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Traveling front of polyhedral shape for a nonlocal delayed diffusion equation

Jia Liu ✉

School of Science, Chang'an University, Xi'an, Shaanxi 710064, People's Republic of China

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Abstract. This paper is concerned with the existence and stability of traveling fronts with convex polyhedral shape for nonlocal delay diffusion equations. By using the existence and stability results of V-form fronts and pyramidal traveling fronts, we first show that there exists a traveling front $V(x,y,z)$ with polyhedral shape of nonlocal delay diffusion equation associated with $z = h(x,y)$. Moreover, the asymptotic stability and other qualitative properties of such traveling front $V(x,y,z)$ are also established.

Keywords: traveling front, polyhedral shape, reaction-diffusion equation, nonlocal delayed.

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1 Introduction

Recently, the study on the nonplanar traveling fronts of reaction-diffusion equations/systems has attracted an increasing attention and many types of nonplanar traveling fronts have been observed. See [5, 6, 9, 10, 14, 15, 28, 31] for V-shaped traveling fronts, see [9, 10, 23, 32] for cylindrically symmetric traveling fronts; see [4, 11, 16, 21, 22, 34] for pyramidal shaped traveling fronts and see [17–19, 23–27, 33] for other related works on multidimensional traveling fronts. It is well known that time delay and nonlocality play very important roles in the study of the population dynamics in biological and epidemiological models. Traveling fronts of reaction-diffusion equations with time delay in one or multidimensional spaces have been extensively studied, see [7, 8, 12, 20, 29, 30, 35]. Nevertheless, a very little attention has been paid to the study of nonplanar traveling fronts for reaction-diffusion equation with delay. As far as we know, Bao and Huang [1] proved that there exists two-dimensional V-shaped traveling fronts of bistable reaction-diffusion equation with delay, also see [3] for the existence of pyramidal traveling fronts. In [2], the author and Bao have established the existence of N-dimensional pyramidal traveling fronts of nonlocal delayed diffusion equation for $N \geq 3$ and see [13] for asymptotic stability of such pyramidal traveling fronts in the three-dimensional whole space.

✉ Email: liujia@chd.edu.cn

Motivated by [19, 23], in the current paper, we consider the existence, uniqueness and stability of three-dimensional traveling fronts with convex polyhedral shape for the following nonlocal delayed diffusion equation

$$\frac{\partial u}{\partial t}(x,y,z,t) = D\Delta u(x,y,z,t) - du(x,y,z,t) + \int_{\mathbb{R}} b\left(u(x,y,z_1,t-\tau)\right) f(z-z_1)dz_1, \tag{1.1}$$

where $D > 0$ and $d > 0$ denote the diffusion rate and death rate of the adult population, respectively, $\tau \geq 0$ is the maturation time for the species, $b(\cdot)$ is related to the birth function. The convolution in space term represents the nonlocal interaction in one direction and the kernel function $f(\cdot) \in C^\infty(\mathbb{R},\mathbb{R})$ satisfies

$$f(x) \geq 0, \int_{\mathbb{R}} f(y)dy = 1 \quad \text{and} \quad \int_{\mathbb{R}} e^{\lambda y} f(y)dy < +\infty \quad \text{for some } \lambda \geq 0. \tag{1.2}$$

Assume that

(A1) $b(\cdot) \in C^1(\mathbb{R},\mathbb{R})$ and there exists a constant $K > 0$ such that $b(0) = dK - b(K) = 0$;

(A2) $b'(u) \geq 0$ for $u \in [0,K]$ and $d > C\max\{b'(0), b'(K)\}$ for some constant $C > 1$;

(A3) there exits $u^* \in (0,K)$ such that $du^* - b(u^*) = 0$, $b'(u^*) > d$ and $du - b(u) \neq 0$ for $u \in (0,u^*) \cup (u^*,K)$.

By assumption (A1), (1.1) has at least two spatially homogeneous equilibria 0 and K and (1.1) is of nonlocal bistable structure if $b(u)$ satisfies (A1)–(A3). It is known from [12] that, under the assumption (A1)–(A3), there exists a unique solution pair (c,U) of (1.1) satisfying

$$DU''(\xi) - dU(\xi) - cU'(\xi) + \int_{\mathbb{R}} b(U(\xi - c\tau - y))f(y)dy = 0$$

and

$$U(-\infty) = 0, \quad U(+\infty) = K,$$

where $U(\cdot)$ is the monotone increasing wave profile and $c \in \mathbb{R}$ is the speed. Moreover, following from Wang et al. [30], if (A1)–(A3) hold, there exist positive constants β_1 and C_1 such that

$$\max\left\{U(-\xi), |U(\xi)-1|, |U'(\pm\xi)|, |U''(\pm\xi)|\right\} \leq C_1 e^{-\beta_1\xi}, \quad \forall\, \xi \geq 0.$$

