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Linear even order homogenous difference equation with delay in coefficient

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Abstract. We use many classical results known for the self-adjoint second-order linear equation and extend them for a three-term even order linear equation with a delay applied to coefficients. We derive several conditions concerning the oscillation and the existence of positive solutions. Our equation for a choice of parameter is disconjugate, and for a different choice can have positive and oscillatory solutions at the same time. However, it is still, in a sense, disconjugate if we use a weaker definition of oscillation.

Keywords: coefficient delayed equations, separately disconjugate, oscillation theory, minimal solution, difference equation.

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1 Introduction

This paper is divided into two parts. In the first part, we analyse the linear second-order homogeneous difference equation with a delay in a coefficient

$$a_{n-k}y_{n-1} + b_n y_n + a_n y_{n+1} = 0, \quad n \in \mathbb{Z}. \tag{1.1}$$

Equations with a delay in term y_{n-1} are usually considered. Nevertheless, we did not find a situation where the considered delay is in the coefficient a_n. This may be because Eq. (1.1) for $k = 1$ is often discussed together with its self-adjoint form $\triangle(p_n \triangle y_n) + q_n y_{n+1} = 0$.

Properties of this special case were discussed many times. Some necessary and sufficient conditions for the equation to be oscillatory were derived in [6, 8, 10, 19, 20, 22, 29] and for a matrix case in [7]. Properties of eventually positive solutions were observed in [28]. Minimal solutions of the special case were discussed in [14]. Recessive solutions and their connection to oscillation were discussed in [27], for a matrix case in [3], and for nonoscillatory symplectic systems in [33]. Notion of generalized zero was developed in [15] and the Sturm comparison theorem on $\mathbb{Z}$ together with the existence of a recessive solutions was discussed in [2, 5]. Many classic results about this special case can be found in [21]. Boundedness and growth of the special case were investigated in [30,31]. Generalization of the special case were considered

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for example in [24–26, 32]. If we consider a continuous case, criteria for oscillation can be found, for example, in [11], and the existence of a principal solution of a $2n$-order self-adjoint equation was recently discussed in [34]. Some ideas about how to extend the results for the fourth-order equation can be found in [9].

In Section 2, we would like to extend the results from [14], where the special case is also considered. The results from [14] were already extended in [12, 13, 17] and for the time scales in [18], but there was used the symmetrical case for $k = 1$. Arbitrary choice of $k \in \mathbb{Z}$ will lead to the generalization of some already known results.

We derive equivalent conditions for which the equation has a positive solution, and later through the deriving of a suitable version of the Sturm comparison theorem, we will get criteria of disconjugacy for Eq. (1.1). These results will be used in Section 3 as a tool, as well.

In Section 3 we analyse the linear even order homogeneous difference equation with a delay in a coefficient

$$a_{n-kH}y_n + b_{n+H}y_{n+H} + a_{n+H}y_{n+2H} = 0, \quad n \in \mathbb{Z}, \tag{1.2}$$

which is a generalization of Eq. (1.1). For $k = 0$ we get a equation discussed in [16]. We can assume that results obtained in Section 2 can be extended for Eq. (1.2) in the similar way as in [16].

We derive conditions under which Eq. (1.2) can or cannot have positive or eventually positive solutions. We also discuss a situation when Eq. (1.2) has recessive and dominant solutions. Among others, we use a combination of ideas as were established in [19, 27]. We find that Eq. (1.2) can have both positive and sign-changing solutions. A situation where an equation has oscillatory and nonoscillatory solutions at the same time was discussed for example in [1]. The same situation can appear in our equation, but we use a weaker version of oscillation to avoid this situation.

2 Second-order linear coefficient delayed equation

Let real valued sequences a_n, b_n satisfy $a_n < 0$, $b_n > 0$, for every $n \in \mathbb{Z}$. In the first part we study the equation

$$a_{n-k}y_{n-1} + b_ny_n + a_ny_{n+1} = 0, \quad k \in \mathbb{Z}. \tag{2.1}$$

If we consider a solution y_n of Eq. (2.1), then we have a solution $x_n = (-1)^n y_n$ of the equation

$$a_{n-k}x_{n-1} + d_nx_n + a_nx_{n+1} = 0,$$

where sequence $d_n < 0$ for every n. In a similar sense if we consider the equation

$$c_{n-p}x_{n-1} + b_nx_n + c_{n+l}x_{n+1} = 0,$$

where $c_n < 0$ for every n. Then we can take $a_n = c_{n+l}$ and this will result in Eq. (2.1) for $k = -l - p$.

There is a natural relation of Eq. (2.1) to the infinite matrix operator, whose truncations for

$n \le p$, $n, p \in \mathbb{Z}$, are the matrices

$$d_{n,p} = \begin{pmatrix} b_n & a_n & 0 & \dots & 0 \\ a_{n-k+1} & b_{n+1} & a_{n+1} & \ddots & \vdots \\ 0 & a_{n-k+2} & b_{n+2} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & a_{p-1} \\ 0 & \dots & \dots & a_{p-k} & b_p \end{pmatrix}$$

and we denote their determinants by $D_{n,p} = \det(d_{n,p})$. Note that for $k = 1$ is $d_{n,p}$ symmetrical.

For simplification of formulas, we take $D_{i+1,i} = 1$ and $D_{i+j,i} = 0$ for any $i \in \mathbb{Z}$ and $j > 1$, as well as $\prod_i^{i-1} x_i = 1$. Moreover, we will use recurrence relations

$$D_{n,p} = b_n D_{n+1,p} - a_{n-k+1} a_n D_{n+2,p}, \tag{2.2}$$
$$D_{n,p} = b_p D_{n,p-1} - a_{p-k} a_{p-1} D_{n,p-2}, \tag{2.3}$$

for $n \le p$.

Lemma 2.1. *Let $n < p$ and real vectors $\mathbf{X} = (x_n, \dots, x_p)^T$, $\mathbf{B} = (y, 0 \dots, 0, z)^T$, then the equation*

$$d_{n,p}\mathbf{X} = \mathbf{B},$$

implies

$$x_h D_{n,p} = y D_{h+1,p} \prod_{j=n+1}^{h} (-a_{j-k}) + z D_{n,h-1} \prod_{j=h}^{p-1} (-a_j), \tag{2.4}$$

where $n \le h \le p$.

Proof. The proof follows from the Cramer's rule. Signs at $-a_j$ and $-a_{j-k}$ follow from comparing the sign and number of terms in a given product. □

Lemma 2.2. *Let*

$$D_{i,j} > 0, \quad \text{for } i \le j, \tag{2.5}$$

and let x_n^1, x_n^2 be two solutions of Eq. (2.1), which satisfy $x_m^1 = x_m^2$ for some $m \in \mathbb{Z}$. If also $x_h^1 > x_h^2$ (respectively $x_h^1 = x_h^2$) for some $h > m$, then it holds that $x_j^1 > x_j^2$ (respectively $x_j^1 = x_j^2$) for all $j > m$.

