thus, we obtain $u_n\to u$ in X_0. The proof is complete. □

Lemma 3.3. Suppose that (f_1), $(f_7)_1$ and (f_{10}) hold. Then for all $\lambda>0$, any $(C)_c$-sequence of φ_λ has a convergent subsequence in X_0.

Proof. Similarly to the proof of Lemma 3.1, we only prove that $\{u_n\}$ is bounded in X_0. Suppose by contradiction that $\|u_n\|\to\infty$ as $n\to\infty$. Let $v_n=\frac{u_n}{\|u_n\|}$, then $\|u_n\|=1$ and $|v_n|_q\le c_q\|v_n\|=c_q$ for $q\in[1,2_s^*)$. Going if necessary to a subsequence, we may assume that

$$\begin{aligned}
&v_n \rightharpoonup v \quad \text{in } X_0; \\
&v_n \to v \quad \text{in } L^q(\Omega), \quad 1\le q<2_s^*; \\
&v_n(x)\to v(x) \quad \text{a.e. on } \Omega.
\end{aligned} \tag{3.14}$$

By (3.1) and (f_{10}), one has

$$\begin{aligned} c+1 &\geq \varphi_\lambda(u_n) - \frac{1}{\mu}\langle \varphi'_\lambda(u_n), u_n\rangle \\ &\geq \frac{\mu-2}{2\mu}\|u_n\|^2 - \frac{\lambda C_4}{\mu}|u_n|_2^2, \end{aligned} \tag{3.15}$$

for $n \in \mathbb{N}$, which implies

$$1 \leq \frac{2\lambda C_4}{\mu-2} \limsup_{n\to\infty} |v_n|_2^2. \tag{3.16}$$

In view of (3.14), $v_n \to v$ in $L^p(\Omega)$. Hence, we deduce from (3.16) that $v \neq 0$. By a similar fashion as (3.9), we can conclude a contradiction. Thus, $\{u_n\}$ is bounded in X_0. The rest of the proof is the same as that in Lemma 3.2. □

Proof of Theorem 1.1. Let $E = X_0$, $I = \varphi$, $G = J$ and $H = \Psi$. We know that φ_λ satisfies the (C)-condition from Lemma 3.2 and $J(0) = \Psi(0) = 0$. In view of Lemma 2.3, it suffices to show that if,

(a) the functional φ_λ is unbounded from below,

(b) for given $\nu > 0$, there exists $\lambda_0 > 0$ such that

$$\sup_{u\in J^{-1}((-\infty,1))} \Psi(u) \leq \frac{1}{\lambda_0}.$$

Verification of (a). By the assumption $(f_7)_1$, for any $M > 0$, there exists a constant $\delta > 0$ such that

$$F(x,t) = |F(x,t)| \geq M|t|^2$$

for $|t| > \delta$ and for almost all $x \in \Omega$. Let $\delta_0 = \max\{\delta, r_0\}$. Then

$$F(x,t) = |F(x,t)| \geq M|t|^2, \qquad \forall |t| > \delta_0,\ \forall x \in \Omega.$$

Hence, from (f_1), there exists a constant $C_M > 0$ such that

$$F(x,t) \geq M|t|^2 - C_M, \quad \text{for a.e. } x \in \Omega,\ t \in \mathbb{R}. \tag{3.17}$$

Take $v \in X_0$ with $v > 0$ on Ω and $\tau > 1$. Then, for any $\lambda > 0$, the relation (3.17) implies that

$$\begin{aligned} \varphi_\lambda(\tau v) &= \frac{\tau^2}{2}\int_{\Omega\times\Omega} |v(x)-v(y)|^2 K(x-y)dxdx - \lambda\int_\Omega F(x,\tau v)dx \\ &\leq \frac{\tau^2}{2}\|v\|^2 - \lambda\tau^2 M\int_\Omega |v|^2 dx + \lambda C_M \operatorname{meas}(\Omega). \end{aligned} \tag{3.18}$$

If M is large enough that

$$\frac{1}{2}\|v\|^2 - \lambda M\int_\Omega |v|^2 dx < 0.$$

This means that

$$\lim_{\tau\to+\infty} \varphi_\lambda(\tau v) = -\infty.$$

Hence the functional φ is unbounded from below.

Verification of (b). Using assumption (f_1) and Lemma 2.2, we deduce

$$\begin{aligned}\Psi(u) &= \int_\Omega F(x,u)dx \\ &\le \int_\Omega \left(a_1|u| + \frac{a_2}{q}|u|^q\right)dx \\ &= a_2|u|_1 + \frac{a_2}{q}|u|_q^q \\ &\le a_1c_1\|u\| + \frac{a_2}{q}c_q\|u\|^q,\end{aligned} \tag{3.19}$$

where c_1, c_q is given in Lemma 2.2.

On the other hand, for each $u \in J^{-1}((-\infty, \nu))$, It follows that

$$2\nu > 2J(u) = \int_{\Omega\times\Omega} |u(x) - u(y)|^2 K(x-y)dxdx = \|u\|^2.$$

This implies that

$$\|u\| < \sqrt{2\nu}. \tag{3.20}$$

Let us denote

$$\lambda_0 := \left(a_1c_1\sqrt{2\nu} + a_2c_q(2\nu)^{\frac{q}{2}}\right)^{-1}.$$

Taking into account (3.19) we assert that

$$\sup_{u\in J^{-1}((-\infty,\nu))} \Psi(u) \le a_1c_1\sqrt{2\nu} + a_2c_q(2\nu)^{\frac{q}{2}} = \frac{1}{\lambda_0} < \frac{1}{\lambda}. \tag{3.21}$$

Therefore, all the assumptions of Lemma 2.3 are satisfied, so that, for each $\lambda \in (0, \lambda_0)$, the problem (P) admits at least two distinct weak solutions in E. This completes the proof. □

Proof of Theorem 1.2. Let $E = X_0$, $I = \varphi$, $G = J$ and $H = \Psi$. We know that φ_λ satisfies the (C)-condition from Lemma 3.3 and $J(0) = \Psi(0) = 0$. The rest proof is the same as that of Theorem 1.1. Hence, the problem (1.1) admits at least two distinct weak solutions in X_0. This completes the proof. □

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