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Ground state solution for a class of supercritical nonlocal equations with variable exponent

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Abstract. In this paper, we obtain the existence of positive critical point with least energy for a class of functionals involving nonlocal and supercritical variable exponent nonlinearities by applying the variational method and approximation techniques. We apply our results to the supercritical Schrödinger–Poisson type systems and supercritical Kirchhoff type equations with variable exponent, respectively.

Keywords: Schrödinger–Poisson type system, Kirchhoff type equations, supercritical exponent, variational method.

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1 Introduction and main results

We divide this section into two parts. In the first part, we present a critical point theory of abstract functional inspired by the article of Marcos do Ó, Ruf and Ubilla [21]. The second part is devoted to introduce its applications to a class of Schrödinger–Poisson type systems and a class of Kirchhoff type equations.

1.1 Abstract critical point theory

In the pioneering article [8], Brézis and Nirenberg considered the existence of solution to the following nonlinear elliptic equation

$$\begin{cases} -\Delta u = u^5 + f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$. If $f(x,u) = 0$ and Ω is star shaped, a well-known nonexistence result of Pohozaev [26] asserts that (1.1) has no solution. But the lower-order terms perturbation can reverse this situation. Brézis and Nirenberg [8] proved the existence of solutions to (1.1) under the assumptions on the lower-order perturbation term $f(x,u)$. On

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the other hand, the topology and the shape of the domain can affect the existence of solution for (1.1) with $f(x,u)=0$. For example, Coron [12] used a variational approach to prove that (1.1) is solvable if Ω exhibits a small hole. Rey [27] established existence of multiple solutions if Ω exhibits several small holes. As Ω is an annulus, Kazdan and Warner [17] observed that there exists a solution to (1.1) without any constraint by critical exponent.

It is worth noticing that there are also a few papers concerning on the supercritical equations except adding lower-order perturbation terms or changing the topology of region Ω. The papers in [10,21] considered the following nonlinear supercritical elliptic problem

$$\begin{cases} -\Delta u = |u|^{4+|x|^\alpha}u, & \text{in } B, \\ u=0, & \text{on } \partial B, \end{cases} \tag{1.2}$$

where $B\subset\mathbb{R}^3$ is the unit ball and $0<\alpha<1$. By using the mountain pass lemma and approximation techniques, a radial positive solution for (1.2) is obtained by Marcos do Ó, Ruf and Ubilla in [21]. Cao, Li and Liu [10] considered the existence of infinitely many nodal solutions to (1.2) by looking for a minimizer of a constrained minimization problem in a special space.

Let H be the subspace of $H_0^1(B)$ consisting of radially symmetric functions. From [21], we know that (1.2) possesses a variational structure, its solutions can be found as critical points of the functional

$$I_0(u)=\frac{1}{2}\int_B|\nabla u|^2-\int_B\frac{1}{6+|x|^\alpha}|u|^{6+|x|^\alpha}, \qquad u\in H.$$

The solutions to this kind of supercritical elliptic equations involving nonlocal nonlinearities can be found to look for the critical points of a suitable perturbation of I_0,

$$J(u)=\frac{1}{2}\int_B|\nabla u|^2+\lambda R(u)-\int_B\frac{1}{6+|x|^\alpha}|u|^{6+|x|^\alpha}, \qquad u\in H,$$

where $\lambda\in\mathbb{R}$ and $R\in C(H,\mathbb{R})$. In order to obtain the nontrivial critical point of J, we need to consider the approximation functional $I: H\to\mathbb{R}$ associated to J given by

$$I(u)=\frac{1}{2}\int_B|\nabla u|^2+\lambda R(u)-\frac{1}{6}\int_B|u|^6.$$

In this paper, we are interested in researching the least energy critical point of J, the following assumptions are needed:

(i) $R\in C^1(H,\mathbb{R}^+)$ with $\mathbb{R}^+=[0,+\infty)$;

(ii) there exist $C,q>0$ such that for $t>0$,

$$R(tu)=t^qR(u), \qquad R(u)\le C\|u\|^q, \qquad \forall\, u\in H;$$

(iii) $qR(u)=\langle R'(u)u\rangle$, $u\in H$;

(iv) if $\{u_n\}$ is a $(PS)_c$ sequence of J for some $c>0$ and $u_n\rightharpoonup u$ weakly in H as $n\to\infty$, then $J'(u)=0$.

Inspired by above papers, the main purpose of this paper is to consider the existence of ground state for the functional J. Our main result reads as follows.

Theorem 1.1. *Assume that $\lambda > 0$, $2 < q < 6$ or $\lambda < 0$, $q > 6$ and the assumptions (i)–(iv) hold. Then the functional J possesses a $(PS)_c$ sequence with some $c > 0$. Moreover if the functional I satisfies the $(PS)_c$ condition, then J admits a nontrivial critical point.*

Theorem 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. If R is even and weakly lower semicontinuous, then the functional J possesses a least energy critical point.*

Remark 1.3. The variable exponent function $p(x) = 6 + |x|^\alpha$ has a strictly supercritical growth except the origin and a critical growth in the origin. Hence, the functional J can be regarded as the supercritical perturbation of the functional I.

Remark 1.4. In each case of $\lambda > 0$, $0 < q < 6$ or $\lambda < 0$, $q > 6$, we can show that J possesses the mountain pass structure. Hence, a minimax level for the functional J can be constructed. It is important to verify that this level lies below the non-compactness level of the functional I. It is worthwhile pointing out that the term R affects the non-compactness level of the functional I. In most cases, it is difficult to calculate the level of the non-compactness level accurately.

Remark 1.5. Since the method of proving (iv) is different when R is different, the condition (iv) is needed. The weak lower semicontinuity of R guarantees the existence of a ground state for functional J.

Remark 1.6. Relatively speaking, the condition (iv) is easy to get for some functional J involving nonlocal nonlinearities. It is obvious to see from (iv) that u is a critical point of the functional J. Hence, we just need to show that u is nontrivial.

As an application, we apply the case of $\lambda < 0$ to a class of Schrödinger–Poisson type systems and the case of $\lambda > 0$ to a class of Kirchhoff type equations, respectively.

1.2 Applications to two nonlocal problems

As a first application, we consider the existence of nontrivial solution to the supercritical Schrödinger–Poisson type systems with variable exponent

$$\begin{cases} -\Delta u - \phi|u|^3 u = |u|^{4+|x|^\alpha} u & \text{in } B, \\ -\Delta\phi = |u|^5 & \text{in } B, \\ u = \phi = 0 & \text{on } \partial B, \end{cases} \tag{1.3}$$

where $B \subset \mathbb{R}^3$ is the unit ball and $0 < \alpha < 1$. The Schrödinger–Poisson system as a model describing the interaction of a charge particle with an electromagnetic field arises in many mathematical physics context (we refer to [7] for more details on the physical aspects). There are a few references which investigated the well-known Schrödinger–Poisson system with nonlocal critical growth in a bounded domain (see e.g. [3–5]). Azzollini, d'Avenia [3] considered the following problem involving the nonlocal critical growth

$$\begin{cases} -\Delta u - \phi|u|^3 u = \lambda u & \text{in } B, \\ -\Delta\phi = |u|^5 & \text{in } B, \\ u = \phi = 0 & \text{on } \partial B. \end{cases} \tag{1.4}$$

They proved the existence of positive solution depending on the value of λ and (1.4) has no solution for $\lambda \leq 0$ via Pohozaev's identity. Later, Azzollini, d'Avenia and Vaira [5] improved the results in [3]. They proved existence and nonexistence results of positive solutions for (1.4) when λ is in proper region. By applying the variational arguments and the cut-off function technique, Azzollini, d'Avenia and Luisi [4] studied the following generalized Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \varepsilon q \phi f(u) = \eta |u|^{p-1} u & \text{in } \Omega, \\ -\Delta \phi = 2qF(u) & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $1 < p < 5$, $q > 0$, $\varepsilon, \eta = \pm 1$, $f \in C(\mathbb{R}, \mathbb{R})$, $F(s) = \int_0^s f(t)dt$. In the case where f is critical growth, they obtained the existence and nonexistence results.

In the recent years, there have been a lot of researches dealing with the Schrödinger–Poisson systems

$$\begin{cases} -\Delta u + \phi u = f(x, u) & \text{in } \Omega, \\ -\Delta \phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

When $f(x, u) = |u|^{p-1}u$ with $p \in (1, 5)$, Ruiz and Siciliano [29] considered the existence, nonexistence and multiplicity results by using variational methods. Alves and Souto [2] studied system (1.5) when f has a subcritical growth. They obtained the existence of least energy nodal solution by using variational methods. Ba and He [6] proved the existence of ground state solution for system (1.5) with a general 4-superlinear nonlinearity f by the aid of the Nehari manifold. Pisani and Siciliano [25] proved the existence of infinitely many solutions of (1.5) by means of variational methods. In [1], Almuaalemi, Chen and Khoutir obtained the existence of nontrivial solutions for (1.5) when f has a critical growth via variational methods.

Motivated by above papers, by applying Theorems 1.1 and 1.2, we obtain the existence of positive ground state solution for system (1.3) with both nonlinearity supercritical growth and nonlocal critical growth. From the technical point of view, there are two difficulties to prove our result. Firstly, the supercritical nonlinearity in the system sets an obstacle since the bounded (PS) sequence could not converge. Secondly, due to the system has two critical terms, it is difficult to estimate the critical level of mountain pass. In order to overcome these difficulties, by employing the ideas of [21], we first estimate the critical level of the mountain pass for the functional corresponding to (1.3) via approximation techniques and then show that the level is below the non-compactness level of the functional. Finally, the existence of positive ground state solution is obtained by applying the Nehari manifold method and regularity theory. Hence, we have the following result:

Theorem 1.7. *System (1.3) possesses at least a positive ground state solution.*

Remark 1.8. By the Pohozaev's identity used in [3], we can deduce that (1.3) has no nontrivial solution if $|x|^\alpha = 0$. Hence, our result is interesting phenomena due to the nonlinearity $|u|^{4+|x|^\alpha}u$ has supercritical growth everywhere in B except in the origin and critical growth in the origin.

Next, as the second application, we consider the following Kirchhoff type equations:

$$\begin{cases} -\left(1 + b\int_B |\nabla u|^2 dx\right)\Delta u = |u|^{4+|x|^\alpha}u, & \text{in } B \\ u = 0, & \text{on } \partial B, \end{cases} \tag{1.6}$$

where $b > 0$, $0 < \alpha < 1$. This kind of equation is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2l} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff in [18]. The equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The solvability of the Kirchhoff type equations has been well studied in a general dimension by many authors after Lions [20] introduced an abstract framework to this problem. By using new analytical skills and non-Nehari manifold method, Tang and Cheng [31] obtained the ground state sign-changing solutions for a class of Kirchhoff type problems in bounded domains. In [11], Chen, Zhang and Tang considered the existence and non-existence results for Kirchhoff-type problems with convolution nonlinearity based on variational and some new analytical techniques. There are also many papers devoted to the existence and multiplicity of solutions for the following critical Kirchhoff type equations with subcritical disturbance

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u) + u^5 & \text{in } \Omega \\ u > 0 & \text{in} \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where a, b are positive constants. By using concentration-compactness principle and variational method, Naimen in [22] obtained the existence and multiplicity of (1.7) with $f(x,u) = \lambda u$. Xie, Wu and Tang [34] derived the existence and multiplicity of solutions to (1.7) via variational method by discussing the sign of a and b and adding different conditions on f. By controlling concentrating Palais–Smale sequences, Naimen and Shibata [23] proved the existence of two positive solutions for (1.7) with $f(x,u) = u^q$, $1 \le q < 5$.

