Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **42**, 1–11; https://doi.org/10.14232/ejqtde.2019.1.42 www.math.u-szeged.hu/ejqtde/

Existence of solutions to discrete and continuous second-order boundary value problems via Lyapunov functions and *a priori* bounds

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Received 22 March 2019, appeared 27 June 2019
Communicated by John R. Graef

Abstract. This article analyzes nonlinear, second-order difference equations subject to "left-focal" two-point boundary conditions. Our research questions are:

RQ1: What are new, sufficient conditions under which solutions to our "discrete" problem will exist?;

RQ2: What, if any, is the relationship between solutions to the discrete problem and solutions of the "continuous", left-focal analogue involving second-order ordinary differential equations?

Our approach involves obtaining new *a priori* bounds on solutions to the discrete problem, with the bounds being independent of the step size. We then apply these bounds, through the use of topological degree theory, to yield the existence of at least one solution to the discrete problem. Lastly, we show that solutions to the discrete problem will converge to solutions of the continuous problem.

Keywords: existence of solutions, boundary value problems, *a priori* bound, difference equation, ordinary differential equation.

2010 Mathematics Subject Classification: 34B15, 39A12.

1 Introduction

This paper considers the nonlinear, second-order difference equation

$$\frac{\Delta\nabla x_i}{h^2} = f\left(t_i, x_i, \frac{\Delta x_i}{h}\right), \quad i = 1, \dots, n-1; \tag{1.1}$$

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subject to the "left-focal" boundary conditions

$$\frac{\Delta x_0}{h} = C, \quad x_n = D. \tag{1.2}$$

Our research questions are:

RQ1: What are new, sufficient under which solutions to the "discrete" problem (1.1), (1.2) will exist?;

RQ2: What, if any, is their relationship to solutions of the "continuous", left-focal analogue involving the following second-order ordinary differential equation

$$x'' = f(t, x, x'), \qquad t \in [0, N]; \tag{1.3}$$
$$x'(0) = C, \qquad x(N) = D? \tag{1.4}$$

Part of our motivation for posing and exploring these research questions may be found by drawing on the works of Franklin and Bell. For example, "Perhaps the most deep-rooted contrast [in mathematics] is that between discrete and continuous. It is so ubiquitous in mathematics that the lack of a straightforward overview of the whole topic and explanation of its significance is astonishing" [4, p. 356]. In addition, "A major task of mathematicians today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both" [1, pp. 13–14]. Thus, by investigating our research questions and the connection between difference equations and differential equations, our work aims to illuminate these particular areas.

Above, $f : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous, nonlinear function; C and D are constants; $N > 0$ is a constant; the step size is $h = N/n$ with $h \leq N/2$; and the grid points are $t_i = ih$ for $i = 0, \dots, n$. The differences are given by:

$$\Delta x_i := \begin{cases} x_{i+1} - x_i, & \text{for } i = 0, \dots, n-1, \\ 0, & \text{for } i = n; \end{cases}$$

$$\nabla x_i := \begin{cases} x_i - x_{i-1}, & \text{for } i = 1, \dots, n, \\ 0, & \text{for } i = 0; \end{cases}$$

$$\Delta\nabla x_i := \begin{cases} x_{i+1} - 2x_i + x_{i-1}, & \text{for } i = 1, \dots, n-1, \\ 0, & \text{for } i = 0 \text{ or } i = n. \end{cases}$$

Equations (1.1), (1.2) are collectively termed as a "discrete", two-point boundary value problem (BVP) with left-focal boundary conditions; while (1.3), (1.4) are altogether known as a "continuous", two-point boundary value problem (BVP) with left-focal boundary conditions. Both these equations can for useful tools in mathematical modelling [3,21,22].

Knowing an equation has one or more solutions is important from both a modelling and theoretical point of view [19, p. 794]. Gaines [5], Lasota [8] and Myjak [10] were pioneers in advancing our knowledge of the existence, uniqueness and approximation of solutions to discrete equations. They each creatively applied fixed-point methods to discrete boundary value problems, including approaches involving: contractive maps; *a priori* bounds on solutions; and lower and upper solutions. In more recent times, authors such: as Henderson and Thompson [6, 7]; Thompson [14], Thompson and Tisdell [15–17]; Rachůnková and Tisdell [11, 12]; and Tisdell [18] have approached the challenges of existence, uniqueness and approximation of solutions to discrete boundary value problems through topological degree and monotone iterative methods. Bohner [2] has explored discretizations of the Sturm–Liouville eigenvalue problem for linear equations and the asymptotic behaviour of solutions.

