Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **42**, 1–16; https://doi.org/10.14232/ejqtde.2020.1.42 www.math.u-szeged.hu/ejqtde/

p-biharmonic equation with Hardy–Sobolev exponent and without the Ambrosetti–Rabinowitz condition

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Received 6 January 2020, appeared 27 June 2020
Communicated by Dimitri Mugnai

Abstract. This paper is concerned with the existence and multiplicity to p-biharmonic equation with Sobolev–Hardy term under Dirichlet boundary conditions and Navier boundary conditions, respectively. We focus on the case of the nonlinear terms without the Ambrosetti–Rabinowitz conditions. Our method is based on the variational method.

Keywords: variational methods, p-biharmonic equation, Sobolev–Hardy inequality, Fountain Theorem.

2020 Mathematics Subject Classification: 35J35, 35J62, 35J75, 35D30.

1 Introduction

We consider the following p-biharmomic equations with clamped Dirichlet boundary conditions

$$\begin{cases} \triangle_p^2 u = \frac{\mu|u|^{r-2}u}{|x|^s} + f(x,u) & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{PD}$$

and p-biharmomic equations with hinged Navier boundary conditions

$$\begin{cases} \triangle_p^2 u = \frac{\mu|u|^{r-2}u}{|x|^s} + f(x,u) & \text{in } \Omega, \\ u = \triangle u = 0 & \text{on } \partial\Omega \end{cases} \tag{PNa}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $0 \in \Omega$, $2 < 2p < N$, $p \leq r < p^*(s) = \frac{(N-s)p}{N-2p} \leq p^*(0) := p^*$, $\mu \geq 0$.

Since Lazer and McKenna [11] provided a model for discussing the traveling waves in suspension bridges, existence and multiple of solutions for nonlinear biharmonic equations and p-biharmonic equations have been studied under the framework of nonlinear functional analysis. Bhakta [4] studied existence, multiplicity and qualitative properties of entire solutions

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of the p-biharmonic equations with Hardy term. Huang and Liu [16] obtained sign-changing solutions for p-biharmonic equations with Hardy potential. Bueno et al. [5] get multiplicity of solutions for p-biharmonic problems with with concave-convex nonlinearities. Wang and Zhao [25] studied the existence and multiplicity of solutions of p-biharmonic type equations with critical growth. On this topic, we also refer to [3,6,22,26] and references therein.

Ghoussoub and Yuan [14] obtained multiple solutions for $-\triangle_p u = \mu \frac{|u|^{r-2}u}{|x|^s} + \lambda |u|^{q-2}u$ with homogeneous Dirichlet boundary conditions in $W_0^{1,p}(\Omega)$. Perera and Zou [23] studied the multiplicity, and bifurcation results for p-Laplacian problems involving critical Hardy–Sobolev exponents in $W_0^{1,p}(\Omega)$. One of the starting points of this paper is to generalize the part results in [14,23] to the fourth-order elliptic equation.

Definition 1.1. The function u in $W_0^{2,p}(\Omega)$ is called a weak solution of Problem (PD), if

$$\int_\Omega \left[|\triangle u|^{p-2}\triangle u \triangle \phi - \frac{\mu |u|^{r-2}u\phi}{|x|^s} - f(x,u)\phi \right] dx = 0 \quad \text{for any } \phi \in W_0^{2,p}(\Omega);$$

u in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ is said to be a weak solution of Problem (PNa), in case

$$\int_\Omega \left[|\triangle u|^{p-2}\triangle u \triangle \phi - \frac{\mu |u|^{r-2}u\phi}{|x|^s} - f(x,u)\phi \right] dx = 0, \qquad \forall \phi \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega).$$

Since Problem (PNa) is handled similarly to Problem (PD), we discuss the problem (PD) and only give a simple explanation for Problem (PNa).

The starting point for the variational methods of the questions (PD) and (PNa) is the following Sobolev–Hardy inequality (we refer to Lemma 2.2 in Section 2). Let $2 < 2p < N$, $r \leq p^*(s)$, then

$$\left(\int_\Omega \frac{|u|^r}{|x|^s} dx \right)^{\frac{1}{r}} \lesssim \left(\int_\Omega |\triangle u|^p dx \right)^{\frac{1}{p}}, \qquad \forall u \in C_0^\infty(\Omega \setminus \{0\})$$

Therefore, we may define

$$\mu_{s,r}(\Omega) = \inf_{\substack{u \in W_0^{2,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\triangle u|^p dx}{\left(\int_\Omega \frac{|u|^r}{|x|^s} dx \right)^{\frac{p}{r}}}$$

and

$$\widetilde{\mu}_{s,r}(\Omega) = \inf_{\substack{u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\triangle u|^p dx}{\left(\int_\Omega \frac{|u|^r}{|x|^s} dx \right)^{\frac{p}{r}}}.$$

We replace $|u|^{q-2}u$ in [14,23] by a more general nonlinear perturbation $f(x,t)$, and we impose naturally some structural conditions on the nonlinear term $f(x,t)$, so that the associated Euler-Lagrange functional is expected to have some mountain pass geometry and compactness results. Specifically, we consider the following assumptions:

$f_1)$ $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(x,0) = 0$ for all $x \in \overline{\Omega}$;

$f_2)$ $\lim_{|t| \to +\infty} \frac{F(x,t)}{t^p} = +\infty$ uniformly on $x \in \overline{\Omega}$, where $F(x,t) = \int_0^t f(x,\tau) \mathrm{d}\tau$;