In particular, there are some papers considered the equations with critical and supercritical growth by adding the smallness of the coefficient in front of critical and supercritical which is used to overcome the difficulty provoked by supercritical growth. By combining an appropriate method of truncation function with Moser's iteration technique, Corrêa and Figueiredo [13, 14] considered the existence of positive solution for a class of p-Kirchhoff type equations and Kirchhoff type equations with supercritical growth, respectively.

Motivated by the above fact, we study the existence of positive ground state solution for (1.6) with variable exponential perturbation by using the similar method introduced by Marcos do Ó, Ruf and Ubilla in [21]. The result reads as follows.

Theorem 1.9. *The equation* (1.6) *possesses at least a positive ground state solution.*

Remark 1.10. Recall that in [22], if $|x|^\alpha = 0$, (1.6) has no nontrivial solution by Pohozaev's identity. Hence, our result is interesting phenomena for this kind of Kirchhoff type equations due to the nonlinearity $|u|^{4+|x|^\alpha}u$ has supercritical growth everywhere in B except the origin and critical growth in the origin.

Remark 1.11. Throughout the paper we denote by $C > 0$ various positive constants which may vary from line to line and are not essential to the problem.

The paper is organized as follows: in Section 2, some notations and preliminary results are presented. We obtain the existence of nontrivial critical point to the functional J in Section 3. By using Nehari manifold method, the least energy critical point of the functional J is derived in Section 4. Sections 5 and 6 are devoted to show that the Theorems 1.1 and 1.2 can be applied to the nonlinear Schrödinger–Poisson type systems and the Kirchhoff type equations, respectively.

2 Preliminary

In this Section, we will give some notations and lemmas which will be used throughout this paper. Let $B \subset \mathbb{R}^3$ denote the unit ball, $H = H^1_{0,rad}(B) = \{u \in H^1_0(B) : u(x) = u(|x|)\}$ be the Sobolev space of radial functions, with respect to the norm

$$\|u\| = \left(\int_B |\nabla u|^2 \right)^{1/2}.$$

Let $C_+(\bar{B}) = \{h : h \in C(\bar{B}), h(x) > 1, x \in \bar{B}\}$. For any $h \in C_+(\bar{B})$, we denote

$$h^+ = \sup_{x \in B} h(x), \qquad h^- = \inf_{x \in B} h(x).$$

Then for each $p \in C_+(\bar{B})$, the variable exponent function space $L^{p(x)}(B)$ is defined as follows

$$L^{p(x)}(B) = \left\{ u : u \text{ is a measurable function in } B \text{ such that } \int_B |u(x)|^{p(x)} dx < \infty \right\}$$

with the norm defined by

$$\|u\|_{L^{p(x)}} = \inf \left\{ \lambda > 0, \int_B \left| \frac{u}{\lambda} \right|^{p(x)} \leq 1 \right\}.$$

We denote by $L^{p'(x)}(B)$ the conjugate space of $L^{p(x)}(B)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(B)$ and $v \in L^{p'(x)}(B)$, there holds the Hölder type inequality

$$\left| \int_B uv \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{L^{p(x)}} \|v\|_{L^{p'(x)}}.$$

Lemma 2.1 ([15]). *Set $\rho(u) = \int_B |u(x)|^{p(x)}$. For $u \in L^{p(x)}(B)$, we have*

(1) $\|u\|_{L^{p(x)}} < 1\ (= 1;\ > 1) \Leftrightarrow \rho(u) < 1\ (= 1;\ > 1)$;

(2) If $\|u\|_{L^{p(x)}} > 1$, then $\|u\|^{p^-}_{L^{p(x)}} \leq \rho(u) \leq \|u\|^{p^+}_{L^{p(x)}}$;

(3) If $\|u\|_{L^{p(x)}} < 1$, then $\|u\|^{p^+}_{L^{p(x)}} \leq \rho(u) \leq \|u\|^{p^-}_{L^{p(x)}}$.

Lemma 2.2 ([21]). *Let $q(x) = 6 + \beta|x|^\alpha$, $x \in B$ and $\alpha, \beta > 0$. The following embedding is continuous:*

$$H \hookrightarrow L^{q(x)}(B).$$

It is easy to check by (i), Lemma 2.2 and Hölder type inequality that J is well defined on H and $J \in C^1(H, \mathbb{R})$, and

$$\langle J'(u), v \rangle = \int_B \nabla \cdot u \nabla v + \lambda \langle R'(u), v \rangle - \int_B |u|^{4+|x|^\alpha} uv, \qquad u, v \in H.$$

In the following we define the best embedding constant S by

$$S = \inf_{u \in H \setminus \{0\}} \frac{\int_B |\nabla u|^2}{\left(\int_B |u|^6\right)^{\frac{1}{3}}}. \tag{2.1}$$

Let $\chi \in C_0^\infty(B)$ be a cut-off function with $\chi = 1$ on $B_{1/2}(0)$ and $\eta \in [0,1]$ on B. Let us define the function

$$U_\varepsilon(x) = (3\varepsilon^2)^{1/4}(\varepsilon^2 + |x|^2)^{-1/2}, \qquad \varepsilon > 0,$$

which satisfies the equation

$$-\Delta u = u^5 \quad \text{on } \mathbb{R}^3.$$

Then define $u_\varepsilon = \chi(x)U_\varepsilon(x)$, the following estimates can be deduced via standard arguments as $\varepsilon \to 0^+$ (see [33]),

$$\int_B |\nabla u_\varepsilon|^2 = S^{\frac{3}{2}} + O(\varepsilon), \qquad \int_B u_\varepsilon^6 = S^{\frac{3}{2}} + O(\varepsilon^3). \tag{2.2}$$

3 The nontrivial critical point

In this section, we first show that the functional J possesses the mountain pass structure under the assumption $\lambda < 0$, $q > 6$ or $\lambda > 0$, $0 < q < 6$, respectively. And hence J has a $(PS)_c$ sequence $\{u_n\}$ with some $c > 0$. Then we prove that $\{u_n\}$ is bounded and is also a $(PS)_c$ sequence of I, which is a key in the existence of nontrivial critical point.

Lemma 3.1. *Assume that $\lambda < 0$, $q > 2$ and the assumptions (i) and (ii) hold.*

(a) There exist $\rho_1 > 0$, $\eta_1 > 0$ such that $\inf\{J(u) : u \in H, \text{ with } \|u\| = \rho_1\} > \eta_1$.

(b) There exists $e_1 \in H$ with $\|e_1\| > \rho_1$ such that $J(e_1) < 0$.

Proof. (a) For $\rho_1 > 0$, let

$$\Sigma_{\rho_1} = \{u \in H : \|u\| \le \rho_1\}.$$

We deduce, from the Sobolev inequality and Lemma 2.1, that for $u \in \partial\Sigma_{\rho_1}$ and $C > 0$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \lambda R(u) - \int_B \frac{1}{6+|x|^\alpha}|u|^{6+|x|^\alpha}. \\ &\ge \frac{1}{2}\|u\|^2 + C\lambda\|u\|^q - C(\|u\|^6 + \|u\|^7) \\ &= \frac{1}{2}\rho_1^2 + C\lambda\rho_1^q - C\rho_1^6 - C\rho_1^7. \end{aligned}$$

Hence, by letting $\rho_1 > 0$ small enough, it is easy to see that there is $\eta_1 > 0$ such that (a) holds.

(b) By [21, Lemma 3.1], we know that there exists a constant $C > 0$ such that for $\varepsilon > 0$ small,

$$\begin{aligned} \int_B |u_\varepsilon|^{6+|x|^\alpha} &\ge \int_B |u_\varepsilon|^6 + C|\log\varepsilon|\varepsilon^\alpha + O(\varepsilon) \\ &= S^{3/2} + C|\log\varepsilon|\varepsilon^\alpha + O(\varepsilon). \end{aligned} \tag{3.1}$$

This together with (2.2) implies that for $t \geq 1$ and $\varepsilon > 0$ small enough,

$$\begin{aligned} J(tu_\varepsilon) &= \frac{t^2}{2}\|u_\varepsilon\|^2 + \lambda t^q R(u_\varepsilon) - \int_B \frac{t^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\ &\leq \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^6}{7}\int_B |u_\varepsilon|^{6+|x|^\alpha} \\ &\leq S^{3/2}t^2 - \frac{S^{3/2}}{14}t^6 \to -\infty \end{aligned}$$

as $t \to +\infty$. Let $T > 0$ and define a path $\tilde{h} : [0,1] \to H$ by $\tilde{h}(t) = tTu_\varepsilon$. For $T > 0$ large enough, we have

$$\int_B |\nabla \tilde{h}(1)|^2 > \rho_1^2, \qquad J(\tilde{h}(1)) < 0.$$

By taking $e_1 = \tilde{h}(1)$, then (b) is valid. The proof is completed. □

Lemma 3.2. *Assume that $\lambda > 0$, $0 < q < 6$ and the assumptions (i) and (ii) hold.*

(a) There exist $\rho_2 > 0$, $\eta_2 > 0$ such that $\inf\{J(u) : u \in H,$ with $\|u\| = \rho_2\} > \eta_2$.

(b) There exists $e_2 \in H$ with $\|e_2\| > \rho$ such that $J(e_2) < 0$.

Proof. (a) Let us define

$$\Sigma_{\rho_2} = \{u \in H : \|u\| \leq \rho_2\}, \qquad \rho_2 > 0.$$

It follows from the Sobolev inequality and Lemma 2.1 that for $u \in \partial\Sigma_{\rho_2}$ and $C > 0$,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \lambda R(u) - \int_B \frac{1}{6+|x|^\alpha}|u|^{6+|x|^\alpha} \\ &\geq \frac{1}{2}\|u\|^2 - C(\|u\|^6 + \|u\|^7) \\ &= \frac{1}{2}\rho_2^2 - C\rho_2^6 - C\rho_2^7. \end{aligned}$$

Hence, by letting $\rho_2 > 0$ small enough, it is easy to see that there is $\eta_2 > 0$ such that (a) holds.

