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NONLINEAR AND OBLIQUE BOUNDARY VALUE PROBLEMS FOR THE STOKES EQUATIONS

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Abstract. In this paper we consider the nonlinear boundary value problem governed by a stationary perturbed Stokes system with mixed boundary conditions (Dirichlet- maximal monotone graph), in a smooth domain. We first establish the existence result and some estimates for weak solutions of its approached problem. A specific regularity of the velocity and the pressure are obtained. The proof is based on the approach of maximal monotone graph by its Yosida regularization and the contraction method.

1. Introduction and formulation of the problem

This paper concerns the study of the existence and regularity for the solution of the following problem. Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n = 2, 3)$ of class C^2. The boundary $\Gamma = \partial\Omega$ is assumed to be composed of two portions $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$, with measure $(\Gamma_1) > 0$. The notation β will stand for a maximal monotone graph such that $0 \in \beta(0)$. For given body forces $f \in L^2(\Omega)^n$, we look for a solution (u, p) in $H^2(\Omega)^n \times H^1(\Omega)$ of the following problem:

$$\begin{cases} -\nu\Delta u + k^2 u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_1, \\ (Lu - \sigma(u)\eta) \in \beta(u) & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

where p, u, η and ν are ,respectively, the pressure, the velocity field, the unit outword normal to Γ and the viscosity. We will note by L the first order differential operator with libschitzian coefficients (for example $Lu = \sum\limits_{i=1}^{n} a_i(x)\ \dfrac{\partial u}{\partial x_i}$ with $u = (u_i)$, $1 \le i \le n$), k is a real number to be fixed lateron. We recall that the components of the stress tensor itself are

$$\sigma_{ij}(u) = -p\delta_{ij} + 2\nu\varepsilon_{ij}(u),\ \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (1 \le i, j \le n).$$

The formulation of boundary conditions, with maximal monotone graphs, involves several types of conditions resulting from physical problems, such as the Dirichlet, Neumann or Signorini conditions (see, [18]), a boundary condition involved in elasticity with friction in a problem of air conditioning (see, [17]).

In the last years, some research papers have been written dealing with both the existence, uniqueness and regularity of solutions of Stokes system in different domains but with the usual boundary conditions (Dirichlet, Neumann, Signorini, ...), see for example [5, 6, 10, 13, 16, 19] and the references cited therein. The case of the elliptic equation with a single nonlinear condition on an convex bounded open to boundary eventually nonregular is treated in [9]. The results about regularity for the solution of elliptic boundary value problems with mixed conditions were studied by [11]. In the case of the Lamé system where Ω is an open subset of $\mathbb{R}^2$ with two maximal monotone graphs is treated in [2].

More recently, in [4] the regularity of a stationary equation for a non-isothermal Newtonian and incompressible fluids, in a three-dimensional bounded domain is studied. The problem is governed by a coupled system involving a balance of linear momentum and the heat energy with Treska free boundary conditions. The authors in [3] have proved the singular behavior of solutions of a boundary value problem

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with mixed conditions in a neighborhood of an edge in the general framework of weighted Sobolev spaces. Existence theorem and regularity of Stokes equations with the leak and slip boundary conditions of friction type have been obtained in [16].

The plan of this paper is as follows: In section 2 we give some preliminaries which will be needed below, while in section 3, we introduce a non-decreasing function β_λ which is regularized in the sense of Yosida. Then we obtain a new nonlinear problem whose the fixed point method is not well adapted. We introduce an intermediate problem for which the Banach fixed point theorem is adapted. Finally, the a priori estimate allows us to pass to the limit when λ tends to zero, we prove our main results of existence and regularity of the solution to initial problem (1.1). We achieve this work by a conclusion and perspectives in section 4.

2. Preliminaries

In this part, we introduce some lemmas and results which will be used in the next section. The detailed description be found in [7].

Lemma 2.1. *Let L a tangent operator of the first order, for all $\varphi \in L^1(\Gamma)^n$ such that $L(\varphi) \in L^1(\Gamma)^n$ we get*:

$$\left| \int_\Gamma L(\varphi) ds \right| \leq c_1 \int_\Gamma |\varphi| \, ds, \tag{2.1}$$

where $c_1 = c_1(\Omega, L)$ is a constant.

