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On the well-posedness of the nonlocal boundary value problem for elliptic-parabolic equations

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Abstract. The abstract nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + sign(t)Au(t) = g(t), (0 \le t \le 1), \\ \frac{du(t)}{dt} + sign(t)Au(t) = f(t), (-1 \le t \le 0), \\ u(1) = u(-1) + \mu \end{cases}$$

for the differential equation in a Hilbert space H with the self-adjoint positive definite operator A is considered. The well-posedness of this problem in Hölder spaces without a weight is established. The coercivity inequalities for solutions of the boundary value problem for elliptic-parabolic equations are obtained.

Key Words: Elliptic-parabolic equation, Nonlocal boundary-value problem, Well-posedness
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1 A nonlocal boundary value problem. Well-posedness

Methods of solutions of the nonlocal boundary value problems for partial differential equations have been studied extensively by many researchers (see, e.g., [4]- [6], [8], [11]- [35], and the references given therein)

The role played by coercivity inequalities (well-posedness) in the study of boundary-value problems for partial differential equations is well known (see, e.g., [1]-[3]). In the present paper we study the well-posedness of the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2u(t)}{dt^2} + sign(t)Au(t) = g(t), (0 \le t \le 1), \\ \frac{du(t)}{dt} + sign(t)Au(t) = f(t), (-1 \le t \le 0), \\ u(1) = u(-1) + \mu \end{cases} \tag{1.1}$$

for the differential equation in a Hilbert space H with the self-adjoint positive definite operator A and $A \ge \delta I, \delta > 0$.

First of all, let us give some estimates that will be needed below.

Lemma 1.1 *[41]. The following estimates hold:*

$$\| \left(A^{\frac{1}{2}}\right)^{\alpha} e^{-tA^{\frac{1}{2}}} \|_{H\to H} \le t^{-\alpha} \left(\frac{\alpha}{e}\right)^{\alpha}, 0 \le \alpha \le e, t > 0, \tag{1.2}$$

$$\| A^{\alpha} e^{-tA} \|_{H\to H} \le t^{-\alpha} \left(\frac{\alpha}{e}\right)^{\alpha}, 0 \le \alpha \le e, t > 0, \tag{1.3}$$

$$\| \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \|_{H\to H} \le M(\delta), \tag{1.4}$$

$$\| \left(I+e^{-2A^{\frac{1}{2}}}+A^{\frac{1}{2}}(I-e^{-2A^{\frac{1}{2}}})-2e^{-(A^{\frac{1}{2}}+A)} \right)^{-1} \|_{H\to H}\leq M(\delta), \tag{1.5}$$

$$\| A^{\frac{1}{2}} \left(I+e^{-2A^{\frac{1}{2}}}+A^{\frac{1}{2}}(I-e^{-2A^{\frac{1}{2}}})-2e^{-(A^{\frac{1}{2}}+A)} \right)^{-1} \|_{H\to H}\leq M(\delta). \tag{1.6}$$

With the help of the self-adjoint positive definite operator B in a Hilbert space H, the Banach space $E_\alpha = E_\alpha(B,H)$ $(0<\alpha<1)$ consists of those $v \in H$ for which the norm (see [38]-[39])

$$\| v \|_{E_\alpha} = \sup_{z>0} z^{1-\alpha} \| Bexp\{-zB\}v \|_H + \| v \|_H$$

is finite. By the definition of $E_\alpha(B,H)$

$$D(B) \subset E_\alpha(B,H) \subset E_\beta(B,H) \subset H \tag{1.7}$$

for all $\beta < \alpha$.

Lemma 1.2 *[37]. For $0<\alpha<1$ the norms of the spaces $E_\alpha(A^{\frac{1}{2}},H)$ and $E_{\frac{\alpha}{2}}(A,H)$ are equivalent.*

Lemma 1.3 . *For $0<\alpha<1$ the following estimates hold:*

$$\| e^{-A^{\frac{1}{2}}} \|_{H\to E_\alpha(A^{\frac{1}{2}},H)} \leq 2, \| e^{-A} \|_{H\to E_{\frac{\alpha}{2}}(A,H)} \leq 2, \tag{1.8}$$

$$\| e^{-A^{\frac{1}{2}}} \|_{H\to E_{\frac{\alpha}{2}}(A,H)} \leq 2, \tag{1.9}$$

$$\| e^{-A} \|_{H\to E_\alpha(A^{\frac{1}{2}},H)} \leq 2. \tag{1.10}$$

Proof. Estimate (1.8) is obvious. Using estimates (1.2)-(1.3), we get

$$\begin{aligned} z^{1-\alpha} \left\| Aexp\{-zA\} e^{-A^{\frac{1}{2}}} v \right\|_H &\leq z^{1-\alpha} \left\| A^\alpha e^{-A^{\frac{1}{2}}} \right\|_{H\to H} \\ \times \left\| A^{1-\alpha} exp\{-zA\} \right\|_{H\to H} \|v\|_H &\leq \|v\|_H , \end{aligned}$$

$$\begin{aligned} z^{1-\alpha} \left\| A^{\frac{1}{2}} exp\{-zA^{\frac{1}{2}}\} e^{-A} v \right\|_H &\leq \left\| A^{\frac{\alpha+1}{2}} e^{-A} \right\|_{H\to H} \\ \times z^{1-\alpha} \left\| A^{\frac{1-\alpha}{2}} exp\{-zA^{\frac{1}{2}}\} \right\|_{H\to H} \|v\|_H &\leq \|v\|_H \end{aligned}$$

for all $z, z>0$ and $v \in H$. From that estimates (1.9)-(1.10) follow. Lemma 1.3 is proved.