The present work differs from the above papers by formulating novel inequalities on the right-hand side of our difference equation. This is based on using a nonstandard Lyapunov function that involves the square of a difference of a solution, rather than the standard approach that employs the square of a solution. It is through this approach that we establish novel *a priori* bounds on solutions. "*A priori* bounds on potential solutions to differential equations give us an estimate on the size of the solutions without having to explicitly compute the solutions" [20, p. 1088]. These ideas are then applied to address our first research question RQ1.

Because our new bounds are independent of the step size, the ideas yield a computational procedure for approximating solutions to the continuous problem (1.3), (1.4), enabling us to present a connection involving the convergence of solutions between the discrete problem and the continuous problem, addressing our second research question RQ2. In this way, we aim to illuminate the connection between the discrete and continuous, responding to the earlier quotes of Bell and Franklin, and also the work of Bohner and Peterson [3] on time scales.

2 Preliminaries

In this section some notation and results are provided that will be used throughout this work.

A solution to (1.1) is a vector $\tilde{x} = \{x_i\}_{i=0}^{n} \in \mathbb{R}^{n+1}$ that satisfies (1.1) for each $i = 1, \dots, n-1$.

A solution to (1.1) is a continuously twice-differentiable function $x : [0, N] \to \mathbb{R}$ (denoted $x \in C^2([0, N]))$ that satisfies (1.1) for each $t \in [0, N]$.

The following well known result transforms the analysis of BVPs to the analysis of equivalent integral/summation equations.

Lemma 2.1. *Let $f : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. The discrete BVP* (1.1), (1.2) *has the equivalent summation equation representation*

$$x_i = h \sum_{j=1}^{n-1} G(t_i, t_j) f\left(t_j, x_j, \frac{\Delta x_j}{h}\right) + D - C(N - t_i), \quad i = 0, \dots, n \tag{2.1}$$

where

$$G(t_i, t_j) := \begin{cases} -(N - t_i), & \text{for } 1 \le j \le i - 1 \le n - 1; \\ -(N - t_j), & \text{for } 1 \le i \le j \le n - 1. \end{cases} \tag{2.2}$$

Similarly, the continuous BVP (1.3), (1.4) *has the equivalent integral equation representation*

$$x(t) = \int_0^N G(t, s) f(s, x(s), x'(s)) + D - C(N - t)\, ds, \quad t \in [0, N]. \tag{2.3}$$

Proof. Both (2.1) and (2.3) are well known and can be verified directly. □

3 Main results

This section contains the main results on *a priori* bounds and existence of solutions to (1.1), (1.2). Our approach involves formulating new bounds via discrete (or difference) inequalities and then applying these bounds to our boundary value problem.

Theorem 3.1. *If there is a constant $K \geq 0$ such that $\tilde{x} \in \mathbb{R}^{n+1}$ satisfies*

$$\left(\frac{\Delta x_i}{h}\right)\left(\frac{\Delta\nabla x_i}{h^2}\right) \leq K, \quad \text{for } i = 1, 2, \dots, n-1 \tag{3.1}$$

and $\tilde{x}$ satisfies (1.2)*, then*

$$|x_i| \leq |C| + N\sqrt{C^2 + 2KN}, \quad \text{for } i = 0, \dots, n \tag{3.2}$$

$$\left|\frac{\Delta x_i}{h}\right| \leq \sqrt{C^2 + 2KN}, \quad \text{for } i = 0, \dots, n-1. \tag{3.3}$$

Proof. We prove the bound (3.3) first. Then we use (3.3) to obtain (3.2). Let $\tilde{x}$ satisfy (3.1) and (1.2). Define the discrete Lyapunov function $\tilde{r}$ by

$$r_i := (\Delta x_i)^2, \quad \text{for } i = 0, 1, \dots, n-1.$$

By the discrete product rule we have

$$\begin{aligned}\nabla r_i &= (\Delta\nabla x_i)(\Delta x_i) + (\Delta\nabla x_i)(\Delta x_{i-1})\\ &= 2(\Delta\nabla x_i)(\Delta x_i) - (\Delta\nabla x_i)^2\\ &\leq 2(\Delta\nabla x_i)(\Delta x_i).\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{\nabla r_i}{h^2} &\leq 2\left(\frac{\Delta\nabla x_i}{h^2}\right)\left(\frac{\Delta x_i}{h}\right)h\\ &\leq 2Kh\end{aligned}$$

where we have used (3.1). Summing the previous inequality we obtain

$$\begin{aligned}\frac{1}{h^2}\sum_{k=1}^{i}\nabla r_k &\leq \sum_{k=1}^{i} 2Kh\\ &= 2Khi\\ &\leq 2KN.\end{aligned}$$