Proof. Obviously, two solutions x_n^1, x_n^2 of Eq. (2.1) have to also satisfy Lemma 2.1 where

$$y = -a_{m-k+1} x_m^1 = -a_{m-k+1} x_m^2,$$
$$z^1 = -a_{h-1} x_h^1 > -a_{h-1} x_h^2 = z^2.$$

Where for $i \in \{1, 2\}$ we have $\mathbf{X}^i = \left(x_{m+1}^i, \dots, x_{h-1}^i\right)^T$ and $\mathbf{B}^i = (y, 0 \dots, 0, z^i)^T$. Together with (2.5), we obtain from (2.4) that

$$x_j^1 D_{m+1,h-1} = y D_{j+1,h-1} \prod_{i=m+2}^{j} (-a_{i-k}) + z^1 D_{m+1,j-1} \prod_{i=j}^{h-2} (-a_i)$$
$$> y D_{j+1,h-1} \prod_{i=m+2}^{j} (-a_{i-k}) + z^2 D_{m+1,j-1} \prod_{i=j}^{h-2} (-a_i) = x_j^2 D_{m+1,h-1},$$

holds for all $n < j < h$ and thus $x_j^1 > x_j^2$. Taking $x_j^1 < x_j^2$ for some $j > h$ leads to a contradiction with $x_h^1 > x_h^2$ in the same manner. Therefore, $x_j^1 > x_j^2$ for all $j > m$. The case of $x_h^1 = x_h^2$ follows analogously. □

Similarly, we get a version of Lemma 2.2 for some $h < m$ and all $j < m$. It means that if two solutions of Eq. (2.1) are equal at two points, then they are equal everywhere.

Lemma 2.3. *Assume (2.5), then for any $h < p$ it holds that*

$$\frac{1}{b_h} < \frac{D_{h+1,p}}{D_{h,p}} < \frac{b_{h-1}}{a_{h-k}a_{h-1}}, \tag{2.6}$$

and the sequence $x_p = \frac{D_{h+1,p}}{D_{h,p}}$ is increasing for any h where $h < p$.

Proof. Because of (2.2) we get

$$D_{h,p} = b_h D_{h+1,p} - a_{h-k+1}a_h D_{h+2,p} < b_h D_{h+1,p},$$

which implies the left inequality of (2.6). Further, we compute

$$\begin{aligned} 0 < D_{h-1,p} &= b_{h-1}D_{h,p} - a_{h-k}a_{h-1}D_{h+1,p}, \\ a_{h-k}a_{h-1}D_{h+1,p} &< b_{h-1}D_{h,p}, \\ \frac{D_{h+1,p}}{D_{h,p}} &< \frac{b_{h-1}}{a_{h-k}a_{h-1}}, \end{aligned}$$

which implies the right inequality in (2.6).

In the second part of the proof, we will proceed by induction. First, we assume $p = h + 1$ and we get

$$\begin{aligned} \frac{D_{h+1,h+2}}{D_{h,h+2}} - \frac{D_{h+1,h+1}}{D_{h,h+1}} &= \frac{D_{h,h+1}D_{h+1,h+2} - D_{h+1,h+1}D_{h,h+2}}{D_{h,h+2}D_{h,h+1}} \\ &= \frac{a_{h-k+1}a_{h-k+2}a_h a_{h+1}}{D_{h,h+2}D_{h,h+1}} > 0. \end{aligned}$$

Next, again by (2.2), we get

$$\frac{D_{h,p}}{D_{h+1,p}} - \frac{D_{h,p+1}}{D_{h+1,p+1}} = a_{h-k+1}a_h \left(\frac{D_{h+2,p+1}}{D_{h+1,p+1}} - \frac{D_{h+2,p}}{D_{h+1,p}} \right) > 0,$$

by the induction assumption, which together with (2.5) results in

$$\begin{aligned} \frac{D_{h,p}}{D_{h+1,p}} &> \frac{D_{h,p+1}}{D_{h+1,p+1}}, \\ \frac{D_{h+1,p}}{D_{h,p}} &< \frac{D_{h+1,p+1}}{D_{h,p+1}}. \end{aligned}$$

Therefore, the sequence is increasing and the proof is complete. □

Similarly, using (2.3), we get for $n < h$ that

$$\frac{1}{b_h} < \frac{D_{n,h-1}}{D_{n,h}} < \frac{b_{h+1}}{a_{h-k+1}a_h},$$

and the sequence $x_n = \frac{D_{n,h-1}}{D_{n,h}}$ is decreasing for any h which $n < h$.

Now, thanks to Lemma 2.3, we can define the sequences

$$c_n^+ = \lim_{p\to\infty} \frac{D_{n+1,p}}{D_{n,p}},$$
$$c_n^- = \lim_{p\to-\infty} \frac{D_{p,n-1}}{D_{p,n}},$$

and

$$u(j,n) = \begin{cases} 1, & j = n, \\ \prod_{h=n}^{j-1}(-a_h)c_h^-, & n < j, \\ \prod_{h=j}^{n-1}(-a_{h-k+1})c_{h+1}^+, & n > j. \end{cases}$$

Notice that by Lemma 2.3 together with $a_i < 0$ for every i, we get that $u(j,n) > 0$ for any j,n.

Definition 2.4. We say that a solution u_n of Eq. (2.1) is minimal on $[j+1,\infty)\cap\mathbb{Z}$ if any linearly independent solution v_n of Eq. (2.1) such that $u_j = v_j$ satisfies $u_k < v_k$ for every $k \geq j+1$. The minimal solution on $(-\infty, j-1]\cap\mathbb{Z}$ is defined analogously.

Lemma 2.5. *Assume* (2.5)*, then $\alpha_n = u(j,n)$ is a positive minimal solution of Eq.* (2.1) *on the interval $[j+1,\infty)\cap\mathbb{Z}$ and also on the interval $(-\infty, j-1]\cap\mathbb{Z}$.*

Proof. Using Lemma 2.1 with $y = -a_{j-k+1}$ and $z = 0$ we obtain that

$$v_n(j,p) = \begin{cases} 1, & n = j, \\ \prod_{h=j}^{n-1}(-a_{h-k+1})\frac{D_{n+1,p}}{D_{j+1,p}}, & j+1 \leq n \leq p, \\ 0, & n = p+1, \end{cases}$$

is a solution on the interval $[j+1,p]\cap\mathbb{Z}$. Moreover, it holds that $u(j,n) = \lim_{p\to\infty} v_n(j,p)$ and so $\alpha_n = u(j,n)$ is a solution on the interval $[j+1,\infty)\cap\mathbb{Z}$, where $\alpha_j = u(j,j) = 1$.

Next, we assume that there is a positive solution v_n such that $v_j = \alpha_j$ and which is also linearly independent on α_n. Then we know that $v_{p+1} > v_{p+1}(j,p) = 0$ and $v_j = v_j(j,p) = 1$, for every p. Therefore, due to Lemma 2.2, we know that $v_n > v_n(j,p)$ for all p. Because $\alpha_n = \lim_{p\to\infty} v_n(j,p)$, we get that $v_n \geq \alpha_n$. But v_n is linearly independent and, again by Lemma 2.2, this inequality must hold strictly, i.e. $v_n > \alpha_n$.

