Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **11**, 1–11; https://doi.org/10.14232/ejqtde.2020.1.11 www.math.u-szeged.hu/ejqtde/

Uniform boundedness and extinction results of solutions to a predator–prey system

Mokhtar Kirane[1], **Salah Badraoui**[✉2] and **Mohammed Guedda**[3]

[1]LaSIE, Fac. des Sciences et Technologies, Univ. La Rochelle, 17042 La Rochelle, France
[2]Laboratory of Analysis and Control of Differential Equations "ACED", Fac. MISM, Dept. Maths, Univ. 8 Mai 1945 Guelma, Algeria
[3]LAMFA, CNRS UMR 7352, Dept. Mathematics, Univ. Picardie Jules Verne, Amiens, France

Received 26 May 2019, appeared 3 February 2020
Communicated by Péter L. Simon

Abstract. Global existence, positivity, uniform boundedness and extinction results of solutions to a system of reaction-diffusion equations on unbounded domain modeling two species on a predator–prey relationship is considered.

Keywords: reaction-diffusion, global existence, positivity, predator–prey, extinction of solutions.

2010 Mathematics Subject Classification: 35A01, 35A02, 35B09, 35B50, 35K57.

1 Introduction

We consider the qualitative theory of the Cauchy problem for a system of reaction-diffusion equations modeling two species interacting with predator–prey relationship. The system in consideration is

$$L_{a,\nu} \equiv u_t - au_{xx} - \nu u_x = -pu + quv \equiv f(u,v), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$L_{b,\mu} \equiv v_t - bv_{xx} - \mu v_x = +rv - suv \equiv g(u,v), \qquad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}. \tag{1.3}$$

The functions $u = u(x,t)$ and $v = v(x,t)$ represent the densities of predators and preys in time t and at position x, respectively. The coefficient of diffusion a and b are positive constants which describe the rate of movement of predators and prey respectively. The nonnegative constants p and r are the coefficients of evolution, and the coefficients q and s are related to the increase of the density of predators, and the decrease of the density of preys due to the presence of predators, respectively. The initial conditions u_0 and v_0 are two bounded and uniformly continuous functions on $\mathbb{R}$.

✉Corresponding author. Email: badraoui.salah@univ-guelma.dz

For a biomathematical discussion of these factors and for a background of the equations, see see [10] and [15].

For the modeling of this system see for example [12], page 53: if we have a lack where there are two species of fish: A, which lives on plants of which there is a plentiful supply, and B (the predator) which subsists by eating A (the prey), where $u = u(x,t)$ represents the population of B and $v = v(x,t)$ represents the population of A.
Further, we suppose the domain is unbounded without boundary and no flux boundary conditions, instead of this we can suppose that the initial species distribution are describing by functions of finite support u_0 and v_0; namely, the initial conditions are of the form

$$\begin{aligned} u(x,0) &= u_0(x) \quad \text{for } -\Delta_u < x < \Delta_u, \quad \text{otherwise } u(x,0) = 0. \\ v(x,0) &= v_0(x) \quad \text{for } -\Delta_v < x < \Delta_v, \quad \text{otherwise } v(x,0) = 0. \end{aligned}$$

where Δ_u and Δ_v give the radius of the initially invaded domain, see [16].

The problem could be treated in the realistic two spatial dimension setting, in order to simplify the mathematics we are to treat it by one dimension space.

When the initial data are continuous, uniformly bounded, and nonnegative, it is shown that (1.1)–(1.2)–(1.3) has a classical positive global solution. Under some conditions on the coefficients or on the initial data, we show that this solution is in fact globally bounded. Moreover, if

- $r = 0$, $p > q\,\|v_0\|$, then v is bounded and $u \to 0$ exponentially as $t \to \infty$.
- $p = 0$, $u_0 \geq k > r/s$ or $u_0^* = \min\left\{u_0^-, u_0^+\right\} > r/s$, where $u_0^\pm = \lim_{x\to\pm\infty} u_0(x)$ then $u(t)$ is bounded and $v \to 0$ exponentially as $t \to \infty$.

On the other hand, we study the behaviour of (u,v) when $x \to \pm\infty$ whenever u_0 and v_0 have limits at $\pm\infty$. We show that $u(\pm\infty,t)$ and $v(\pm\infty,t)$ satisfy an ordinary differential system (ODS) in t. The qualitative behaviour of solutions to (1.1)–(1.2)–(1.3), as $x \to \pm\infty$, can then be obtained from the ODS associated to it [7].