Define

$$[0,K]_C := \left\{\phi \in C(\mathbb{R}^3 \times [-\tau,0],\mathbb{R}) : 0 \leq \phi(x,y,z,r) \leq K, r \in [-\tau,0]\right\}.$$

Due to the effect of nonlocality in (1.1), the solution travel towards z direction. Set $z_1 = z + st$ and $u(x,y,z,t) = w(x,y,z_1,t)$. For simplicity, we still denote $w(x,y,z_1,t)$ by $w(x,y,z,t)$. Substituting w into (1.1), we have

$$\begin{cases} \frac{\partial w}{\partial t} = D\Delta w - s\frac{\partial w}{\partial z} - dw + \int_{\mathbb{R}} b(w(x,y,z-s\tau-z_1,t-\tau))f(z_1)dz_1, \\ w(x,y,z,r) = \phi(x,y,z,r), \quad (x,y,z) \in \mathbb{R}^3,\ r \in [-\tau,0]. \end{cases} \tag{1.3}$$

Let $w(x,y,z,t;\phi)$ be the solution of (1.3) with $w(x,y,z,r) = \phi(x,y,z,r) \in [0,K]_C$. Hereafter, we always assume $s > c > 0$. The objective of this paper is to seek for the solution $V(x,y,z) \in [0,K]_C$ of

$$\mathcal{L}[V] := -D\Delta V + s\frac{\partial V}{\partial z} + dV - \int_{\mathbb{R}} b\left(V(\mathbf{x}', z - s\tau - z_1)\right) f(z_1)dz_1 = 0 \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Let

$$m_* = \frac{\sqrt{s^2 - c^2}}{c}.$$

Given $n \geq 3$, assume that $\{\theta_j\}_{1\leq j\leq n}$ satisfies

$$0 < \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi \quad \text{and} \quad \max_{1\leq j\leq n} (\theta_{j+1} - \theta_j) < \pi,$$

where $\theta_{n+1} = \theta_1 + 2\pi$. Given s_j with

$$\min_{1\leq j\leq n} s_j \geq 0 \quad \text{for } 1 \leq j \leq n.$$

Then

$$\mu_j := \frac{1}{\sqrt{1+m_*^2}} \begin{pmatrix} m_* \cos\theta_j \\ m_* \sin\theta_j \\ -1 \end{pmatrix}$$

is the unit normal vector of a surface $\{z = m_*(x\cos\theta_j + y\sin\theta_j)\}$. Putting

$$\begin{aligned} h_j(x,y) &:= m_*(x\cos\theta_j + y\sin\theta_j - s_j), \\ h(x,y) &:= \max_{1\leq j\leq n} h_j(x,y) = m_* \max_{1\leq j\leq n} (x\cos\theta_j + y\sin\theta_j - s_j). \end{aligned} \tag{1.5}$$

Then $\{(x,y,z) \in \mathbb{R}^3 | -z \geq h(x,y)\}$ is a convex polyhedron. If $(s_1, ..., s_n) = (0, 0, ..., 0)$, the polyhedron becomes a pyramid in $\mathbb{R}^3$.

Define

$$\Theta := \max_{2\leq j\leq n-1} \frac{s_j \sin(\theta_{j+1} - \theta_{j-1}) - s_{j-1}\sin(\theta_{j+1} - \theta_j) - s_{j+1}\sin(\theta_j - \theta_{j-1})}{\sin(\theta_{j+1} - \theta_j) + \sin(\theta_j - \theta_{j-1}) - \sin(\theta_{j+1} - \theta_{j-1})}. \tag{1.6}$$

For $j = 1, 2, \ldots, n$, define

$$\Omega_j := \{(x,y) \in \mathbb{R}^2 \mid h(x,y) = h_j(x,y),\, h(x,y) \geq m_*\Theta\}.$$

We note that $\Omega_j \neq \emptyset$ for all $1 \leq j \leq n$. Here $\Omega_1, \ldots, \Omega_n$ are located counterclockwise.