Similarly, we get that $\alpha_n = u(j,n)$ is a solution on interval $(-\infty, j-1]\cap\mathbb{Z}$ using function

$$v_n(j,m) = \begin{cases} 1, & n = j, \\ \prod_{h=n}^{j-1}(-a_h)\frac{D_{m,n-1}}{D_{m,j-1}}, & m \leq n \leq j-1, \\ 0, & n = m-1. \end{cases}$$

□

Further, we will use the following notation. We define

$$u_n^+ = \begin{cases} 1, & n = 0, \\ u(0,n), & n \in \mathbb{N}, \\ u(n,0)^{-1}, & -n \in \mathbb{N}, \end{cases} \quad \text{and} \quad u_n^- = \begin{cases} 1, & n = 0, \\ u(n,0)^{-1}, & n \in \mathbb{N}, \\ u(0,n), & -n \in \mathbb{N}. \end{cases}$$

Lemma 2.6. *Assume* (2.5)*, then $u_n^\pm$ are positive solutions of Eq.* (2.1) *on $\mathbb{Z}$.*

Proof. From Lemma 2.5 we know, that u_n^+ is a solution on $\mathbb{N}$. Moreover, for arbitrary $n, B \in \mathbb{N} \cup \{0\}$, $n < B$, it holds

$$u(-B,0) = u(-B,-n)u(-n,0),$$

and so

$$u_{-n}^+ = \frac{1}{u(-n,0)} = \frac{u(-B,-n)}{u(-B,0)}.$$

Using Lemma 2.5 we obtain that u_n^+ is a solution on interval $[-B+1,\infty) \cap \mathbb{Z}$. Because B is arbitrary, we have that u_n^+ is a solution on $\mathbb{Z}$. The second part involving u_n^- is done in the similar way. □

Theorem 2.7. *Condition (2.5) holds if and only if there is a positive solution of Eq. (2.1).*

Proof. The sufficiency of (2.5) comes directly from Lemma 2.6. For the second part, we assume the existence of a positive solution u_n. Then, using Lemma 2.1 for arbitrary n, $n < p$, with $y = -a_{n-k}u_{n-1}, z = -a_p u_{p+1}$, we get from (2.4) that

$$u_n D_{n,p} = -a_{n-k}u_{n-1}D_{n+1,p} - a_p u_{p+1} \prod_{j=n}^{p-1}(-a_j).$$

If we put $p = n+1$, then because $D_{n+1,n+1} = b_{n+1} > 0$ we obtain that the right-hand side is positive which implies the positivity of $D_{n,n+1} > 0$. Next, by induction we obtain that if $D_{n+1,p} > 0$, then also $D_{n,p} > 0$ through the same procedure. Therefore, the condition (2.5) is satisfied. □

We emphasize that for $k = 1$ is $d_{n,p}$ symmetrical, thus condition (2.5) gives the positive definiteness of all $d_{n,p}$. Now we recall the definitions of generalized zero and disconjugacy.

Definition 2.8. Solution y_n has a generalized zero at n_0 if $y_{n_0} = 0$ or $y_{n_0-1}y_{n_0} < 0$.

Definition 2.9. The given difference equation is disconjugate on an interval I if every nontrivial solution has at most one generalized zero on I.

Lemma 2.10. *Let Eq. (2.1) be disconjugate on interval $[a,b]$ then the boundary value problem*

$$\begin{aligned} &a_{n-k}y_{n-1} + b_n y_n + a_n y_{n+1} = 0, \\ &y_{n_1} = A, \quad y_{n_2} = B, \end{aligned}$$

where $a \leq n_1 < n_2 \leq b$ and $A, B \in \mathbb{R}$, has an unique solution.

Proof. General solution of Eq. (2.1) is

$$y_n = Cz_n^1 + Dz_n^2,$$

for some linearly independent z_n^1 and z_n^2. The boundary conditions result in the system

$$\begin{aligned} Cz_{n_1}^1 + Dz_{n_1}^2 &= A, \\ Cz_{n_2}^1 + Dz_{n_2}^2 &= B. \end{aligned}$$

We see that the boundary value problem has a solution whenever

$$\det \begin{pmatrix} z_{n_1}^1 & z_{n_1}^2 \\ z_{n_2}^1 & z_{n_2}^2 \end{pmatrix} \neq 0.$$

Now assume that this determinant is equal to zero. Then there would exist constants $C, D \in \mathbb{R}$ such that

$$\begin{aligned} Cz^1_{n_1} + Dz^2_{n_1} &= 0, \\ Cz^1_{n_2} + Dz^2_{n_2} &= 0. \end{aligned}$$

Thus, $y_{n_1} = y_{n_2} = 0$. This contradicts that Eq. (2.1) is disconjugate. □

Theorem 2.11. *Let Eq.* (2.1) *be disconjugate on* $\mathbb{Z}$*, then* (2.5) *holds.*

Proof. We will show that $D_{i,i+k-1} > 0$ by induction on $k \in \mathbb{N}$ for arbitrary i. Because $b_i > 0$ we have that $D_{i,i} > 0$.

Let y_n be a solution of

$$\begin{aligned} &a_{n-k}y_{n-1} + b_n y_n + a_n y_{n+1} = 0, \\ &y_{i-1} = 0, \quad y_{i+k+1} = 1, \end{aligned}$$

and assume that $D_{i,i+k-1} > 0$. By Lemma 2.10, we know that such y_n exists and it must satisfy system

$$d_{i,i+k}\mathbf{y} = \mathbf{b},$$

where $\mathbf{y} = (y_i, \dots, y_{i+k})^T$, $\mathbf{b} = (0, \dots, 0, -a_{i+k})$. Now, using Lemma 2.1 we get that

$$y_{i+k}D_{i,i+k} = -a_{i+k}D_{i,i+k-1}.$$

By disconjugacy we know that $y_{i+k} > 0$ and together with the assumption $D_{i,i+k-1} > 0$ we see that $D_{i,i+k} > 0$, as well. □

Corollary 2.12. *Let Eq.* (2.1) *be disconjugate on* $\mathbb{Z}$*, then there exists a positive solution of Eq.* (2.1)*.*

Proof. This is a direct consequence of Theorem 2.7. □

The natural question is whether the converse statement is valid as well. We will solve this problem by formulating an appropriate version of Sturm's comparison theorem. Nevertheless, it can be solved using Theorem 2.7 and Lemma 2.6 together with $u_n^{\pm}$ being minimal solutions as well. Note that we have two separate situations where $u_n^+ = u_n^-$ and $u_n^+ \neq u_n^-$.