(b) By using (2.2) and (3.1) again, we have for $t \geq 1$ and $\varepsilon > 0$ small enough,

$$\begin{aligned} J(tu_\varepsilon) &= \frac{t^2}{2}\|u_\varepsilon\|^2 + \lambda t^q R(u_\varepsilon) - \int_B \frac{t^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\ &\leq \frac{t^2}{2}\|u_\varepsilon\|^2 + C\lambda t^q\|u_\varepsilon\|^q - \frac{t^6}{7}\int_B |u_\varepsilon|^{6+|x|^\alpha} \\ &\leq S^{3/2}t^2 + 2C\lambda S^{3q/4}t^q - \frac{t^6}{14}S^{3/2} \to -\infty \end{aligned}$$

as $t \to +\infty$. Let $T > 0$ and define a path $\hat{h} : [0,1] \to H$ by $\hat{h}(t) = tTu_\varepsilon$. For $T > 0$ large enough, we have

$$\int_B |\nabla \hat{h}(1)|^2 > \rho_2^2, \qquad J(\hat{h}(1)) < 0.$$

By taking $e_2 = \hat{h}(1)$, we proof (b). The proof is completed. □

From Lemmas 3.1 and 3.2, we know that the functional J possesses the mountain pass geometry. Then there is a $(PS)_c$ sequence $\{u_n\} \subset H$ for J with the property that

$$J(u_n) \to c, \qquad \|J'(u_n)\|_{H^{-1}} \to 0, \qquad n \to \infty,$$

where c is given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)), \tag{3.2}$$

and $\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0, J(\gamma(1)) < 0\}$.

Lemma 3.3. *Assume that $\lambda < 0$, $q > 6$ or $\lambda > 0$, $0 < q < 6$ and the assumption (iii) holds. If $\{u_n\} \subset H$ is a $(PS)_c$ sequence for J with $c > 0$, then $\{u_n\}$ is bounded in H.*

Proof. For n large enough, it is easy to deduce from (iii) that

$$\begin{aligned} c+1 &\geq J(u_n) - \frac{1}{6}\langle J'(u_n), u_n\rangle \\ &= \frac{1}{3}\|u_n\|^2 + \lambda\left(\frac{1}{q} - \frac{1}{6}\right)\langle R'(u_n), u_n\rangle + \int_B \left(\frac{1}{6} - \frac{1}{6+|x|^\alpha}\right)|u_n|^{6+|x|^\alpha} \\ &\geq \frac{1}{3}\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in H. The proof is completed. □

Lemma 3.4 ([21])**.** *Assume that $u \in H$. Then*

$$|u(r)| \leq r^{-1/2}\|u\|, \qquad r > 0.$$

Proof of Theorem 1.1. By using Lemmas 3.1 and 3.2 respectively, there exists a sequence $\{u_n\} \subset H$ satisfying $J(u_n) \to c$, $J'(u_n) \to 0$ as $n \to \infty$, where c is given in (3.2). By Lemma 3.3, $\{u_n\}$ is a bounded sequence in H. Passing to a subsequence if necessary, we may assume that there exists $u \in H$ such that

$$u_n \rightharpoonup u \text{ in } H, \text{ and } u_n(x) \to u(x), \ a.e.\ x \in B.$$

If $u \neq 0$, then u is a nontrivial critical point of the functional J follows from the assumption (iv). In what follows, we will deal with the case of $u = 0$ and show that this is impossible. In fact, since $H^1_r(B\backslash B_\delta) \hookrightarrow\hookrightarrow L^p(B\backslash B_\delta)$, for $\delta \in (0,1)$ and $p \geq 1$, there holds

$$\int_\delta^1 |u_n|^{6+r^\alpha} r^2 \to 0, \text{ as } n \to \infty \tag{3.3}$$

and

$$\int_\delta^1 |u_n|^6 r^2 \to 0, \text{ as } n \to \infty. \tag{3.4}$$

In the following, we will show that $\{u_n\}$ is also a $(PS)_c$ sequence of I. Hence, it is sufficient to prove

(a) $J(u_n) = I(u_n) + o(1)$;

(b) $\langle J'(u_n), v\rangle = \langle I'(u_n), v\rangle + o(1)\|v\|$, $v \in H$.

We first claim that (a) is valid, indeed we only need to estimate

$$\begin{aligned}&\int_B\left(\frac{1}{6}|u_n|^6-\frac{1}{6+|x|^\alpha}|u_n|^{6+|x|^\alpha}\right)\\&\quad=\int_B\left(\frac{1}{6}|u_n|^6-\frac{1}{6+|x|^\alpha}|u_n|^6\right)+\int_B\left(\frac{1}{6+|x|^\alpha}|u_n|^6-\frac{1}{6+|x|^\alpha}|u_n|^{6+|x|^\alpha}\right).\end{aligned}\tag{3.5}$$

For any $\varepsilon>0$, there exist $\delta>0$ and $n_1\in\mathbb{N}$ such that for any $n\geq n_1$, we have, by (3.4),

$$\begin{aligned}\int_B\left(\frac{1}{6}|u_n|^6-\frac{1}{6+|x|^\alpha}|u_n|^6\right)&\leq\frac{\omega}{36}\int_0^1|u_n|^6r^{2+\alpha}\\&=\frac{\omega}{36}\int_0^\delta|u_n|^6r^{2+\alpha}+\frac{\omega}{36}\int_\delta^1|u_n|^6r^{2+\alpha}\\&\leq\frac{\|u_n\|^6}{36\alpha}\omega\delta^\alpha+\frac{\omega}{36}\int_\delta^1|u_n|^6r^2\leq\frac{\varepsilon}{2},\end{aligned}\tag{3.6}$$

where ω is the surface area of the unit sphere in $\mathbb{R}^3$. Similarly, for above $\varepsilon>0$, there exist $\delta_1>0$ small enough and $n_2\in\mathbb{N}$ such that for any $n\geq n_2$, it follows from (3.3) and (3.4) that

$$\begin{aligned}&\left|\int_B\left(\frac{1}{6+|x|^\alpha}|u_n|^6-\frac{1}{6+|x|^\alpha}|u_n|^{6+|x|^\alpha}\right)\right|\\&\quad\leq\frac{\omega}{6}\int_{[0,\delta_1]\cap\{|u_n|>1\}}|u_n|^6\left||u_n|^{r^\alpha}-1\right|r^2+\frac{\omega}{6}\int_{[0,\delta_1]\cap\{|u_n|\leq1\}}|u_n|^6\left||u_n|^{r^\alpha}-1\right|r^2\\&\qquad+\frac{\omega}{6}\left|\int_{\delta_1}^1|u_n|^6(|u_n|^{r^\alpha}-1)r^2\right|\\&\quad\leq\frac{\omega}{6}\int_0^{\delta_1}|u_n|^6r^2\left|\exp[-\frac{r^\alpha}{2}\log(Cr)]-1\right|+\frac{\omega}{18}\delta_1^3+\frac{\omega}{6}\left|\int_{\delta_1}^1|u_n|^6(|u_n|^{r^\alpha}-1)r^2\right|\\&\quad\leq C\omega\int_0^{\delta_1}|u_n|^6r^2r^\alpha|\log Cr|+\frac{\omega}{18}\delta_1^3+\frac{\omega}{6}\left|\int_{\delta_1}^1|u_n|^6(|u_n|^{r^\alpha}-1)r^2\right|\\&\quad\leq C_1\omega{\delta_1}^\alpha|\log C\delta_1|+\frac{\omega}{18}\delta_1^3+\frac{\omega}{6}\left|\int_{\delta_1}^1|u_n|^6(|u_n|^{r^\alpha}-1)r^2\right|\leq\frac{\varepsilon}{2}.\end{aligned}\tag{3.7}$$

Hence, combining (3.5), (3.6) and (3.7), we have for above $\varepsilon>0$, there exists $n_0=\max\{n_1,n_2\}$, such that for any $n\geq n_0$,

$$\left|\int_B\left(\frac{1}{6}|u_n|^6-\frac{1}{6+|x|^\alpha}|u_n|^{6+|x|^\alpha}\right)\right|\leq\varepsilon,$$

which implies that (a) is true.

Secondly, we will devoted to verify that (b) is correct. In fact, by Lemma 3.4, for $0<\eta<1$

small enough and $v \in H$,

$$\begin{aligned}
&\left|\int_0^\eta |u_n|^5|v|(|u_n|^{r^\alpha}-1)r^2\right| \\
&\quad\le \left|\int_{[0,\eta]\cap\{|u_n|>1\}} |u_n|^5|v|(|u_n|^{r^\alpha}-1)r^2\right| + \left|\int_{[0,\eta]\cap\{|u_n|\le 1\}} |u_n|^5|v|(|u_n|^{r^\alpha}-1)r^2\right| \\
&\quad\le \int_0^\eta |u_n|^5|v|\left|(Cr)^{-r^\alpha/2}-1\right| r^2 + C\eta^{3/2}\|v\| \\
&\quad\le \int_0^\eta |u_n|^5|v|\left|\exp(r^\alpha/2\log(Cr)^{-1})-1\right| r^2 + C\eta^{3/2}\|v\| \\
&\quad\le C\int_0^\eta |u_n|^5|v|r^\alpha|\log(Cr)|r^2 + C\eta^{3/2}\|v\| \\
&\quad\le C\eta^\alpha|\log(C\eta)|\int_0^1 |u_n|^5|v|r^2 + C\eta^{3/2}\|v\| \\
&\quad\le C\eta^\alpha|\log(C\eta)|\|u_n\|^5\|v\| + C\eta^{3/2}\|v\|.
\end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ sufficiently small such that

$$C\eta^\alpha|\log(C\eta)|\|u_n\|^5\|v\| + C\eta^{3/2}\|v\| < \frac{\varepsilon}{3}\|v\|,$$

and then

$$\left|\int_0^\eta |u_n|^5|v|(|u_n|^{r^\alpha}-1)r^2\right| < \frac{\varepsilon}{3}\|v\|. \tag{3.8}$$

On the other hand, it follows that for above $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for $n > n_1$,

$$\int_\eta^1 |u_n|^{5+r^\alpha}|v|r^2 \le C\left(\int_\eta^1 |u_n|^{6+r^\alpha}r^2\right)^{5/7}\|v\| \le \frac{\varepsilon}{3}\|v\|. \tag{3.9}$$

Similarly, we have for above $\varepsilon > 0$, there exists $n_2 \in \mathbb{N}$ such that for $n > n_2$,

$$\int_\eta^1 |u_n|^5|v|r^2 \le C\left(\int_\eta^1 |u_n|^6 r^2\right)^{5/6}\|v\| \le \frac{\varepsilon}{3}\|v\|. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we obtain for $\varepsilon > 0$, there exists $n_0 = \max\{n_1, n_2\}$ such that for $n > n_0$,

$$\begin{aligned}
&\left|\int_0^1 |u_n|^{4+r^\alpha}u_n v r^2 - \int_0^1 |u_n|^4 u_n v r^2\right| \\
&\quad\le \int_0^1 |u_n|^5|v|\left||u_n|^{r^\alpha}-1\right| r^2 \\
&\quad\le \int_0^\eta |u_n|^5|v|\left||u_n|^{r^\alpha}-1\right| r^2 + \int_\eta^1 |u_n|^5|v|r^2 + \int_\eta^1 |u_n|^5|v||u_n|^{r^\alpha}r^2 \le \varepsilon\|v\|, \qquad v \in H,
\end{aligned}$$

which ensures that (b) is valid. Thereby, it is obvious that $\{u_n\}$ is also a $(PS)_c$ sequence for the functional I. Recall that I satisfies $(PS)_c$ condition, we have that $u_n \to u = 0$ strongly in H, which is a contradiction to $I(u_n) \to c > 0$. The proof is completed. □

