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Notes on the linear equation with Stieltjes derivatives

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Abstract. In this paper we continue the study of the linear equation with Stieltjes derivatives in [M. Frigon, R. López Pouso, *Adv. Nonlinear Anal.* **6**(2017), 13–36]. Specifically, we revisit some of the results there presented, removing some of the required conditions as well as amending some mistakes. Furthermore, following the classical setting, we use the connection between the linear equation and the Gronwall inequality to obtain a new version of this type of inequalities in the context of Lebesgue–Stieltjes integrals. From there, we obtain a uniqueness criterion for initial value problems.

Keywords: Stieltjes integration, Stieltjes differentiation, linear equation, uniqueness, Gronwall inequality.

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1 Introduction

In this paper we explore the linear equation with Stieltjes derivatives in its homogeneous and nonhomogeneous formulation. Specifically, we will be looking at the initial value problem

$$x'_g(t) + d(t)x(t) = h(t), \quad t \in [t_0, t_0 + T), \quad x(t_0) = x_0, \tag{1.1}$$

where $t_0, T, x_0 \in \mathbb{R}$, $T > 0$, are fixed, $d, h : [t_0, t_0 + T) \to \mathbb{R}$ are given functions and x'_g stands for the Stieltjes derivative of x with respect to a nondecreasing and left-continuous function $g : \mathbb{R} \to \mathbb{R}$, usually called derivator, see [3, 10]. Note that [3] provides some information regarding (1.1) in its homogeneous form (i.e. $h = 0$) as well as for the nonhomogeneous case. Nevertheless, the results obtained there present some limitations, as the authors make use of the product rule for Stieltjes derivatives in [10] which, unfortunately, is wrongly stated. Here, we amend the mistakes in [10] as well as we simplify the required hypotheses for the existence and uniqueness of solution of (1.1).

Furthermore, given the close relation existing between the Gronwall inequality and the linear equation in the setting of ordinary differential equations, we will prove a new version

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of the inequality in the context of Stieltjes integrals, generalizing the classical result in [5], as well as other existing formulations in the context of Stieltjes integrals, see [6,8,11,12,17].

The paper is structure as follows. In Section 2 we gather and revisit some of the information available in [3, 10] regarding the definition of Stieltjes derivatives and its properties, as well as some other basic definitions necessary for this paper. In particular, it is at this point that we correct the formula for the product and the quotient rule in [10]. Next, in Section 3 we study the linear equation (1.1), providing explicit expressions for its solutions as well as some of their properties. Finally, in Section 4 we establish a Gronwall-type inequality for the Lebesgue–Stieltjes integral using the solution of the homogeneous linear equation. Then, we discuss the relations with other existing inequalities available in the literature and we complete the revision of the results in [3] for (1.1) through a uniqueness result based on our version of Gronwall's inequality.

2 Preliminaries

Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function. Let us introduce some notation before including the definition of Stieltjes derivative in [10]. In what follows, we will consider μ_g to be the Lebesgue–Stieltjes measure associated to g, given by

$$\mu_g([a,b)) = g(b) - g(a), \quad a,b \in \mathbb{R},\ a < b,$$

see [1, 13, 15]; we will use the term "g-measurable" for a set or function to refer to μ_g-measurability in the corresponding sense; and we will denote the integration with respect to μ_g as

$$\int_X f(s)\,\mathrm{d}\,g(s).$$

Similarly, we will talk about properties holding g-almost everywhere in a set X, shortened to g-a.e. in X, as a simplified way to express that they hold μ_g-almost everywhere in X. In an analogous way, we will write that a property holds for g-almost all (or simply, g-a.a.) $x \in X$ meaning that it holds for μ_g-almost all $x \in X$. Along those lines, we find the following interesting set:

$$C_g := \{t \in \mathbb{R} : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\}.$$