Lemma 2.13. *If y_n is a nontrivial solution of Eq.* (2.1) *such that $y_{n_0} = 0$, then $y_{n_0-1}y_{n_0+1} < 0$.*

Proof. If y_n is a nontrivial solution and $y_{n_0} = 0$ for some $n_0 \in \mathbb{Z}$, then $y_{n_0-1} \neq 0 \neq y_{n_0+1}$. The rest follows from y_n being a solution of Eq. (2.1). □

Lemma 2.14. *Assume* (2.5)*. If a nontrivial solution y_n of Eq.* (2.1) *has two generalized zeros at n_1 and n_2, then any other linearly independent solution has a generalized zero in $[n_1, n_2]$.*

Proof. Without loss of generality assume that there are not other generalized zeros of y_n on (n_1, n_2). Now by contradiction, we assume that $y_n > 0$ on (n_1, n_2) and that there is a linearly independent solution z_n such that $z_n > 0$ on $[n_1, n_2]$ and $z_{n_1-1} \geq 0$, i.e. it does not have a generalized zero on $[n_1, n_2]$. We consider some n_0 from (n_1, n_2) and we can find $K \in \mathbb{R}$ such that $Kz_{n_0} = y_{n_0}$. Because $y_{n_2} \leq 0$ and it has to hold that $y_{n_1} = 0$ or $y_{n_1-1} < 0$ we can use Lemma 2.2 to get that $Kz_n > y_n$. Moreover, $u_n = Kz_n - y_n$ is also a solution of Eq. (2.1) and $u_{n_0} = 0$, $u_n > 0$ for $n \neq n_0$. Finally, $u_{n_0-1}u_{n_0+1} > 0$ gives us a contradiction with Lemma 2.13. □

Theorem 2.15. *Eq. (2.1) is disconjugate on $\mathbb{Z}$ if and only if it has a positive solution on $\mathbb{Z}$.*

Proof. We already have the first part from Corollary 2.12. Next, assume that Eq. (2.1) has a positive solution. By Theorem 2.7 we know, that (2.5) holds and so does Lemma 2.14. However, because we have a positive solution, then by Lemma 2.14, we know that there cannot be a solution with more than one generalized zero. □

3 Even order linear coefficient delayed equation

In this section we will focus on the equation

$$a_{n-kH}y_n + b_{n+H}y_{n+H} + a_{n+H}y_{n+2H} = 0, \tag{3.1}$$

for $n \in \mathbb{Z}$, with the parameters $H \in \mathbb{N}$, $k \in \mathbb{Z}$.

Lemma 3.1. *If $a_i < 0$ for every i and there is a subsequence b_{n_l} such that $b_{n_l} \leq 0$ for $n_l \to \infty$ then Eq. (3.1) cannot have an eventually positive solution (i.e. a solution y_n, where $y_n > 0$ for all $n \geq N$, for some $N \in \mathbb{Z}$).*

Proof. Suppose that there exist an eventually positive solution y_n. It implies

$$a_{n_l-k\cdot H}y_{n_l} + b_{n_l+H}y_{n_l+H} + a_{n_l+H}y_{n_l+2H} < 0,$$

for $n_l \to \infty$. This is a contradiction with y_n being a solution of Eq. (3.1). □

Similar statement holds even if $n_l \to -\infty$ and $y_n > 0$ for all $n \leq N$ for some $N \in \mathbb{Z}$. Because of this, we will again assume that $a_j < 0$, $b_j > 0$ for every j.

Theorem 3.2. *The following statements are true.*

1. *Let H be an even number, then Eq. (3.1) has a solution y_n if and only if it has a solution $(-1)^n y_n$.*
2. *Let H be an odd number, then Eq. (3.1) cannot have a solution $(-1)^n p_n$ where $p_n > 0$ for all $|n| \geq N$ and some $N \in \mathbb{N}$.*

Proof. For the first part, it suffices to use $z_n = (-1)^n y_n$ in Eq. (3.1) and the rest follows from H being even.

To prove the second part, we suppose that Eq. (3.1) has a solution $(-1)^n p_n$. Then we have that

$$a_{n-k\cdot H}p_n + b_{n+H}(-1)^H p_{n+H} + a_{n+H}p_{n+2H} = 0.$$

For $|n|$ sufficiently large, the terms are negative, hence the left-hand side cannot be equal zero and such a solution cannot exist. □

Corollary 3.3. *Let H be an even number, then Eq. (3.1) has at least on solution, which is not eventually positive.*

Proof. Assume that all solutions of Eq. (3.1) are eventually positive. Then there is a solution y_n, which is positive for n greater than some N. However, because H is an even number, then $(-1)^n y_n$ is also a solution of Eq. (3.1) and is not eventually positive. Thus we arrive to a contradiction. □

We obtain further generalization if we let p_n^k be real sequences and consider a linear equation

$$\sum_{k=0}^{m} p_n^k y_{n+2k} = 0. \tag{3.2}$$

Then Eq. (3.2) has a solution, which is not eventually positive.

We see that, in some cases, the studied equation cannot have a positive solution. Later we show that there is an equation that has positive and sign-changing solutions at the same time, which is a case that for $k = 0$ cannot occur. For this reason, it is more useful to focus on the situation when the equation has a positive solution. Nevertheless, we start by reminding us of the lemma, which can be found in [21].

Lemma 3.4. *Let us consider the equation*

$$\sum_{k=0}^{m} p_n^k u_{n+k} = 0, \tag{3.3}$$

where p_n^k, $k \in \{0, \ldots, m\}$, are real sequences, for some $m \in \mathbb{N}$. If Eq. (3.3) has a solution u_n, then Eq. (3.3) has another solution in the form $v_n u_n$, where v_n solves the equation

$$\sum_{k=0}^{m-1} \left(\sum_{i=0}^{k} p_n^i u_{n+i} \right) \triangle v_{n+k} = 0. \tag{3.4}$$

Proof. We expand the sum $\sum_{k=0}^{m} p_n^k v_{n+k} u_{n+k}$ by Abel's summation formula and use the fact that u_n is a solution of Eq. (3.3) to obtain Eq. (3.4). □

Assume that we have a solution u_n of Eq. (3.1) and using Lemma 3.4 we obtain other solution as $v_n u_n$, where v_n solves

$$a_{n-k\cdot H} u_n \sum_{j=0}^{H-1} \triangle v_{n+j} + (a_{n-k\cdot H} u_n + b_{n+H} u_{n+H}) \sum_{j=0}^{H-1} \triangle v_{n+H+j} = 0.$$

Using the substitution $z_n = v_{n+H} - v_n$ we get using u_n being a solution of Eq. (3.1) that

$$0 = a_{n-kH} u_n z_n + (a_{n-k\cdot H} u_n + b_{n+H} u_{n+H}) z_{n+H} = a_{n-kH} u_n z_n - a_{n+H} u_{n+2H} z_{n+H}. \tag{3.5}$$

Whenever $u_n \neq 0$ for all n, then the solution of Eq. (3.5) is

$$z_n = \frac{D \prod_{j=1}^{-k-1} a_{n+jH}}{u_n u_{n+H} \prod_{j=-k}^{0} a_{n+jH}},$$

for some $D \in \mathbb{R}$. Finally, we can use the fact that $z_n = v_{n+H} - v_n$. Hence,

$$v_n = -\sum_{g=0}^{\infty} z_{n+gH}, \tag{3.6}$$

$$v_n = \sum_{g=1}^{\infty} z_{n-gH}.$$

Definition 3.5. We say that a solution u_n of Eq. (3.1) is minimal on $[\mu, \infty) \cap \mathbb{Z}$ if any linearly independent solution v_n of Eq. (3.1) with $u_\mu = v_\mu, \ldots, u_{\mu+H-1} = v_{\mu+H-1}$ satisfies $v_n > u_n$, for every $n \geq \mu + H$.