Let us denote by $C^\alpha([-1,1],H), C^{\frac{\alpha}{2}}([-1,0],H), C^\alpha([0,1],H), 0<\alpha<1$ the Banach spaces obtained by completion of the set of all smooth H-valued functions $\varphi(t)$ in the norms

$$\begin{aligned} &\| \varphi \|_{C^\alpha([-1,1],H)} = \|\varphi\|_{C([-1,1],H)} \\ &+ \sup_{-1<t<t+\tau<0} \frac{\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\frac{\alpha}{2}}} + \sup_{0<t<t+\tau<1} \frac{\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^\alpha}, \end{aligned}$$

$$\| \varphi \|_{C^{\frac{\alpha}{2}}([-1,0],H)} = \|\varphi\|_{C([-1,0],H)} + \sup_{-1<t<t+\tau<0} \frac{\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\frac{\alpha}{2}}}$$

$$\| \varphi \|_{C^{\alpha}([0,1],H)} = \|\varphi\|_{C([0,1],H)} + \sup_{0<t<t+\tau<1} \frac{\|\varphi(t+\tau)-\varphi(t)\|_H}{\tau^{\alpha}},$$

where $C([a,b],H)$ stands for the Banach space of all continuous functions $\varphi(t)$ defined on $[a,b]$ with values in H equipped with the norm

$$||\varphi||_{C([a,b],H)} = \max_{a\le t\le b} \|\varphi(t)\|_H.$$

Lemma 1.4 . *Suppose $g(t) \in C^{\alpha}([0,1],H)$ and $f(t) \in C^{\frac{\alpha}{2}}([-1,0],H), 0<\alpha<1$. Then the following estimates hold:*

$$\| \int_0^1 A^{\frac{1}{2}} e^{-sA^{\frac{1}{2}}} \left(g(s)-g(0)\right) ds||_{E_\alpha(A^{\frac{1}{2}},H)} \le \frac{1}{\alpha\,(1-\alpha)} \|g\|_{C^\alpha([0,1],H)}\,, \tag{1.11}$$

$$\| \int_0^1 A^{\frac{1}{2}} e^{-(1-s)A^{\frac{1}{2}}} \left(g(s)-g(1)\right) ds||_{E_\alpha(A^{\frac{1}{2}},H)} \le \frac{1}{\alpha\,(1-\alpha)} \|g\|_{C^\alpha([0,1],H)}\,, \tag{1.12}$$

$$\| \int_{-1}^0 A e^{-(s+1)A} \left(f(s)-f(-1)\right) ds||_{E_{\frac{\alpha}{2}}(A,H)} \le \frac{1}{\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)} \|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}\,, \tag{1.13}$$

$$\| \int_{-1}^0 A e^{-(s+1)A} \left(f(s)-f(-1)\right) ds||_{E_\alpha(A^{\frac{1}{2}},H)} \le \frac{M}{\alpha(1-\alpha)} \|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}\,, \tag{1.14}$$

$$\| \int_0^1 A^{\frac{1}{2}} e^{-sA^{\frac{1}{2}}} \left(g(s)-g(0)\right) ds||_{E_{\frac{\alpha}{2}}(A,H)} \le \frac{M}{\alpha\,(1-\alpha)} \|g\|_{C^\alpha([0,1],H)}\,, \tag{1.15}$$

$$\| \int_0^1 A^{\frac{1}{2}} e^{-(1-s)A^{\frac{1}{2}}} \left(g(s)-g(1)\right) ds||_{E_{\frac{\alpha}{2}}(A,H)} \le \frac{M}{\alpha\,(1-\alpha)} \|g\|_{C^\alpha([0,1],H)}\,, \tag{1.16}$$

where M does not depend on α, $f(t)$ and $g(t)$.

Proof. Using estimates (1.2)-(1.3), we get

$$z^{1-\alpha} \left\| A^{\frac{1}{2}} exp\{-zA^{\frac{1}{2}}\} \int_0^1 A^{\frac{1}{2}} e^{-sA^{\frac{1}{2}}} \left(g(s)-g(0)\right) ds \right\|_H$$

$$\le z^{1-\alpha} \int_0^1 \left\| A e^{-(s+z)A^{\frac{1}{2}}} \right\|_{H\to H} \|g(s)-g(0)\|_H \, ds$$

$$\leq z^{1-\alpha}\int_0^1 \frac{s^\alpha}{(s+z)^2}ds\,\|g\|_{C^\alpha([0,1],H)} \leq \frac{1}{1-\alpha}\|g\|_{C^\alpha([0,1],H)} \tag{1.17}$$

for all z, $z>0$ and $g(t)\in C^\alpha([0,1],H)$. Using estimates (1.2)-(1.3), we get

$$\left\|\int_0^1 A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds\right\|_H \leq \int_0^1 \left\|Ae^{-sA^{\frac{1}{2}}}\right\|_{H\to H}\|g(s)-g(0)\|_H\,ds$$