$f_3)$ $\limsup_{|t| \to 0} \frac{pF(x,t)}{|t|^p} < \lambda_1(\text{or } \tilde{\lambda}_1)$ uniformly on $x \in \overline{\Omega}$, where $\lambda_1 > 0$ is the first eigenvalue of the operator $\triangle_p^2$ in Ω with homogeneous Dirichlet boundary conditions (or homogeneous Navier boundary conditions);

(SCPI) $f(x,t)$ has subcritical polynomial growth, i.e.

$$\lim_{|t|\to+\infty|} \frac{f(x,t)}{|t|^{p^*-1}} = 0.$$

The critical point theory is based on the existence of some linking structure and deformation lemmas. To obtain such deformation results, some compactness condition of the functional is necessary. In order to get compactness, the standard approach is to apply the Ambrosetti–Rabinowitz conditions ((A–R) for short) to $f(x,t)$ and $F(x,t)$ due to Ambrosetti–Rabinowitz [1]:

(A–R) $\exists\, R_0 > 0, \theta > p$ such that $0 < \theta F(x,s) \le sf(x,s)$ for any $(|s|,x)$ in $[R_0,+\infty)\times\Omega$.

The main role of (A–R) condition is to ensure the boundedness of Palais–Smale or Cerami sequence of Euler–Lagrange functional associated to Eq. (PD) and (PNa). But (A–R) condition is a relatively restrictive eliminating many nonlinearities, for example, $f(x,t) = t\log t^2$. The absence of (A–R) condition in the second order elliptic equation goes back to Costa, Magalhães [7], Miyagaki, Souto [24], Li, Yang [19] and Liu [20], and was improved by Mugnai and Papageorgiou [21]. On this topic, we also refer to [2,8,13,17] and references therein. Inspired by [19,21], we assume the following conditions (without the (A–R) condition):

f_4) There exist $C_* \ge 0, \theta \ge 1$ such that

$$H(x,t) \le \theta H(x,\tau) + C_* \qquad \forall t,\tau\in\mathbb{R},\ 0<|t|<|\tau|,\ \forall x\in\Omega,$$

where $H(x,t) = \frac{1}{p}tf(x,t) - F(x,t)$.

Theorem 1.2. *Assume that $f(x,t)$ satisfies (f_1)–(f_3) and* (SCPI) *condition. Then*

- *Problem* (PD) *admits at least a nontrivial weak solution $u \in W_0^{2,p}(\Omega)$;*
- *Problem* (PNa) *admits at least a nontrivial weak solution $u \in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$.*

For convenience, we first define the Euler–Lagrange functional I_μ as follows:

$$I_\mu(u) = \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega F(x,u)dx,$$

where $F(x,t) = \int_0^t f(x,\tau)d\tau$.

Additionally if we assume that $f(x,t)$ is an odd function in t, then we can prove the existence of infinitely many weak solutions to Problem (PD) and (PNa). Specifically, we can get the following results:

Theorem 1.3. *Suppose that (f_1)–(f_3) hold and*

f_5) there exist $a,b>0$ and $q\in(p,p^*)$ such that

(SCP) $$|f(x,t)| \le a + b|t|^{q-1} \quad \text{for any } (x,t)\in\Omega\times\mathbb{R};$$

f_6) $f(x,-t) = -f(x,t)$, $\forall (x,t)\in\Omega\times\mathbb{R}$,

in addition, if $p = r$, then

- *Problem* (PD) *possesses a sequence of solutions* $\{u_n\} \in W_0^{2,p}(\Omega)$ *such that* $I_\mu(u_n) \to +\infty$ *provided* $0 \le \mu < \mu_{s,r}(\Omega)$*;*

- *Problem* (PNa) *contains a sequence of solutions* $\{u_n\} \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ *such that* $I_\mu(u_n) \to +\infty$ *in case* $0 \le \mu < \widetilde{\mu}_{s,r}(\Omega)$*.*

This paper is organized as follows: Section 2 is devoted to review some necessary mathematical knowledge about function spaces, embedding and associated functional settings. In Section 3, we gets the existence of solution to Eq. (PD) and (PNa) under $g(x,t)$ with A–R condition. In Section 4, we obtain the multiplicity of Eq. (PD) and (PNa). Section A is an appendix.

2 Functional framework

In this paper, $W_0^{2,p}(\Omega)$ and $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ are equipped with norm

$$\|u\| = \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{1}{p}},$$

then $W_0^{2,p}(\Omega)$ and $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ are all Banach space.

Davies [9] extends the Rellich inequality to L^p spaces. But we only need one special case here.

Lemma 2.1 ([9, Corollary 14])**.** *For any* $p \in (1, \frac{N}{2})$ *and* $u \in C_0^\infty(\Omega \setminus \{0\})$*, the following inequality*

$$\int_\Omega |\triangle u|^p dx \ge \left(\frac{(p-1)N(N-2p)}{p^2}\right)^p \int_\Omega \frac{|u|^p}{|x|^{2p}} dx$$

is established.

Next, we will prove the corresponding Sobolev–Hardy inequality in the space $W^{2,p}(\Omega)$. Our method is derived from the proof method of Lemma 2.1 in [28] and Lemma 3.2 in [14].