4 The least energy critical point

In this section, we will use the Nehari manifold method to show the existence of nontrivial nonnegative ground state of the functional J. In order to obtain the ground state, we need the Nehari manifold associated with J given by

$$\mathcal{N} = \{u \in H \setminus \{0\} : \langle J'(u), u\rangle = 0\}.$$

Lemma 4.1. *Assume that $\lambda < 0$, $q > 2$ or $\lambda > 0$, $2 < q < 6$ and the assumptions (i)–(ii) hold. Then, for each $u \in H \setminus \{0\}$, there exists a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}$. Moreover, $J(t(u)u) = \max_{t\geq 0} J(tu)$.*

Proof. (a) Let $u \in H \setminus \{0\}$ be fixed. For convenience, we define the function $h(t) = J(tu)$ for $t > 0$. Note that $h'(t) = \langle J'(tu), u\rangle = 0$ if and only if $tu \in \mathcal{N}$. By simple calculation, we see that when $\lambda < 0$, $q > 2$

$$\begin{aligned} h'(t) &= t\|u\|^2 + \lambda q t^{q-1} R(u) - \int_B t^{5+|x|^\alpha} |u|^{6+|x|^\alpha} \\ &= t\left(\|u\|^2 + \lambda t^{q-2} R(u) - \int_B t^{4+|x|^\alpha} |u|^{6+|x|^\alpha}\right) \\ &= t\xi(t). \end{aligned}$$

It is obvious that ξ is a non-increasing function for $t > 0$ and $\lim_{t\to 0^+} \xi(t) = \|u\|^2 > 0$, $\lim_{t\to\infty} \xi(t) = -\infty$. Hence, there exists a unique $t(u) > 0$ such that $h'(t(u)) = 0$ and $t(u)u \in \mathcal{N}$. Moreover, $J(t(u)u) = \max_{t\geq 0} J(tu)$.

(b) By simple calculation, we see that for $\lambda > 0$, $2 < q < 6$,

$$\begin{aligned} h'(t) &= t\|u\|^2 + \lambda q t^{q-1} R(u) - \int_B t^{5+|x|^\alpha} |u|^{6+|x|^\alpha} \\ &= t^{q-1}\left(\frac{1}{t^{q-2}}\|u\|^2 + \lambda q R(u) - \int_B t^{6-q+|x|^\alpha} |u|^{6+|x|^\alpha}\right) \\ &= t^{q-1}\xi(t). \end{aligned}$$

It is easy to see that ξ is a non-increasing for $t > 0$ and $\lim_{t\to 0^+} \xi(t) = \infty$, $\lim_{t\to\infty} \xi(t) = -\infty$. Hence, there exists a unique $t(u) > 0$ such that $h'(t(u)) = 0$ and $t(u)u \in \mathcal{N}$. In addition, $J(t(u)u) = \max_{t\geq 0} J(tu)$. The proof is completed. □

Lemma 4.2. *Assume that $\lambda < 0$, $q > 6$ or $\lambda > 0$, $2 < q < 6$ and the assumptions (i)–(iii) hold. Then J is bounded from below on $\mathcal{N}$.*

Proof. For $u \in \mathcal{N}$, it follows from (i) and (ii) that

$$\begin{aligned} \|u\|^2 &= -\lambda q R(u) + \int_B |u|^{6+|x|^\alpha} \\ &\leq C(\|u\|^6 + \|u\|^7 + \|u\|^q), \end{aligned}$$

which implies that there exists a positive constant C such that $\|u\| \geq C$. On the other hand, we have

$$\begin{aligned} J(u) &= J(u) - \frac{1}{6}\langle J'(u), u\rangle \\ &= \frac{1}{3}\|u\|^2 + \lambda\left(\frac{1}{q} - \frac{1}{6}\right)\langle R'(u), u\rangle + \int_B \left(\frac{1}{6} - \frac{1}{6+|x|^\alpha}\right)|u|^{6+|x|^\alpha} \\ &\geq \frac{1}{3}\|u\|^2, \ u \in \mathcal{N}. \end{aligned}$$

Hence, J is bounded below. The proof is completed. □

By Lemmas 4.1 and 4.2, we can define

$$c^* = \inf_{u \in \mathcal{N}} J(u), \qquad c^{**} = \inf_{u \in H \setminus \{0\}} \max_{t \geq 0} J(tu).$$

Lemma 4.3. *Assume that $\lambda < 0$, $q > 6$ or $\lambda > 0$, $2 < q < 6$ and the assumptions (i)–(iii) hold. Then $c = c^* = c^{**}$.*

Proof. It follows from Lemma 4.1 that $c^* = c^{**}$. In the following, we will show that $c = c^*$. Indeed, let $u \in \mathcal{N}$, by Lemmas 3.1 and 3.2 there exists some $t_0 > 1$ such that $J(t_0 u) < 0$. Thus, $J(u) = \max_{t>0} J(tu) \geq \max_{t \in [0,1]} J(tt_0 u) \geq c$, which leads to $c^* \geq c$.

On the other hand, we find for $u \in H$ that

$$\begin{aligned} J(u) - \frac{1}{6}\langle J'(u), u\rangle &= \frac{1}{3}\|u\|^2 + \lambda\left(\frac{1}{q} - \frac{1}{6}\right)\langle R'(u), u\rangle + \int_B \left(\frac{1}{6} - \frac{1}{6+|x|^\alpha}\right)|u|^{6+|x|^\alpha} \\ &\geq \frac{1}{3}\|u\|^2 \geq 0. \end{aligned} \tag{4.1}$$

Let $\gamma \in \Gamma$, then it follows from (4.1) that $\langle J'(\gamma(1)), \gamma(1)\rangle \leq 6J(\gamma(1)) < 0$. Let us define $t_1 = \inf\{t \in [0,1) : \langle J'(\gamma(s)), \gamma(s)\rangle < 0,\ s \in (t,1]\}$. Then $\langle J'(\gamma(t_1)), \gamma(t_1)\rangle = 0$ and $\gamma(s) \neq 0$ for all $s \in (t_1, 1]$. We now show that $\gamma(t_1) \neq 0$. Otherwise, $\gamma(t_1) = 0$ then Lemma 3.1 implies that $\langle J'(\gamma(s)), \gamma(s)\rangle > 0$ as $s \to t_1^+$, thus there exists $\delta > 0$ such that $t_1 + \delta < 1$ and $\langle J'(\gamma(t_1 + \delta)), \gamma(t_1 + \delta)\rangle > 0$. Note that the definition of t_1, there holds $\langle J'(\gamma(t_1 + \delta)), \gamma(t_1 + \delta)\rangle < 0$. This comes to a contradiction. Thus, we conclude that $\gamma(t_1) \in \mathcal{N}$ and $c \geq c^*$. The proof is completed. □

The following lemma can be also obtained by Implicit Function Theorem or by the Lusternik Theorem. We give the other proof by applying the Lagrange multiplier method.

Lemma 4.4. *Assume that $\lambda < 0$, $q > 6$ or $\lambda > 0$, $2 < q < 6$ and the assumptions (i)–(iii) hold. If c^* is attained at some $u \in \mathcal{N}$, then u is a critical point of J in H.*

Proof. Let $G(u) = \langle J'(u), u\rangle$, then $G \in C^1(H, \mathbb{R})$. By Lemma 4.1, $\mathcal{N} \neq \emptyset$. We claim that $0 \notin \partial\mathcal{N}$. In fact,

$$\begin{aligned} G(u) &= \|u\|^2 + \lambda R'(u)u - \int_B |u|^{6+|x|^\alpha} \\ &\geq \frac{1}{2}\|u\|^2 - C(\|u\|^6 + \|u\|^7) > 0 \end{aligned}$$

for any $u \in H$ with $\|u\|$ small. Note that for any $u \in \mathcal{N}$

$$\begin{aligned} \langle G'(u), u\rangle &= \langle G'(u), u\rangle - 6G(u) \\ &= -4\|u\|^2 + \lambda q(q-6)R(u) - \int_B |x|^\alpha |u|^{6+|x|^\alpha} < 0. \end{aligned} \tag{4.2}$$

Hence, $G'(u) \neq 0$ for any $u \in \mathcal{N}$. Then the implicit function theorem implies that $\mathcal{N}$ is a C^1 manifold. Recall that u is minimizer of J on $u \in \mathcal{N}$. Then by the Lagrange multiplier method, there exists $\lambda \in \mathbb{R}$ such that

$$J'(u) = \lambda G'(u). \tag{4.3}$$

Combining (4.2) and (4.3), we can find $J'(u) = 0$. The proof is completed. □

Proof of Theorem 1.2. Recall that Theorem 1.1 shows that $u \in \mathcal{N}$ and hence $J(u) \geq c^*$. Then by applying Lemma 4.3, Fatou's lemma and weak semicontinuity of the norm, we derive

$$\begin{aligned} c^* &= \liminf_{n\to\infty}[J(u_n) - \frac{1}{6}\langle J'(u_n), u_n\rangle] \\ &= \liminf_{n\to\infty}\left[\frac{1}{3}\|u_n\|^2 + \lambda q\left(\frac{1}{q}-\frac{1}{6}\right)R(u_n) + \int_B\left(\frac{1}{6}-\frac{1}{6+|x|^\alpha}\right)|u_n|^{6+|x|^\alpha}\right] \\ &\geq \frac{1}{3}\|u\|^2 + \lambda q\left(\frac{1}{q}-\frac{1}{6}\right)R(u) + \int_B\left(\frac{1}{6}-\frac{1}{6+|x|^\alpha}\right)|u|^{6+|x|^\alpha} \\ &= J(u) - \frac{1}{6}\langle J'(u), u\rangle = J(u). \end{aligned}$$

This shows that $J(u) = c^*$. It is easy to see that $J(|u|) = J(u) = c^*$. Thus, Lemma 4.4 implies that $|u|$ is a ground state of J. The proof is completed. □

5 The Schrödinger–Poisson type system

This section is devoted to apply the Theorems 1.1 and 1.2 to a class of Schrödinger–Poisson type system. We first estimate the critical level of mountain pass of the functional $\tilde{J}$ associated to (1.3) and then show that the critical level of mountain pass is below the non-compactness level of $\tilde{J}$. Secondly, we are devoted to verify that the (PS) sequence of the functional $\tilde{J}$ is also the one of the approximation functional associated to $\tilde{J}$ by using approximation techniques. Finally, by using the regularity theory, the positive ground state solution of (1.3) is obtained. We establish the following lemmas, which guarantee that the conditions in the Theorems 1.1 and 1.2 are valid.

We observe that by [3], for given $u \in H$, there exists a unique solution $\phi = \phi_u \in H$ satisfying $-\Delta\phi_u = |u|^5$ in B, $u = 0$ on ∂B in a weak sense and it has the following properties.