Lemma 2.2. *Let Ω be open bounded subset of $\mathbb{R}^n$ with Lipschtzienne boundary Γ, if $u \in H^1(\Omega)^n$ and if β is function uniformly Lipschtzienne then $\beta(u)$ belongs to $H^1(\Gamma)^n$.*

Theorem 2.1. *For Ω of class $C^{0,1}$ and for any $\gamma > 0$, there exists a constant $c_2(\gamma)$ depending only on γ such that*:

$$\|\varphi\|^2_{L^2(\Gamma)^n} \leq \gamma \|\nabla\varphi\|^2_{L^2(\Omega)^{n\times n}} + c_2(\gamma) \|\varphi\|^2_{L^2(\Omega)^n}, \forall \varphi \in H^1(\Omega)^n. \tag{2.2}$$

$$\|\nabla\varphi\|^2_{L^2(\Gamma)^{n\times n}} \leq \gamma \|\varphi\|^2_{H^2(\Omega)^n} + c_2(\gamma) \|\varphi\|^2_{H^1(\Omega)^n}, \forall \varphi \in H^2(\Omega)^n. \tag{2.3}$$

Throughout this paper we assume that Ω is the bounded open written in paragraph 1.

3. Main Results

In this section and for the study of the considered problem, we approach the maximal monotone graph β by a function, in order to have quasilinear boundary conditions on Γ_2. To reach the desired goal, let us introduce a non-decreasing function β_λ which are regularized in the sense of Yosida of β defined by: $\beta_\lambda = \lambda^{-1}(I + J_\lambda)$, where $J_\lambda = -(I + \lambda\beta)^{-1}$ is the resolvante of β.
At first time, we considere the following approached problem:

$$\begin{cases} -\nu\Delta u_\lambda + k^2 u_\lambda + \nabla p = f & \text{in } \Omega, \\ \operatorname{div}(u_\lambda) = 0 & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \Gamma_1, \\ (Lu_\lambda - \sigma(u_\lambda)\eta) = \beta_\lambda(u_\lambda) & \text{on } \Gamma_2. \end{cases} \tag{3.1}$$

The nonlinear problem (3.1) is not well adapted to the fixed point method, an other difficulty is to give the priori estimate.

So we introduce the following intermediate problem for wich the Banach fixed point theorem holds.

$$\left\{\begin{array}{ll} -\nu\Delta v + k^2 v + \nabla p = f & \text{in } \Omega, \\ \operatorname{div}(v) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ (Lv - \sigma(v)\eta) - \dfrac{v}{\lambda} = \dfrac{J_\lambda(u)}{\lambda} & \text{on } \Gamma_2, \end{array}\right. \tag{3.2}$$

where $u \in H^{\frac{1}{2}}(\Gamma_2)^n$.
It is clear that each fixed point of (3.2) (ie. a solution v is found as $v_{/\Gamma_2} = u$) gives a solution of (3.1).

3.1. Study of the intermediate problem (3.2)

To get a weak formulation, we introduce

$$\begin{aligned} V_{\text{div}} &= \left\{\varphi \in H^1(\Omega)^n : v_{/\Gamma_1} = 0 \textit{ and } \operatorname{div}(\varphi) = 0\right\}; \\ L_0^2(\Omega) &= \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}. \end{aligned}$$

Theorem 3.1. *There exists a unique $v \in V_{\text{div}}$ and $p \in L_0^2(\Omega)$ (up to an additive constant) solution to problem* (3.2).
Proof. The variational formulation of the linearized problem (3.2) leads to for any $\varphi \in V_{\text{div}}$:

$$2\nu \int_\Omega \varepsilon(v)\varepsilon(\varphi)dx + k^2 \int_\Omega v.\varphi dx - \int_{\Gamma_2} Lv.\varphi ds + \frac{1}{\lambda}\int_{\Gamma_2} v.\varphi ds = \int_\Omega f.\varphi dx - \frac{1}{\lambda}\int_{\Gamma_2} J_\lambda(u).\varphi ds.$$

Let

$$\begin{aligned} a(v,\varphi) &= 2\nu \int_\Omega \varepsilon(v)\varepsilon(\varphi)dx + k^2 \int_\Omega v.\varphi dx - \int_{\Gamma_2} Lv.\varphi ds + \frac{1}{\lambda}\int_{\Gamma_2} v.\varphi ds, \\ l(\varphi) &= \int_\Omega f.\varphi dx - \frac{1}{\lambda}\int_{\Gamma_2} J_\lambda(u).\varphi ds. \end{aligned}$$