Lemma 2.2 (Sobolev–Hardy inequality)**.** *Suppose that* $2 < 2p < N$*, then*

(1) If $0 < r < p^*(s)$*, there exists a constant* $C > 0$ *such that*

$$\left(\int_\Omega \frac{|u|^r}{|x|^s} dx\right)^{\frac{1}{r}} \le C \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{1}{p}} \tag{2.1}$$

for any $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$*.*

(2) If $p \le r < p^*(s)$*, then the map* $u \to \frac{u}{|x|^{\frac{s}{r}}}$ *is compact from* $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ *to* $L^r(\Omega)$*.*

Proof. (1) When $s = 0$ or $s = 2p$, (2.1) is Sobolev's inequality or Rellich's inequality, respectively. Since $p^*(s) \ge p$, we only need to consider the scenario of $0 < s < 2p$. According to

Rellich's inequality, Sobolev's inequality and Hölder's inequality, we can get

$$
\begin{aligned}
\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}dx &= \int_\Omega \frac{|u|^{\frac{s}{2}}}{|x|^s}|u|^{p^*(s)-\frac{s}{2}}dx \\
&\le \left(\int_\Omega \frac{|u|^p}{|x|^{2p}}dx\right)^{\frac{s}{2p}} \left(\int_\Omega |u|^{p^*}dx\right)^{\frac{2p-s}{2p}} \\
&\le \left(\frac{p^2}{(p-1)N(N-2p)}\right)^{\frac{s}{2}} \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{s}{2p}} S_2 \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{2p-s}{2p}\cdot\frac{p^*}{p}} \\
&= C_1 \left(\int_\Omega |\triangle u|^p dx\right)^{\frac{N-s}{N-2p}},
\end{aligned}
$$

where

$$
C_1 = \left(\frac{p^2}{(p-1)N(N-2p)}\right)^{\frac{s}{2}} S_2, S_2 = \inf_{\substack{u\in W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)\\ u\neq 0}} \frac{\int_\Omega |\triangle u|^p dx}{(\int_\Omega |u|^{p^*}dx)^{\frac{p}{p^*}}}
$$

is the corresponding optimal Sobolev constant.

(2) Let $\{u_n\}$ be a bounded sequence in $W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega)$, then there is a convergent subsequence of $\{u_n\}$ (still represented by $\{u_n\}$) such that

$$
\begin{aligned}
u_n &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega), \\
u_n &\to u \quad \text{strongly in } L^r(\Omega),\ \ p\le r< p^*(s).
\end{aligned}
$$

On the other hand,

$$
\int_\Omega \frac{|u_n-u|^r}{|x|^s}dx \le C\int_{B_\delta(0)} \frac{|u_n-u|^r}{|x|^s}dx + C\|u_n-u\|^r_{L^r(\Omega)}, \quad \text{where } B_\delta(0)=B(0,\delta).
$$

In the light of Hölder's inequality, we have

$$
\begin{aligned}
\int_\Omega \frac{|u_n-u|^r}{|x|^s}dx &\le C\left(\int_\Omega |u_n-u|^{p^*}dx\right)^{\frac{r}{p^*}} \left(\int_{B_\delta(0)} |x|^{-\frac{p^*s}{p^*-r}}dx\right)^{1-\frac{r}{p^*}} + C\|u_n-u\|^r_{L^r(\Omega)} \\
&\le C\left(\delta^{-\frac{p^*s}{p^*-r}+N}\right)^{1-\frac{r}{p^*}} + C\|u_n-u\|^r_{L^r(\Omega)}.
\end{aligned}
$$

Considering $p\le r<p^*(s)$ and $N-\frac{p^*s}{p-r}>0$ and let $\delta\to 0, n\to\infty$, we can get immediately inequalities

$$
\int_\Omega \frac{|u_n-u|^r}{|x|^s}dx \to 0.
$$

□

In order to study Eq. (PD) and (PNa), we need to discuss some properties of operator $\triangle_p^2$ on $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.

Proposition 2.3. *For any bounded Ω in $\mathbb{R}^N$ and any p in $(1,+\infty)$, $\triangle_p^2$ satisfies the following properties:*

1) ([10]) $\triangle_p^2 : W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \to (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))^*$ is a hemicontinuous operator;

2) $\triangle_p^2$ is a bounded continuous and uniformly convex coercive operator;

3) $\triangle_p^2 : W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \to (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))^*$ is homeomorphic.

Proof. 2) Obviously, $\triangle_p^2$ is bounded continuously coercive. And the strict monotonicity of $\triangle_p^2$ can be derived from the following inequality [15, Lemma 5.1 and Lemma 5.2]:

Let $x,\ y \in \mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^N$, then

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y \rangle \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2, \\ C_p \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } 1 < p < 2. \end{cases} \tag{2.2}$$

3) Applying the Browder–Minty theorem, 1) and 2), we known that $\triangle_p^2$ is surjection. Similar to [12, Lemma 3.1 (iii)], it is not difficult to prove $\triangle_p^2$ is a homeomorphism. □

Remark 2.4. If $\triangle_p^2$ is an operator from $W_0^{2,p}(\Omega)$ to $(W_0^{2,p}(\Omega))^*$, Proposition 2.3 is also valid [18, Proposition 2.1].

Since $f(x,t)$ satisfies the condition (SCPI), $I_\mu(u)$ is well-posed on $W^{2,p}(\Omega)$ and is C^1, the weak solution to the problem (PD) is the critical point of $I_\mu(u)$ in $W_0^{2,p}(\Omega)$. Because the boundary condition $\triangle u|_{\partial\Omega} \equiv 0$ in Problem (PNa) is not included in natural space $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, so Problem (PNa) must be considered in another way. Specifically, we need the regularity of the critical point to $I_\mu(u)$ in space $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ to ensure this boundary condition.