$$\leq \int_0^1 \frac{ds}{s^{1-\alpha}}\|g\|_{C^\alpha([0,1],H)} = \frac{1}{\alpha}\|g\|_{C^\alpha([0,1],H)} \tag{1.18}$$

for $g(t)\in C^\alpha([0,1],H)$. From (1.17)- (1.18) estimate (1.11) follows. In a similar manner one establishes estimates (1.12) and (1.13). Using estimates (1.2)-(1.3), we get

$$z^{1-\alpha}\left\|A^{\frac{1}{2}}exp\{-zA^{\frac{1}{2}}\}\int_{-1}^0 Ae^{-(s+1)A}\left(f(s)-f(-1)\right)ds\right\|_H$$

$$\leq z^{1-\alpha}\int_0^1 \left\|A^{\frac{3}{2}}e^{-zA^{\frac{1}{2}}}e^{-(s+1)A}\right\|_{H\to H}\|f(s)-f(-1)\|_H\,ds$$

$$\leq z^{1-\alpha}\left(\frac{3}{e}\right)^3\int_{-1}^0 \frac{2^{\frac{3}{2}}(s+1)^{\frac{\alpha}{2}}}{(z^2+s+1)^{\frac{3}{2}}}ds\,\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)} \tag{1.19}$$

$$\leq \frac{Mz^{1-\alpha}}{(1-\alpha)(z^2)^{\frac{1-\alpha}{2}}}\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)} = \frac{M}{1-\alpha}\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}$$

for all z, $z>0$ and $f(t)\in C^{\frac{\alpha}{2}}([-1,0],H)$. Using estimates (1.2)-(1.3), we get

$$\left\|\int_{-1}^0 Ae^{-(s+1)A}\left(f(s)-f(-1)\right)ds\right\|_H \leq \int_{-1}^0 \left\|Ae^{-(s+1)A}\right\|_{H\to H}\|f(s)-f(-1)\|_H\,ds$$

$$\leq \int_{-1}^0 \frac{ds}{(s+1)^{1-\frac{\alpha}{2}}}\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)} = \frac{2}{\alpha}\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}. \tag{1.20}$$

From (1.19)-(1.20) estimate (1.14) follows. Using estimates (1.2)-(1.3), we get

$$z^{1-\frac{\alpha}{2}}\left\|Aexp\{-zA\}\int_0^1 A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds\right\|_H$$

$$\leq z^{1-\frac{\alpha}{2}}\int_0^1 \left\|A^{\frac{3}{2}}e^{-zA}e^{-sA^{\frac{1}{2}}}\right\|_{H\to H}\|g(s)-g(0)\|_H\,ds$$

$$\leq z^{1-\frac{\alpha}{2}}\int\limits_0^1\left\|A^{\frac{3}{2}}e^{-zA}e^{-sA^{\frac{1}{2}}}\right\|_{H\to H}s^{\alpha}ds\,\|g\|_{C^{\alpha}([0,1],H)},$$

for all z, $z>0$ and $g(t)\in C^{\alpha}([0,1],H)$. Since

$$\left\|A^{\frac{3}{2}}e^{-zA}e^{-sA^{\frac{1}{2}}}\right\|_{H\to H}\leq\min\left\{\frac{1}{z^3},\left(\frac{3}{e}\right)^3\frac{1}{s^{\frac{3}{2}}}\right\}$$

for all z, $z>0$ and all s, $s>0$, we have the bounded

$$\int\limits_0^1\left\|A^{\frac{3}{2}}e^{-zA}e^{-sA^{\frac{1}{2}}}\right\|_{H\to H}s^{\alpha}ds\leq\int\limits_0^1\frac{M}{(\sqrt{z}+s)^{3-\alpha}}ds\leq\frac{M_1}{(\sqrt{z})^{2-\alpha}}.$$

Then

$$z^{1-\frac{\alpha}{2}}\left\|Aexp\{-zA\}\int\limits_0^1A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds\right\|_H\leq M_1\,\|g\|_{C^{\alpha}([0,1],H)}\tag{1.21}$$

for all z, $z>0$ and $g(t)\in C^{\alpha}([0,1],H)$. Using estimates (1.2)-(1.3), we get

$$\left\|\int\limits_0^1A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds\right\|_H\leq\int\limits_0^1\left\|A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\right\|_{H\to H}\|g(s)-g(0)\|_H\,ds$$

$$\leq\int\limits_0^1\frac{ds}{s^{1-\alpha}}\,\|g\|_{C^{\alpha}([0,1],H)}=\frac{1}{\alpha}\,\|g\|_{C^{\alpha}([0,1],H)}.\tag{1.22}$$

From (1.21)-(1.22) estimate (1.15) follows. In a similar manner one establishes estimate (1.16). Lemma 1.4 is proved.

A function $u(t)$ is called a solution of problem (1.1) if the following conditions are satisfied:

i. $u(t)$ is a twice continuously differentiable in the segment $[0,1]$ and continuously differentiable on the segment $[-1,1]$.

ii. The element $u(t)$ belongs to $D(A)$ for all $t\in[-1,1]$, and the function $Au(t)$ is continuous on $[-1,1]$.

iii. $u(t)$ satisfies the equation and nonlocal boundary condition (1.1).