Some systems of predator–prey were studied in bounded domains, see [9, 19] and in the references therein. Also, some results about global existence of solutions for systems of reaction-diffusion systems were established in [4, 5, 8, 13, 14].

In the following, u_0 and v_0 will be taken nonnegative and are elements of the Banach space $X = (BUC(\mathbb{R}), \|\cdot\|)$, the space of bounded and uniformly continuous functions on $\mathbb{R}$ endowed with the supremum norm $\|u\| = \sup_{x\in\mathbb{R}} |u(x)|$.

Note here that every continuous function of finite support is a uniformly continuous function on $\mathbb{R}$.

2 Existence, positivity and a priori bounds

We denote by A_1 and A_2 the linear operators $a(\cdot)_{xx} + \nu(\cdot)_x$ and $b(\cdot)_{xx} + \mu(\cdot)_x$, respectively. It is well known that A_j, $j = 1, 2$, generates an analytic semigroup of contractions on the Banach space X given explicitly by the expression

$$\left[S_j(t)u\right](x) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{+\infty} \left[\exp\left(-\frac{|x + \sigma t - \xi|^2}{4\alpha t}\right)\right] u(\xi)d\xi, \tag{2.1}$$

where $\alpha = a$ and $\sigma = \nu$ for $j = 1$, and $\alpha = b$ and $\sigma = \mu$ for $j = 2$.

Moreover, for any integer n there is a positive constant $c = c(n)$ such that for any $u \in X$, any positive t we have $D_n S_j(t)u \in X$ and the estimates

$$\left\|D_n S_j(t)u\right\| \leq ct^{-n/2} \|u\|, \tag{2.2}$$

where $D_n = d^n/(dx)^n$, and $j = 1, 2$, holds true [6].

Our first result provides the existence of a global positive solution.

Theorem 2.1. *Suppose that u_0, $v_0 \in X$, there exists a unique global classical nonnegative solution to the problem (1.1)–(1.2)–(1.3).*

Proof. Local existence and uniqueness follow from standard arguments of abstract parabolic theory or from fixed point arguments involving the heat kernel and the Duhamel principle; whence, there exists a $t_0 > 0$ such that the problem (1.1)–(1.2)–(1.3) has a unique local mild solution $(u, v) \in \mathcal{C}([0, t_0]; X) \times \mathcal{C}([0, t_0]; X)$, i.e.

$$u(t) = S_1(t)u_0 + \int_0^t S_1(t-s) f(u(s), v(s))\, ds, \qquad t \in [0, t_0],$$
$$v(t) = S_2(t)v_0 + \int_0^t S_2(t-s) g(u(s), v(s))\, ds, \qquad t \in [0, t_0].$$

From the Lebesgue theory and the fact the functions $(x, y) \longmapsto f(x, y)$ and $(x, y) \longmapsto g(x, y)$ are of class $\mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R})$ we can conclude that the solution $(u, v) \in \mathcal{C}^\infty(]0, T]; X) \times \mathcal{C}^\infty(]0, T]; X)$ for all $0 < T < T_{\max}$, and $(u(t), v(t)) \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$ for all $t \in]0, T]$; where $T_{\max}$ is the maximal time of existence of the solution.

The continuous dependence of the solution on the initial data makes use of the local existence result and the Gronwall lemma.

The nonnegativity of the solution can be proved as follows: let $\lambda_1 = \sup\{\|u(t)\|,\ 0 \leq t \leq T\}$ and $\lambda_2 = \sup\{\|v(t)\|,\ 0 \leq t \leq T\}$ where $0 < T < T_{\max}$ ($T_{\max}$ is the maximal time of existence of (u, v)), and $\lambda_0 \geq \sup\{r + s\lambda_1, p + q\lambda_2\}$. The substitutions $u = e^{\lambda_0 t}\varphi$ and $v = e^{\lambda_0 t}\psi$ transform system (1.1)–(1.2)–(1.3) into

$$\varphi_t - a\varphi_{xx} - \nu\varphi_x + (p - qv + \lambda_0)\varphi \equiv 0, \qquad x \in \mathbb{R},\ 0 < t \leq T,$$
$$\psi_t - b\psi_{xx} - \mu\varphi_x + (-r + su + \lambda_0)\psi \equiv 0, \qquad x \in \mathbb{R},\ 0 < t \leq T,$$

with

$$\varphi(x, 0) = e^{-\lambda_0 t}u_0(x) \geq 0 \quad \text{and} \quad \psi(x, 0) = e^{-\lambda_0 t}v_0(x) \geq 0, \qquad x \in \mathbb{R}.$$

As u, $v \in \mathcal{C}([0, T]; X)$ and $p - qv + \lambda_0 \geq 0$ and $-r + su + \lambda_0 \geq 0$ for all $t \in [0, T]$, we can use Theorem 9 on page 43 of the maximum principle in [11] to get that φ and ψ are nonnegative which in turn implies the nonnegativity of u and v.