Proposition 2.5 ([26], Proposition 4.7)**.** *Suppose that $f(x,t)$ satisfies the condition* (SCPI) *and $|\mu| \leq \widetilde{\mu}_{s,r}(\Omega)$, every critical point u of I_μ satisfies $\triangle u|_{\partial\Omega} \equiv 0$ in the sense of the trace in $W_0^{1,p}(\Omega)$ $\cap W^{2,p}(\Omega)$.*

3 Proof of Theorem 1.2

In order to use Theorem A.2 to study Eq. (PD) and (PNa), we need to verify that the functionals I_μ satisfies the mountain pass geometry structure and compactness conditions.

Lemma 3.1. *Let f satisfies conditions (f_1)–(f_3) and* (SCPI)*. Then the functional I_μ satisfies mountain pass geometry:*

1. *$I_\mu(0) = 0$.*
2. *There exist positive constants ρ and η such that $I_\mu(u)|_{\partial B_\rho} \geq \eta$.*
3. *There exists e with $\|e\| > \rho$ such that $I_\mu(e) < 0$.*

Proof. 1. $I_\mu(0) = 0$ is straightforward by the condition (f_1). For 2, it follows from (f_3) and (SCPI) that there exist C_2, λ such that

$$F(x,t) \leq \frac{1}{p}(\lambda_1 - \lambda)|t|^p + C_2|t|^{p^*} \quad \text{for any } (x,t) \text{ in } \Omega \times \mathbb{R}.$$

Considering the Sobolev embedding theorem and Lemma 2.2, we obtain

$$\begin{aligned} I_\mu(u) &= \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s} dx - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{p}\left(1 - \frac{\lambda_1 - \lambda}{\lambda_1}\right)\|u\|^p - \frac{\mu}{r}C^r\|u\|^r - \mathfrak{C}_2\|u\|^{p^*} \\ &\geq \frac{\lambda}{p\lambda_1}\|u\|^p - \frac{\mu}{r}(\mu_{s,r}(\Omega))^{-\frac{r}{p}}\|u\|^r - \mathfrak{C}_2\|u\|^{p^*}, \end{aligned}$$

where C is the constant in Lemma 2.2.

Thanks to $\lambda > 0$, $p \leq r$ and $p < p^*$, we may take an enough small positive ρ and a positive constant η such that $I_\mu(u)|_{\partial B_\rho} \geq \eta$.

Next, we give the proof of 3. According to the condition (f_2), for all $M > 0$, there is $\delta > 0$ such that $F(x,t) > M|t|^p$ for all (x,t) in $\bar{\Omega} \times [-\delta,\delta]^c$.

On the other hand, considering the continuity of F, we may get

$$m := \min_{(x,t)\in\bar{\Omega}\times[-\delta,\delta]} F(x,t) \leq F(x,0) = 0.$$

Therefore, we take $M > \frac{\|u\|^p}{p\|u\|^p_{L^p}} > 0$ especially, then there is an $A > 0$ such that

$$F(x,t) \geq M|t|^p - A \quad \text{for any } (x,t) \text{ in } \bar{\Omega} \times \mathbb{R}. \tag{3.1}$$

Hence,

$$\begin{aligned} I_\mu(tu) &= \frac{1}{p}\int_\Omega |\triangle tu|^p dx - \frac{\mu}{r}\int_\Omega \frac{|tu|^r}{|x|^s}dx - \int_\Omega F(x,tu)dx \\ &\leq \frac{1}{p}|t|^p \int_\Omega |\triangle u|^p dx - \frac{\mu}{r}|t|^r \int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega (M|t|^p|u|^p - A)dx \\ &= |t|^p(\frac{1}{p}\|u\|^p - M\|u\|^p_{L^p}) - \frac{\mu}{r}|t|^r \int_\Omega \frac{|u|^r}{|x|^s}dx + A|\Omega|. \end{aligned}$$

Thence $\lim_{t\to+\infty} I_\mu(tu) = -\infty$. □

Lemma 3.2. *Assume that f satisfies (f_1)–(f_4) and* (SCPI)*, then the energy functional I_μ satisfies the Cerami condition for all c in $\mathbb{R}$.*

Proof. Let $\{u_n\}_n^\infty$ be in $W_0^{2,p}(\Omega)$ such that

$$I_\mu\left(u_n\right) \to c$$

and

$$\left(1 + \|u_n\|_{W_0^{2,p}}\right)\left\|I'\left(u_n\right)\right\|_{(W_0^{2,p})^*} \to 0,$$

that is to say,

$$\frac{1}{p}\int_\Omega |\triangle u_n|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u_n|^r}{|x|^s}dx - \int_\Omega F(x,u_n)dx \to c, \tag{3.2}$$

and

$$\left(1 + \|u_n\|_{W_0^{2,p}}\right)\sup_{\|\varphi\|=1}\left|\langle I'(u_n),\varphi\rangle\right| \to 0. \tag{3.3}$$

Step 1. *The sequence $\{u_n\}$ is bounded in $W_0^{2,p}(\Omega)$.*

For if not, i.e. $\|u_n\| \to +\infty$ as $n \to +\infty$. Let $v_n =: \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ (Bounded). Hence, up to a subsequence, $v_n \rightharpoonup v$ in $W_0^{2,p}(\Omega)$. Therefore,

$$\begin{aligned} v_n \to v \quad & \text{in } L^q(\Omega),\ q < p^*, \\ v_n(x) \to v(x) \quad & a.e. \text{ in } \Omega, \\ \frac{v_n}{|x|^{\frac{s}{r}}} \to \frac{v}{|x|^{\frac{s}{r}}} \quad & \text{in } L^r(\Omega),\ r < p^*(s). \end{aligned}$$

We discuss v in two cases.