Theorem 3.6. *Let Eq. (3.1) have a positive solution u_n on $\mathbb{Z}$, which is minimal on an interval $[l,\infty)$, where $l \in \mathbb{Z}$. Then for every $\mu \in \mathbb{Z}$ it holds*

$$\sum_{g=0}^{\infty} \frac{\prod_{j=g+1}^{g-k-1}(-a_{\mu+jH})}{u_{\mu+gH}u_{\mu+(g+1)H}\prod_{j=g-k}^{g}(-a_{\mu+jH})} = \infty. \tag{3.7}$$

Proof. Assume that for some $\mu \in \mathbb{Z}$ the sum in (3.7) is finite. Since u_n is a positive solution, by (3.6) we know that also

$$w_n = \begin{cases} u_n \sum_{g=0}^{\infty} \frac{\prod_{j=g+1}^{g-k-1}(-a_{n+jH})}{u_{n+gH}u_{n+(g+1)H}\prod_{j=g-k}^{g}(-a_{n+jH})}, & n \equiv \mu (\text{mod } H), \\ u_n, & n \not\equiv \mu (\text{mod } H), \end{cases}$$

is a positive solution.

Next, we introduce

$$w_n^* = \frac{w_n}{w_\mu} u_\mu, \qquad \text{when} \qquad n \equiv \mu (\text{mod } H).$$

Therefore, w_n^* is also a solution where values of w_n^* and u_n are equal for H consecutive indices around μ. Because the sum in (3.7) is finite, we get

$$\liminf_{n\to\infty} \frac{w_n^*}{u_n} = \frac{u_\mu}{w_\mu} \lim_{n\to\infty} \sum_{g=0}^{\infty} \frac{\prod_{j=g+1}^{g-k-1}(-a_{n+jH})}{u_{n+gH}u_{n+(g+1)H}\prod_{j=g-k}^{g}(-a_{n+jH})} = 0.$$

It means that from some $N > l$ we have $w_N^* < u_N$ which is a contradiction with u_n being a minimal solution on $[l,\infty)$. □

Through similar means as were used in [27], we can deduce the following statements. But first, we have to define a generalization of Casoratian as

$$\omega_{n,\mu} = \det \begin{pmatrix} u_{\mu+nH} & v_{\mu+nH} \\ u_{\mu+(n+1)H} & v_{\mu+(n+1)H} \end{pmatrix}.$$

Lemma 3.7. *Let u_n, v_n be two solutions of Eq. (3.1), then $\omega_{n,\mu}$ satisfies for all $\mu \in \mathbb{Z}$ the equation*

$$\omega_{n+1,\mu} = \frac{-a_{\mu+(n-k)H}}{-a_{\mu+(n+1)H}} \omega_{n,\mu}.$$

Proof. Because u_n, v_n are solutions of (3.1) we have

$$\begin{aligned} \omega_{n,\mu} &= \det \begin{pmatrix} -\frac{a_{\mu+(n+1)H}}{a_{\mu+(n-k)H}} u_{\mu+(n+2)H} & -\frac{a_{\mu+(n+1)H}}{a_{\mu+(n-k)H}} v_{\mu+(n+2)H} \\ u_{\mu+(n+1)H} & v_{\mu+(n+1)H} \end{pmatrix} \\ &= (-1)\left(-\frac{a_{\mu+(n+1)H}}{a_{\mu+(n-k)H}}\right)\omega_{n+1,\mu} = \frac{-a_{\mu+(n+1)H}}{-a_{\mu+(n-k)H}}\omega_{n+1,\mu}. \end{aligned}$$

□

Hence, we can compute for some $D \in \mathbb{R}$ that

$$\omega_{n,\mu} = \frac{D}{\prod_{j=n-k}^{n}(-a_{\mu+jH})} \prod_{j=n+2}^{n-k}\left(-a_{\mu+(j-1)H}\right).$$

Note that if for some $\omega_{n,\mu}$ is $D < 0$, we get by swapping values of u_n and v_n on the set $\{\mu + jH | j \in \mathbb{Z}\}$ that u_n and v_n are still solutions of Eq. (3.1) and $D > 0$.

Theorem 3.8. *If Eq.* (3.1) *has two independent eventually positive solutions, then there are two independent eventually positive solutions u_n, v_n for which $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$. Moreover, for arbitrary $\mu \in \mathbb{Z}$ sufficiently large*

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{u_{\mu+nH}u_{\mu+(n+1)H}\prod_{j=n-k}^{n}(-a_{\mu+jH})} = \infty, \tag{3.8}$$

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{v_{\mu+nH}v_{\mu+(n+1)H}\prod_{j=n-k}^{n}(-a_{\mu+jH})} < \infty. \tag{3.9}$$

Proof. We can expect that u_n, v_n are linearly independent, eventually positive and also that in $\omega_{n,\mu}$ is $D < 0$, for all μ. Considering μ sufficiently large we have

$$\begin{aligned}\triangle\left(\frac{u_{\mu+nH}}{v_{\mu+nH}}\right) &= \frac{u_{\mu+nH}v_{\mu+(n+1)H} - u_{\mu+(n+1)H}v_{\mu+nH}}{v_{\mu+nH}v_{\mu+(n+1)H}} \\ &= \frac{D}{v_{\mu+nH}v_{\mu+(n+1)H}\prod_{j=n-k}^{n}\left(-a_{\mu+jH}\right)} \prod_{j=n+2}^{n-k}\left(-a_{\mu+(j-1)H}\right).\end{aligned} \tag{3.10}$$

Hence, (3.10) is negative, therefore $\frac{u_{\mu+nH}}{v_{\mu+nH}}$ is strictly decreasing in n, but $\frac{u_{\mu+nH}}{v_{\mu+nH}}$ is also positive and thus bounded from below. We have that $\lim_{n\to\infty} \frac{u_{\mu+nH}}{v_{\mu+nH}} = L_\mu \geq 0$. In case that for some μ is $L_\mu > 0$, we replace u_n by $u_n - L_\mu v_n$, for $n \in \{\mu + jH | j \in \mathbb{Z}\}$. Hence, u_n will still be a solution and we get that $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$.