Lemma 5.1 ([5])**.** *For every fixed $u \in H$, we have*

(i) $\phi_u \geq 0$ *a.e. in* B;

(ii) $\phi_{tu} = t^5\phi_u$ *for all* $t > 0$;

(iii) $\|\phi_u\| \leq S^{-3}\|u\|^5$ *and*

$$\int_B \phi_u|u|^5 \leq S^{-6}\|u\|^{10}, \tag{5.1}$$

where S is defined in (2.1);

(iv) *if $u_n \rightharpoonup u$ in H, then, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in H.*

Moreover, (1.3) is variational and its solutions are the critical points of the functional defined in H by

$$\tilde{J}(u) = \frac{1}{2}\int_B|\nabla u|^2 - \frac{1}{10}\int_B\phi_u|u|^5 - \int_B\frac{1}{6+|x|^\alpha}|u|^{6+|x|^\alpha}.$$

It is easy to check by Lemmas 2.2 and 5.1 that $\tilde{J}$ is well defined on H and $\tilde{J} \in C^1(H,\mathbb{R})$, and

$$\langle\tilde{J}'(u), v\rangle = \int_B\nabla u\nabla v - \int_B\phi_u|u|^3uv - \int_B|u|^{4+|x|^\alpha}uv, \qquad u, v \in H.$$

Lemma 5.2. *Let $\alpha_1, \beta_1, \gamma_1 > 0$ and define $f_1 : [0, \infty) \to \mathbb{R}$ as*

$$f_1(t) = \frac{\alpha_1}{2} t^2 - \frac{\beta_1}{10} t^{10} - \frac{\gamma_1}{6} t^6.$$

Then

$$\sup_{t \in [0,\infty)} f_1(t) = \left(\frac{\sqrt{\gamma_1^2 + 4\alpha_1\beta_1} - \gamma_1}{2\beta_1} \right)^{1/2} \frac{12\alpha_1\beta_1 + \gamma_1^2 - \gamma_1\sqrt{\gamma_1^2 + 4\alpha_1\beta_1}}{30\beta_1}.$$

Proof. For $t \geq 0$, we have

$$f_1'(t) = \alpha_1 t - \beta_1 t^9 - \gamma_1 t^5 = t(\alpha_1 - \beta_1 t^8 - \gamma_1 t^4).$$

Set $h(t) = \alpha_1 - \beta_1 t^8 - \gamma_1 t^4 = 0$, we write at

$$t^4 = \frac{\sqrt{\gamma_1^2 + 4\alpha_1\beta_1} - \gamma_1}{2\beta_1}.$$

Substituting it into $f_1(t)$, the result is obtained. The proof is completed. □

Lemma 5.3. *Let*

$$g_1(t) = \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{10}}{10}\int_B \phi_{u_\varepsilon}|u_\varepsilon|^5 - \frac{t^6}{6}\int_B |u_\varepsilon|^6,$$

then we have, as $\varepsilon \to 0^+$,

$$\sup_{t \geq 0} g_1(t) \leq \frac{13 - \sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{1/2} S^{3/2} + O(\varepsilon) =: \Lambda + O(\varepsilon).$$

Proof. Since $-\Delta\phi_{u_\varepsilon} = |u_\varepsilon|^5$, we have

$$\begin{aligned}\int_B |u_\varepsilon|^6 &= \int_B \nabla\phi_{u_\varepsilon}\nabla|u_\varepsilon| \\ &\leq \frac{1}{2}\int_B |\nabla|u_\varepsilon||^2 + \frac{1}{2}\int_B |\nabla\phi_{u_\varepsilon}|^2 \\ &= \frac{1}{2}\int_B \phi_{u_\varepsilon}|u_\varepsilon|^5 + \frac{1}{2}\int_B |\nabla u_\varepsilon|^2.\end{aligned}$$

Then thanks to (2.2) we derive that, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}\int_B \phi_{u_\varepsilon}|u_\varepsilon|^5 &\geq 2\int_B |u_\varepsilon|^6 - \int_B |\nabla u_\varepsilon|^2 \\ &= S^{\frac{3}{2}} + O(\varepsilon).\end{aligned}$$

This together with Lemma 5.2 and the estimate (2.2) implies that

$$\begin{aligned}g_1(t) &= \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{10}}{10}\int_B \phi_{u_\varepsilon}|u_\varepsilon|^5 - \frac{t^6}{6}\int_B |u_\varepsilon|^6 \\ &\leq \frac{t^2}{2}(S^{3/2} + O(\varepsilon)) - \frac{t^{10}}{10}(S^{3/2} + O(\varepsilon)) - \frac{t^6}{6}(S^{3/2} + O(\varepsilon)) \\ &\leq \frac{13 - \sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{1/2} S^{3/2} + O(\varepsilon),\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. The proof is completed. □

From Lemma 3.1, we know that the functional $\tilde{J}$ possesses the mountain pass geometry. Then there is a $(PS)_{c_1}$ sequence $\{u_n\} \subset H$ for $\tilde{J}$ with the property that

$$\tilde{J}(u_n) \to c_1, \qquad \|\tilde{J}'(u_n)\|_{H^{-1}} \to 0, \qquad n \to \infty,$$

where c_1 is given by

$$c_1 = \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{J}(\gamma(t)), \tag{5.2}$$

and $\tilde{\Gamma} = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \tilde{J}(\gamma(1)) < 0\}$.

In the following we give an estimate of the upper bound of the critical level c_1 by using above two lemmas.

Lemma 5.4. *Let c_1 be defined by (5.2), then $0 < c_1 < \Lambda$.*

Proof. It follows from (3.1) that, for ε small enough,

$$\begin{aligned}
\tilde{J}(tu_\varepsilon) &= \frac{t^2}{2} \int_B |\nabla u_\varepsilon|^2 - \frac{t^{10}}{10} \int_B \phi_{u_\varepsilon} |u_\varepsilon|^5 - \int_B \frac{t^{6+|x|^\alpha}}{6+|x|^\alpha} |u_\varepsilon|^{6+|x|^\alpha} \\
&\le \frac{t^2}{2} \|u_\varepsilon\|^2 - \frac{t^6}{7} \int_B |u_\varepsilon|^{6+|x|^\alpha} \\
&\le S^{3/2} t^2 - \frac{S^{3/2}}{14} t^6 := \varphi(t).
\end{aligned}$$

Thus, there exists $R_1 > 0$ sufficiently large which is independent of ε, such that $\varphi(R_1) = 0$ and $\tilde{J}(R_1 u_\varepsilon) \le 0$ for ε small enough. Hence, we can find $0 < t_\varepsilon < R_1$ satisfying

$$0 < \eta_1 \le c_1 \le \max_{t \in [0,R_1]} \tilde{J}(tu_\varepsilon) = \tilde{J}(t_\varepsilon u_\varepsilon).$$

Since $\frac{d}{dt} \tilde{J}(tu_\varepsilon)|_{t=t_\varepsilon} = 0$, we have

$$t_\varepsilon \|u_\varepsilon\|^2 = t_\varepsilon^9 \int_B \phi_{u_\varepsilon} |u_\varepsilon|^5 + \int_B t_\varepsilon^{5+|x|^\alpha} |u|^{6+|x|^\alpha}.$$

Hence we deduce from (2.2) that

$$\begin{aligned}
S^{\frac{3}{2}} + O(\varepsilon) &= t_\varepsilon^8 \int_B \phi_{u_\varepsilon} |u_\varepsilon|^5 + t_\varepsilon^4 \int_B |u_\varepsilon|^6 + t_\varepsilon^4 \int_B \left(t_\varepsilon^{|x|^\alpha} |u_\varepsilon|^{6+|x|^\alpha} - |u_\varepsilon|^6 \right) \\
&= t_\varepsilon^8 \int_B \phi_{u_\varepsilon} |u_\varepsilon|^5 + t_\varepsilon^4 [S^{\frac{3}{2}} + O(\varepsilon^3) + A_\varepsilon] \\
&= t_\varepsilon^8 \int_B \phi_{u_\varepsilon} |u_\varepsilon|^5 + t_\varepsilon^4 [S^{\frac{3}{2}} + O(\varepsilon^3) + O(\varepsilon^\alpha |\log \varepsilon|) + O(\varepsilon^{3/2})],
\end{aligned} \tag{5.3}$$

where $A_\varepsilon = O(\varepsilon^\alpha |\log \varepsilon|) + O(\varepsilon^{3/2})$ is given in [21]. For convenience, we set $A = S^{\frac{3}{2}} + O(\varepsilon)$, $B = \int_B \phi_{u_\varepsilon} |u_\varepsilon|^5$ and $C = S^{\frac{3}{2}} + O(\varepsilon^3) + O(\varepsilon^\alpha |\log \varepsilon|) + O(\varepsilon^{3/2})$. Thus, (5.3) can be rewritten as $A = Bt_\varepsilon^8 + Ct_\varepsilon^4$. It is easy to see that for ε small,

$$t_\varepsilon^4 = \frac{\sqrt{C^2 + 4AB} - C}{2B} = \frac{\sqrt{5S^3 + O(\varepsilon) + O(\varepsilon^\alpha |\log \varepsilon|)} - S^{\frac{3}{2}} - O(\varepsilon^\alpha |\log \varepsilon|) - O(\varepsilon^{3/2})}{2S^{3/2} + O(\varepsilon)}.$$

Thereby, for ε small enough, there holds

$$(\sqrt{5} - 1)/4 < t_\varepsilon^2 < 4/5. \tag{5.4}$$

In what follows, we will estimate the term

$$\begin{aligned}
&\int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&= \int_B \left(\frac{t_\varepsilon^6}{6} - \frac{t_\varepsilon^6}{6+|x|^\alpha}\right)|u_\varepsilon|^6 + \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}(|u_\varepsilon|^6 - |u_\varepsilon|^{6+|x|^\alpha}) \\
&\quad + \int_B \left(\frac{t_\varepsilon^6}{6+|x|^\alpha} - \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}\right)|u_\varepsilon|^6 \\
&= I + II + III.
\end{aligned} \tag{5.5}$$

By [21, page 16] and (5.4), we can find

$$I = \int_B \left(\frac{t_\varepsilon^6}{6} - \frac{t_\varepsilon^6}{6+|x|^\alpha}\right)|u_\varepsilon|^6 \le C\varepsilon^\alpha \tag{5.6}$$

and

$$II = \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}(|u_\varepsilon|^6 - |u_\varepsilon|^{6+|x|^\alpha}) \le -C\varepsilon^\alpha|\log\varepsilon|. \tag{5.7}$$

It follows from (5.4) again that

$$\begin{aligned}
III &= \int_B \left(\frac{t_\varepsilon^6}{6+|x|^\alpha} - \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}\right)|u_\varepsilon|^6 \le C\int_B \left(1 - t_\varepsilon^{|x|^\alpha}\right)|u_\varepsilon|^6 \\
&= C\int_B (1 - \exp(|x|^\alpha \log t_\varepsilon))|u_\varepsilon|^6 \le C\int_B |x|^\alpha|u_\varepsilon|^6 \\
&\le C\omega\int_0^\varepsilon r^\alpha\varepsilon^{-3}r^2 + C\omega\int_\varepsilon^1 r^\alpha\varepsilon^3 r^{-4} \\
&\le C\varepsilon^\alpha + C(\varepsilon^\alpha - \varepsilon^3) \le C\varepsilon^\alpha.
\end{aligned} \tag{5.8}$$