Thus

$$\frac{r_i - r_0}{h^2} \leq 2KN$$

which we can rearrange to form

$$\begin{aligned}\frac{r_i}{h^2} &\leq \frac{r_0}{h^2} + 2KN\\ &= C^2 + 2KN.\end{aligned}$$

where we have used (1.2). The estimate (3.3) now follows.

To prove the *a priori* bound (3.2) consider

$$\begin{aligned}|x_i| - |x_n| &\leq |x_n - x_i|\\ &= \left|h\sum_{k=i}^{n-1}\frac{\Delta x_k}{h}\right|\\ &\leq h\sum_{k=i}^{n-1}\sqrt{C^2 + 2KN}\\ &= h(n-i)\sqrt{C^2 + 2KN}\\ &\leq N\sqrt{C^2 + 2KN}\end{aligned}$$

where we have applied the bound (3.3). Thus, (3.2) holds. □

In the next result we apply the findings from the previous theorem to produce *a priori* bounds on all possible solutions to (1.1), (1.2) with the bounds being independent of the step size $h > 0$.

Theorem 3.2. *Let $f : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$. If there is a $K \geq 0$ such that*

$$qf(t, p, q) \leq K, \quad \text{for all } (t, p, q) \in [0, N] \times \mathbb{R}^2 \tag{3.4}$$

then all solutions $\tilde{x}$ to (1.1), (1.2) satisfy the a priori *bounds (3.2) and (3.3).*

Proof. Let $\tilde{x}$ solve (1.1), (1.2). If (3.4) holds then for $i = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} K &\geq \left(\frac{\Delta x_i}{h}\right) f\left(t_i, x_i, \frac{\Delta x_i}{h}\right) \\ &= \left(\frac{\Delta x_i}{h}\right)\left(\frac{\Delta \nabla x_i}{h^2}\right). \end{aligned}$$

Thus the conditions of Theorem 3.1 hold. Hence the *a priori* bounds (3.2) and (3.3) hold with both bounds independent of the step size $h > 0$. □

We are now in a position to apply the preceding results to obtain the existence of at least one solution to (1.1), (1.2).

Theorem 3.3. *Let $f : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and consider (1.1), (1.2). If there is a constant $K \geq 0$ such that (3.4) holds then the discrete BVP (1.1), (1.2) has at least one solution $\tilde{x} \in \mathbb{R}^{n+1}$.*

Proof. In view of Lemma 2.1, consider the operator $\tilde{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by

$$(\tilde{T}\tilde{x})_i = h \sum_{j=1}^{n-1} G(t_i, t_j) f\left(t_j, x_j, \frac{\Delta x_j}{h}\right) + D - C(N - t_i), \quad i = 0, \ldots, n \tag{3.5}$$

so that the equation

$$\tilde{T}\tilde{x} = \tilde{x} \tag{3.6}$$

is equivalent to the problem (1.1), (1.2). Consider the family of problems associated with (3.6), namely

$$\lambda \tilde{T}\tilde{x} = \tilde{x}, \quad \lambda \in [0, 1]. \tag{3.7}$$

Consider the set Ω defined by

$$\Omega := \left\{\tilde{y} \in \mathbb{R}^{n+1} : |y_i| \leq |C| + N\sqrt{C^2 + 2KN} + 1, \left|\frac{\Delta y_i}{h}\right| \leq \sqrt{C^2 + 2KN} + 1\right\}.$$

We show that for each fixed $\lambda \in [0, 1]$, all potential solutions to (3.7) must lie in the interior of Ω.