The bilinear form $a(.,.)$ is continuous.
For $\varphi \in H^2(\Omega)^n \cap V_{\text{div}}$, we have

$$a(\varphi,\varphi) = 2\nu \|\varepsilon(\varphi)\|^2_{L^2(\Omega)^{n\times n}} + k^2 \|\varphi\|^2_{L^2(\Omega)^n} - \int_{\Gamma_2} L\varphi.\varphi ds + \frac{1}{\lambda} \|\varphi\|^2_{L^2(\Gamma_2)^n},$$

Using lemma 2.1, we obtain

$$\int_{\Gamma_2} L\varphi.\varphi ds = \int_{\Gamma_2} L(\frac{1}{2}\varphi.\varphi)ds \quad \text{and} \quad \int_{\Gamma_2} L\varphi.\varphi ds \le c_1 \int_{\Gamma_2} \varphi.\varphi ds.$$

As measure $(\Gamma_1) > 0$, using Korn's inequality there existe $c_3 > 0$, such that

$$\|\varepsilon(\varphi)\|^2_{L^2(\Omega)^{n\times n}} \ge c_3 \|\varphi\|^2_{H^1(\Omega)^n}. \tag{3.3}$$

We apply (3.3) and we use the theorem 2.1, it follows that

$$a(\varphi,\varphi) \geqslant (2\nu c_3 - c_1\gamma) \|\nabla\varphi\|^2_{L^2(\Omega)^{n\times n}} + \left(k^2 - c_1 c_2(\gamma)\right) \|\varphi\|^2_{L^2(\Omega)^n}. \tag{3.4}$$

Choosing γ and k such that

$$\gamma < \frac{2\nu c_3}{c_1}, \qquad k \ge \sqrt{c_1 c_2(\gamma)} \qquad \text{and} \qquad \alpha = \min\left((2\nu c_3 - c_1\gamma),\ \left(k^2 - c_1 c_2(\gamma)\right)\right) \tag{3.5}$$

we obtain

$$a(\varphi,\varphi) \geqslant \alpha \left\|\varphi\right\|^2_{H^1(\Omega)^n}, \ \forall\varphi \in H^2(\Omega)^n \cap V_{\mathrm{div}}.$$

As $H^2(\Omega)^n \cap V_{\mathrm{div}}$ is dense in V_{div} one has

$$a(\varphi,\varphi) \geqslant \alpha \left\|\varphi\right\|^2_{H^1(\Omega)^n}, \ \forall\varphi \in V_{\mathrm{div}}.$$

This shows the coercivity of the form $a(.,.)$.
The form l is linear and continuous, so, by the Lax-Milgram theorem, there exists a unique solution $v \in V_{\mathrm{div}}$ of $a(v,\varphi) = l(\varphi)$, $\forall\varphi \in V_{\mathrm{div}}$, and then as in [1] the existence of p is obtained by using a duality results of convex optimization ([12], Theorem 4.1, p58 and remark 4.2. pp. 59-61).
Therefore, there exists $(v,p) \in V_{\mathrm{div}} \times L^2_0(\Omega)$ solution of the problem (3.2). $\square$

Now we establish the solution of a nonlinear problem (3.1).
Theorem 3.2. *Under the assumption of* (3.5), *there exists a unique* $u_\lambda \in V_{\mathrm{div}}$, *and a unique* (*up to an additive constant*) $p \in L^2_0(\Omega)$, *solution to the problem* (3.1).
Proof. We use the Banach fixed point theorem. For this, we introduce the mapping defined by

$$\begin{gathered}\Lambda : L^2(\Gamma_2)^n \to L^2(\Gamma_2)^n \\ u \to \Lambda(u) = v_{/\Gamma_2},\end{gathered}$$

where v is the solution of (3.2).
We will show that Λ is a strict contraction, we can take $\Gamma \in C^{0,1}$ only, let (v_i,p_i), $i=1,2$ be solutions of the following problems:

$$\left\{\begin{array}{ll} -\nu\Delta v_i + k^2 v_i + \nabla p_i = f & \text{in } \Omega, \\ \operatorname{div}(v_i) = 0 & \text{in } \Omega, \\ v_i = 0 & \text{on } \Gamma_1, \\ (Lv_i - \sigma(v_i)\eta) - \dfrac{v_i}{\lambda} = \dfrac{J_\lambda(u_i)}{\lambda} & \text{on } \Gamma_2, \end{array}\right. \quad ,i=1,2.$$