Combining (5.5)-(5.8) and using Lemma 5.3, we derive

$$\begin{aligned}
\tilde{J}(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 - \frac{t_\varepsilon^{10}}{10}\int_B \phi_{u_\varepsilon}|u_\varepsilon|^5 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&= \frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 - \frac{t_\varepsilon^{10}}{10}\int_B \phi_{u_\varepsilon}|u_\varepsilon|^5 - \int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 + \int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&\le \sup_{t\ge 0} g_1(t) + \int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&\le \frac{13-\sqrt{5}}{30}\left(\frac{\sqrt{5}-1}{2}\right)^{1/2} S^{3/2} + O(\varepsilon) + C\varepsilon^\alpha - C\varepsilon^\alpha|\log\varepsilon|.
\end{aligned} \tag{5.9}$$

By choosing $\varepsilon > 0$ small enough, we derive by (5.9),

$$0 < \eta_1 \le c_1 \le \tilde{J}(t_\varepsilon u_\varepsilon) < \Lambda.$$

The proof is finished. □

Lemma 5.5. *If $\{u_n\}$ is a $(PS)_{c_1}$ sequence of $\tilde{J}$, then there exists $u \in H$ such that, up to a subsequence, $u_n \rightharpoonup u$ and $\tilde{J}'(u) = 0$.*

Proof. From Lemma 3.3 we see that $\{u_n\}$ is bounded in H. Then, up to a subsequence, we can assume that $\{u_n\}$ converges to u weakly in H and $u_n \to u$ a.e. in B. By taking $\varphi \in C_0^\infty(B)$, we find

$$\langle \tilde{J}'(u_n), \varphi\rangle = \int_B \nabla u_n \nabla \varphi - \int_B \phi_{u_n}|u_n|^3 u_n\varphi - \int_B |u_n|^{4+|x|^\alpha} u_n\varphi.$$

It follows from Lemma 5.1 that $\phi_{u_n} \rightharpoonup \phi_u$ in H, which implies $\phi_{u_n} \rightharpoonup \phi_u$ in $L^6(B)$. Then

$$\int_B (\phi_{u_n} - \phi_u)|u|^3 u\varphi \to 0, \qquad n \to \infty. \tag{5.10}$$

Since $u_n \to u$ a.e. in B and

$$\int_B |\phi_{u_n}(|u_n|^3 u_n - |u|^3 u)|^{\frac{6}{5}} \le C(|\phi_{u_n}|_6^{\frac{6}{5}}|u_n|_6^{\frac{24}{5}} + |\phi_{u_n}|_6^{\frac{6}{5}}|u|_6^{\frac{24}{5}}) \le C,$$

we have $\phi_{u_n}(|u_n|^3 u_n - |u|^3 u) \rightharpoonup 0$ in $L^{\frac{6}{5}}(B)$ and thus

$$\int_B \phi_{u_n}(|u_n|^3 u_n - |u|^3 u)\varphi \to 0, \qquad n \to \infty,$$

which together with (5.10) ensures that

$$\int_B \phi_{u_n}|u_n|^3 u_n\varphi \to \int_B \phi_u|u|^3 u\varphi, \qquad n \to \infty. \tag{5.11}$$

For any measurable subset $Q \subset B$, we have

$$\begin{aligned}\left|\int_Q (|u_n|^{4+|x|^\alpha} u_n - |u|^{4+|x|^\alpha} u)\varphi\right| &\le \int_Q (|u_n|^{5+|x|^\alpha} + |u|^{5+|x|^\alpha})|\varphi| \\ &\le \| |u_n|^{5+|x|^\alpha} + |u|^{5+|x|^\alpha} \|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(Q)} \|\varphi\|_{L^{p(\cdot)}(Q)},\end{aligned}$$

where $p(x) = 6 + |x|^\alpha$. Hence, Vitali's theorem (see [28]) implies

$$\int_B |u_n|^{4+|x|^\alpha} u_n\varphi \to \int_B |u|^{4+|x|^\alpha} u\varphi, \quad \text{as } n \to \infty. \tag{5.12}$$

Combining (5.10), (5.11) and (5.12), there holds

$$\langle \tilde{J}'(u), \varphi\rangle = \lim_{n\to\infty} \langle \tilde{J}'(u_n), \varphi\rangle = 0.$$

Therefore, by density, we derive that $\tilde{J}'(u) = 0$. The proof is completed. □

In order to obtain the nontrivial solution of (1.3), we need define the approximation functional $\tilde{I} : H \to \mathbb{R}$ associated to $\tilde{J}$ given by

$$\tilde{I}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{10}\int_B \phi_u|u|^5 - \frac{1}{6}\int_B |u|^6.$$

Lemma 5.6. *The functional $\tilde{I}$ satisfies the $(PS)_{c_1}$ condition with $c_1 \in (0, \Lambda)$.*

Proof. Suppose that $\{u_n\}$ is a $(PS)_{c_1}$ sequence of $\tilde{I}$ for $c_1 \in (0, \Lambda)$, i.e.

$$\tilde{I}(u_n) \to c_1, \qquad \tilde{I}'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Similarly to Lemma 3.3, it is easy to see that $\{u_n\}$ is bounded in H. Going if necessary to a subsequence, we can find $u \in H$ such that $u_n \rightharpoonup u$ in H. By the same argument used in Lemma 5.5, we deduce that $\tilde{I}'(u) = 0$, hence

$$\begin{aligned} \tilde{I}(u) &= \tilde{I}(u) - \frac{1}{6}\langle \tilde{I}'(u), u\rangle \\ &= \frac{1}{3}\|u\|^2 + \frac{1}{15}\int_B \phi_u |u|^5 \geq 0. \end{aligned} \tag{5.13}$$

Now, let $v_n = u_n - u$, it is obvious to see that

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1).$$

From Brézis–Lieb Lemma in [9,19], we have

$$\int_B |u_n|^6 dx = \int_B |v_n|^6 dx + \int_B |u|^6 dx + o(1)$$

and

$$\int_B \phi_{u_n} |u_n|^5 = \int_B \phi_{v_n} |v_n|^5 + \int_B \phi_u |u|^5 + o(1).$$

These three equalities imply that

$$\begin{aligned} c_1 - \tilde{I}(u) &= \tilde{I}(u_n) - \tilde{I}(u) + o(1) \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 - \frac{1}{10}\int_B \phi_{u_n}|u_n|^5 + \frac{1}{10}\int_B \phi_u |u|^5 \\ &\quad - \frac{1}{6}\int_B |u_n|^6 + \frac{1}{6}\int_B |u|^6 + o(1) \\ &= \frac{1}{2}\|v_n\|^2 - \frac{1}{10}\int_B \phi_{v_n}|v_n|^5 - \frac{1}{6}\int_B |v_n|^6 + o(1), \end{aligned} \tag{5.14}$$

and similarly

$$\begin{aligned} o(1) &= \langle \tilde{I}'(u_n), u_n\rangle - \langle \tilde{I}'(u), u\rangle \\ &= \|u_n\|^2 - \|u\|^2 - \int_B \phi_{u_n}|u_n|^5 + \int_B \phi_u|u|^5 - \int_B |u_n|^6 + \int_B |u|^6 \\ &= \|v_n\|^2 - \int_B \phi_{v_n}|v_n|^5 - \int_B |v_n|^6 + o(1). \end{aligned} \tag{5.15}$$

We will show that $\|v_n\| \to 0$. Otherwise, there exists a subsequence still denoted by $\{v_n\}$ such that $\|v_n\|^2 \to l > 0$. For convenience, let $a_n = \int_B \phi_{v_n}|v_n|^5$ and $b_n = \int_B |v_n|^6$. Without loss of generality, we may assume $a_n \to a_1$ and $b_n \to b_1$, as $n \to \infty$. Notice that

$$\begin{aligned} \int_B |v_n|^6 &= \int_B \nabla\phi_{v_n} \nabla |v_n| \\ &\leq \frac{\varepsilon^2}{2}\int_B |\nabla|v_n||^2 + \frac{1}{2\varepsilon^2}\int_B |\nabla\phi_{v_n}|^2 \\ &= \frac{1}{2\varepsilon^2}\int_B \phi_{v_n}|v_n|^5 + \frac{\varepsilon^2}{2}\int_B |\nabla v_n|^2, \end{aligned}$$

then as $n \to \infty$ passing to the limit, we conclude that

$$b_1 \leq \frac{1}{2\varepsilon^2} a_1 + \frac{\varepsilon^2}{2} l.$$

Taking $\varepsilon^2 = \frac{\sqrt{5}-1}{2}$, and combining with (5.15) leads to

$$a_1 \geq \frac{3-\sqrt{5}}{2} l,$$

from which we get by (5.13), (5.14) and (5.15) that

$$c_1 \geq c_1 - \tilde{I}(u) = \frac{2}{5} a_1 + \frac{1}{3} b_1 + o(1) = \frac{1}{3} l + \frac{1}{15} a_1 + o(1) \geq \frac{13-\sqrt{5}}{30} l + o(1). \tag{5.16}$$

On the other hand, (5.1) and (5.15) yield

$$l \leq S^{-6} l^5 + S^{-3} l^3.$$

Therefore we get $l^2 \geq \frac{-1+\sqrt{5}}{2} S^3$. This together with (5.16) implies that $c_1 \geq \Lambda$, which will come to a contradiction. Therefore $v_n \to 0$ strongly in H, or equivalently, $u_n \to u$ in H as $n \to \infty$. The proof is completed. □

Lemma 5.7 ([30]). *Let Ω be a domain in $\mathbb{R}^3$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratéodory function such that for almost every $x \in \Omega$, there holds*

$$|g(x, u)| \leq a(x)(1 + |u|).$$

If $0 \leq a \in L^{\frac{3}{2}}(\Omega)$ and $u \in H_0^1(\Omega)$ is a weak solution of equation $-\Delta u = g(\cdot, u)$ in Ω. Then, $u \in L^p(\Omega)$ for all $p < \infty$.