If one can establish the existence of a priori bounds for the solution components u, v on $[0, T_{\max}[$, standard continuation arguments yield global well posedness.

The solution to (1.1)–(1.2)–(1.3) can be written in integral form as follows

$$u(t) = e^{-pt}S_1(t)u_0 + \int_0^t e^{-p(t-\tau)}S_1(t-\tau)qu(\tau)v(\tau)d\tau, \tag{2.3}$$
$$v(t) = e^{+rt}S_2(t)u_0 - \int_0^t e^{+r(t-\tau)}S_2(t-\tau)su(\tau)v(\tau)d\tau. \tag{2.4}$$

Using the nonnegativity of (u,v) we get

$$\|v(t)\| \leq e^{rt}\|v_0\|, \quad \text{for all } t \geq 0. \tag{2.5}$$

Using (2.3) and (2.5) we obtain

$$\|u(t)\| \leq \|u_0\| + q\|v_0\| \int_0^t e^{r\tau}\|u(\tau)\|\,d\tau, \quad \text{for all } t \geq 0. \tag{2.6}$$

Gronwall's inequality yields

$$\|u(t)\| \leq \|u_0\|\, e^{q\|v_0\|k(t)}, \quad \text{for all } t \geq 0, \tag{2.7}$$

where

$$k(t) = \begin{cases} \frac{1}{r}\left(e^{rt}-1\right), & \text{if } r>0, \\ t, & \text{if } r=0. \end{cases}$$

Estimates (2.5) and (2.7) imply that the solution is global, i.e., $T_{\max} = +\infty$. □

3 Boundedness and extinction results

The solution to (1.1)–(1.2)–(1.3) established in Theorem 2.1 is not always bounded as is shown in the following proposition.

Proposition 3.1. *Assume $v_0 \not\equiv 0$ (v_0 is not identically null) and r is sufficiently large, then (u,v) is unbounded. More precisely, v grows exponentially as t goes to ∞.*

Proof. Assume the contrary that (u,v) is a globally bounded solution, i.e., $\|u(t)\| \leq C$ and $\|v(t)\| \leq C$ for any $t \geq 0$ and some constant $C > 0$. As $v_0 \neq 0$, there exists a constant $\delta > 0$ such that $S_2(t)v_0 \geq \delta$ for any $t \geq 0$. Furthermore, we use (2.4) to obtain

$$v(t) \geq \left(\delta - sC^2/r\right)e^{rt} + sC^2/r, \quad \text{for all } t \geq 0.$$

Choosing $r > sC^2/\delta$ we clearly have $\|v(t)\| \longrightarrow +\infty$ as t goes to $+\infty$. Whence (u,v) could not be bounded. □

It is now clear that to get bounded solutions we have to impose some restrictions either on the coefficients of the system or on the initial data.

Theorem 3.2. *If u_0, $v_0 \in X$ then we have the estimates*

$$\|v(t)\| \leq \|v_0\|\, e^{rt}, \quad \textit{for all } t \geq 0, \tag{3.1}$$

$$\|u(t)\| \leq e^{\left(q\|v_0\|e^{rT}-p\right)t}\|u_0\|, \quad \textit{for all } t \in [0,T]. \tag{3.2}$$