Set

$$S_j = \{(x,y,z) \in \mathbb{R}^3 \mid -z = h_j(x,y),\, (x,y) \in \Omega_j\}, \quad j = 1, \ldots, n.$$

Let

$$\Gamma_j = \{(x,y,z) \in \mathbb{R}^3 \mid -z = h_j(x,y) = h_{j+1}(x,y) \geq m_*\Theta\}, \quad j = 1, \ldots, n$$

be a part of an edge of a polyhedron $\{(x,y,z) \in \mathbb{R}^3 \mid -z \geq h(x,y)\}$. If $(s_1, \ldots, s_n) = (0, \ldots, 0)$ and $\Theta = 0$, Γ_j and $\cup_{j=1}^n \Gamma_j$ stand for an edge and the set of all edges of a pyramid, respectively.

For each $\gamma > 0$, we define

$$D(\gamma) := \{(x,y,z) \in \mathbb{R}^3 \mid \mathrm{dist}((x,y,z), \cup_{j=1}^n \Gamma_j) > \gamma\}.$$

Define

$$v^-(x,y,z) = U\left(\frac{c}{s}(z + h(x,y))\right) = \max_{1\leqslant j\leqslant n} U\left(\frac{c}{s}(z + h_j(x,y))\right).$$

Theorem 1.1. *Let $s > c > 0$ and $h(x,y)$ be given by (1.5). Under the assumption (A1)–(A3), there exists a solution $V(x,y,z)$ of (1.4) such that*

$$\lim_{\gamma\to\infty}\sup_{(x,y,z)\in D(\gamma)}\left|V(x,y,z)-U\left(\frac{c}{s}(z+h(x,y))\right)\right|=0, \tag{1.7}$$

$$0<U\left(\frac{c}{s}(z+h(x,y))\right)<V(x,y,z)<K \quad \text{for all } (x,y,z)\in\mathbb{R}^3,$$

$$\lim_{R\to\infty}\sup_{|z+h(x,y)|>R}|V_z(x,y,z)|=0,$$

$$\inf_{\delta\le V(x,y,z)\le K-\delta}V_z(x,y,z)>0 \quad \text{for } \delta>0 \text{ small}$$

and

$$\lim_{R\to\infty}\sup_{|\mathbf{x}|>R}\left|V(x,y,z)-\max_{1\le j\le n}E_j(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho)\right|=0,$$

where E_j is the two-dimensional V-shaped traveling front defined by (2.4) in Section 2 and $\rho\in(\Theta,\infty)$.

Theorem 1.2. *Let $V(x,y,z)$ be given by Theorem 1.1, $\widehat{s}=\max_{1\le j\le n}s_j>0$, $\widetilde{V}$ is the pyramidal traveling front given in Theorem 2.3, $X_j(-\widehat{s})$, $Y_j(-\widehat{s})$ and $X_j(\rho),Y_j(\rho)$ satisfy $h(X_j(-\widehat{s}),Y_j(-\widehat{s}))=-m_*\widehat{s}$ and $h(X_j(\rho),Y_j(\rho))=m_*\rho$ for $\rho\in(\Theta,\infty)$, respectively. If the initial value $\phi\in C(\mathbb{R}^3\times[-\tau,0],\mathbb{R})$ satisfies $\phi(x,y,z,r)\ge v^-(x,y,z)$ and*

$$\begin{aligned}&\max_{1\le j\le n}\widetilde{V}(x-X_j(-\widehat{s}),y-Y_j(-\widehat{s}),z-m_*\widehat{s})\\ \le&\phi(x,y,z,r)\le\min_{1\le j\le n}\widetilde{V}(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho),\end{aligned}$$

then the solution $w(x,y,z,t;\phi)$ of (1.3) satisfies

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}^3}|w(x,y,z,t;\phi)-V(x,y,z)|=0.$$

Note that the set $\{(x,y,z)\in\mathbb{R}^3|-z\ge h(x,y)\}$ is a convex polyhedron for given $h(x,y)$ in (1.5). Then $V(x,y,z)$ given in Theorem 1.1 is called traveling front with convex polyhedral shape associated with $z=h(x,y)$. Since the polyhedron becomes a pyramid in $\mathbb{R}^3$ if $(s_1,\ldots,s_n)=(0,0,\ldots,0)$, then traveling front with convex polyhedral shape $V(x,y,z)$ becomes the pyramidal shape traveling front when $s_j=0\,(j=1,2,\ldots,n)$. Theorem 1.2 implies that such traveling front $V(x,y,z)$ is also asymptotically stable and uniquely determined by (1.4) and (1.7).

The rest of this paper is organized as follows. In Section 2, we state some preliminaries on the V-form traveling fronts and pyramidal traveling fronts. We study the existence and asymptotic stability of traveling fronts with convex polyhedral shape in Section 3.