Taking $\omega = v_2 - v_1$ and we see that ω is solution of

$$\left\{\begin{array}{ll} -\nu\Delta\omega + k^2\omega + \nabla(p_2 - p_1) = 0 & \text{in } \Omega, \\ \operatorname{div}(\omega) = 0 & \text{in } \Omega, \\ \omega = 0 & \text{on } \Gamma_1, \\ (L\omega - \sigma(\omega)\eta) - \dfrac{\omega}{\lambda} = \dfrac{J_\lambda(u_2) - J_\lambda(u_1)}{\lambda} & \text{on } \Gamma_2, \end{array}\right. \quad i=1,2$$

by variational formulation and as $\int\limits_\Omega \nabla(p_2 - p_1).\omega dx = \int\limits_\Omega (p_2 - p_1).\operatorname{div}\omega dx = 0$, we obtain

$$\begin{aligned} \frac{1}{\lambda}\int\limits_{\Gamma_2} (J_\lambda(u_2) - J_\lambda(u_1))\,\omega ds &= 2\nu\int\limits_\Omega |\varepsilon(\omega)|^2\,dx + k^2\int\limits_\Omega |\omega|^2\,dx - \int\limits_{\Gamma_2} L\omega.\omega ds + \frac{1}{\lambda}\int\limits_{\Gamma_2} |\omega|^2\,ds \\ &\geqslant 2\nu\int\limits_\Omega |\varepsilon(\omega)|^2\,dx + k^2\int\limits_\Omega |\omega|^2\,dx - c_1\gamma\int\limits_\Omega |\nabla\omega|^2\,dx + \\ &\quad -c_1c_2(\gamma)\int\limits_\Omega |\omega|^2\,dx + \frac{1}{\lambda}\int\limits_{\Gamma_2} |\omega|^2\,ds. \end{aligned}$$

Applying Korn's inequality, we get

$$\int\limits_{\Gamma_2} (J_\lambda(u_2) - J_\lambda(u_1))\,\omega ds \geqslant \lambda\,(2\nu c_3 - c_1\gamma)\int\limits_\Omega |\nabla\omega|^2\,dx + \lambda\left(k^2 - c_1c_2(\gamma)\right)\int\limits_\Omega |\omega|^2\,dx + \int\limits_{\Gamma_2} |\omega|^2\,ds,$$

then we use the Young inequality, we have

$$\frac{1}{2}\int\limits_{\Gamma_2}\left(J_\lambda(u_2)-J_\lambda(u_1)\right)^2 ds+\frac{1}{2}\int\limits_{\Gamma_2}\omega^2 ds \geqslant \lambda\left(2\nu c_3-c_1\gamma\right)\int\limits_{\Omega}\left|\nabla\omega\right|^2 dx+\lambda\left(k^2-c_1c_2(\gamma)\right)\int\limits_{\Omega}\left|\omega\right|^2 dx+\int\limits_{\Gamma_2}\left|\omega\right|^2 ds.$$

Since J_λ is a contracting mapping and if (3.5) is verified, then

$$2\left(\lambda\alpha\right)\left\|\omega\right\|^2_{H^1(\Omega)^n}+\int\limits_{\Gamma_2}\left|\omega\right|^2 ds\leq\int\limits_{\Gamma_2}\left|u_2-u_1\right|^2 ds.$$

From traces theorems, we deduce that:

$$\left(c_4+1\right)\left\|\omega\right\|^2_{L^2(\Gamma_2)^n}\leq\left\|u_2-u_1\right\|^2_{L^2(\Gamma_2)^n},$$

this implies

$$\left\|v_2-v_1\right\|_{L^2(\Gamma_2)^n}\leq\frac{1}{\sqrt{c_4+1}}\left\|u_2-u_1\right\|_{L^2(\Gamma_2)^n}.$$

The mapping Λ is strictly contracting, then there exists one and only one element $u\in L^2(\Gamma_2)^n$ such that $\Lambda(u)=u=v_{/\Gamma_2}$ and v is solution of (3.2). Finally, we have proved the existence of (u_λ, p) in $V_{\text{div}}\times L^2_0(\Omega)$ solution of (3.1). This completes the proof. □

In order to study problem (1.1) we need to establish the regularity result of (u_λ, p) solution of problem (3.1).

3.2. Regularity of the solution for the problem (3.1)

This subsection is devoted only to the proof of the following theorem:

Theorem 3.3. *If k verify* (3.5), *the solution* (u_λ, p) *of the nonlinear problem* (3.1) *satisfies* $u_\lambda\in H^2(\Omega)^n$ *and* $p\in H^1(\Omega)$.