Moreover, if $r = 0$ and $p > q\|v_0\|$ we have

$$\lim_{t\to\infty} \|u(t)\| = 0. \tag{3.3}$$

Proof. Setting

$$u = \varphi \exp(-pt), \tag{3.4}$$
$$v = \psi \exp(rt), \tag{3.5}$$

the system (1.1)–(1.2) becomes

$$\varphi_t - a\varphi_{xx} - \nu\varphi_x = qe^{rt}\varphi\psi, \tag{3.6}$$
$$\psi_t - b\psi_{xx} - \mu\psi_x = -se^{-pt}\varphi\psi, \tag{3.7}$$

with the initial data satisfying

$$\varphi_0(x) = u_0(x), \tag{3.8}$$
$$\psi_0(x) = v_0(x). \tag{3.9}$$

As $\varphi \geq 0$ and $\psi \geq 0$, we first have from (3.7) and (3.9)

$$\psi(t) = S_2(t)v_0 - s\int_0^t S_2(t-\tau)e^{-p\tau}\varphi(\tau)\psi(\tau)d\tau \leq S_2(t)v_0 \leq \|v_0\|, \tag{3.10}$$

for all $(x,t) \in \mathbb{R} \times [0,T]$. Whence $v(t) \leq \|v_0\| e^{rt}$, for all $t \geq 0$.
Substituting (3.10) into (3.6) yields

$$\varphi_t - a\varphi_{xx} - \nu\varphi_x \leq q\|v_0\| e^{rt}\varphi. \tag{3.11}$$

If we set $\varphi = e^{Mt}w$, where $M = q\|v_0\| e^{rT}$, then we have over $\mathbb{R} \times [0,T]$

$$w_t - aw_{xx} - \nu w_x \leq 0, \qquad w(x,0) = \varphi_0(x) = u_0(x). \tag{3.12}$$

Furthermore

$$w(t) = S_1(t)u_0 \leq \|u_0\|, \quad \text{for all } t \geq 0. \tag{3.13}$$

Whence $\varphi \leq e^{Mt}\|u_0\|$ and then

$$u(t) \leq e^{Mt}\|u_0\| e^{-pt} = e^{\left(q\|v_0\|e^{rT}-p\right)t}\|u_0\|, \quad \text{for all } t \in [0,T].$$

Thus we obtain (3.2).

We deduce from (3.1)–(3.2) that if $r = 0$ and $p > q\|v_0\|$ we will have

$$\|v(t)\| \leq \|v_0\|, \quad \text{for all } t \geq 0 \quad \text{and} \quad \lim_{t\to\infty}\|u(t)\| = 0. \qquad \square$$

Theorem 3.3. *If $r = 0$, $a \leq b$ and $\nu = \mu$, then the solution to (1.1)–(1.2) is globally bounded. We have the estimates*

$$\|v(t)\| \leq \|v_0\|, \quad \textit{for all } t \geq 0, \tag{3.14}$$

and

$$\|u(t)\| \leq \|u_0\| + \frac{q}{s}\sqrt{b/a}\,\|v_0\|, \quad \textit{for all } t \geq 0. \tag{3.15}$$

Proof. Let Y and Z be the solutions to

$$Y_t - aY_{xx} - \nu Y_x + pY = uv, \qquad Y(x,0) = 0, \tag{3.16}$$

and

$$Z_t - bZ_{xx} - \mu Z_x = uv, \qquad Z(x,0) = 0, \tag{3.17}$$

respectively, where (u,v) is the solution to (1.1)–(1.2)–(1.3) with $r = 0$, $a \leq b$ and $\mu = \nu$. Then (u,v) can be written in terms of (Y,Z) as follows

$$u(x,t) = e^{-pt}S_1(t)u_0(x) + qY(x,t), \qquad t \geq 0, \tag{3.18}$$

$$v(x,t) = S_2(t)v_0(x) - sZ(x,t), \qquad t \geq 0. \tag{3.19}$$

Using the positivity of $Z(x,t)$ we deduce (3.14) from (3.19). By the explicit formulas of Y and Z:

$$Y(t) = \int_0^t e^{-p(t-\tau)}S_1(t-\tau)u(\tau)v(\tau)d\tau \leq \int_0^t S_1(t-\tau)u(\tau)v(\tau)d\tau, \tag{3.20}$$

$$Z(t) = \int_0^t S_2(t-\tau)u(\tau)v(\tau)d\tau. \tag{3.21}$$

As $a \leq b$, $\nu = \mu$ and (2.1), it is easy (see [1]) to deduce that

$$\sqrt{a}S_1(t)w \leq \sqrt{b}S_2(t)w, \quad \text{for all } w \in X$$

and then

$$S_1(t)w \leq \sqrt{\frac{b}{a}}S_2(t)w, \quad \text{for all } t \geq 0. \tag{3.22}$$

From (3.20)–(3.22) we obtain

$$Y(t) \leq \sqrt{\frac{b}{a}}\int_0^t S_2(t-\tau)u(\tau)v(\tau)d\tau = \sqrt{\frac{b}{a}}Z(t), \quad \text{for all } t \geq 0. \tag{3.23}$$