2 Preliminary

In this section, we recall some results established in [2] and [13] including comparison principle, the existence and stability of V-form fronts and pyramidal traveling front in two dimensional space and three dimensional space, respectively.

Let $X=BUC(\mathbb{R}^3,\mathbb{R})$ be the Banach norm of all bounded and uniformly continuous functions from $\mathbb{R}^3$ to $\mathbb{R}$ with the usual supremum $|\cdot|_X$, and $X^+=\{\phi\in X:\phi(x,y,z)\ge$

$0, \forall (x,y,z) \in \mathbb{R}^3\}$. Let $\phi \in [-\delta_0, K+\delta_0]_{\mathcal{C}} = \{\phi \in \mathcal{C} : \phi(x,y,z,s) \in [-\delta_0, K+\delta_0], s \in [-\tau, 0], (x,y,z) \in \mathbb{R}^3\}$ for some $\delta_0 > 0$.

Then, from [2, Theorem 2.1], we have the following existence and comparison theorem.

Theorem 2.1. *Assume that* (A1)–(A3) *hold. Then for any* $\phi \in [-\delta_0, K+\delta_0]_{\mathcal{C}}$, (1.3) *has a unique mild solution* $w(x,y,z,t;\phi)$ *on* $[0,\infty)$ *with* $-\delta_0 \le w(x,y,z,t;\phi) \le K+\delta_0$ *for* $(x,y,z,t) \in \mathbb{R}^3 \times [-\tau,\infty)$*, and* $w(x,y,z,t;\phi)$ *is a classical solution of* (1.3) *for* $(x,y,z,t) \in \mathbb{R}^3 \times [\tau,\infty)$*. Moreover, suppose that* $w^+(x,y,z,t)$ *and* $w^-(x,y,z,t)$ *are supersolution and subsolution of* (1.3) *on* $\mathbb{R}^3 \times \mathbb{R}^+$*, respectively, and satisfy* $-\delta_0 \le w^{\pm}(x,y,z,t) \le K+\delta_0$ *for* $t \in [-\tau,\infty)$ *and* $(x,y,z) \in \mathbb{R}^N$*, and* $w^-(x,y,z,s) \le w^+(x,y,z,s)$ *for any* $(x,y,z) \in \mathbb{R}^3$ *and* $s \in [-\tau,0]$*. Then there holds* $w^+(x,y,z,t) \ge w^-(x,y,z,t)$ *for* $(x,y,z) \in \mathbb{R}^3, t \ge 0$.

Next, we state the existence and stability of V-form front of nonlocal delayed diffusion equation in two-dimensional space, see [2,13].

Let $\widehat{w}(\xi,\eta,t;\widehat{\phi})$ be the solution of

$$\begin{cases} \frac{\partial \widehat{w}}{\partial t} - D(\widehat{w}_{\xi\xi} + \widehat{w}_{\eta\eta}) + s\widehat{w}_\eta - d\widehat{w} + \int_{\mathbb{R}} b(\widehat{w}(\xi,\eta - s\tau - \eta_1, t-\tau)) f(\eta_1) d\eta_1 = 0, \\ \widehat{w}(\xi,\eta,r) = \widehat{\phi}(\xi,\eta,r), \quad (\xi,\eta) \in \mathbb{R}^2,\ r \in [-\tau,0]. \end{cases} \tag{2.1}$$

Theorem 2.2. (See [2, Corollary 3.1]) *For any* $s > c$*, there exists a solution* $\widehat{V}(\xi,\eta)$ *satisfying*

$$-\widehat{V}_{\xi\xi} - \widehat{V}_{\eta\eta} + s\widehat{V}_\eta + d\widehat{V} - \int_{\mathbb{R}} b(\widehat{V}(\xi,\eta - s\tau - \eta_1)) f(\eta_1) d\eta_1 = 0 \tag{2.2}$$

for any $(\xi,\eta) \in \mathbb{R}^2$*. Moreover, there hold*

$$\widehat{V}(\xi,\eta) > U\left(\frac{c}{s}(\eta + m^*|\xi|)\right) \quad \textit{for } (\xi,\eta) \in \mathbb{R}^2$$

and

$$\lim_{R\to\infty} \sup_{\xi^2+\eta^2>R^2} \left| \widehat{V}(\xi,\eta) - U\left(\frac{c}{s}(\eta + m^*|\xi|)\right) \right| = 0.$$