Proof. Let us see what is happening locally.
$(a)-$ We know that inside Ω and on its boundary Γ_1, $(u_\lambda, p)\in H^2(\Omega)^n\times H^1(\Omega)$.
$(b)-$ We prove the regularity of the solution on the boundary Γ_2.
Since β_λ are lipschitzian functions and the variational solution $u_\lambda\in H^1(\Omega)^n$, then according to lemma 2.2 one has $\beta_\lambda(u_\lambda)\in H^1(\Gamma_2)^n$, and by [8] there exists a lifting $\widetilde{v}\in H^2(\Omega)^n$ verify the incompressibility equation such that

$$\begin{cases}\widetilde{v}=0 & \text{on }\Gamma_1,\\ \left(L\widetilde{v}-\sigma(\widetilde{v})\eta\right)=\beta_\lambda(u_\lambda) & \text{on }\Gamma_2.\end{cases}$$

Let $w=u_\lambda-\widetilde{v}$, then w satisfies the problem

$$\begin{cases}-\nu\Delta w+k^2w+\nabla p=f+\nu\Delta\widetilde{v}-k^2\widetilde{v} & \text{in }\Omega,\\ \operatorname{div}(w)=0 & \text{in }\Omega,\\ w=0 & \text{on }\Gamma_1,\\ \left(Lw-\sigma(w)\eta\right)=0 & \text{on }\Gamma_2,\end{cases}\tag{3.6}$$

where $\left(f+\nu\Delta\widetilde{v}-k^2\widetilde{v}\right)\in L^2(\Omega)^n$.
This problem is therefore homogeneous: there exists a unique solution (w, p) in $H^2(\Omega)^n\times H^1(\Omega)$ of this problem (see for example [5, 6, 10, 13, 16, 19]). By combining the results (a) and (b) we obtain the total regularity of (u_λ, p) in $H^2(\Omega)^n\times H^1(\Omega)$. This finishes the proof. □

3.3. A priori estimate

In this section, we will obtain the estimates on u_λ and ∇p. These estimates will be useful in order to prove the convergence of (3.1) toward the initial problem (1.1).

Let (u_λ, p) solution of (3.1) we have the:

Theorem 3.4. *For* $k \geq \sqrt{2c_1c_2(\gamma)}$ and $\gamma < \dfrac{2\nu c_3}{c_1}$, *there exists a constant C independent of λ such that the following estimates holds*

$$\|u_\lambda\|^2_{H^2(\Omega)^n} \leq C\,\|f\|_{L^2(\Omega)^n} \;\; \textit{and} \;\; \|\nabla p\|_{L^2(\Omega)^n} \leq C\,\|f\|_{L^2(\Omega)^n}\,. \tag{3.7}$$

Proof. Let u_λ be solution of (3.1) and $(-\operatorname{div}\sigma(u_\lambda)) = h$ in Ω then

$$\begin{aligned} \langle f, f\rangle &= \|h\|^2_{L^2(\Omega)^n} - 2k^2\langle \operatorname{div}\sigma(u_\lambda), u_\lambda\rangle + k^4\|u_\lambda\|^2_{L^2(\Omega)^n} \\ &= \|h\|^2_{L^2(\Omega)^n} + 4k^2\nu\int_\Omega |\varepsilon(u_\lambda)|^2\,dx - 2k^2\int_{\Gamma_2}(\sigma(u_\lambda)\eta)\,u_\lambda ds + k^4\|u_\lambda\|^2_{L^2(\Omega)^n}\,. \end{aligned}$$

Using the fact that B_λ is increasing lipschitzian with $B_\lambda(0) = 0$ and lemma 2.1, we obtain:

$$\int_{\Gamma_2}(\sigma(u_\lambda)\eta)\,u_\lambda ds = \int_{\Gamma_2} Lu_\lambda.u_\lambda ds - \int_{\Gamma_2}\beta_\lambda(u_\lambda).u_\lambda ds \leq c_1\int_{\Gamma_2}|u_\lambda|^2\,ds. \tag{3.8}$$

On the other hand, using the results of [4, for $T = 1$] and [15, 16, 19], we show the existence of a constant $c_5 > 0$ such that

$$c_5\,\|u_\lambda\|^2_{H^2(\Omega)^n} \leq \|h\|^2_{L^2(\Omega)^n}\,. \tag{3.9}$$