As v is nonnegative, from (3.19) we get

$$Z(x,t) \leq \frac{1}{s}S_2(t)v_0, \quad \text{for all } t \geq 0. \tag{3.24}$$

Using (3.24) in (3.23) we get

$$Y(t) \leq \frac{1}{s}\sqrt{\frac{b}{a}}S_2(t)v_0, \quad \text{for all } t \geq 0. \tag{3.25}$$

Finally, from (3.25) in (3.18) we get (3.15). □

Theorem 3.4. *Assume $p = 0$ and $u_0 \geq r/s$ for all $x \in \mathbb{R}$. Then we have*

$$\|v(t)\| \leq \|v_0\|, \quad \textit{for all } t \geq 0. \tag{3.26}$$

Moreover, if there is a constant $k > r/s$ such that $u_0 > k$ for all $x \in \mathbb{R}$, then

$$\|u(t)\| \leq \left(1 + \frac{q}{ks - r}\|u_0\|\right)\|v_0\|, \quad \textit{for all } t \geq 0, \tag{3.27}$$

and

$$\|v(t)\| \leq e^{-(ks-r)t}\|v_0\|, \quad \textit{for all } t \geq 0. \tag{3.28}$$

In particular, $v \longrightarrow 0$ uniformly in $x \in \mathbb{R}$ as $t \longrightarrow \infty$.

Proof. For $p = 0$ and $u_0 \geq r/s$, from (2.1) we get

$$u(t) \geq r/s, \quad \text{for all } t \geq 0. \tag{3.29}$$

Setting $B(t) = r - su(t)$, we have

$$v_t = [A_2 + B(t)]\, v(t). \tag{3.30}$$

As the linear operator $B(t)$ is dissipative on X [18], $A_2 + B(t)$ generates for each t fixed a semigroup of contractions. Whence $A_2 + B(t)$ generates on X a system of evolution $P(t,\tau)$ of contractions [18]. Whence the solution to (3.17)–(1.3) is

$$v(t) = P(t,0)v_0, \quad \text{for all } t \geq 0. \tag{3.31}$$

This implies (3.26).

If $u_0 \geq k > r/s$, then from (1.1) we get $u(t) \geq k$, and consequently $r - su(t) \leq r - ks < 0$ for any $t \geq 0$. Setting $\omega := ks - r$ ($\omega > 0$), equation (1.2) can be written in the form

$$v(t) = [A_2 + B(t) + \omega I]\, v(t) - \omega v(t). \tag{3.32}$$

The dissipative operator $B(t) + \omega I$ generates on X a system of evolution $G(t,\tau)$ of contractions. Consequently, $A_2 + B(t)$ generates a system of evolution $U(t,\tau)$ given by

$$U(t,\tau) = e^{-\omega(t-\tau)} G(t,\tau).$$

Hence the solution $v(t)$ of (3.32)–(1.3) can be written in the form

$$v(t) = U(t,0)v_0 = e^{-\omega t} G(t,0)v_0, \quad \text{for all } t \geq 0. \tag{3.33}$$

This implies estimate (3.13). Using (1.1), (3.33) and Gronwall's lemma we get (3.15). □

In what follows, we denote by $C_\pm$ the closed subspaces of X defined as follows

$$C_\pm := \left\{ u \in X \text{ such that} : \lim_{x\to\pm\infty} u(x) \text{ exists} \right\}.$$

Lemma 3.5. *Let $f \in C_\pm$ be such that $f^+, f^- > 0$. Then for any $\varepsilon > 0$ there exists $t^* > 0$ such that*

$$\left[S_j(t)f\right](x) \geq f^* - \varepsilon, \quad \textit{for all } x \in \mathbb{R},$$

where $f^* := \min\left(f^+, f^-\right)$.

Proof. The proof is similar to that of [4, Lemma 5.3]. □

In what follows we denote $u_0^\pm = \lim_{x\to\pm\infty} u_0(x)$ and $u_0^* = \min\left\{u_0^-, u_0^+\right\}$.