One also has

$$\inf_{\delta \le \widehat{V}(\xi,\eta) \le K-\delta} \widehat{V}_\eta(\xi,\eta) > 0 \quad \textit{for any } \delta \in (0,\delta^*]$$

and

$$\widehat{V}(\xi+\xi_0,\eta) \le \widehat{V}(\xi,\eta+\eta_0) \quad \forall (\xi,\eta) \in \mathbb{R}^2, \xi_0,\eta_0 \in \mathbb{R} \quad \textit{with } \eta_0 \ge m_*|\xi_0|.$$

The solution $\widehat{w}(\xi,\eta,t;\phi)$ *of* (2.1) *satisfies*

$$\lim_{t\to\infty} \|\widehat{w}(\xi,\eta,t;\phi) - \widehat{V}(\xi,\eta)\|_{L^\infty(\mathbb{R}^2)} = 0$$

for any initial value $\phi(\xi,\eta,r) \in [0,K]_{\mathcal{C}}$ *satisfying* $\widehat{\phi}(\xi,\eta,r) \ge v^-(\xi,\eta)$ *and*

$$\lim_{\gamma\to\infty} \sup_{(\xi,\eta)\in D(\gamma), r\in[-\tau,0]} |\widehat{\phi}(\xi,\eta,r) - v^-(\xi,\eta)| = 0.$$

Set

$$\begin{aligned} p_j(x,y) &:= m_*(x\cos\theta_j + y\sin\theta_j), \\ p(x,y) &= \max_{1\le j\le n} h_j(x,y) = m_* \max_{1\le j\le n} (x\cos\theta_j + y\sin\theta_j) \end{aligned} \tag{2.3}$$

and

$$k_j := \cos\left(\frac{\theta_{j+1}-\theta_j}{2}\right) > 0, \quad \phi_j := \frac{\theta_{j+1}+\theta_j}{2}, \quad 1 \le j \le n.$$

Define

$$E_j(x,y,z) := \widehat{V}\left(x\sin\phi_j - y\cos\phi_j, \frac{z - m_* k_j(x\sin\phi_j + y\cos\phi_j)}{\sqrt{m_*^2 k_j^2 + 1}}\right). \tag{2.4}$$

It is easy to check that every $E_j(x,y,z)$ is a V-shaped traveling front with speed $\frac{s}{\sqrt{1+m_*^2k_j^2}} > 0$ for any $1 \le j \le n$, that is, $E_j(x,y,z)$ satisfies (2.2) in Theorem 2.2. By [2, Theorem 1.1] and [13, Theorem 1.2], we have the following existence and stability of pyramidal traveling front $\widetilde{V}(x,y,z)$ associated with a pyramid $z = p(x,y)$.

Theorem 2.3. *Assume that (A1)–(A3) hold true. Let $s > c > 0$ and $p(x,y)$ be given by (2.3). Then there exists a solution $\widetilde{V}(x,y,z)$ of (1.4) with*

$$\lim_{\gamma\to\infty} \sup_{(x,y,z)\in D(\gamma)} \left|\widetilde{V}(x,y,z) - U\left(\frac{c}{s}(z+p(x,y))\right)\right| = 0,$$

$$U\left(\frac{c}{s}(z+p(x,y))\right) < \widetilde{V}(x,y,z) < K \quad \textit{for all } (x,y,z) \in \mathbb{R}^3,$$

$$\frac{\partial \widetilde{V}}{\partial z}(x,y,z) > 0 \quad \textit{for all } (x,y,z) \in \mathbb{R}^3,$$

$$\lim_{R\to\infty} \sup_{|z+p(x,y)|\ge R} |\widetilde{V}_z(x,y,z)| = 0 \quad \textit{and} \quad \inf_{\delta \le \widetilde{V}(\xi,\eta) \le K-\delta} \widetilde{V}_\eta(x,y,z) > 0 \quad \textit{for any } \delta \in (0,\delta^*].$$

Suppose that the initial value $\phi(x,y,z,r) \in C(\mathbb{R}^3 \times [-\tau,0],\mathbb{R})$ satisfies $\phi(x,y,z,r) \ge v^-(x,y,z)$ and

$$\lim_{\gamma\to+\infty} \sup_{(x,y,z)\in D(\gamma), r\in[-\tau,0]} |\phi(x,y,z,r) - \widetilde{V}(x,y,z)| = 0,$$

then the solution $w(x,y,z,t;\phi)$ of (1.3) satisfies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^3} |w(x,y,z,t;\phi) - \widetilde{V}(x,y,z)| = 0.$$

Furthermore, by [13], we have the following useful lemmas.