A solution of problem (1.1) defined in this manner will from now on be referred to as a solution of problem (1.1) in the space $C(H)=C([-1,1],H)$.

We say that the problem (1.1) is well-posed in $C(H)$, if there exists the unique solution $u(t)$ in $C(H)$ of problem (1.1) for any $g(t)\in C([0,1],H)$, $f(t)\in C([-1,0],H)$ and $\mu\in D(A)$ and the following coercivity inequality is satisfied:

$$\|u''\|_{C([0,1],H)}+\|u'\|_{C([-1,0],H)}+\|Au\|_{C(H)}\tag{1.23}$$

$$\leq M[\|g\|_{C([0,1],H)} + \|f\|_{C([-1,0],H)} + \|A\mu\|_H] \ ,$$

where M does not depend on $\mu,\ f(t)$ and $g(t)$.

In fact, inequality (1.23) does not, generally speaking, hold in an arbitrary Hilbert space H and for the general unbounded self-adjoint positive definite operator A. Therefore, the problem (1.1) is not well- posed in $C(H)$[8]. The well-posedness of the boundary value problem (1.1) can be established if one considers this problem in certain spaces $F(H)$ of smooth H-valued functions on $[-1,1]$.

A function $u(t)$ is said to be a solution of problem (1.1) in $F(H)$ if it is a solution of this problem in $C(H)$ and the functions $u''(t)\ (t\in[0,1]), u'(t)(t\in[-1,1])$ and $Au(t)(t\in[-1,1])$ belong to $F(H)$.

As in the case of the space $C(H)$, we say that the problem (1.1) is well-posed in $F(H)$, if the following coercivity inequality is satisfied:

$$\begin{aligned} &\|u''\|_{F([0,1],H)} + \|u'\|_{F([-1,0],H)} + \|Au\|_{F(H)} \\ &\leq M[\|g\|_{F([0,1],H)} + \|f\|_{F([-1,0],H)} + \|A\mu\|_H] \ , \end{aligned} \tag{1.24}$$

where M does not depend on $\mu,\ f(t)$ and $g(t)$.

In paper [41] the well-posedness of problem (1.1) in Hölder spaces $C^{\alpha,\alpha}([-1,1],H),(0<\alpha<1)$ with a weight was established. The coercivity inequalities for the solution of boundary value problems for elliptic-parabolic equations were obtained. The first order of accuracy difference scheme for the approximate solution of the nonlocal boundary value problem (1.1) was presented. The well-posedness of this difference scheme in Hölder spaces with a weight was established. In applications, the coercivity inequalities for the solution of difference scheme for elliptic-parabolic equations were obtained.

Note that the coercivity inequality (1.24) fails if we set $F(H)$ equal to $C^\alpha(H)=C^\alpha([-1,1],H),(0<\alpha<1)$. Nevertheless, we can establish the following coercivity inequality.

Theorem 1.5 . *Suppose* $A\mu\in E_\alpha\left(A^{\frac{1}{2}},H\right)$, $f(0)+g(0)\in E_{\frac{\alpha}{2}}(A,H)$, $f(-1)+g(1)\in E_\alpha\left(A^{\frac{1}{2}},H\right)$ *and* $g(t)\in C^\alpha([0,1],H)$, $f(t)\in C^{\frac{\alpha}{2}}([-1,0],H), 0<\alpha<1$. *Then the boundary value problem (1.1) is well-posed in a Holder space* $C^\alpha(H)$ *and the following coercivity inequality holds:*

$$\begin{aligned} &\|u'\|_{C^{\frac{\alpha}{2}}([-1,0],H)} + ||Au||_{C^\alpha([-1,1],H)} + \|u''\|_{C^\alpha([0,1],H)} \\ &\leq \frac{M}{\alpha(1-\alpha)}\left[\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)} + \|g\|_{C^\alpha([0,1],H)}\right] \ + M\left[\|A\mu\|_{E_\alpha(A^{\frac{1}{2}},H)}\right. \\ &\left. + \|f(0)+g(0)\|_{E_{\frac{\alpha}{2}}(A,H)} + \|f(-1)+g(1)\|_{E_\alpha\left(A^{\frac{1}{2}},H\right)}\right], \end{aligned} \tag{1.25}$$

where M *does not depend on* $\alpha,\ f(t),\ g(t)$ *and* μ.

Proof. First, we will obtain the formula for solution of the problem (1.1). It is known that (see, e.g., [7]) for smooth data of the problems

$$\begin{cases} -u''(t) + Au(t) = g(t), (0\leq t\leq 1), \\ u(0)=u_0,\ u(1)=u_1, \end{cases} \tag{1.26}$$

$$\begin{cases} u'(t) - Au(t) = f(t), (-1 \le t \le 0), \\ u(0) = u_0, \end{cases} \tag{1.27}$$

there are unique solutions of the problems (1.26), (1.27), and the following formulas hold:

$$u(t) = \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \left[\left(e^{-tA^{\frac{1}{2}}} - e^{-(-t+2)A^{\frac{1}{2}}}\right) u_0 \right. \tag{1.28}$$

$$\left. + \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) u_1 \right] - \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1}$$

$$\times \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(1-s)A^{\frac{1}{2}}} - e^{-(s+1)A^{\frac{1}{2}}}\right) g(s) ds$$

$$- \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(t+s)A^{\frac{1}{2}}} - e^{-|t-s|A^{\frac{1}{2}}}\right) g(s) ds, \ 0 \le t \le 1,$$

and

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(s) ds, \quad -1 \le t \le 0. \tag{1.29}$$