Theorem 3.6. *Assume $p = 0$ and $u_0 \in C_\pm$. If $u_0^* > r/s$, then there exists $t^* > 0$ and three positive constants C_1, C_2 and ω^* such that*

$$\|v(t)\| \leq C_1 e^{-\omega^*(t-t^*)}, \quad \textit{for all } t \geq t^*, \tag{3.34}$$
$$\|u(t)\| \leq C_2, \quad \textit{for all } t \geq t^*. \tag{3.35}$$

Proof. Choose $\varepsilon > 0$ such that $u_0^* - \varepsilon > r/s$, then by Lemma 3.5, there exists $t^* > 0$ such that $[S_1(t)u_0](x) \geq u_0^* - \varepsilon$, for any $x \in \mathbb{R}$. We then have $u(t) \geq u^* - \varepsilon$, for any $t \geq t^*$. Using Theorem 3.4 with initial data $(u(t^*), v(t^*))$ and $k = u_0^* - \varepsilon$, $\omega^* = ks - r$, we then have

$$\|v(t)\| \leq \|v(t^*)\| \, e^{-\omega^*(t-t^*)}, \quad \text{for all } t \geq t^*.$$

We get (3.34) by setting $C_1 = \|v(t^*)\|$.

Now, combining (2.3) and (3.34) we infer

$$\|u(t)\| \leq \|u_0\| + qC_1 e^{\omega^* t^*} \int_0^t e^{-\omega^* \tau} \|u(\tau)\| \, d\tau, \quad \text{for all } t \geq t^*.$$

The Gronwall inequality yields

$$\|u(t)\| \leq \|u_0\| \, e^{\frac{qC_1}{\omega^*} e^{\omega^* t^*}} = C_2, \text{ for all } t \geq t^*.$$

Whence (3.35). □

4 Stability of the solution

Definition 4.1. We say that the solution to the problem (1.1)–(1.2)–(1.3) is unconditionally stable on $\mathbb{R}_+$, if for all $T > 0$ and all $\varepsilon > 0$, there exist $\delta = \delta(T, \varepsilon) > 0$ such that for all solution $(\overline{u}, \overline{v})$ with initial condition $(\overline{u}_0, \overline{v}_0)$ to the same problem satisfying $\|\overline{u}_0 - u_0\| < \delta$ and $\|\overline{v}_0 - v_0\| < \delta$ we have $\|\overline{u}(t) - u(t)\| < \varepsilon$ and $\|\overline{v}(t) - v(t)\| < \varepsilon$ for all $t \in [0, T]$.

Proposition 4.2. *The solution of the problem* (1.1)–(1.2) *is unconditionally stable on* $\mathbb{R}_+$.

Proof. From the integral writin of the solution (u, v) and $(\overline{u}, \overline{v})$ we get

$$\|\overline{u}(t) - u(t)\| \leq \|\overline{u}_0 - u_0\| + \int_0^t \{p \, \|\overline{u}(\tau) - u(\tau)\| + q \, \|u(\tau)v(\tau) - \overline{u}(\tau)\overline{v}(t)\|\} \, d\tau, \tag{4.1}$$

$$\|\overline{v}(t) - v(t)\| \leq \|\overline{v}_0 - v_0\| + \int_0^t \{r \, \|\overline{v}(\tau) - v(\tau)\| + s \, \|u(\tau)v(\tau) - \overline{u}(\tau)\overline{v}(t)\|\} \, d\tau. \tag{4.2}$$

Setting $\Phi = (u, v)$, $\overline{\Phi} = (\overline{u}, \overline{v})$, $\Phi_0 = (u_0, v_0)$, $\overline{\Phi}_0 = (\overline{u}_0, \overline{v}_0)$ and define $\|\Phi(t)\| = \|(u(t), v(t))\| = \|u(t)\| + \|v(t)\|$; then from (4.1)–(4.2) we get

$$\begin{aligned} \left\|\overline{\Phi}(t) - \Phi(t)\right\| \leq \left\|\overline{\Phi}_0 - \Phi_0\right\| &+ (p + r) \int_0^t \|\overline{u}(\tau) - u(\tau)\| \, d\tau \\ &+ (q + s) \int_0^t \|\overline{u}(\tau)\overline{v}(\tau) - u(\tau)v(t)\| \, d\tau. \end{aligned} \tag{4.3}$$

Let $\varepsilon > 0$ and $T > 0$. As u, v, $\overline{u}$, $\overline{v} \in C(\mathbb{R}^+; X)$; then, they are bounded over $[0, T]$. Define

$$\|u\|_\infty = \sup_{t \in [0,T]} \|u(t)\|, \quad \text{for all } u \in \mathcal{C}\left(\mathbb{R}^+; X\right), \tag{4.4}$$

then we have

$$\|\overline{u}\overline{v} - uv\|_\infty \leq M \left\|\overline{\Phi}(t) - \Phi(t)\right\|, \quad \text{for all } t \in [0, T], \tag{4.5}$$

where $M = \|u\|_\infty + \|v\|_\infty$.