Now, (3.7) is equivalent to the family of discrete BVPs

$$\frac{\Delta \nabla x_i}{h^2} = \lambda f\left(t_i, x_i, \frac{\Delta x_i}{h}\right), \quad i = 1, \ldots, n-1; \tag{3.8}$$

$$\frac{\Delta x_0}{h} = \lambda C, \quad x_n = \lambda D. \tag{3.9}$$

We show that the right-hand side of (3.8) satisfies (3.4). By assumption, f satisfies (3.4) so that for all $\lambda \in [0,1]$ and all $(t,p,q) \in [0,N] \times \mathbb{R}^2$ we have

$$\begin{aligned} q\lambda f(t,p,q) &\le \lambda K \\ &\le K. \end{aligned}$$

Thus, by Theorem 3.2, for each fixed $\lambda \in [0,1]$, all potential solutions to (3.8), (3.9) satisfy

$$|x_i| \le |\lambda C| + N\sqrt{(\lambda C)^2 + 2KN} \le |C| + N\sqrt{C^2 + 2KN} \tag{3.10}$$

$$\left|\frac{\Delta x_i}{h}\right| \le \sqrt{(\lambda C)^2 + 2KN} \le \sqrt{C^2 + 2KN}. \tag{3.11}$$

Hence all potential solutions to (3.7) lie within the interior of Ω.

Thus, if I is the identity operator, then the Brouwer degree $d(I - \lambda\tilde{T}, \Omega, \tilde{0})$ is well defined and independent of λ [9, Chap. 3]. Thus,

$$\begin{aligned} d(I - \lambda\tilde{T}, \Omega, \tilde{0}) &= d(I - \tilde{T}, \Omega, \tilde{0}) \\ &= d(I, \Omega, \tilde{0}) \\ &\ne 0. \end{aligned}$$

We have $d(I, \Omega, \tilde{0}) \ne 0$ because $\tilde{0} \in \Omega$.

Thus, we have shown $d(I - \tilde{T}, \Omega, \tilde{0}) \ne 0$ and so by the nonzero property of Brouwer degree we conclude that there exists at least one solution to (3.7) that lies in Ω. □

Let us discuss a simple example to illustrate one way of applying our new results.

Example 3.4. Consider the discrete problem with $N = 1$:

$$\frac{\Delta\nabla x_i}{h^2} = -\left(\frac{\Delta x_i}{h}\right)^3 x_i^2, \quad i = 1, \dots, n-1 \tag{3.12}$$

$$\frac{\Delta x_0}{h} = 1, \quad x_n = 1. \tag{3.13}$$

Consider

$$\begin{aligned} qf(t,p,q) = q[-q^3p^2] &= -q^4p^2 \\ &\le 0. \end{aligned}$$

Thus we see the conditions of Theorem 3.3 hold with $K = 0$. The existence of at least one solution to our example (3.12), (3.13) follows.

Remark 3.5. We can see from (3.4) that our class of $f(t,p,q)$ is sensitive to dependency in its third variable q. While this may show one limitation of Theorem 3.2, our inclusion of Example 3.4 illustrates that the ideas do enjoy tangible applications to examples never-the-less.

4 A discrete approach to differential equations

In this final section we build a relationship between solutions to the discrete BVP (1.1), (1.2) and solutions to the continuous BVP (1.3), (1.4). We construct a sequence of continuous functions that are based on the solutions to (1.1), (1.2) and furnish some conditions under which they will converge to a function as $h \to 0$, with this limit function being a solution to (1.3), (1.4). This approach leverages the discrete problem to produce existence results for the continuous problem.

Our next result involves a bound on the solutions and their differences to (1.1), (1.2), with the bounds being independent of h.

We will need the following notation from [5] which we reproduce for completeness and convenience. Denote the sequence $n_m \to \infty$ as $m \to \infty$; let $0 < h_m = N/n_m$; and let $t_i^m = ih_m$ for $i = 0, \dots, n$. If (1.1), (1.2) has a solution for $h = h_m$ and $m \geq m_0$ that we denote by

$$\tilde{x}^m := (x_0^m, \dots, x_n^m) \tag{4.1}$$

then we construct the following sequence of continuous functions from (4.1) via linear interpolation to form

$$x^m(t) := x_i^m + \frac{(x_{i+1}^m - x_i^m)}{h_m}(t - t_i^m), \quad t_i^m \leq t \leq t_{i+1}^m; \tag{4.2}$$

for $m \geq m_0$ and $t \in [0, N]$. Note that $x^m(t_i^m) = x_i^m$ for $i = 0, \dots, n$.