Case (i): If $v \neq 0$, then let $\Omega_{\neq} := \{x \in \Omega : v(x) \neq 0\}$.

$$|u_n(x)| = |v_n(x)|\|u_n\| \to +\infty \quad a.e. \text{ in } \Omega_{\neq}.$$

Since $I_\mu(u_n) \to c$, we get $\frac{I_\mu(u_n)}{\|u_n\|} \to 0$, i.e.

$$o(1) = \frac{1}{p} - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s\|u_n\|^p}dx - \int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx - \int_{\Omega\setminus\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx. \tag{3.4}$$

In accordance to (f_2), we have

$$\frac{F(x,u_n)}{\|u_n\|^p} = \frac{F(x,u_n)}{|u_n|^p}\cdot\frac{|u_n|^p}{\|u_n\|^p} = \frac{F(x,u_n)}{|u_n|^p}|v_n|^p \to +\infty \quad a.e. \text{ in } \Omega_{\neq} \text{ as } n \to +\infty,$$

which implies $\int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx \to +\infty$.

We claim that

$$\int_{\Omega\setminus\Omega_{\neq}} \frac{F\,(x,u_n)}{\|u_n\|^p}\mathrm{d}x \geqslant -\frac{K}{\|u_n\|^p}\,|\Omega\setminus\Omega_{\neq}| \tag{3.5}$$

for some positive constant K.

In fact, from the condition (f_2), we get $\lim_{|t|\to+\infty} F(x,t) = +\infty$ uniformly in $x \in \bar\Omega$, which implies

$$F(x,t) \geq -K \text{ for any } (x,t) \quad \text{in } \bar\Omega\times\mathbb{R}. \tag{3.6}$$

(The proof for (3.6) is similar to the process of deriving the inequality (3.1) by the condition (f_2). These details are omitted and left to the reader.)

From the inequality (3.6), we may obtain the inequality (3.5).

Since $\|u_n\| \to +\infty$, combining (3.5) and (3.6), we get

$$\begin{aligned} \frac{I_\mu(u_n)}{\|u\|^p} &= \frac{1}{p} - \frac{\mu}{r\|u\|^p}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx - \int_{\Omega\setminus\Omega_{\neq}} \frac{F\,(x,u_n)}{\|u_n\|^p}\mathrm{d}x \\ &\leq \frac{1}{p} - \int_{\Omega_{\neq}} \frac{F(x,u_n)}{\|u_n\|^p}dx - \int_{\Omega\setminus\Omega_{\neq}} \frac{F\,(x,u_n)}{\|u_n\|^p}\mathrm{d}x \\ &\to -\infty, \end{aligned}$$

which contradicts inequality (3.4).

Case (ii): When $v \equiv 0$. Because $t \mapsto I_\mu(tu_n)$ is continuous in $[0,1]$, thence for all n in $\mathbb{N}$ there exists t_n in $[0,1]$ such that

$$I_\mu(t_nu_n) = \max_{t\in[0,1]} I_\mu(tu_n). \tag{3.7}$$

According to the condition (SCPI), for any $R > 0$, there exists $C_3 > 0$ such that

$$F(x,t) \leq C_3|t| + \frac{|t|^{p^*}}{R^{p^*}} \text{ for all } (x,t) \quad \text{in } \Omega \times \mathbb{R}.$$

Owing to $\frac{R}{\|u_n\|}$ in $[0,1]$ for n large enough, we get

$$I_\mu(t_n u_n) = \max_{t\in[0,1]} I_\mu(tu_n) \geq I_\mu\left(R\frac{u_n}{\|u_n\|}\right) = I_\mu(Rv_n)$$

and

$$\begin{aligned} I_\mu(Rv_n) &= \frac{1}{p}\int_\Omega |\triangle Rv_n|^p dx - \frac{\mu}{r}\int_\Omega \frac{|Rv_n|^r}{|x|^s}dx - \int_\Omega F(x,Rv_n)dx \\ &\geq \frac{1}{p}R^p - \frac{\mu}{r}R^r\int_\Omega \frac{|v_n|^r}{|x|^s}dx - C_3R\int_\Omega |v_n|dx - \int_\Omega |v_n|^{p^*}dx. \end{aligned} \tag{3.8}$$

Due to $v_n \rightharpoonup v \equiv 0$ in $W_0^{2,p}(\Omega)$, then $\int_\Omega |v_n(x)|\,\mathrm{d}x \to 0$, $\int_\Omega \frac{|v_n|^r}{|x|^s}dx \to 0$ and $\int_\Omega |v_n(x)|^{p*}\,\mathrm{d}x < C(\Omega)$. Therefore, let $n \to +\infty$ in (3.8), and then let $R \to +\infty$, we have

$$I_\mu(t_n u_n) \geq I_\mu(Rv_n) \to +\infty \quad \text{as } n \to +\infty. \tag{3.9}$$

In addition, it is not difficult to infer that $0 < t_n < 1$ from $I_\mu(0) = 0$ and $I_\mu(u_n) \to c < +\infty$ as $n \to +\infty$.