The set C_g is the set of points around which g is constant and, as pointed out in [10, Proposition 2.5], we have that $\mu_g(C_g) = 0$. Hence, this set can be disregarded when it comes to properties holding g-almost everywhere in a set. Observe that, as pointed out in [10], the set C_g is open in the usual topology, so it can be uniquely expressed as the countable union of open disjoint intervals, say

$$C_g = \bigcup_{n\in\mathbb{N}} (a_n, b_n). \tag{2.1}$$

Another fundamental set for the work that lies ahead is the set D_g of all discontinuity points of g. Observe that, given that g is nondecreasing, we can write

$$D_g = \{t \in \mathbb{R} : \Delta^+ g(t) > 0\},$$

where $\Delta^+ g(t) = g(t^+) - g(t)$, $t \in \mathbb{R}$, and $g(t^+)$ denotes the right-hand side limit of g at t. Recall that Froda's Theorem, [4], ensures that the set D_g is at most countable. Finally, given the previous definitions, we can define the sets N_g^- and N_g^+ introduced in [9] as

$$N_g^- = \{a_n : n \in \mathbb{N}\} \setminus D_g, \quad N_g^+ = \{b_n : n \in \mathbb{R}\} \setminus D_g,$$

where $a_n, b_n \in \mathbb{R}$ are as in (2.1). We denote $N_g = N_g^- \cup N_g^+$.

We have now all the information required to properly introduce the definition of Stieltjes derivative in [10]. In order to clarify its definition, we have included a brief remark explaining the limits involved.

Definition 2.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function and consider a map $f : \mathbb{R} \to \mathbb{R}$. We define the *Stieltjes derivative*, or *g-derivative*, of f at a point $t \in \mathbb{R} \setminus C_g$ as

$$f'_g(t) = \begin{cases} \displaystyle\lim_{s\to t} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g, \\ \displaystyle\lim_{s\to t^+} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g, \end{cases}$$

provided the corresponding limits exist. In that case, we say that f is *g-differentiable at t*.

Remark 2.2. Given a function $f : \mathbb{R} \to \mathbb{R}$ and a point $t \in \mathbb{R}$, we define the function

$$F_t(\cdot) = \frac{f(\cdot)-f(t)}{g(\cdot)-g(t)},$$

which we will assume to be defined in a neighbourhood of t in which the expression makes sense, namely, at the points s such that $g(s) - g(t) \neq 0$. The limits in Definition 2.1 are well-defined when t is an accumulation point of the domain of the function F_t. This explains why the points of C_g are excluded in the definition as if $t \in C_g$, then there exists $\varepsilon_t > 0$ such that the expression of F_t does not make sense for any neighbourhood $(t-\varepsilon, t+\varepsilon)$, $\varepsilon \in (0, \varepsilon_t)$. Moreover, the limits in Definition 2.1 should be properly understood at some other conflicting points. For example, imagine there exists $\delta > 0$ such that $g(s) = g(t)$ for $s \in (t-\delta, t)$, and $g(s) > g(t)$ for $s > t$. Then

$$\lim_{s\to t} F_t(s) = \lim_{s\to t^+} F_t(s),$$

since F_t is not defined at the left of t. Similarly, if there exists $\delta > 0$ such that $g(s) = g(t)$ for $s \in (t, t+\delta)$, the function F_t is not defined at the right of t, so if $g(s) < g(t)$ for $s < t$, then

$$\lim_{s\to t} F_t(s) = \lim_{s\to t^-} F_t(s).$$

Therefore, the g-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $t \in \mathbb{R} \setminus C_g$ is computed as

$$f'_g(t) = \begin{cases} \displaystyle\lim_{s\to t} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup N_g, \\ \displaystyle\lim_{s\to t^-} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \in N_g^-, \\ \displaystyle\lim_{s\to t^+} \frac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g \cup N_g^+, \end{cases}$$

provided the corresponding limits exist.