Moreover, by summing equality (3.10) we obtain

$$\begin{aligned} D\sum_{g=k}^{n-1} \frac{1}{v_{\mu+gH}v_{\mu+(g+1)H}\prod_{j=g-k}^{g}\left(-a_{\mu+jH}\right)} \prod_{j=g+2}^{g-k}\left(-a_{\mu+(j-1)H}\right) &= \frac{u_{\mu+nH}}{v_{\mu+nH}} - \frac{u_{\mu+kH}}{v_{\mu+kH}}, \\ \xrightarrow{n\to\infty} D\sum_{g=k}^{\infty} \frac{1}{v_{\mu+gH}v_{\mu+(g+1)H}\prod_{j=g-k}^{g}\left(-a_{\mu+jH}\right)} \prod_{j=g+2}^{g-k}\left(-a_{\mu+(j-1)H}\right) &= -\frac{u_{\mu+kH}}{v_{\mu+kH}}, \end{aligned}$$

which confirms the validity of (3.9). Using the unboundedness of $\frac{v_{\mu+nH}}{u_{\mu+nH}}$, we get (3.8). □

Corollary 3.9. *Let for some μ be*

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{\prod_{j=n-k}^{n}(-a_{\mu+jH})} = \infty,$$

and every solution of Eq. (3.1) *be eventually bounded, then Eq.* (3.1) *has at most one linearly independent eventually positive solution.*

Proof. Suppose that Eq. (3.1) has two such solutions. Then from Theorem 3.8 there has to be a solution v_n such that $0 < v_n < M$ for n sufficiently large and some M. Moreover, for ν sufficiently large and satisfying $\nu \equiv \mu \pmod{H}$ we get from (3.9) that

$$\infty > \sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\nu+(j-1)H})}{v_{\nu+nH}v_{\nu+(n+1)H}\prod_{j=n-k}^{n}(-a_{\nu+jH})} > \frac{1}{M^2}\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\nu+(j-1)H})}{\prod_{j=n-k}^{n}(-a_{\nu+jH})}.$$

Which is a contradiction. □

As an example we consider the equation

$$-\frac{1}{2}y_n + y_{n+2} - \frac{1}{2}y_{n+4} = 0. \tag{3.11}$$

It has two solutions $u_n = K$, $v_n = Kn$ of eventually one sigh as well as two sign changing ones $(-1)^n u_n$, $(-1)^n v_n$. Moreover, it holds that

$$\sum^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+2(j-1)})}{u_{\mu+2n}u_{\mu+2(n+1)}\prod_{j=n-k}^{n}(-a_{\mu+2j})} = \infty,$$
$$\sum^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+2(j-1)})}{v_{\mu+2n}v_{\mu+2(n+1)}\prod_{j=n-k}^{n}(-a_{\mu+2j})} < \infty,$$

where $a_i \equiv -1/2$ and we can choose k arbitrarily. According to [16] Eq. (3.11) has a minimal solution on intervals $[2,\infty)$ and $(-\infty,-2]$.

We define the Riccati transformation through the substitution

$$s_n = \frac{b_{n+H}y_{n+H}}{(-a_{n-kH})y_n}, \quad \text{and} \quad q_n = \frac{a_n a_{n-kH}}{b_n b_{n+H}}, \tag{3.12}$$

to obtain

$$\begin{aligned} a_{n-k\cdot H}y_n + b_{n+H}y_{n+H} + a_{n+H}y_{n+2H} &= 0, \\ \frac{a_{n-k\cdot H}y_n}{b_{n+H}y_{n+H}} + 1 + \frac{a_{n+H}y_{n+2H}}{b_{n+H}y_{n+H}} &= 0, \\ -\frac{1}{s_n} + 1 - \frac{a_{n+H}a_{n-(k-1)H}}{b_{n+H}b_{n+2H}}s_{n+H} &= 0, \\ q_{n+H}s_{n+H} + \frac{1}{s_n} &= 1. \end{aligned} \tag{3.13}$$

We emphasize that $q_n > 0$ for all n.

Theorem 3.10. *Eq. (3.13) has a positive solution if and only if Eq. (3.1) has also a positive solution.*

Proof. First, if Eq. (3.1) has a positive solution y_n then via the transformation $s_n = \frac{b_{n+H}y_{n+H}}{(-a_{n-kH})y_n}$ we can see that s_n is also a positive solution of Eq. (3.13).

Second, if s_n is a positive solution of Eq. (3.13) then we can consider the initial conditions $y_N = 1, \ldots, y_{N+H-1} = 1$ for some $N \in \mathbb{Z}$ and the recurrence relation

$$y_{n+H} = \frac{(-a_{n-kH})s_n}{b_{n+H}}y_n.$$

Then, for $n \geq N$, y_n is a positive solution of Eq. (3.1). The rest of y_n is computed through the relation

$$y_n = \frac{b_{n+H}y_{n+H}}{(-a_{n-kH})s_n}. \qquad \square$$

Note that the Theorem 3.10 holds even if we consider eventually positive solutions instead of positive ones. Moreover, at this place, we can see a connection to Theorem 3.2. If H is an even number, then solutions y_n and $(-1)^n y_n$ give the same positive solution s_n of Eq. (3.13). For H being an odd number, the existence of a solution $(-1)^n y_n$ would give a solution s_n of Eq. (3.13) that is eventually negative. Nevertheless, such s_n cannot exist.

Lemma 3.11. *Let $q_n \geq p_n > 0$ and let s_n be a positive solution of*

$$q_{n+H}s_{n+H} + \frac{1}{s_n} = 1$$

on $[N, \infty)$, where $N \in \mathbb{Z}$. Then the equation

$$p_{n+H}u_{n+H} + \frac{1}{u_n} = 1,$$

has a solution u_n such that $u_n \geq s_n > 1$ on $[N, \infty)$.

Proof. If s_n is a positive solution, then also $q_{n+H}s_{n+H} > 0$, and so $\frac{1}{s_n} = 1 - q_{n+H}s_{n+H} < 1$ implies that $s_n > 1$ on $[N, \infty)$.

Now we consider initial conditions such that $u_N \geq s_N, \ldots, u_{N+H-1} \geq s_{N+H-1}$ and we get that if $u_n \geq s_n$ then

$$p_{n+H}u_{n+H} = 1 - \frac{1}{u_n} = q_{n+H}s_{n+H} + \frac{1}{s_n} - \frac{1}{u_n} \geq q_{n+H}s_{n+H}.$$

Therefore, $u_{n+H} \geq \frac{q_{n+H}s_{n+H}}{p_{n+H}} \geq s_{n+H}$ and the statement of the lemma holds by induction. □

Theorem 3.12. *If q_n of (3.12) satisfy $1/(4-\varepsilon) \leq q_n$ for some $\varepsilon > 0$ and for all n sufficiently large, then Eq. (3.1) cannot have an eventually positive solution.*

Proof. If $\varepsilon \geq 4$ it would mean that $\frac{b_n b_{n+H}}{a_n a_{n-kH}} \leq (4-\varepsilon) \leq 0$, however because $a_i < 0$, $b_i > 0$ this cannot be true. Here the statement shadows Lemma 3.1.