Proof of Theorem 1.7. The Lemmas 5.4, 5.5 and Theorem 1.2 imply that (1.3) admits a nonnegative nontrivial ground state solution $u \in H$, which satisfies the following equation in weak sense

$$-\Delta u = \phi_u |u|^3 u + u^{5+|x|^\alpha} \quad \text{in } B.$$

Let us define

$$\tilde{g}(u(x)) = \phi_u |u|^3 u + u^{5+|x|^\alpha}, \qquad x \in B.$$

Then thanks to Lemma 2.2, we have $\int_B u^{6+\frac{3}{2}|x|^\alpha} \leq C$. The fact $\phi_u \in D^{1,2}(B)$ that implies $\phi_u \in L^6(B)$. On the other hand, it is easy to see that $|\phi_u|^{\frac{3}{2}} \in L^4(B)$ and $|u|^{\frac{9}{2}} \in L^{\frac{4}{3}}(B)$. Thus we derive from the Hölder inequality that $\phi_u |u|^3 \in L^{\frac{3}{2}}(B)$, which implies

$$a = \frac{\tilde{g}(u)}{1+|u|} \in L^{\frac{3}{2}}(B).$$

Thereby, we deduce immediately from Lemma 5.7 that $u \in L^q(B)$ for any $1 < q < \infty$. Hence, there holds $\tilde{g}(u) \in L^q(B)$ for any $1 < q < \infty$. Now, arguing by the Calderón–Zygmund inequality and L^p estimate given in [16,30], we derive $u \in W^{2,q}(B)$, whence also $u \in C^{1,\alpha_1}(B)$ by Sobolev embedding theorem for any $0 < \alpha_1 < 1$. Moreover, the Harnack inequality [32] implies $u(x) > 0$ for all $x \in B$. The proof is completed. □

6 The Kirchhoff type equation

In this section, we obtain the existence of positive ground state solution of (1.6) by using Theorem 1.2 with $\lambda = 1$, $q = 4$. Similarly to Section 4, we first estimate the level of mountain critical of the functional $\hat{J}$ corresponding to (1.6) and show that the critical level is below the non-compactness level of $\hat{J}$ by using approximation techniques. Then we are devoted to verify that the (PS) sequence of the functional $\hat{J}$ is also the one of the approximation functional associated to $\hat{J}$. Finally, by the regularity theory of the elliptic equation, the positive ground state solution of (1.6) is obtained. In order to find the weak solutions to (1.6) and it is natural to consider the energy functional on H:

$$\hat{J}(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_B \frac{1}{6+|x|^\alpha}|u|^{6+|x|^\alpha}.$$

Then we have from Lemma 2.2 that $\hat{J}$ is well defined on H and is of C^1, and

$$(\hat{J}'(u), v) = (1 + b\|u\|^2)\int_B \nabla u \nabla v - \int_B |u|^{4+|x|^\alpha} uv, \qquad u, v \in H.$$

It is standard to verify that the weak solutions of (1.6) correspond to the critical points of the functional $\hat{J}$.

Lemma 6.1. *Let $\alpha_2, \beta_2, \gamma_2 > 0$ and define $f_2 : [0,\infty) \to \mathbb{R}$ as*

$$f_2(t) = \frac{\alpha_2}{2}t^2 + \frac{\beta_2}{4}t^4 - \frac{\gamma_2}{6}t^6.$$

Then

$$\sup_{t\in[0,\infty)} f_2(t) = \frac{6\alpha_2\beta_2\gamma_2 + \beta_2^3 + 4\alpha_2\gamma_2\sqrt{\beta_2^2 + 4\alpha_2\gamma_2} + \beta_2^2\sqrt{\beta_2^2 + 4\alpha_2\gamma_2}}{24\gamma_2^2}.$$

Proof. For $t \geq 0$, we have

$$f_2'(t) = \alpha_2 t + \beta_2 t^3 - \gamma_2 t^5 = t(\alpha_2 + \beta_2 t^2 - \gamma_2 t^4).$$

Let $\alpha_2 + \beta_2 t^2 - \gamma_2 t^4 = 0$, we write at

$$t^2 = \frac{\sqrt{\beta_2^2 + 4\alpha_2\gamma_2} + \beta_2}{2\gamma_2}.$$

Substituting it into $f_2(t)$, the result is valid. The proof is completed. □

Lemma 6.2. *Let*

$$g_2(t) = \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{t^6}{6}\int_{\mathbb{R}^3} |u_\varepsilon|^6,$$

then we have, as $\varepsilon \to 0^+$,

$$\sup_{t\geq 0} g_2(t) \leq \Lambda_1 + O(\varepsilon),$$

where $\Lambda_1 = \frac{b}{4}S^3 + \frac{b^3}{24}S^6 + \frac{1}{6}S\sqrt{S^4b^2 + 4S} + \frac{b^2}{24}S^4\sqrt{S^4b^2 + 4S}$.

Proof. It follows from Lemma 6.1 and the estimate (2.2) that

$$\begin{aligned} g_2(t) &= \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{t^6}{6}\int_{\mathbb{R}^3}|u_\varepsilon|^6 \\ &= \frac{t^2}{2}(S^{3/2}+O(\varepsilon)) + \frac{bt^4}{4}(S^3+O(\varepsilon)) - \frac{t^6}{6}(S^{3/2}+O(\varepsilon^3)) \\ &\le \frac{b}{4}S^3 + \frac{b^3}{24}S^6 + \frac{1}{6}S\sqrt{S^4b^2+4S} + \frac{b^2}{24}S^4\sqrt{S^4b^2+4S} + O(\varepsilon), \end{aligned}$$

for $\varepsilon > 0$ sufficiently small. The proof is completed. □

From Lemma 3.2, we know that the functional $\hat{J}$ possesses the mountain pass geometry. Then there is a $(PS)_{c_2}$ sequence $\{u_n\} \subset H$ for $\hat{J}$ with the property that

$$\hat{J}(u_n) \to c_2, \qquad \|\hat{J}'(u_n)\|_{H^{-1}} \to 0, \qquad n \to \infty,$$

where c_2 is given by

$$c_2 = \inf_{\hat{\gamma}\in\Gamma} \max_{t\in[0,1]} \hat{J}(\gamma(t)),$$

and $\hat{\Gamma} = \{\gamma \in C([0,1], H) : \gamma(0) = 0, \hat{J}(\gamma(1)) < 0\}$.

In the following we give an estimate of the upper bound of the critical level c_2 by using above two lemmas.

Lemma 6.3. *There holds* $0 < c_2 < \Lambda_1$.

Proof. Similar to Lemma 5.4, there exists $R_2 > 0$ sufficiently large, such that $\hat{J}(R_2u_\varepsilon) \le 0$ for ε small enough, hence, we can find $0 < t_\varepsilon < R_2$ satisfying

$$0 < \eta_2 \le c_2 \le \max_{t\in[0,R_2]} \hat{J}(tu_\varepsilon) = \hat{J}(t_\varepsilon u_\varepsilon).$$

Since $\frac{d}{dt}\hat{J}(tu_\varepsilon)|_{t=t_\varepsilon} = 0$, we have

$$t_\varepsilon\|u_\varepsilon\|^2 + bt_\varepsilon^3\|u_\varepsilon\|^4 = \int_B t_\varepsilon^{5+|x|^\alpha}|u|^{6+|x|^\alpha}.$$

Hence we deduce from (2.2) that

$$\begin{aligned} S^{\frac{3}{2}} + O(\varepsilon) + bt_\varepsilon^2(S^3+O(\varepsilon)) &= t_\varepsilon^4\int_B |u_\varepsilon|^6 + t_\varepsilon^4\int_B \left(t_\varepsilon^{|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} - |u_\varepsilon|^6\right) \\ &= t_\varepsilon^4[S^{\frac{3}{2}} + O(\varepsilon^3) + A_\varepsilon] \\ &= t_\varepsilon^4[S^{\frac{3}{2}} + O(\varepsilon^3) + O(\varepsilon^\alpha|\log\varepsilon|) + O(\varepsilon^{3/2})], \end{aligned} \tag{6.1}$$

where $A_\varepsilon = O(\varepsilon^\alpha|\log\varepsilon|) + O(\varepsilon^{3/2})$ is given in [21, page 14]. For convenience, we set $A = S^{\frac{3}{2}} + O(\varepsilon)$, $B = b(S^3 + O(\varepsilon))$ and $C = S^{\frac{3}{2}} + O(\varepsilon^3) + O(\varepsilon^\alpha|\log\varepsilon|) + O(\varepsilon^{3/2})$. Thus, (6.1) can be rewritten as $A + Bt_\varepsilon^2 = Ct_\varepsilon^4$. It is easy to see that

$$t_\varepsilon^2 = \frac{B+\sqrt{B^2+4AC}}{2C} = \frac{bS^3 + O(\varepsilon) + \sqrt{b^2S^6 + 4S^3 + O(\varepsilon) + O(\varepsilon^\alpha|\log\varepsilon|)}}{2S^{3/2} + O(\varepsilon^{3/2}) + O(\varepsilon^\alpha|\log\varepsilon|)}.$$

Thereby, $t_\varepsilon^2 > 1$ for ε small enough, which implies

$$\begin{aligned}
&\int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&= \int_B \left(\frac{t_\varepsilon^6}{6} - \frac{t_\varepsilon^6}{6+|x|^\alpha}\right)|u_\varepsilon|^6 + \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}(|u_\varepsilon|^6 - |u_\varepsilon|^{6+|x|^\alpha}) \\
&\quad + \int_B \left(\frac{t_\varepsilon^6}{6+|x|^\alpha} - \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}\right)|u_\varepsilon|^6 \\
&\leq \int_B \left(\frac{t_\varepsilon^6}{6} - \frac{t_\varepsilon^6}{6+|x|^\alpha}\right)|u_\varepsilon|^6 + \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}(|u_\varepsilon|^6 - |u_\varepsilon|^{6+|x|^\alpha}).
\end{aligned} \tag{6.2}$$

By [21, page 16] and using the fact that $t_\varepsilon < R_2$, we have

$$\int_B \left(\frac{t_\varepsilon^6}{6} - \frac{t_\varepsilon^6}{6+|x|^\alpha}\right)|u_\varepsilon|^6 \leq C\varepsilon^\alpha \tag{6.3}$$

and

$$\int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}(|u_\varepsilon|^6 - |u_\varepsilon|^{6+|x|^\alpha}) \leq -C\varepsilon^\alpha|\log\varepsilon|. \tag{6.4}$$

Combining (6.3), (6.4) with (6.2) and using Lemma 6.2, we derive

$$\begin{aligned}
\hat{J}(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 + \frac{bt_\varepsilon^4}{4}\|u_\varepsilon\|^4 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&= \frac{t_\varepsilon^2}{2}\|u_\varepsilon\|^2 + \frac{bt_\varepsilon^4}{4}\|u_\varepsilon\|^4 - \int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 + \int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&= \sup_{t\geq 0} g_2(t) + \int_B \frac{t_\varepsilon^6}{6}|u_\varepsilon|^6 - \int_B \frac{t_\varepsilon^{6+|x|^\alpha}}{6+|x|^\alpha}|u_\varepsilon|^{6+|x|^\alpha} \\
&\leq \frac{b}{4}S^3 + \frac{b^3}{24}S^6 + \frac{1}{6}S\sqrt{S^4b^2+4S} + \frac{b^2}{24}S^4\sqrt{S^4b^2+4S} + O(\varepsilon) + C\varepsilon^\alpha - C\varepsilon^\alpha|\log\varepsilon|.
\end{aligned} \tag{6.5}$$

By choosing $\varepsilon > 0$ small enough, we derive by (6.5),

$$0 < \eta_2 \leq c_2 \leq \hat{J}(t_\varepsilon u_\varepsilon) < \Lambda_1.$$

The proof is completed. □

Lemma 6.4. *If $\{u_n\}$ is a $(PS)_{c_2}$ sequence of $\hat{J}$, then there exists $u \in H$ such that, up to a subsequence, $u_n \rightharpoonup u$ and $\hat{J}'(u) = 0$.*