Lemma 2.4. *Let $\widetilde{V}(x,y,z)$ be as in Theorem 2.3 associated with pyramid $z = p(x,y)$. Then one has*

$$\lim_{R\to\infty} \sup_{|\mathbf{x}|\ge R} |\widetilde{V}(x,y,z) - \max_{1\le j\le n} E_j(x,y,z)| = 0,$$

$$\max_{1\le j\le n} E_j(x,y,z) < \widetilde{V}(x,y,z) \quad \textit{for all } (x,y,z) \in \mathbb{R}^3$$

and

$$\lim_{\gamma\to\infty} \sup_{\mathbf{x}\in D(\gamma), t\in[0,T]} \left|\max_{1\le j\le n} E_j(x,y,z) - U\left(\frac{c}{s}(z+p(x,y))\right)\right| = 0.$$

Lemma 2.5 (See [13, Lemma 3.1]). *There exist a positive constant ρ sufficiently large and a positive constant β small enough such that for any $0 < \delta < \frac{\delta^*}{2}e^{-\beta\tau}$, the function*

$$w^+(x,y,z,t) := \widetilde{V}(x,y,z+\rho\delta(1-e^{-\beta t})) + \delta e^{-\beta t}$$

is a supersolution of (1.3) and the function

$$w^-(x,y,z,t) := \widetilde{V}(x,y,z-\rho\delta(1-e^{-\beta t})) - \delta e^{-\beta t}$$

is a subsolution of (1.3) for any $(x,y,z) \in \mathbb{R}^3$ and $t \ge 0$, where $\widetilde{V}(x,y,z)$ be as in Theorem 2.3.

3 Traveling front with polyhedral shape

In this section, we study the existence and asymptotic stability of traveling fronts with convex polyhedral shape of (1.1) and prove Theorems 1.1–1.2.

We first recall that $\{(x,y,z) \in \mathbb{R}^3 | -z \geq h(x,y)\}$ is a convex polyhedron. For any $\zeta \in \mathbb{R}$ and $1 \leq j \leq n$, let $(X_j(\zeta), Y_j(\zeta))$ be defined by

$$h_j(X_j(\zeta), Y_j(\zeta)) = h_{j+1}(X_j(\zeta), Y_j(\zeta)) = m_*\zeta.$$

Direct computations give

$$\begin{pmatrix} X_j(\zeta) \\ Y_j(\zeta) \end{pmatrix} = \frac{1}{\sin(\theta_{j+1} - \theta_j)} \begin{pmatrix} (\zeta + s_j)\sin\theta_{j+1} - (\zeta + s_{j+1})\sin\theta_j \\ -(\zeta + s_j)\cos\theta_{j+1} + (\zeta + s_{j+1})\cos\theta_j \end{pmatrix}.$$

As point in [23], a set $\{(x,y) \in \mathbb{R}^2 | h(x,y) \leq \zeta\}$ is either an empty set or a nonempty convex closed set in $\mathbb{R}^2$. By [23, Lemma 3.1], the set $\{(x,y) \in \mathbb{R}^2 | h(x,y) \leq m_*\rho\}$ is a convex $n-$polygon in the $x-y$ plane with vertices $\{(X_j(\rho), Y_j(\rho))\}_{1\leq j\leq n}$ for any fixed number $\rho \in (0, +\infty)$.

Proof of Theorem 1.1. Since $h(X_j(\rho), Y_j(\rho)) = m_*\rho$ for all $1 \leq j \leq n$, then we obtain

$$h(x,y) \leq m_*\rho + p(x - X_j(\rho), y - Y_j(\rho))$$

for all $(x,y) \in \mathbb{R}^N$, $1 \leq j \leq n$, where $h(x,y)$ and $p(x,y)$ are defined in (1.5) and (2.3), respectively. Set

$$v^-(x,y,z) = U\left(\frac{c}{s}(z + h(x,y))\right) = \max_{1\leq j\leq n} U\left(\frac{c}{s}(z + h_j(x,y))\right).$$

Note that the function $v^-(x,y,z)$ is a subsolution of (1.4) and the pyramidal traveling front $\widetilde{V}(x,y,z)$ defined in Theorem 2.3 is a solution of (1.4). Thus, we have

$$v^-(x,y,z) < \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)$$

for all $(x,y,z) \in \mathbb{R}^3$ and $1 \leq j \leq n$. This shows that

$$\min_{1\leq j\leq n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)$$

is a supersolution of (1.4) for all $(x,y,z) \in \mathbb{R}^3$. Define

$$V(x,y,z) := \lim_{t\to\infty} w(x,y,z,t;v^-), \quad \forall (x,y,z) \in \mathbb{R}^3.$$