Furthermore, define $v_i^m := (x_i^m - x_{i-1}^m)/h$ and similarly construct the sequence of continuous functions v^m on $[0, N]$ by

$$v^m(t) := \begin{cases} v_i^m + \dfrac{v_{i+1}^m - v_i^m}{h_m}(t - t_i^m), & \text{for } t_i^m \leq t \leq t_{i+1}^m; \\ v_1^m, & \text{for } 0 \leq t \leq t_1^m. \end{cases} \tag{4.3}$$

Lemma 4.1. *Let $f : [0, N] \times \mathbb{R} \to \mathbb{R}$ be continuous and let $R \geq 0$ and $T \geq 0$ be constants. If* (1.1), (1.2) *has a solution for $h \leq h_m$ and $m \geq m_0$ that we denote by $\tilde{x}^m$ with*

$$\max_{i=0,\dots,n} |x_i^m| \leq R, \quad m \geq m_0; \tag{4.4}$$

$$\max_{i=0,\dots,n-1} \left| \frac{\Delta x_i^m}{h} \right| \leq T, \quad m \geq m_0; \tag{4.5}$$

then (1.3), (1.4) *has a solution $x = x(t)$ that is the limit of a subsequence of* (4.2).

Proof. The proof is quite similar to that of [5, Lemma 2.4] and so is only sketched.

For $m \geq m_0$ consider the sequence of functions $x^m(t)$ for $t \in [0, 1]$ in (4.2). We show that the sequence of functions x^m is uniformly bounded and equicontinuous on $[0, 1]$. For $t \in [t_i^m, t_{i+1}^m]$ and $m \geq m_0$ we have

$$\begin{aligned} |x^m(t)| &\leq |x_i^m| + \left| \frac{(x_{i+1}^m - x_i^m)}{h_m} \right| |t - t_i^m| \\ &\leq R + TN. \end{aligned}$$

Similar calculations show that v^m is uniformly bounded on $[0, N]$.

For $\beta, \gamma \in [0, N]$ and given $\varepsilon > 0$, consider

$$\begin{aligned} |x^m(\beta) - x^m(\gamma)| &\leq \left| \frac{(x_{i+1}^m - x_i^m)}{h_m} \right| |\beta - \gamma| \\ &\leq T|\beta - \gamma| \\ &< \varepsilon \end{aligned}$$

whenever $|\beta - \gamma| < \delta(\varepsilon) := \varepsilon/T$. Thus, x^m is equicontinuous on $[0, N]$.

A similar argument shows v^m is equicontinuous on $[0, N]$.

The convergence theorem of Arzelà–Ascoli [13, p. 527] guarantees that the sequence of continuous functions $x^m = x^m(t)$ has a subsequence $x^{k(m)}(t)$ which converges uniformly to a continuous function $x = x(t)$ for $t \in [0, N]$. That is,

$$\max_{t\in[0,N]} |x^{k(m)}(t) - x(t)| \to 0, \quad \text{as } m \to \infty$$

Similarly, $v^m = v^m(t)$ has a subsequence $v^{k(m)}(t)$ that converges uniformly to a continuous function $y = y(t)$ for $t \in [0, N]$. That is,

$$\max_{t\in[0,N]} |v^{k(m)}(t) - y(t)| \to 0, \quad \text{as } m \to \infty.$$

Additionally, it can be shown that $x' = y$ on $[0, N]$.

The continuity of f ensures that the above limit function will be a solution to (1.3), (1.4). □

The next theorem is motivated by [5, Theorem 2.5] and needs the following notation. If (1.1), (1.2) has a solution $\tilde{x}$ for $0 < h \le h_0$ then we define the continuous function $x(t, \tilde{x})$ by

$$x(t, \tilde{x}) := x_i + \frac{(x_{i+1} - x_i)}{h}(t - t_i), \quad t_i \le t \le t_{i+1}$$

and define the continuous function $v(t, \tilde{x})$ by

$$v(t, \tilde{x}) := \begin{cases} \dfrac{x_i - x_{i-1}}{h} + \dfrac{x_{i+1} - 2x_i + x_{i-1}}{h^2}(t - t_i), & \text{for } t_i \le t \le t_{i+1}; \\ \dfrac{x_1 - x_0}{h}, & \text{for } 0 \le t \le t_1. \end{cases} \tag{4.6}$$