Furthermore, in the light of (3.7), we have $\frac{d}{dt}(I_\mu(tu_n))|_{t=t_n} = 0$. Therefore,

$$\begin{aligned} \langle I'_\mu(t_n u_n), t_n u_n\rangle &= t_n\langle I'_\mu(t_n u_n), u_n\rangle \\ &= t_n\frac{d}{d\tau}(I_\mu(t_n u_n + \tau u_n))|_{\tau=0} \\ &= t_n\frac{d}{d\tau}(I_\mu(tu_n + \tau u_n))|_{\tau=0,t=t_n} \\ &= t_n\frac{d}{dt}(I_\mu(tu_n + \tau u_n))|_{t=t_n,\tau=0} \\ &= t_n\frac{d}{dt}(I_\mu(tu_n))|_{t=t_n} = 0. \end{aligned}$$

And considering the condition (f_4), we have

$$\begin{aligned} \frac{1}{\theta}I_\mu(t_n u_n) &= \frac{1}{\theta}\left(I_\mu(t_n u_n) - \frac{1}{p}\langle I'_\mu(t_n u_n), t_n u_n\rangle\right) \\ &= \frac{1}{\theta}\mu\left(\frac{1}{p} - \frac{1}{r}\right)|t_n|^r\int_\Omega \frac{|u_n|^r}{|x|^s}dx \\ &\quad + \frac{1}{\theta}\int_\Omega \left(\frac{1}{p}f(x,t_n u_n)\,t_n u_n - F(x,t_n u_n)\right)\mathrm{d}x \\ &= \frac{1}{\theta}\mu\left(\frac{1}{p} - \frac{1}{r}\right)|t_n|^r\int_\Omega \frac{|u_n|^r}{|x|^s}dx + \frac{1}{\theta}\int_\Omega H(x,t_n u_n)dx \\ &= \frac{1}{\theta}\mu\left(\frac{1}{p} - \frac{1}{r}\right)|t_n|^r\int_\Omega \frac{|u_n|^r}{|x|^s}dx + \frac{1}{\theta}\int_\Omega (\theta H(x,u_n) + C_*)dx \\ &= \mu\left(\frac{1}{p} - \frac{1}{r}\right)\left(\frac{|t_n|^r}{\theta} - 1\right)\int_\Omega \frac{|u_n|^r}{|x|^s}dx \\ &\quad + I_\mu(u_n) - \frac{1}{p}\langle I'_\mu(u_n), u_n\rangle + \frac{C_*}{\theta}|\Omega| \end{aligned}$$

$$\begin{aligned}&\leq I_\mu(u_n)-\frac{1}{p}\langle I'_\mu(u_n),u_n\rangle+\frac{C_*}{\theta}|\Omega|\\&\to c+\frac{C_*}{\theta}|\Omega|.\end{aligned}$$

Thence,

$$\limsup_{n\to+\infty} I_\mu(t_nu_n)\leq \theta c+C_*|\Omega|<+\infty,$$

which is contradictive to (3.7).

Step 2. *$\{u_n\}$ admits a convergent subsequence in $W_0^{2,p}(\Omega)$.*

Since $\{u_n\}$ is bounded in the reflexive Bananch space $W_0^{2,p}(\Omega)$, up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{2,p}(\Omega)$. Therefore,

$$\begin{aligned}u_n\to u &\quad \text{in } L^q(\Omega), q<p^*,\\ u_n(x)\to u(x) &\quad a.e. \text{ in } \Omega,\\ \frac{u_n}{|x|^{\frac{s}{r}}}\to\frac{u}{|x|^{\frac{s}{r}}} &\quad \text{in } L^r(\Omega), r<p^*(s),\\ \frac{|u_n|^{r-2}u_n}{|x|^s}\rightharpoonup\frac{|u|^{r-2}u}{|x|^s} &\quad \text{weakly in } L^r(\Omega), r<p^*(s).\end{aligned}$$

According to the condition (SCPI), for every $\varepsilon>0$, there is a $C(\varepsilon)>0$ such that $|f(x,t)|\leq C(\varepsilon)+\varepsilon|t|^{p^*-1}$ for any (x,t) in $\Omega\times\mathbb{R}$. Therefore, we get

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x\right| &\leq C(\epsilon)\int_\Omega|u_n-u|\,\mathrm{d}x+\epsilon\int_\Omega|u_n-u|\,|u_n|^{p^*-1}\,\mathrm{d}x\\ &\leq C(\epsilon)\int_\Omega|u_n-u|\,\mathrm{d}x\\ &\quad+\epsilon\left(\int_\Omega|u_n|^{p^*}\,\mathrm{d}x\right)^{\frac{p^*-1}{p^*}}\left(\int_\Omega|u_n-u|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}\\ &\leq C(\epsilon)\int_\Omega|u_n-u|\,\mathrm{d}x+\epsilon C(\Omega).\end{aligned}$$