Using the condition $u(1) = u(-1) + \mu$ and formulas (1.28), (1.29), we can write

$$u(t) = \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \left[\left(e^{-tA^{\frac{1}{2}}} - e^{-(-t+2)A^{\frac{1}{2}}}\right) u_0 \right. \tag{1.30}$$

$$\left. + \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) \left(e^{-A} u_0 + \int_0^{-1} e^{-(1+s)A} f(s) ds + \mu\right)\right] - \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1}$$

$$\times \left(e^{-(1-t)A^{\frac{1}{2}}} - e^{-(t+1)A^{\frac{1}{2}}}\right) \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(1-s)A^{\frac{1}{2}}} - e^{-(s+1)A^{\frac{1}{2}}}\right) g(s) ds$$

$$- \int_0^1 A^{-\frac{1}{2}} 2^{-1} \left(e^{-(t+s)A^{\frac{1}{2}}} - e^{-|t-s|A^{\frac{1}{2}}}\right) g(s) ds, \ 0 \le t \le 1.$$

For u_0, using the condition $u'(0+) = Au(0) + f(0)$ and formula (1.30), we obtain the operator equation

$$Au(0) + f(0) = \left(I - e^{-2A^{\frac{1}{2}}}\right)^{-1} \left[-A^{\frac{1}{2}} \left(I + e^{-2A^{\frac{1}{2}}}\right) u_0 \right. \tag{1.31}$$

$$\left. + 2A^{\frac{1}{2}} e^{-A^{\frac{1}{2}}} \left(e^{-A} u_0 + \int_0^{-1} e^{-(1+s)A} f(s) ds + \mu\right)\right] + \int_0^1 e^{-sA^{\frac{1}{2}}} g(s) ds$$

$$-\left(I-e^{-2A^{\frac{1}{2}}}\right)^{-1}2A^{\frac{1}{2}}e^{-A^{\frac{1}{2}}}\int\limits_{0}^{1}A^{-\frac{1}{2}}2^{-1}\left(e^{-(1-s)A^{\frac{1}{2}}}-e^{-(s+1)A^{\frac{1}{2}}}\right)g(s)ds.$$

Since the operator

$$I+e^{-2A^{\frac{1}{2}}}+A^{\frac{1}{2}}(I-e^{-2A^{\frac{1}{2}}})-2e^{-(A^{\frac{1}{2}}+A)}$$

has an inverse

$$T=\left(I+e^{-2A^{\frac{1}{2}}}+A^{\frac{1}{2}}(I-e^{-2A^{\frac{1}{2}}})-2e^{-(A^{\frac{1}{2}}+A)}\right)^{-1}$$

it follows that

$$u_0=T\left[e^{-A^{\frac{1}{2}}}\left[2\int\limits_{0}^{-1}e^{-(1+s)A}f(s)ds\right.\right. \tag{1.32}$$

$$\left.\left.-\int\limits_{0}^{1}A^{-\frac{1}{2}}\left(e^{-(1-s)A^{\frac{1}{2}}}-e^{-(s+1)A^{\frac{1}{2}}}\right)g(s)ds\right]+2e^{-A^{\frac{1}{2}}}\mu\right]$$

$$+\left(I-e^{-2A^{\frac{1}{2}}}\right)T\left[-A^{-\frac{1}{2}}f(0)+\int\limits_{0}^{1}A^{-\frac{1}{2}}e^{-sA^{\frac{1}{2}}}g(s)ds\right]$$

for the solution of the operator equation (1.31). Hence, for the solution of the nonlocal boundary value problem (1.1), we have formulas (1.29), (1.30) and (1.32).

Second, we will establish estimate (1.25). It is based on the estimates

$$\|u'\|_{C^{\frac{\alpha}{2}}([-1,0],H)}+||Au||_{C^{\frac{\alpha}{2}}([-1,0],H)} \tag{1.33}$$

$$\leq\frac{M}{\frac{\alpha}{2}(1-\frac{\alpha}{2})}\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}+M\left\|Au_0+f(0)\right\|_{E_{\frac{\alpha}{2}}(A,H)}$$

for the solution of an inverse Cauchy problem (1.27) and on the estimates

$$\|u''\|_{C^{\alpha}([0,1],H)}+||Au||_{C^{\alpha}([0,1],H)}\leq\frac{M}{\alpha(1-\alpha)}\|g\|_{C^{\alpha}([0,1],H)} \tag{1.34}$$

$$+M[\|Au_0-g(0)\|_{E_{\alpha}(A^{\frac{1}{2}},H)}+\|Au_1-g(1)\|_{E_{\alpha}(A^{\frac{1}{2}},H)}]$$

for the solution of the boundary value problem (1.26) and on the estimates

$$\|Au_0+f(0)\|_{E_{\frac{\alpha}{2}}(A,H)}\leq\frac{M}{\alpha\,(1-\alpha)}[\|g\|_{C^{\alpha}([0,1],H)}+\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}] \tag{1.35}$$