Theorem 4.2. *Let $f : [0, N] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and let $R \ge 0$ and $T \ge 0$ be constants. Assume (1.1), (1.2) has a solution for $h \le h_0$ that we denote by $\tilde{x}$ with*

$$\max_{i=0,\dots,n} |x_i| \le R. \tag{4.7}$$

$$\max_{i=0,\dots,n-1} \left|\frac{\Delta x_i}{h}\right| \le T \tag{4.8}$$

Given any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ such that if $h \le \delta$ then (1.3), (1.4) has a solution $x = x(t)$ with

$$\max_{t\in[0,N]} |x(t, \tilde{x}) - x(t)| \le \varepsilon \tag{4.9}$$

$$\max_{t\in[0,N]} |v(t, \tilde{x}) - x'(t)| \le \varepsilon \tag{4.10}$$

Proof. Suppose, for some $\varepsilon > 0$, there is a sequence h_m such that $h_m \to 0$ as $m \to \infty$ and for $h = h_m = N/n_m$ (1.1), (1.2) has a solution $\tilde{x}^m$ with every solution $x = x(t)$ to (1.3), (1.4) satisfying at least one of

$$\max_{t\in[0,N]} |x(t, \tilde{x}) - x(t)| > \varepsilon \tag{4.11}$$

$$\max_{t\in[0,N]} |v(t, \tilde{x}) - x'(t)| > \varepsilon. \tag{4.12}$$

By assumption, for m sufficiently large, there is a $R \geq 0$ and $T \geq 0$ such that the solution $\tilde{x}^m$ to (1.1), (1.2) satisfies

$$\max_{i=0,\dots,n} |x_i^m| \leq R$$
$$\max_{i=0,\dots,n-1} |v_i^m| \leq T.$$

Thus, the conditions of Lemma 4.1 are satisfied and so we obtain a subsequence $x^{k(m)}(t)$ of $x^m(t)$ and a subsequence $v^{k(m)}(t)$ of $v^m(t)$ that converge uniformly on $[0, N]$ to a solution x of (1.3), (1.4). Thus, the inequalities (4.11) or (4.12) cannot hold. □

We now relate the above abstract results to the ideas from earlier sections.

Theorem 4.3. *Let the conditions of Theorem 3.3 hold. Given any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that if $h \leq \delta$ then (1.3), (1.4) has a solution x that satisfies (4.9) and (4.10).*

Proof. We claim that the conditions of Theorem 4.2 are satisfied. The solution $\tilde{x}$ to (1.1), (1.2) ensured to exist by Theorem 3.3 satisfies $|x_i| \leq R$ for $i = 0, \dots, n$ and $|\Delta x_i/h| \leq T$ for $i = 0, \dots, n-1$ with R the bound in (3.2) and T the bound in (3.3). Thus (4.7) and (4.8) hold. All of the conditions of Theorem 4.2 hold and the result follows. □

Let us conclude with an example to illustrate the ideas of this section.

Example 4.4. Consider the following continuous problem with $N = 1$:

$$x'' = -\left(x'\right)^3 x^2, \tag{4.13}$$
$$x'(0) = 1, \qquad x(1) = 1. \tag{4.14}$$

This is the continuous cousin of the problem discussed in Example 3.4 where we verified that the conditions of Theorem 3.3 were satisfied. Thus, we see that we may apply Theorem 4.3 to (4.13), (4.14). That is, given any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that if $h \leq \delta$ then the continuous problem (4.13), (4.14) will admit at least one solution $x = x(t)$ that satisfies (4.9) and (4.10).

Acknowledgements

The work was supported by NNSF of China (No. 11671101), NSF of Guangxi (No. 2018GXNSFDA138002) and National Natural Science Foundation of China (Grant No. 11561069), Natural Science Foundation of Guangxi Province (Grant No. 2018GXNSFDA281028), the High Level Innovation Team Program from Guangxi Higher Education Institutions of China (Document No. [2018] 35). This project has received funding from the European Unions Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie Grant Agreement No. 823731 CONMECH.