In line with $u_n\rightharpoonup u$ in $W_0^{2,p}(\Omega)$, $\int_\Omega|u_n-u|\,\mathrm{d}x\to 0$, and the arbitrariness of ϵ, we may infer that

$$\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x\to 0.$$

On the other hand,

$$\int_\Omega\frac{|u_n|^{r-2}u_n}{|x|^s}(u_n-u)dx\to 0.$$

Hence,

$$\begin{aligned}0\leftarrow&\langle I'_{u_n},u_n-u\rangle\\ =&\int_\Omega|\triangle u_n|^{p-2}\triangle u_n(\triangle u_n-\triangle u)dx\\ &-\mu\int_\Omega\frac{|u_n|^{r-2}u_n}{|x|^s}(u_n-u)dx-\int_\Omega f(x,u_n)(u_n-u)\,\mathrm{d}x\\ =&\int_\Omega|\triangle u_n|^{p-2}\triangle u_n(\triangle u_n-\triangle u)dx+o(1).\end{aligned}$$

Therefore,

$$\int_\Omega |\triangle u_n|^{p-2}\triangle u_n(\triangle u_n - \triangle u)dx \to 0,$$

which implies that $u \to u$ strongly in $W_0^{2,p}(\Omega)$, that is to say, the functional I_μ satisfies the Cerami condition for any c in $\mathbb{R}$. □

Proof of Theorem 1.2. According to Theorem A.2, Lemma 3.1 and Lemma 3.2, we know that Problem (PD) admits a nontrivial weak solution in $W_0^{2,p}(\Omega)$.

From Proposition 2.5, we obtain Lemma 3.1 and Lemma 3.2 when $W_0^{2,p}(\Omega)$ is replaced by $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. Hence Problem (PNa) has also a nontrivial weak solution in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. □

4 Proof of Theorem 1.3

In this section, we apply Theorem A.3 to prove Theorem 1.3. First of all, because $W_0^{2,p}(\Omega)$ is a Banach space, we formulate Y_k and Z_k as in (A.1). The condition (f_6) means $I_\mu(-u) = -I_\mu(u)$. Since the condition (SCP) indicates the condition (SCPI), I_μ contents the Cerami condition for any c in $\mathbb{R}$ under Lemma 3.2. Here, we mimic part of the proof of Theorem 3.7 in [27] and Theorem 1.2 in [2].

In order to estimate A6) in Theorem 1.3, we need the following lemma.

Lemma 4.1.

$$\beta_k = \sup_{\substack{u\in Z_k\\ \|u\|=1}} \|u\|_{L^q} \to 0 \quad \text{as } k \to \infty$$

provided $1 \le q < p^*$.

Proof. $Z_{k+1} = \overline{\bigoplus_{j\ge k+1} X_j} \subset \overline{\bigoplus_{j\ge k} X_j} = Z_k$ suggests $0 \le \beta_{k+1} \le \beta_k$, thence $\lim_{k\to+\infty} \beta_k = b \ge 0$. According to the definition of supper bound, for any $k > 0$, there exists u_k in Z_k with $\|u\|_{L^q} > \frac{\beta_k}{2}$ on $\partial B_1(0)$ in $W_0^{2,p}(\Omega)$. Since $W_0^{2,p}(\Omega)$ is a real, reflexive, and separable Banach space, we can extract a subsequence of $\{u_k\}$ (still denoted for $\{u_k\}$) such that $u_k \rightharpoonup u$ weakly in $W_0^{2,p}(\Omega)$, i.e. $\langle u_k, \varphi\rangle \to \langle u, \varphi\rangle$ for any φ in $(W_0^{2,p}(\Omega))^*$.

Since each Z_k is convex and closed, hence it is closed for the weak topology, which implies

$$u \in \bigcap_{k=1}^{+\infty} Z_k = \{0\}.$$

Therefore, according to Sobolev embedding theorem, we have

$$0 < \frac{\beta_k}{2} < u_k \to 0 \quad \text{in } L^q(\Omega) \text{ as } k \to +\infty.$$ □

Proof of Theorem 1.3. Rewrite (3.1) to the form we need here: For some $k > 0$, there exist $C_k > 0$ and $A_k > 0$ such that

$$F(x,t) \ge C_k|t|^p - A_k \text{ for every } (x,s) \quad \text{in } \overline{\Omega} \times \mathbb{R}.$$

Step 1. *For any* $k \in \mathbb{N}$, *there exists* $\rho_k > 0$ *such that*

$$a_k = \max_{\substack{u\in Y_k\\ \|u\|=\rho_k}} I_\mu(u) \le 0.$$

In fact, all norms on Y_k are equivalent since Y_k is finite dimensional, hence there exist two positive constants $C_{k,p}$ and $\widetilde{C}_{k,p}$ such that

$$C_{k,p}^{\frac{1}{p}}\|u\|_{L^p} \le \|u\| \le \widetilde{C}_{k,p}^{\frac{1}{p}}\|u\|_{L^p} \quad \text{for all } u \in Y_k.$$

Therefore, for all u in Y_k, we have

$$\begin{aligned} I_\mu(u) &= \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega F(x,u)dx \\ &\le \frac{1}{p}\|u\|^p - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - C_k\|u\|_{L^p}^p + A_k|\Omega| \\ &\le \frac{1}{p}\|u\|^p - \|u\|^p + A_k|\Omega| - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx \\ &\le \frac{1-p}{p}\|u\|^p + A_k|\Omega|. \end{aligned}$$

Thence, we choose u in Y_k with $\|u\| = \rho_k > 0$ large enough and obtain

$$I_\mu(u) \le 0.$$

Step 2. *There exists r_k in $(0, \rho_k)$ such that*

$$b_k = \inf_{\substack{u \in Z_k \\ \|u\| = r_k}} I_\mu(u) \to +\infty, \ \textit{as } k \to \infty.$$