Then the function $V(x,y,z) \in C^2(\mathbb{R}^3)$ is a solution of (1.4). As a result of the comparison principle (see Theorem 2.1), we have

$$v^-(x,y,z) < V(x,y,z) \leq \min_{1\leq j\leq n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho) \tag{3.1}$$

for all $(x,y,z) \in \mathbb{R}^3$. On the other hand, since

$$\max\{h_j(x,y), h_{j+1}(x,y)\} \leq h(x,y) \quad \text{in } \mathbb{R}^2, \forall 1 \leq j \leq n,$$

then

$$U\left(\frac{c}{s}(z+\max\{h_j(x,y),h_{j+1}(x,y)\})\right)\le v^-(x,y,z),\quad (x,y,z)\in\mathbb{R}^3,\ \forall 1\le j\le n.$$

We consider the left-hand side and the right hand side as an initial value of (1.3), respectively. Then Theorem 2.1 yields that

$$w\left(x,y,z,t;U\left(\frac{c}{s}(z+\max\{h_j(x,y),h_{j+1}(x,y)\})\right)\right)\le w(x,y,z,t;v^-(x,y,z)) \tag{3.2}$$

for all $1\le j\le n$. Note that

$$h_j(x,y)=p_j(x-X_j(\rho),y-Y_j(\rho))+m_*\rho.$$

Recall that E_j $(1\le j\le n)$ is defined by (2.3). Let $t\to\infty$ in (3.2), by Lemma 2.4, we obtain

$$E_j(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho)\le V(x,y,z),\quad (x,y,z)\in\mathbb{R}^3.$$

This together with (3.1), there is

$$\max_{1\le j\le n}E_j(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho)\le V(x,y,z)\le\min_{1\le j\le n}\widetilde{V}(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho)$$

for all $(x,y,z)\in\mathbb{R}^3$. By Theorem 2.2 and 2.3, we then have

$$\lim_{R\to\infty}\sup_{|\mathbf{x}|>R}\left|V(x,y,z)-\max_{1\le j\le n}E_j(x-X_j(\rho),y-Y_j(\rho),z+m_*\rho)\right|=0 \tag{3.3}$$

and

$$0<U\left(\frac{c}{s}(z+h(x,y))\right)<V(x,y,z)<K\quad\text{for all }(x,y,z)\in\mathbb{R}^3.$$

We use the Schauder interior estimate to the following equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+s\frac{\partial}{\partial z}\right)(V-E_j)\\&\quad=-d(V-E_j)+\int_{\mathbb{R}}b(V(x,y,z-s\tau-z_1))f(z_1)dz-\int_{\mathbb{R}}b(E_j(x,y,z-s\tau-z_1))f(z_1)dz.\end{aligned}$$

Then by Theorems 2.2–2.3 and (3.3), we obtain

$$\inf_{\delta\le V(x,y,z)\le K-\delta}V_z(x,y,z)>0\quad\text{for }\delta>0\text{ small.}$$

Note that $|z+h(x,y)|\to\infty$ implies $\mathrm{dist}((x,y,z),\Gamma_j)\to\infty$ for $1\le j\le n$. Then we have

$$\lim_{\gamma\to\infty}\sup_{(x,y,z)\in D(\gamma)}\left|V(x,y,z)-U\left(\frac{c}{s}(z+h(x,y))\right)\right|=0.$$

By the interpolation $\|\cdot\|_{C^1}\le 2\sqrt{\|\cdot\|_{C^0}\|\cdot\|_{C^2}}$ and the fact

$$\lim_{R\to\infty}\sup_{|z+h(x,y)|\ge R}\left|U_z\left(\frac{c}{s}(z+h(x,y))\right)\right|=0,$$

we get

$$\lim_{R\to\infty}\sup_{|z+h(x,y)|>R}|V_z(x,y,z)|=0.$$

This completes the proof. □

In the following, we show that the traveling front $V(x,y,z)$ with convex polyhedral shape is asymptotically stable.