References

[1] E. T. Bell, *Men of mathematics*, Touchstone, New York, 2014. MR3728303

[2] M. Bohner, Asymptotic behavior of discretized Sturm–Liouville eigenvalue problems, *J. Differ. Equations Appl.* **3**(1998), 289–295. https://doi.org/10.1080/10236199808808104; MR1616034

[3] M. Bohner, A. C. Peterson, *Dynamic equations on time scales: an introduction with applications*, Birkhäuser, Basel, 2001. MR1843232

[4] J. Franklin, Discrete and continuous: a fundamental dichotomy in mathematics, *J. Humanist. Math.* **7**(2017), 355–378. https://doi.org/10.5642/jhummath.201702.18; MR3680865

[5] R. Gaines, Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations, *SIAM J. Numer. Anal.* **11**(1974), 411–434. https://doi.org/10.1137/0711035; MR0383757

[6] J. Henderson, H. B. Thompson, Existence of multiple solutions for second-order discrete boundary value problems, *Comput. Math. Appl.* **43**(2002), 1239–1248. https://doi.org/10.1016/S0898-1221(02)00095-0; MR1906350

[7] J. Henderson, H. B. Thompson, Difference equations associated with fully nonlinear boundary value problems for second order ordinary differential equations, *J. Differ. Equations Appl.* **7**(2001), 297–321. https://doi.org/10.1080/10236190108808274; MR1923625

[8] A. Lasota, A discrete boundary value problem, *Ann. Polon. Math.* **20**(1968), 183–190. https://doi.org/10.4064/ap-20-2-183-190; MR0226870

[9] N. G. Lloyd, *Degree theory*, Cambridge Tracts in Mathematics, No. 73, Cambridge University Press, Cambridge, 1978. MR0493564

[10] J. Myjak, Boundary value problems for nonlinear differential and difference equations of the second order, *Zeszyty Nauk. Uniw. Jagiello. Prace Mat.* **15**(1971), 113–123. MR0335931

[11] I. Rachůnková, C. C. Tisdell, Existence of non-spurious solutions to discrete boundary value problems, *Austral. J. Math. Anal. Appl.* **3**(2006), No. 2, Art. 6, 9 pp. MR2237260

[12] I. Rachůnková, C. C. Tisdell, Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions, *Nonlinear Anal.* **67**(2007), 1236–1245. https://doi.org/10.1016/j.na.2006.07.010; MR2325376

[13] W. T. Reid, *Ordinary differential equations*, John Wiley & Sons, Inc., New York, 1971. MR0273082

[14] H. B. Thompson, Topological methods for some boundary value problems. Advances in difference equations III, *Comput. Math. Appl.* **42**(2001), 487–495. https://doi.org/10.1016/S0898-1221(01)00172-9; MR1838009

[15] H. B. Thompson, C. Tisdell, Systems of difference equations associated with boundary value problems for second order systems of ordinary differential equations, *J. Math. Anal. Appl.* **248**(2000), 333–347. https://doi.org/10.1006/jmaa.2000.6823; MR1775855

[16] H. B. Thompson, C. Tisdell, Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations, *Appl. Math. Lett.* **15**(2002), 761–766. https://doi.org/10.1016/S0893-9659(02)00039-3; MR1913281

[17] H. B. Thompson, C. C. Tisdell, The nonexistence of spurious solutions to discrete, two-point boundary value problems, *Appl. Math. Lett.* **16**(2003), 79–84. https://doi.org/10.1016/S0893-9659(02)00147-7; MR1938194

[18] C. C. Tisdell, A discrete approach to continuous, second-order boundary value problems via monotone iterative techniques, *Int. J. Difference Equ.* **12**(2017), 145–160. MR3696687

[19] C. C. Tisdell, Rethinking pedagogy for second-order differential equations: a simplified approach to understanding well-posed problems, *Internat. J. Math. Ed. Sci. Tech.* **48**(2017), 794–801. https://doi.org/10.1080/0020739X.2017.1285062; MR3638496

[20] C. C. Tisdell, Improved pedagogy for linear differential equations by reconsidering how we measure the size of solutions, *Internat. J. Math. Ed. Sci. Tech.* **48** (2017), 1087–1095. https://doi.org/10.1080/0020739X.2017.1298856; MR3685618

[21] C. C. Tisdell, Pedagogical perspectives on the method of Wilmer III and Costa for solving second-order differential equations, *Internat. J. Math. Ed. Sci. Tech.* **50**(2019), 629–635. https://doi.org/10.1080/0020739X.2018.1480811; MR3943883

[22] C. C. Tisdell, On Picard's iteration method to solve differential equations and a pedagogical space for otherness, *Internat. J. Math. Ed. Sci. Tech.* **50**(2019), 788–799. https://doi.org/10.1080/0020739X.2018.1507051; MR3960427