Proof. By Lemma 3.3, $\{u_n\}$ is bounded in H and hence, going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u$ in H. Let $A > 0$ be such that $\int_B |\nabla u_n|^2 \to A^2$. If $u = 0$, it is easy to see that $\hat{J}'(u) = 0$. If $u \neq 0$, then by the weakly lower semi-continuity of the norm, $\int_B |\nabla u|^2 \leq A^2$. In the sequel, we will claim that $\int_B |\nabla u|^2 = A^2$. In fact, if it is false, then $\int_B |\nabla u|^2 < A^2$. For any measurable subset $Q \subset B$, we have for $v \in H$,

$$\begin{aligned}
\left|\int_Q (|u_n|^{4+|x|^\alpha}u_n - |u|^{4+|x|^\alpha}u)v\right| &\leq \int_Q (|u_n|^{5+|x|^\alpha} + |u|^{5+|x|^\alpha})|v| \\
&\leq \||u_n|^{5+|x|^\alpha} + |u|^{5+|x|^\alpha}\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(Q)}\|v\|_{L^{p(\cdot)}(Q)},
\end{aligned}$$

where $p(x) = 6 + |x|^\alpha$. Hence, the Vitali theorem (see [28]) leads to

$$\int_B |u_n|^{4+|x|^\alpha} u_n v \to \int_B |u|^{4+|x|^\alpha} uv, \quad \text{as } n \to \infty.$$

This together with the fact that $\hat{J}'(u_n) \to 0$ ensures that

$$(1 + A^2 b) \int_B \nabla u \nabla v = \int_B |u|^{4+|x|^\alpha} uv, \qquad v \in H. \tag{6.6}$$

By taking $v = u$ in (6.6), there holds $\langle \hat{J}'(u), u\rangle < 0$. Similarly to the proof of Lemma 3.1, we have $\langle \hat{J}'(tu), tu\rangle > 0$ for small $t > 0$. Thus, there exists a $t_u \in (0, 1)$ such that $\hat{J}(t_u u) = \max_{t\geq 0} \hat{J}(tu)$ and $\langle \hat{J}'(t_u u), t_u u\rangle = 0$. Then, we deduce by the weak lower semicontinuity of the norm and Fatou's lemma that

$$\begin{aligned}
c_2 &\leq \hat{J}(t_u u) - \frac{1}{6}\langle \hat{J}'(t_u u), t_u u\rangle \\
&= \frac{t_u^2}{3}\|u\|^2 + \frac{t_u^4 b}{12}\|u\|^4 + \int_B \left(\frac{t_u^{6+|x|^\alpha}}{6} - \frac{t_u^{6+|x|^\alpha}}{6+|x|^\alpha}\right)|u|^{6+|x|^\alpha} \\
&< \frac{1}{3}\|u\|^2 + \frac{b}{12}\|u\|^4 + \int_B \left(\frac{1}{6} - \frac{1}{6+|x|^\alpha}\right)|u|^{6+|x|^\alpha} \\
&\leq \liminf_{n\to\infty} \left(\frac{1}{3}\|u_n\|^2 + \frac{b}{12}\|u_n\|^4 + \int_B \left(\frac{1}{6} - \frac{1}{6+|x|^\alpha}\right)|u_n|^{6+|x|^\alpha}\right) \\
&= \liminf_{n\to\infty} \left(J(u_n) - \frac{1}{6}\langle J'(u_n), u_n\rangle\right) = c_2,
\end{aligned}$$

which is impossible. Thus, $\int_B |\nabla u|^2 = A^2$ and $\hat{J}'(u) = 0$. The proof is completed. □

In order to obtain the nontrivial solution of (1.6), we need define the functional $\hat{I} : H \to \mathbb{R}$ by

$$\hat{I}(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_B |u|^6.$$

Lemma 6.5. *Assume that $0 < c_2 < \Lambda_1$. The functional I satisfies the $(PS)_{c_2}$ condition.*

Proof. Suppose that $\{u_n\}$ is a $(PS)_{c_2}$ sequence for $c_2 \in (0, \Lambda_1)$, i.e.

$$\hat{I}(u_n) \to c_2, \qquad \hat{I}'(u_n) \to 0 \quad \text{as } n \to \infty.$$

By repeating the arguments used in Lemma 3.3, it is easy to show that $\{u_n\}$ is bounded in H. Then passing to a subsequence, we can find $u \in H$ such that $u_n \rightharpoonup u$ in H. Now, let $v_n = u_n - u$, we claim that $\|v_n\| \to 0$. In fact, we use an argument of contradiction and suppose that there exists a subsequence still denoted by $\{v_n\}$ such that $\|v_n\| \to \tilde{l} > 0$. It is easy to verify that

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1) \tag{6.7}$$

and

$$\|u_n\|^4 = \|v_n\|^4 + \|u\|^4 + 2\|v_n\|^2\|u\|^2 + o(1). \tag{6.8}$$

From the Brezis–Lieb lemma in [9], we have

$$\int_B |u_n|^6 dx = \int_B |v_n|^6 dx + \int_B |u|^6 dx + o(1). \tag{6.9}$$

Recall that $\hat{I}'(u_n) \to 0$ as $n \to \infty$, there holds by (6.7),

$$\lim_{n\to\infty} \langle \hat{I}'(u_n), u\rangle = \|u\|^2 + b\tilde{l}^2\|u\|^2 + b\|u\|^4 - \int_B |u|^6 dx = 0, \tag{6.10}$$

which yields

$$\begin{aligned} \hat{I}(u) &= \hat{I}(u) - \frac{1}{4}\left(\|u\|^2 + bl^2\|u\|^2 + b\|u\|^4 - \int_B |u|^6 dx\right) \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{12}\int_B |u|^6 dx - \frac{b}{4}\tilde{l}^2\|u\|^2 \\ &\geq -\frac{b}{4}\tilde{l}^2\|u\|^2. \end{aligned} \tag{6.11}$$

On the other hand, combining (6.7), (6.8) with (6.9) leads to

$$\begin{aligned} &\hat{I}(u_n) - \hat{I}(u) + o(1) \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u_n\|^4 - \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_B |u_n|^6 + \frac{1}{6}\int_B |u|^6 + o(1) \\ &= \frac{1}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 + \frac{b}{2}\|v_n\|^2\|u\|^2 - \frac{1}{6}\int_B |v_n|^6 + o(1). \end{aligned} \tag{6.12}$$

Similarly, by using (6.10) again, we deduce

$$\begin{aligned} o(1) &= \langle \hat{I}'(u_n), u_n\rangle - (\|u\|^2 + b\tilde{l}^2\|u\|^2 + b\|u\|^4 - \int_B |u|^6 dx) \\ &= \|u_n\|^2 - \|u\|^2 + b\|u_n\|^4 - b\|u\|^4 - b\tilde{l}^2\|u\|^2 - \int_B |u_n|^6 + \int_B |u|^6 \\ &= \|v_n\|^2 + b\|v_n\|^4 + b\|v_n\|^2\|u\|^2 - \int_B |v_n|^6 + o(1). \end{aligned} \tag{6.13}$$

Then, taking the limit on the both sides in (6.13) as $n \to \infty$, we find $\tilde{l}^2 + b\tilde{l}^4 + b\tilde{l}^2\|u\|^2 \leq S^{-3}\tilde{l}^6$, which implies that

$$\tilde{l}^2 \geq \frac{S^3 b + S\sqrt{S^4 b^2 + 4(1+b\|u\|^2)S}}{2}. \tag{6.14}$$

It follows from (6.12) and (6.13) that

$$\hat{I}(u) = \hat{I}(u_n) - \left(\frac{1}{2} - \frac{1}{6}\right)\|v_n\|^2 - \left(\frac{1}{4} - \frac{1}{6}\right) b\|v_n\|^4 - \left(\frac{1}{2} - \frac{1}{6}\right) b\|v_n\|^2\|u\|^2 + o(1).$$

This together with (6.14) ensures that

$$\begin{aligned} \hat{I}(u) &= c_2 - \left(\frac{1}{3}\tilde{l}^2 + \frac{1}{12}b\tilde{l}^4 + \frac{1}{3}b\tilde{l}^2\|u\|^2\right) \\ &\leq c_2 - \frac{b}{4}S^3 - \frac{1}{24}b^3S^6 - \frac{S}{6}\sqrt{b^2S^4 + 4\left(1+b\|u\|^2\right)S} \\ &\quad - \frac{b^2S^4}{24}\sqrt{b^2S^4 + 4\left(1+b\|u\|^2\right)S} \\ &\quad - \frac{1}{24}\left(3b^2S^3 + S\sqrt{b^2S^4 + 4\left(1+b\|u\|^2\right)S}\right)\|u\|^2 - \frac{b}{4}\tilde{l}^2\|u\|^2 \\ &\leq c_2 - \left(\frac{b}{4}S^3 + \frac{b^3}{24}S^6 + \frac{S}{6}\sqrt{b^2S^4 + 4S} + \frac{b^2}{24}S^4\sqrt{b^2S^4 + 4S}\right) - \frac{b}{4}\tilde{l}^2\|u\|^2 \\ &\leq c_2 - \Lambda - \frac{b}{4}\tilde{l}^2\|u\|^2 < -\frac{b}{4}\tilde{l}^2\|u\|^2, \end{aligned}$$

which contradicts to (6.11). Therefore $v_n \to 0$ strongly in H, or equivalently, $u_n \to u$ in H as $n \to \infty$. The proof is completed. □

Proof of Theorem 1.9. By Lemmas 6.4, 6.5, we know that the assumptions in Theorem 1.2 are valid. Hence, (1.6) possesses a nonnegative nontrivial ground state solution $u \in H$, which satisfies the following equation in weak sense

$$-\left(1 + b\int_B |\nabla u|^2\right)\Delta u = u^{5+|x|^\alpha} \quad \text{in } B.$$

Let us define

$$\hat{g}(u(x)) = \frac{u^{5+|x|^\alpha}}{1 + b\int_B |\nabla u|^2}, \qquad x \in B.$$

It follows from Lemma 2.2 that $\int_B u^{6+\frac{3}{2}|x|^\alpha} \leq C$, which implies

$$a = \frac{\hat{g}(u)}{1+|u|} \in L^{\frac{3}{2}}(B).$$

Hence, we deduce immediately from Lemma 5.7 that $u \in L^q(B)$ for any $1 < q < \infty$. Then, there holds $\hat{g}(u) \in L^q(B)$ for any $1 < q < \infty$. By the Calderón–Zygmund inequality and L^p estimate given in [16, 30], we derive $u \in W^{2,q}(B)$, whence also $u \in C^{1,\alpha_2}(B)$ by Sobolev embedding theorem for any $0 < \alpha_2 < 1$. Moreover, the Harnack inequality [32] implies $u(x) > 0$ for all $x \in B$. The proof is completed. □

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