From (4.3) and (4.5) we get

$$\left\|\overline{\Phi}(t)-\Phi(t)\right\| \leq \left\|\overline{\Phi}_0-\Phi_0\right\| + [p+r+M(q+s)]\int_0^t \left\|\overline{\Phi}(\tau)-\Phi(\tau)\right\| d\tau. \tag{4.6}$$

Using Gronwall inequality we obtain

$$\left\|\overline{\Phi}(t)-\Phi(t)\right\| \leq \left\|\overline{\Phi}_0-\Phi_0\right\| e^{[p+r+M(q+s)]t}, \quad \text{for all } t\in[0,T]\,. \tag{4.7}$$

The estimate (4.6) gives the stability of the solution to the problem (1.1)–(1.2)–(1.3). □

5 Remarks

Remark 5.1. In turns out that if u_0, $v_0 \in \mathcal{C}_+$ then the diffusive system for x large will behave like the system of ordinary differential equations associated to it, and hence, for x large can be replaced by the latter which is simpler to analyze [7]

$$\begin{aligned}
\frac{dU(t)}{dt} &= -pU(t)+qU(t)V(t), \quad \text{for all } t>0,\\
\frac{dV(t)}{dt} &= +rU(t)-sU(t)V(t), \quad \text{for all } t>0,
\end{aligned}$$

satisfying the initial data

$$U(0)=\lim_{x\to+\infty} u_0(x), \qquad V(0)=\lim_{x\to+\infty} v_0(x),$$

where

$$U(t)=\lim_{x\to+\infty} u(x,t), \qquad V(t)=\lim_{x\to+\infty} u(x,t)$$

This result is based on the fact that if $h\in\mathcal{C}_+$ with $h^+=\lim_{x\to+\infty} h(x)$, then $\lim_{x\to+\infty}\left[S_j(t)h\right](x)=h^+$, for $j=1,2$.

The same thing holds if u_0, $v_0\in\mathcal{C}_-$.

Remark 5.2. The same analysis can also be done for $x\in[0,+\infty[$. In this case, the explicit formula associated to (1.1)–(1.2)–(1.3)

$$u(t)=e^{-pt}S_1(t)u_0+\int_0^t e^{-p(t-\tau)}S_1(t-\tau)f\left(u(\tau),v(\tau)\right)d\tau,$$

$$v(t)=e^{+rt}S_2(t)u_0+\int_0^t e^{+r(t-\tau)}S_2(t-\tau)g\left(u(\tau),v(\tau)\right)d\tau,$$

will be

$$\begin{aligned}
u(x,t) = &\int_0^\infty N_1(x,\xi,t)u_0(\xi)d\xi + \int_0^t \frac{x}{t-\tau}K_1(x,t-\tau)u_1(\tau)d\tau\\
&+\int_0^t\int_0^{+\infty} N_1(x,\xi,t-\tau)f(u,v)(\xi,\tau)d\xi d\tau,
\end{aligned}$$

and

$$\begin{aligned}
v(x,t) = &\int_0^\infty N_2(x,\xi,t)v_0(\xi)d\xi + + \int_0^t \frac{x}{t-\tau}K_2(x,t-\tau)v_1(\tau)d\tau\\
&+\int_0^t\int_0^{+\infty} N_2(x,\xi,t-\tau)g(u,v)(\xi,\tau)d\xi d\tau,
\end{aligned}$$

where

$$N_1(x,\xi,t) = K_1(x-\xi,t) - K_1(x+\xi,t),\ K_1(x,t) = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{|x+\nu t|^2}{4at}\right),$$
$$N_2(x,\xi,t) = K_2(x-\xi,t) - K_2(x+\xi,t),\ K_2(x,t) = \frac{1}{\sqrt{4\pi bt}} \exp\left(-\frac{|x+\mu t|^2}{4bt}\right),$$

and

$$u_1(t) = u(0,t), \qquad v_1(t) = v(0,t),$$

with u_1, v_1 bounded. These expressions can be deduced from [17, Chapter 3, Section 3].

It will be interesting to perform the same analysis for the case $x \in [0,+\infty[$ with other boundary conditions.