Now we know that $\varepsilon < 4$ and assume that (3.1) has an eventually positive solution. Then there is an eventually positive solution s_n of Eq. (3.13). By Lemma 3.11 we have that the equation

$$\frac{u_{n+H}}{4-\varepsilon} + \frac{1}{u_n} = 1, \tag{3.14}$$

has a solution $u_n \geq s_n > 1$ on some $[N, \infty)$, for a sufficiently large N. If we take a positive sequence given by $x_N = 1, \ldots, x_{N+H-1} = 1$, and $x_{n+H} = \frac{u_n x_n}{\sqrt{4-\varepsilon}}$, then also $u_n = \sqrt{4-\varepsilon}\frac{x_{n+H}}{x_n}$ and by substituting into (3.14) we get that x_n is a positive solution of

$$x_{n+2H} - \sqrt{4-\varepsilon}x_{n+H} + x_n = 0, \tag{3.15}$$

for $n \geq N$. This is a contradiction because Eq. (3.15) does not have an eventually positive solution. In fact Eq. (3.15) has constant coeficients and we can find all its solutions through the characteristic polynomial and de Moivre's formula. They are $\cos n\theta_k$ and $\sin n\theta_k$ where $\theta_k = \left(\arctan \frac{\varepsilon}{4-\varepsilon} + 2k\pi\right)/H$, for $k = 0, \ldots, H-1$. □

Remark 3.13. We discussed eventually positive solutions, which are positive as $n \to \infty$. We can discuss the same situation if $n \to -\infty$ by taking these results and rewriting Eq. (3.1) appropriately. We emphasize that if an equation does not have and eventually positive solution, hence it even does not have a positive solution. If an equation has a positive solution, it is also an eventually positive solution.

Theorem 3.14. *If q_n of (3.12) satisfy $q_n \leq 1/4$, for all n, then Eq. (3.1) has a positive solution.*

Proof. First, let s_n be a solution of Eq. (3.13). If $s_N \geq 2$ for some N then $q_{N+H}s_{N+H} = 1 - \frac{1}{s_N} \geq 1/2$. Therefore, because $1/q_n \geq 4$, we have $s_{N+H} \geq \frac{1}{2q_{N+H}} \geq 1/2 \cdot 4 = 2$. By induction, we know that $s_n \geq 2$, for all $n \in \{N + lH | l \in [0, \infty) \cap \mathbb{Z}\}$.

Second, let again s_n be a solution of Eq. (3.13). If $0 < s_{N+H} \leq 2$ for some N then $\frac{1}{s_N} = 1 - q_{N+H}s_{N+H} \geq 1 - 1/4 \cdot 2 = 1/2$ and therefore $s_N \leq 2$. But also $1/s_N > 0$ implies that $s_N > 0$. By induction, we know that $0 < s_n \leq 2$ for all $n \in \{N + lH | l \in (-\infty, 1] \cap \mathbb{Z}\}$.

Finally, let s_n be a solution of Eq. (3.13) together with initial conditions $s_N = 2, \dots, s_{N+H-1} = 2$, for some $N \in \mathbb{Z}$. From previous two parts we have, that s_n is a positive solution of Eq. (3.13) on $\mathbb{Z}$ and by Theorem 3.10 we know that Eq. (3.1) has also a positive solution. □

Corollary 3.15. *If $b_n \geq \max\{-a_{n-H}\lambda, -4a_{n-kH}/\lambda\}$ for some $\lambda > 0$ then Eq. (3.1) has a positive solution.*

Proof. The assumption of the corollary implies that $b_n \geq -4a_{n-kH}/\lambda$ and $b_{n+H} \geq -a_n\lambda$. It follows that $b_n b_{n+H} \geq 4a_n a_{n-kH}$ and the rest is due to Theorem 3.14. □

We can connect Eq. (2.1) with Eq. (3.1) for $H = 1$ by shifting it. In the first part, the equivalence condition for Eq. (2.1) to have a positive solution was formulated. One could probably obtain similar relation by extension of the results of [16] for Eq. (3.1).

Moreover, it remains a question how this connects to q_n. By Theorem 3.12 we know that if Eq. (3.1) has a positive solution, then surely $q_n \leq 1/4$ for n sufficiently large. But we can ask whether Eq. (3.1) can have a positive solution even if $q_n > 1/4$ for some n and how Condition (2.5) connects to it.

Using again Eq. (3.11), we see that $q_n = 1/4$ and so by Theorem 3.14, we know that this equation has a positive solution.

Theorem 3.16. *If Eq. (3.1) has a solution y_n such that $y_{\mu+nH}$ is a positive sequence for some $\mu \in \mathbb{Z}$, then for every other solution $\bar{y}_n$ of Eq. (3.1), the sequence $\bar{y}_{\mu+nH}$ must have at most one generalized zero (from Definition 2.8) on $\mathbb{Z}$.*

Proof. Consider the substitution $x_p = y_{\mu+(p+1)H}$ in Eq. (3.1) and by taking $n = \mu + pH$, Eq. (3.1) changes into

$$a_{\mu+(p-k)H}y_{\mu+pH} + b_{\mu+(p+1)H}y_{\mu+(p+1)H} + a_{\mu+(p+1)H}y_{\mu+(p+2)H} = 0.$$

Now if we take $\tilde{a}_p = a_{\mu+(p+1)H}$, $\tilde{b}_p = b_{\mu+(p+1)H}$, it transforms into

$$\tilde{a}_{p-k-1}x_{p-1} + \tilde{b}_p x_p + \tilde{a}_p x_{p+1} = 0,$$

which corresponds to Eq. (2.1) and so by Theorem 2.15 we know that this equation is disconjugate. □

To further refine results obtained in Theorem 3.16, we formulate the definition of the separately nonoscillatory solution. However, let us first recall the following definition, which can be found, for example, in [4].

Definition 3.17. A nontrivial solution y_n of self-adjoint difference equation of order $2m$ has a generalized zero of order m at $n_0 + 1$ if $y_{n_0} \neq 0$, $y_{n_0+1} = \cdots = y_{n_0+m-1} = 0$, and $(-1)^m y_{n_0} y_{n_0+m} \geq 0$.

This definition corresponds to Definition 2.8 if $m = 1$. Nevertheless, for our purposes we need a combination of Definitions 2.8 and 3.17. We start by defining for some $p \in \mathbb{N}$ equivalence relation $x \sim y$ on $\mathbb{Z}$ such that $x \sim y$ if and only if $x = y + jp$ for some $j \in \mathbb{Z}$. From this equivalence we obtain equivalence classes $A_1(p), \dots, A_p(p) \subseteq \mathbb{Z}$ such that $i \in A_i(p)$. Of course $A_1(1) = \mathbb{Z}$.

Next, we define on a linearly ordered set S for $x \in S$ function

$$\rho(x) = \max\{y \in S | y < x\}.$$

Definition 3.18. Solution y_n of a given difference equation has n_0 a generalized zero on a linearly ordered set S if y_n is nontrivial on S and for $n_0 \in S$ is $y_{n_0} = 0$ or $y_{\rho(n_0)} y_{n_0} < 0$ provided that $\rho(n_0)$ exists. Solution y_n is nonoscillatory on $A_i(p) \cap I$ provided that y_n has on $A_i(p) \cap I$ only finitely many generalized zeros.

For example, recall again Eq. (3.11), which is of the fourth-order and has a solution

$$y_n = \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Such solution has infinitely many generalized zeros with respect to both Definition 2.8 and 3.17. On the other hand, such solution does not have a generalized zero on $A_i(2)$ for both $i = 1$ (here y_n is positive) and $i = 2$ (here y_n is trivial). Another solution of Eq. (3.11) is $z_n = 1$ which does not have a generalized zero under any Definition of 2.8, 3.17 and 3.18.