$$+M\left[\|A\mu\|_{E_{\alpha}(A^{\frac{1}{2}},H)}+\|f(0)+g(0)\|_{E_{\frac{\alpha}{2}}(A,H)}\right],$$

$$\|Au_0-g(0)\|_{E_{\alpha}(A^{\frac{1}{2}},H)}\leq\frac{M}{\alpha(1-\alpha)}[\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}+\|g\|_{C^{\alpha}([0,1],H)}] \tag{1.36}$$

$$+M\left[\|A\mu\|_{E_{\alpha}(A^{\frac{1}{2}},H)}+\|f(0)+g(0)\|_{E_{\frac{\alpha}{2}}(A,H)}\right],$$

$$\|Au_1-g(1)\|_{E_\alpha(A^{\frac12},H)}\le \frac{M}{\alpha(1-\alpha)}[\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}+\|g\|_{C^\alpha([0,1],H)}] \tag{1.37}$$

$$+M\left[\|A\mu\|_{E_\alpha(A^{\frac12},H)}+\|f(0)+g(0)\|_{E_{\frac{\alpha}{2}}(A,H)}+\|f(-1)+g(1)\|_{E_\alpha(A^{\frac12},H)}\right]$$

for the solution of the boundary value problem (1.1). Estimates (1.33) and (1.34) were established in [9] and [10]. Now, first step would be to establish (1.35). Using (1.32), we get

$$Au_0+f(0)=Te^{-A^{\frac12}}\left[2\int_0^{-1}Ae^{-(1+s)A}\left(f(s)-f(-1)\right)ds\right.$$

$$-\int_0^1 A^{\frac12}e^{-(1-s)A^{\frac12}}\left(g(s)-g(1)\right)ds+2A\mu$$

$$\left.+2\left(e^{-A}-I\right)f(-1)-\left(I-e^{-A^{\frac12}}\right)g(1)-g(0)+\left(e^{-A^{\frac12}}-2e^{-A}\right)f(0)\right]$$

$$+T\int_0^1 A^{\frac12}e^{-sA^{\frac12}}\left(g(s)-g(0)\right)ds+T\left[g(0)+f(0)\right].$$

Using this formula and estimates (1.2), (1.3), (1.5), (1.9), (1.13), (1.15) and (1.16), we obtain

$$\|Au_0+f(0)\|_{E_{\frac{\alpha}{2}}(A,H)}\le\|T\|_{H\to H}\left\|e^{-A^{\frac12}}\right\|_{H\to H}$$

$$\times\left[2\left\|\int_{-1}^0 Ae^{-(1+s)A}\left(f(s)-f(-1)\right)ds\right\|_{E_{\frac{\alpha}{2}}(A,H)}\right.$$

$$\left.+\left\|\int_0^1 A^{\frac12}e^{-(1-s)A^{\frac12}}\left(g(s)-g(1)\right)ds\right\|_{E_{\frac{\alpha}{2}}(A,H)}\right]$$

$$+\|T\|_{H\to H}\left\|e^{-A^{\frac12}}\right\|_{H\to E_{\frac{\alpha}{2}}(A,H)}\left[2\left(1+\left\|e^{-A}\right\|_{H\to H}\right)\|f(-1)\|_H\right.$$

$$+\left(1+\left\|e^{-A^{\frac12}}\right\|_{H\to H}\right)\|g(1)\|_H$$

$$\left.+2\|A\mu\|_H+\|g(0)\|_H+\left(\left\|e^{-A^{\frac12}}\right\|_{H\to H}+2\left\|e^{-A}\right\|_{H\to H}\right)\|f(0)\|_H\right]$$

$$+\|T\|_{H\to H}\,\|\int_0^1 A^{\frac12}e^{-sA^{\frac12}}\left(g(s)-g(0)\right)ds\|_{E_{\frac{\alpha}{2}}(A,H)}+\|T\|_{H\to H}\,\|g(0)+f(0)\|_{E_{\frac{\alpha}{2}}(A,H)}$$

$$\le\frac{M}{\alpha(1-\alpha)}\left[\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}+\|g\|_{C^\alpha([0,1],H)}\right]\quad+M\left[\|A\mu\|_{E_\alpha(A^{\frac12},H)}\right.$$

$$+\left\|f(0)+g(0)\right\|_{E_{\frac{\alpha}{2}}(A,H)}\Big].$$

Second step would be to establish (1.36). Using (1.32), we get

$$Au_0-g(0)=Te^{-A^{\frac{1}{2}}}\Bigg[2\int\limits_0^{-1}Ae^{-(1+s)A}\left(f(s)-f(-1)\right)ds$$

$$-\int\limits_0^1 A^{\frac{1}{2}}e^{-(1-s)A^{\frac{1}{2}}}\left(g(s)-g(1)\right)ds+2A\mu$$

$$+2\left(e^{-A}-I\right)f(-1)-\left(I-e^{-A^{\frac{1}{2}}}\right)g(1)+\left(-e^{-A^{\frac{1}{2}}}+2e^{-A}\right)g(0)\Bigg]$$

$$+T\int\limits_0^1 A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds-TA^{\frac{1}{2}}\left(I-e^{-2A^{\frac{1}{2}}}\right)\left(f(0)+g(0)\right).$$