Remark 5.3. For $x \in \mathbb{R}^n$ $(n \geq 2)$ and replacing au_{xx} and bv_{xx} in (1.1)–(1.2) by the second order uniform elliptic operators

$$L_1 u = \sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_j}\right) u_{x_i}, \qquad L_2 u = \sum_{i,j=1}^{n} \left(b_{ij}(x)v_{x_j}\right) v_{x_i},$$

the problem deserves to be studied in appropriate functional spaces using the results in Aronson [2] and [3].

References

[1] W. F. Ames, C. Rogers, *Nonlinear equations in the applied sciences*, Mathematics in Science and Engineering, Vol. 185, Academic Press, Inc., Boston, MA, 1992. MR1132036

[2] D. G. Aronson, Nonnegative solutions of linear parabolic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **22**(1968), 607–694. MR0435594; Zbl 0182.13802

[3] D. G. Aronson, Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.* **73**(1967), 890–896. https://doi.org/10.1090/S0002-9904-1967-11830-5; MR217444

[4] J. D. Avrin, Qualitative theory for a model of laminar flames with arbitrary nonnegative initial data, *J. Differential Equations* **84**(1990), 290–308. https://doi.org/10.1016/0022-0396(90)90080-9; MR1047571

[5] J. D. Avrin, M. Kirane, Temperature growth and temperatures bounds in special cases of combustion models, *Appl. Anal.* **50**(1993), 131–144. https://doi.org/10.1080/00036819308840189; MR1281208

[6] S. Badraoui, Z. Mehamedia, Uniform boundedness and decay estimates for a system of reaction-diffusion equations with a triangular diffusion matrix on $\mathbb{R}^n$, *Mediterr. J. Math.* **10**(2013), 241–254. https://doi.org/10.1007/s00009-012-0239-8; MR3019103; Zbl 1270.35254

[7] S. Badraoui, Asymptotic behavior of solutions to 2×2 reaction-diffusion system with a cross diffusion matrix on unbounded domains, *Electron. J. Differential Equations* **2006**, No. 61, 1–13. MR2226934

[8] J. W. Bebernes, W. Fulks, The small heat-loss problem, *J. Differential Equations* **57**(1985), 324–332. https://doi.org/10.1016/0022-0396(85)90059-2; MR0790279

[9] H. Cao, S. Fu, Global existence and convergence of solutions to a cross-diffusion cubic predator-prey system with stage structure for the prey, *Bound. Value Probl.* **2010**, Art. ID 285961, 24 pp. https://doi.org/10.1155/2010/285961; MR2646370

[10] M. Farkas, *Dynamical models in biology*, Academic Press, Inc., San Diego, CA, 2001. MR1836483

[11] A. Friedman, *Partial differential equations of parabolic type*, Robert E. Krieger Publisching Company Malabar, Florida, 1983.

[12] D. W. Jordan, P. Smith, *Nonlinear differential equations*, 4th Edition, Oxford University Press, 2007. MR2348881

[13] J. I. Kanel, On global initial boundary-value problems for reaction-diffusion systems with balance conditions, *Nonlinear Anal.* **37**(1999), 971–995. https://doi.org/10.1016/s0362-546x(97)00685-8; MR1689279

[14] B. Larrouturou, The equations of one-dimensional unsteady flame propagation: existence and uniqueness, *SIAM J. Math. Anal.* **19**(1988), 32–59. https://doi.org/10.1137/0519003; MR0924543

[15] A. Okubo, S. A. Levin, *Diffusion and ecological problems: mathematical models*, Springer-Verlag, Berlin, 1980. MR572962

[16] S. Petrovskii, A. Morozov, B. L. Li, Regimes of biological invasion in a predator-prey system with the Allee effect, *Bull. Math. Biol.* **67**(2005), 637–661. https://doi.org/10.1016/j.bulm.2004.09.003; MR2215557.

[17] A. N. Tikhonov, A. A. Samarskii, *Equations of mathematical physics*, Dover Publications, 1990. MR1149383

[18] K. Yosida, *Functional analysis*, Springer-Verlag, Berlin-Heidelberg, 1965. MR0180824

[19] R. Zhang, L. Guo, S. Fu, Global behavior for a diffusive predator-prey model with stage structure and nonlinear density restriction. II. The case in $\mathbb{R}^1$, *Bound. Value Probl.* **2009**, Art. ID 654539, 19 pp. https://doi.org/10.1155/2009/654539; MR2570329.