Definition 3.19. Solution y_n of a given difference equation is separately i-nonoscillatory on $I(p)$ if there is a set $J \subseteq \{1, \dots, p\}$, $|J| = i$, such that y_n is nonoscillatory on $A_j(p) \cap I$ for all $j \in J$. If all solutions of the equation are separately i-nonoscillatory on $I(p)$, then this equation is called separately i-nonoscillatory on $I(p)$.

In this paper, we consider for I only $\mathbb{Z}$ or $[N, \infty)$ as well as $p = H$, because they make the most sense to us. We assume that we could get some interesting or strange results for a different choice of I and p. Moreover, with the choice of $p = 1$ and $I = [N, \infty)$, we get the usual definition of nonoscillatory solutions used for second-order linear equations through generalized zeros of Definition 2.8. Such solutions are eventually positive or negative. Hence, if a solution is separately nonoscillatory on $I(1)$, then it is also separately nonoscillatory on $I(p)$.

Corollary 3.20. *Assume there is a set $J \subseteq \{1, \dots, H\}$, $|J| = i$ such that q_n of (3.12) satisfies $q_n \leq 1/4$, for all $n \in A_j(H)$ and $j \in J$, then Eq. (3.1) is separately i-nonoscillatory on $\mathbb{Z}(H)$.*

Proof. By the proof of Theorem 3.14 we know that Eq. (3.1) has a solution which is positive on $A_j(H)$, $j \in J$. Hence, by Theorem 3.16 we know that every solution is nonoscillatory on A_j, where $j \in J$. □

Theorem 3.21. *If there is a subsequence q_{n_l} of q_n such that $q_{n_l} \geq 1$ for $n_l \to \infty$, $n_l \in A_i(H)$ and some $i \in \{1, \dots, H\}$, then Eq. (3.1) cannot have y_n a nonoscillatory solution on $A_i(h) \cap [N, \infty)$, for some $N \in \mathbb{N}$.*

Proof. Suppose that there is such a solution, then we can assume that it is positive on $I = A_i(H) \cap [N, \infty)$ for N sufficiently large. Therefore, Eq. (3.13) has a solution s_n such that $s_n > 0$

on I. Moreover, by definition $q_n > 0$ for all n and if $n \in A_i(H)$, then also $n + H \in A_i(H)$. Hence,

$$q_{n+H}s_{n+H} + \frac{1}{s_n} = 1,$$

and we have that $1/s_n < 1$ on I, thus $s_n > 1$ on I. Nevertheless, for the same reason $q_{n+H}s_{n+H} < 1$ on I and so $q_n < 1$, for all $n \geq N + H$, $n \in I$. That is a contradiction with our assumption. □

In such a case, equation cannot be separately H-nonoscillatory on $I(H)$ where $I = [N, \infty)$.

Corollary 3.22. *If*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} q_{i+jH} > 1,$$

then Eq. (3.1) cannot have y_n a nonoscillatory solution on $A_i(H) \cap [N, \infty)$ for some $N \in \mathbb{N}$.

Proof. Suppose there is such a solution. Then by Theorem 3.21, $q_n < 1$ on $A_i(H) \cap [N, \infty)$, for N sufficiently large and let $m \in A_i(H) \cap [N, \infty)$ be arbitrary. Then it holds $\sum_{j=1}^{n} q_{m+jH} < n$ and also $\frac{1}{n}\sum_{j=1}^{n} q_{i+jH} < 1 + \frac{C}{n}$, for some $C \in \mathbb{R}$. Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} q_{i+jH} \leq 1,$$

which is a contradiction. □

Theorem 3.23. *If Eq. (3.1) has a solution $y_n > 0$ on $A_i(H) \cap [N, \infty)$ and $\prod_{j=1}^{n} \frac{b_{i+jH}}{(-a_{i+jH})}$ is a bounded sequence, then y_n is bounded on $A_i(H) \cap [N, \infty)$.*

Proof. Taking $z_n = \frac{y_{n+H}}{y_n}$ on $I = A_i(H) \cap [N, \infty)$, for N sufficiently large, we can see that $z_n > 0$ is a solution of the equation

$$(-a_{n-kH})\frac{1}{z_n} + (-a_{n+H})z_{n+H} = b_{n+H},$$

on I. Because all the terms are positive, it holds that $(-a_{n+H})z_{n+H} < b_{n+H}$ on I. Let $M \in I$ be arbitrary and we have

$$\frac{y_{M+nH}}{y_{M+H}} = \prod_{j=1}^{n-1} \frac{y_{M+(j+1)H}}{y_{M+jH}} = \prod_{j=1}^{n-1} z_{M+jH} < \prod_{j=1}^{n-1} \frac{b_{M+jH}}{(-a_{M+jH})}.$$

Hence, $y_{M+nH} < y_{M+H} \prod_{j=1}^{n-1} \frac{b_{M+jH}}{(-a_{M+jH})}$ for all $n \in \mathbb{N}$ is giving us the result. □

Corollary 3.24. *If Eq. (3.1) has a positive solution y_n and $\prod_{j=1}^{n} \frac{b_{i+jH}}{(-a_{i+jH})}$, $\prod_{j=-n}^{1} \frac{b_{i+jH}}{(-a_{i+(j-k)H})}$ are bounded sequences for every $i \in \{1, \dots, H\}$, then y_n is bounded on $\mathbb{Z}$.*

Proof. By Theorem 3.23 we see that y_n is bounded on all $A_i(H) \cap [N, \infty)$ for $n \to \infty$. Via the same way, we can see that

$$b_{n+H} > (-a_{n-kH})\frac{1}{z_n} = (-a_{n-kH})\frac{y_n}{y_{n+H}},$$

for every n and similarly we see that y_n is bounded even for $n \to -\infty$. □

Corollary 3.25. *If* $\prod_{j=1}^{n} \frac{b_{i+jH}}{(-a_{i+jH})}$ *is a bounded sequence for every* $i \in \{1, \dots, H\}$ *and for some* $\mu \in \mathbb{Z}$ *is*

$$\sum_{n}^{\infty} \frac{\prod_{j=n+2}^{n-k}(-a_{\mu+(j-1)H})}{\prod_{j=n-k}^{n}(-a_{\mu+jH})} = \infty,$$

then Eq. (3.1) has at most one linearly independent eventually positive solution.

Proof. Suppose that there are two such solutions. Then by Theorem 3.23, they are bounded as $n \to \infty$. Using the proof of Corollary 3.9, we get a contradiction. □

It is posible to extend previous ideas to other equations. As an example, we consider the equation

$$c_{n-1}a_n y_n + b_{n+1}y_{n+1} + c_{n+1}a_{n+1}y_{n+2} = 0.$$

It would result in similar but more complicated statements. However, our results can be extended even more in a similar fashion, how [23] extends the results of [27]. It should also be possible to find other criteria of separate oscillation shadowing the approach used for the case of $H = 1$, $k = 0$.

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