Indeed, (SCP) implies that there exists $C' > 0$ such that

$$|F(x,t)| \le C'\left(1 + |t|^q\right).$$

Hence, for any u in Z_k, we get

$$\begin{aligned} I_\mu(u) &= \frac{1}{p}\int_\Omega |\triangle u|^p dx - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - \int_\Omega F(x,u)dx \\ &\ge \frac{1}{p}\|u\|^p - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - C'\|u\|_{L^q}^q - C'|\Omega| \\ &\ge \frac{1}{p}\|u\|^p - \frac{\mu}{r}\int_\Omega \frac{|u|^r}{|x|^s}dx - C'\left\|\frac{u}{\|u\|}\right\|_{L^q}^q\|u\|^q - C'|\Omega| \\ &\ge \frac{1}{p}\|u\|^p - \frac{\mu}{r}(\mu_{s,r}(\Omega))^{-\frac{r}{p}}\|u\|^r - C'\beta_k^q\|u\|^q - C'|\Omega| \\ &= \left[\frac{1}{p}\left(1 - \frac{\mu}{\mu_{s,r}(\Omega)}\right) - C'\beta_k^q\|u\|^{q-p}\right]\|u\|^p - C'|\Omega|. \end{aligned}$$

According to Lemma 4.1, $\lim_{k\to+\infty}\beta_k = +\infty$. Let $r_k = \left(\frac{\mu_{s,r}(\Omega)C'q\beta_k^q}{\mu_{s,r}(\Omega)-\mu}\right)^{-\frac{1}{q-p}}$, then $\lim_{k\to+\infty} r_k = +\infty$. If for $u \in Z_k$ with $\|u\| = r_k$, then we have

$$I_\mu(u) \ge \left(\frac{1}{p} - \frac{1}{q}\right)\left(1 - \frac{\mu}{\mu_{s,r}(\Omega)}\right) r_k^p - C'|\Omega| \to +\infty, \quad \text{as } k \to +\infty,$$

which yields **Step 2**. □

Remark 4.2. If $p < r$, we seem impossible to get $I_\mu(u) \to +\infty$, as $k \to +\infty$. Therefore, in a sense, $p = r$ are sharp.

Appendix A

The machinery of the critical point theory is based on the existence of a linking structure and deformation lemmas. Generally speaking, it is necessary that some compactness condition of the functional in order to derive such deformation results. We use the famous Cerami condition:

Definition A.1 (Cerami (C) condition)**.** Let X be a real Banach space with its dual space X^* and $J \in C^1(X,\mathbb{R})$. For $c \in \mathbb{R}$ we say that J satisfies the $(C)_c$ condition if for any sequence $\{x_n\} \subset X$ with $J(x_n) \to c$ and $(1+\|x_n\|_X)\|J'(x_n)\|_{X^*} \to 0$, then the sequence $\{x_n\}$ admits a subsequence strongly convergent in X.

Theorem A.2 (Mountain Pass Theorem with Cerami condition [8])**.** *Assume that X is a real Banach space and $J \in C^1(X,\mathbb{R})$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, $J(0) = 0$, and, in addition,*

A1) There exist positive constants r and η such that $J(u)|_{\partial B_r} \geq \eta$;

A2) There exists an $u_0 \in X$ with $\|u_0\| > \rho$ such that $J(u_0) \leq 0$.

Then $c = \inf\limits_{\gamma\in\Gamma}\max\limits_{t\in[0,1]} J(\gamma(t)) \geq \alpha$ is a critical value of J, where

$$\Gamma = \left\{\gamma \in C^0([0,1],X) : \gamma(0) = 0, \gamma(1) = u_0\right\}.$$

Let X be a reflexive and separable Banach space, then there exist sequences $\{e_j\} \subset X$ and $\{\varphi_j\} \subset X^*$ with

A3) $\langle\varphi_i, e_i\rangle = \delta_{i,j}$, where $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j; \end{cases}$

A4) $\overline{\operatorname{span}\{e_j\}_{j=1}^{\infty}} = X$ and $\overline{\operatorname{span}^{w^*}\{\varphi_j\}_{j=1}^{\infty}} = X^*$.

Let $X_j = \mathbb{R}e_j$, then $X = \overline{\bigoplus_{j\geq1} X_j}$. And we define

$$Y_k = \bigoplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k = \overline{\bigoplus_{j\geq k} X_j} \tag{A.1}$$

Theorem A.3 (Fountain Theorem with Cerami condition [2])**.** *Suppose that $\varphi \in C^1(X,\mathbb{R})$ satisfies the $(C)_c$ condition for all $c \in \mathbb{R}$ and $\varphi(u) = \varphi(-u)$. If for any $k \in \mathbb{N}$, there exists $\rho_k > r_k$ such that*

A5) $a_k = \max\limits_{\substack{u\in Y_k \\ \|u\|=\rho_k}} \varphi(u) \leq 0$;

A6) $b_k = \inf\limits_{\substack{u\in Z_k \\ \|u\|=r_k}} \varphi(u) \to +\infty$, as $k \to \infty$,

then φ possesses an unbounded sequence of critical values.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 11771423, 11871452) and National Science Foundation of Jiangsu Higher Education Institutions of China, No. 19KJD100007.

The author wishes to express his thanks to Professor Peihao Zhao from the School of Mathematics and Statistics in Lanzhou University for giving a guide to nonlinear functional analysis. And the author would like to express his gratitude to the anonymous reviewers for careful reading and helpful suggestions.

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