Using this formula and estimates (1.2), (1.3), (1.5), (1.6), (1.8), (1.11), (1.12) and (1.14), we obtain

$$\left\|Au_0-g(0)\right\|_{E_\alpha(A^{\frac{1}{2}},H)}\leq\|T\|_{H\to H}\left\|e^{-A^{\frac{1}{2}}}\right\|_{H\to H}$$

$$\times 2\left\|\int\limits_{-1}^0 Ae^{-(1+s)A}\left(f(s)-f(-1)\right)ds\right\|_{E_\alpha(A^{\frac{1}{2}},H)}$$

$$+\left\|\int\limits_0^1 A^{\frac{1}{2}}e^{-(1-s)A^{\frac{1}{2}}}\left(g(s)-g(1)\right)ds\right\|_{E_\alpha(A^{\frac{1}{2}},H)}\Bigg]$$

$$+\|T\|_{H\to H}\left\|e^{-A^{\frac{1}{2}}}\right\|_{H\to E_\alpha(A^{\frac{1}{2}},H)}\left[2\left(1+\left\|e^{-A}\right\|_{H\to H}\right)\|f(-1)\|_H\right.$$

$$+\left(1+\left\|e^{-A^{\frac{1}{2}}}\right\|_{H\to H}\right)\|g(1)\|_H$$

$$\left.+2\|A\mu\|_H+\left(\left\|e^{-A^{\frac{1}{2}}}\right\|_{H\to H}+2\left\|e^{-A}\right\|_{H\to H}\right)\|g(0)\|_H\right]$$

$$+\|T\|_{H\to H}\,\|\int\limits_0^1 A^{\frac{1}{2}}e^{-sA^{\frac{1}{2}}}\left(g(s)-g(0)\right)ds\|_{E_\alpha(A^{\frac{1}{2}},H)}$$

$$+\left\|A^{\frac{1}{2}}T\right\|_{H\to H}\left[1+\left\|e^{-2A^{\frac{1}{2}}}\right\|_{H\to H}\right]\|g(0)+f(0)\|_{E_\alpha(A^{\frac{1}{2}},H)}$$

$$\leq\frac{M}{\alpha(1-\alpha)}\left[\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)}+\|g\|_{C^\alpha([0,1],H)}\right]\;+M\Big[\|A\mu\|_{E_\alpha(A^{\frac{1}{2}},H)}$$

$$+\left\|f(0)+g(0)\right\|_{E_{\frac{\alpha}{2}}(A,H)}\Big].\qquad(1.38)$$

Third step would be to establish (1.37). Using (1.32), we get

$$Au_1 - g(1) = e^{-A}\left[Au_0 - g(0)\right] + e^{-A}\left[g(0) + f(-1)\right] + A\mu$$
$$+ \int\limits_0^{-1} Ae^{-(1+s)A}\left(f(s) - f(-1)\right) ds - \left(f(-1) + g(1)\right).$$

Using this formula and estimates (1.2), (1.16), (1.6), (1.8), (1.11), (1.12) and (1.38), we obtain

$$\|Au_1 - g(1)\|_{E_\alpha(A^{\frac{1}{2}},H)} \le \left\|e^{-A}\right\|_{H\to H}\|Au_0 - g(0)\|_{E_\alpha(A^{\frac{1}{2}},H)}$$
$$+ \left\|e^{-A}\right\|_{H\to E_\alpha(A^{\frac{1}{2}},H)}\left[\|g(0)\|_H + \|f(-1)\|_H\right] + \|A\mu\|_{E_\alpha(A^{\frac{1}{2}},H)}$$
$$+ \left\|\int\limits_{-1}^{0} Ae^{-(1+s)A}\left(f(s) - f(-1)\right) ds\right\|_{E_\alpha(A^{\frac{1}{2}},H)} + \|f(-1) + g(1)\|_{E_\alpha(A^{\frac{1}{2}},H)}$$
$$\le \frac{M}{\alpha(1-\alpha)}\left[\|f\|_{C^{\frac{\alpha}{2}}([-1,0],H)} + \|g\|_{C^{\alpha}([0,1],H)}\right] + M\left[\|A\mu\|_{E_\alpha(A^{\frac{1}{2}},H)}\right.$$
$$\left. + \|f(0) + g(0)\|_{E_{\frac{\alpha}{2}}(A,H)} + \|f(-1) + g(1)\|_{E_\alpha(A^{\frac{1}{2}},H)}\right].$$

Theorem 1.5 is proved.

Remark 1. Theorem 1.5 holds for the solution of the problem (1.1) in an arbitrary Banach space E with strongly positive operator A under the assumptions

$$\left\|\left(I + e^{-2A^{\frac{1}{2}}} + A^{\frac{1}{2}}\left(I - e^{-2A^{\frac{1}{2}}}\right) - 2e^{-(A^{\frac{1}{2}}+A)}\right)^{-1}\right\|_{E\to E} \le M,$$

$$\left\|A^{\frac{1}{2}}\left(I + e^{-2A^{\frac{1}{2}}} + A^{\frac{1}{2}}\left(I - e^{-2A^{\frac{1}{2}}}\right) - 2e^{-(A^{\frac{1}{2}}+A)}\right)^{-1}\right\|_{E\to E} \le M.$$

Remark 2. The nonlocal boundary value problem for the elliptic-parabolic equation

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t), 0 < t < 1, \\ -\frac{d^2u(t)}{dt^2} + Au(t) = g(t), -1 < t < 0, \\ u(1) = u(-1) + \mu \end{cases}$$

in a Hilbert space H with a self-adjoint positive definite operator A is considered in paper [42]. The well-posedness of this problem in Hölder spaces $C^\alpha(H)$ without a weight was established under the strong condition on μ, $f(-1) + g(1)$ and $f(0) + g(0)$.