Proof of Theorem 1.2. Set

$$\widehat{s} := \max_{1\le j\le n} s_j \ge 0.$$

Then there holds

$$-m_*\widehat{s} + p(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s})) \le h(x,y) \quad \text{for } 1 \le j \le n. \tag{3.4}$$

It then follows that

$$U\left(\frac{c}{s}(z - m_*\widehat{s} + p(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s})))\right) \le U\left(\frac{c}{s}(z + h(x,y))\right) \quad \text{for } 1 \le j \le n. \tag{3.5}$$

Consider the left-hand side and the right-hand side of (3.5) as initial values of (1.3) and let $t \to \infty$, we obtain

$$\widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s}) \le V(x,y,z) \quad \text{for } (x,y,z) \in \mathbb{R}^3,\ 1 \le j \le n. \tag{3.6}$$

Together with (3.5) and (3.6), we have

$$\max_{1\le j\le n} \widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s}) \le V(x,y,z) \le \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho) \tag{3.7}$$

for all $(x,y,z) \in \mathbb{R}^3$.

For all $(x,y,z) \in \mathbb{R}^3$, set

$$V^*(x,y,z) := \lim_{t\to\infty} w\left(x,y,z,t; \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)\right).$$

Then the comparison principle gives that

$$V(x,y,z) \le V^*(x,y,z) \le \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho).$$

By (3.3) and Theorem 2.3, we have

$$\lim_{R\to\infty} \sup_{|\mathbf{x}|\ge R} |V^*(x,y,z) - V(x,y,z)| = 0.$$

It then follows the similar way in [23] that, there holds $V^*(x,y,z) \equiv V(x,y,z)$. This implies

$$\lim_{t\to\infty} \left\| w\left(x,y,z,t; \min_{1\le j\le n} \widetilde{V}(x - X_j(\rho), y - Y_j(\rho), z + m_*\rho)\right) - V(x,y,z) \right\|_{L^\infty(\mathbb{R}^3)} = 0.$$

Using the similar process to $\max_{1\le j\le n} \widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s})$, we also have

$$\lim_{t\to\infty} \left\| w\left(x,y,z,t; \max_{1\le j\le n} \widetilde{V}(x - X_j(-\widehat{s}), y - Y_j(-\widehat{s}), z - m_*\widehat{s})\right) - V(x,y,z) \right\|_{L^\infty(\mathbb{R}^3)} = 0.$$

Note that for any fixed $(x,y,z) \in \mathbb{R}^3$ and $t > 0$, $w(x,y,z,t;\cdot)$ is continuous mapping in X. By the continuity of $w(x,y,z,t;\cdot)$ and Theorems 2.1–2.3, we obtain

$$\lim_{t\to\infty} \|w(x,y,z,t;\phi) - V(x,y,z)\|_{L^\infty} = 0.$$

The proof is completed. $\square$

Furthermore, $V(x,y,z)$ also enjoys the following properties, which can be proved by the similar ways as that in [23, Lemma 3.3–3.5] and we omit them here.

Lemma 3.1. *Let $V(x,y,z)$ be as in Theorem 1.1. Then there holds*

(i) Let $h(x,y)$ be defined in (1.5), $\overline{h}(x,y)=\max_{1\le j\le n} h_j(x,y)=m_*\max_{1\le j\le n}(x\cos\theta_j+y\sin\theta_j-\overline{s}_j)$ with $\min_{1\le j\le n}\overline{s}_j\ge 0$. Define $\overline{V}(x,y,z)$ be the traveling front of polyhedral-shape associated with $\overline{h}(x,y)$. If $\overline{h}(x,y)\ge h(x,y)$ for any $(x,y)\in\mathbb{R}^2$, then $\overline{V}(x,y,z)\ge V(x,y,z)$ for all $(x,y,z)\in\mathbb{R}^3$.

(ii) One has $\frac{\partial V}{\partial \nu}(x,y,z)>0$ in $\mathbb{R}^3$ for

$$\nu=\frac{1}{\sqrt{1+t_1^2+t_2^2}}\begin{pmatrix} t_1\\ t_2\\ 1\end{pmatrix}\quad \textit{with } \sqrt{t_1^2+t_2^2}\le\frac{1}{m_*}.$$

(iii) If $h(x,y)=h(|x|,|y|)$, then there holds $V(x,y,z)=V(|x|,|y|,z)$ for all $(x,y,z)\in\mathbb{R}^3$ and

$$\begin{aligned} &V_x(x,y,z)>0 && \textit{for } (x,y,z)\in(0,\infty)\times\mathbb{R}^2,\\ &V_x(0,y,z)=0 && \textit{for } (y,z)\mathbb{R}^2,\\ &V_y(x,y,z)>0 && \textit{for } (x,y,z)\in\mathbb{R}\times(0,\infty)\times\mathbb{R},\\ &V_y(x,0,z)=0 && \textit{for } (x,z)\in\mathbb{R}^2.\end{aligned}$$

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