According to inequality (3.3) and theorem 2.1, we have

$$\langle f, f\rangle \geq c_5\|u_\lambda\|^2_{H^2(\Omega)^n} - 2k^2c_1\gamma\|\nabla u_\lambda\|^2_{L^2(\Omega)^{n\times n}} - 2k^2c_1c_2(\gamma)\|u_\lambda\|^2_{L^2(\Omega)^n} + 4k^2\nu c_3\|u_\lambda\|^2_{H^1(\Omega)^n} + k^4\|u_\lambda\|^2_{L^2(\Omega)^n}$$

this implies

$$\|f\|^2_{L^2(\Omega)^n} \geq c_5\|u_\lambda\|^2_{H^2(\Omega)^n} + 2k^2(2\nu c_3 - c_1\gamma)\|\nabla u_\lambda\|^2_{L^2(\Omega)^{n\times n}} + k^2\left(k^2 - 2c_1c_2(\gamma)\right)\|u_\lambda\|^2_{L^2(\Omega)^n}\,.$$

If we choose

$$\gamma < \frac{2\nu c_3}{c_1} \qquad \text{and} \qquad k \geq \sqrt{2c_1c_2(\gamma)}, \tag{3.10}$$

then there exists a constant $C > 0$ independent of λ such that

$$\|u_\lambda\|^2_{H^2(\Omega)^n} \leq C\,\|f\|^2_{L^2(\Omega)^n}\,.$$

Since

$$\nabla p = f + \nu\Delta u_\lambda - k^2 u_\lambda,$$

whence (3.7) follows. □

Finally, from theorem 3.4, we deduce the following theorem.

Theorem 3.5. *Under the same assumptions of theorem 3.4, there exists a* $u \in H^2(\Omega)^n$, *and a* $p \in H^1(\Omega)$, *solution to the initial problem* (1.1).

Proof. According to (3.7) there exists a sequence λ_j tending to 0 such that

$$\begin{array}{ll} u_{\lambda_j} \to u & \text{weakly in } H^2(\Omega)^n, \\ u_{\lambda_j} \to u & \text{strongly in } H^1(\Omega)^n, \\ \operatorname{div}(u_{\lambda_j}) \to 0 & \text{strongly in } L^2(\Omega)^n, \\ u = 0 & \text{on } \Gamma_1. \end{array}$$

On the other hand, from the lemma 2.2, we have

$$\begin{array}{ll} L(u_{\lambda_j}) \longrightarrow L(u) & \text{in } L^2(\Gamma_2)^n, \\ \sigma(u_{\lambda_j})\eta \longrightarrow \sigma(u)\eta & \text{in } L^2(\Gamma_2)^n, \end{array}$$

thus

$$\beta_{\lambda_j}(u_{\lambda_j}) \to -\sigma(u)\eta + L(u) \text{ strongly in } L^2(\Gamma_2)^n.$$

But since $\beta_{\lambda_j} \subset \beta \circ (-J_\lambda)$, we deduce that

$$u_{\lambda_j} + J_{\lambda_j}(u_{\lambda_j}) = \lambda_j \beta_{\lambda_j}(u_{\lambda_j}) \underset{\lambda_j \to 0}{\to} 0$$

which is equivalent to

$$-J_{\lambda_j}(u_{\lambda_j}) \underset{\lambda_j \to 0}{\to} u.$$

Hence the limit u satisfies (1.1) and then we have the regularity. This finishes the proof. $\square$

4. Conclusion and perspectives

In this research, using the approach of maximal monotone graph by its Yosida regularization and the contraction method of [14], we study the existence and regularity of the weak solution of the nonlinear boundary value problem governed by a stationary perturbed Stokes system with mixed boundary conditions (Dirichlet- maximal monotone graph), in a smooth domain. So this paper is an extension to similary ones where the boundary conditions are usual (Dirichlet, Neumann, Signorini,...).

We will deserve a further paper to a possible generalization, more precisely, we propose the study of the following problem

$$\begin{cases} -\nu\Delta u + k^2 u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \\ \left(-\sigma(u)\eta_j + \psi(\sigma)\dfrac{\partial u}{\partial x_j}\right) \in \beta_j(u) & \text{on } \Gamma_j,\ j = 1, ..., N, \end{cases}$$

where Ω is a convex polygon of sides Γ_j with $\Gamma_{N+1} = \Gamma_N$, $\Gamma = \overset{N}{\underset{j=1}{\cup}} \Gamma_j$, $f \in L^2(\Omega)^2$, $\psi \in C^{0,1}(\overline{\Omega})$, β_j a maximal monotone graph such that $0 \in \beta_j(0)$.

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