2 Applications

First, the mixed boundary value problem for the elliptic-parabolic equation

$$\begin{cases} -u_{tt} - (a(x)u_x)_x + \delta u = g(t,x), 0 < t < 1, 0 < x < 1, \\ u_t + (a(x)u_x)_x - \delta u = f(t,x), -1 < t < 0, 0 < x < 1, \\ f(0,x) + g(0,x) = 0, f(-1,x) + g(1,x) = 0, 0 \le x \le 1, \\ u(t,0) = u(t,1), u_x(t,0) = u_x(t,1), -1 \le t \le 1, \\ u(1,x) = u(-1,x), 0 \le x \le 1, \\ u(0+,x) = u(0-,x), u_t(0+,x) = u_t(0-,x), 0 \le x \le 1 \end{cases} \tag{2.1}$$

generated by the investigation of the motion of gas on the nonhomogeneous space is considered (see [6] and [40]). Problem (2.1) has a unique smooth solution $u(t,x)$ for the smooth $a(x) \geqslant a > 0 (x \in (0,1))$, and $g(t,x)(t \in [0,1],\ x \in [0,1])$, $f(t,x)(t \in [-1,0], x \in [0,1])$ functions and $\delta = \text{const} > 0$. This allows us to reduce the mixed problem(2.1) to the nonlocal boundary value problem (1.1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A defined by (2.1).

Theorem 2.1 . *The solutions of the nonlocal boundary value problem (2.1) satisfy the coercivity inequality*

$$\| u_{tt} \|_{C^\alpha([0,1],L_2[0,1])} + \| u_t \|_{C^{\frac{\alpha}{2}}([-1,0],L_2[0,1])} + \| u \|_{C^\alpha([-1,1],W_2^2[0,1])}$$
$$\le \frac{M}{\alpha(1-\alpha)} \left[\| g \|_{C^\alpha([0,1],L_2[0,1])} + \| f \|_{C^{\frac{\alpha}{2}}([-1,0],L_2[0,1])} \right].$$

Here M does not depend on α, $f(t,x)$ and $g(t,x)$.

The proof of Theorem 2.1 is based on the abstract Theorem 1.5 and the symmetry properties of the space operator generated by the problem (2.1).

Second, let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1,\ \ 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[-1,1] \times \Omega$, the mixed boundary value problem for multi-dimensional mixed equation

$$\begin{cases} -u_{tt} - \sum\limits_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} = g(t,x), 0 < t < 1, x \in \Omega, \\ u_t + \sum\limits_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} = f(t,x), -1 < t < 0, x \in \Omega, \\ f(0,x) + g(0,x) = 0, f(-1,x) + g(1,x) = 0, x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S,\ -1 \le t \le 1; u(1,x) = u(-1,x), x \in \overline{\Omega}, \\ u(0+,x) = u(0-,x), u_t(0+,x) = u_t(0-,x), x \in \overline{\Omega} \end{cases} \tag{2.2}$$

is considered. The problem (2.2) has a unique smooth solution $u(t,x)$ for the smooth $a_r(x) \geqslant a > 0\ (x \in \Omega)$ and $g(t,x)\ (t \in (0,1),\ x \in \overline{\Omega}), f(t,x)\ (t \in (-1,0),\ x \in \overline{\Omega})$ functions. This allows us to reduce the mixed problem (2.2) to the nonlocal boundary value problem (1.1) in a Hilbert space $H = L_2(\overline{\Omega})$ of the all integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\| f \|_{L_2(\overline{\Omega})} = \{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \}^{\frac{1}{2}}$$

with a self- adjoint positive definite operator A defined by (2.2).

Theorem 2.2 . *The solutions of the nonlocal boundary value problem (2.2) satisfy the coercivity inequality*

$$\| u_{tt} \|_{C^{\alpha}([0,1],L_2(\overline{\Omega}))} + \| u_t \|_{C^{\frac{\alpha}{2}}([-1,0],L_2(\overline{\Omega})])} + \| u \|_{C^{\alpha}([-1,1],W_2^2(\overline{\Omega}))}$$
$$\leq \frac{M}{\alpha(1-\alpha)} \left[\| g \|_{C^{\alpha}([0,1],L_2(\overline{\Omega}))} + \| f \|_{C^{\frac{\alpha}{2}}([-1,0],L_2(\overline{\Omega}))} \right].$$

Here M does not depend on α, $f(t,x)$ and $g(t,x)$.

The proof Theorem 2.2 is based on the abstract Theorem 1.5 and the symmetry properties of the space operator A generated by the problem (2.2) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 2.3 . *For the solutions of the elliptic differential problem*

$$\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = \omega(x), x \in \overline{\Omega}, \tag{2.3}$$

$$u(x) = 0, x \in S$$

the following coercivity inequality [36]

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M ||\omega||_{L_2(\overline{\Omega})}$$

is valid.

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