Remark 2.3. Since g is a regulated function, it follows that the g-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $t \in D_g$ exists if and only if the limit of f from the right of t, $f(t^+)$, exists. In that case, we have that

$$f'_g(t) = \frac{f(t^+)-f(t)}{\Delta^+ g(t)}.$$

First, we include some information available in [10] regarding the Stieltjes derivatives of functions. Specifically, we include a result about the continuity of differentiable functions, [10, Proposition 2.1], that we will use to revisit the product and quotient rule in [10], as the formulas there included are not correct.

Proposition 2.4. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function and f be a real-valued function defined on a neighborhood of t such that $f_g'(t)$ exists. Then $t \notin C_g$ and, if g is continuous at t:*

- *f is continuous from the left at t provided that*

$$g(s) < g(t) \quad \text{for all } s < t; \tag{2.2}$$

- *f is continuous from the right at t provided that*

$$g(s) > g(t) \quad \text{for all } s > t. \tag{2.3}$$

Proposition 2.4 is a fundamental tool for the proof of [10, Proposition 2.2], where the authors included some basic properties of the Stieltjes derivatives, such as the linearity of the derivative or the product and the quotient rule. However, the authors did not include the proof of the result, which led to an incorrect formulation of the product and the quotient rule. Here, we amend these mistakes and, later, we show the limitations of the formulas in [10, Proposition 2.2].

Proposition 2.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function, $t \in \mathbb{R}$, and f_1, f_2 be two real-valued functions defined on a neighborhood of t, U_t. If f_1 and f_2 are g-differentiable at t, then:*

(i) *The product $f_1 f_2$ is g-differentiable at t and*

$$(f_1 f_2)_g'(t) = (f_1)_g'(t) f_2(t) + (f_2)_g'(t) f_1(t) + (f_1)_g'(t)(f_2)_g'(t)\Delta^+ g(t). \tag{2.4}$$

(ii) *If $(f_2(t))^2 + (f_2)_g'(t) f_2(t)\Delta^+ g(t) \neq 0$, the quotient f_1/f_2 is g-differentiable at t and*

$$\left(\frac{f_1}{f_2}\right)_g'(t) = \frac{(f_1)_g'(t) f_2(t) - f_1(t)(f_2)_g'(t)}{(f_2(t))^2 + (f_2)_g'(t) f_2(t)\Delta^+ g(t))}. \tag{2.5}$$

Proof. First, observe that $t \notin C_g$ since $(f_1)_g'(t)$ and $(f_2)_g'(t)$ exist. Hence, we have that (2.2) and/or (2.3) hold.

Let us show that (2.4) holds. First, observe that we can rewrite $f_1 f_2(s) - f_1 f_2(t)$, $s \in U_t$, as

$$\frac{(f_1(s) - f_1(t))(f_2(t) + f_2(s)) + (f_2(s) - f_2(t))(f_1(t) + f_1(s))}{2}, \quad s \in U_t. \tag{2.6}$$

Assume that (2.3) holds. Then, it follows from (2.6) that the following limit exists and

$$\lim_{s \to t^+} \frac{f_1 f_2(s) - f_1 f_2(t)}{g(s) - g(t)} = \frac{(f_1)_g'(t)(f_2(t) + f_2(t^+)) + (f_2)_g'(t)(f_1(t) + f_1(t^+))}{2}. \tag{2.7}$$

Now, if $t \in D_g$, it follows from Remark 2.3 that

$$f_i(t^+) = (f_i)_g'(t)\Delta^+ g(t) + f_i(t), \quad i = 1, 2. \tag{2.8}$$

Thus, (2.7) yields that

$$\lim_{s\to t^+}\frac{f_1f_2(s)-f_1f_2(t)}{g(s)-g(t)}=(f_1)'_g(t)f_2(t)+(f_2)'_g(t)f_1(t)+(f_1)'_g(t)(f_2)'_g(t)\Delta^+g(t). \tag{2.9}$$

On the other hand, if $t\notin D_g$, it follows from Proposition 2.4 and (2.7) that

$$\lim_{s\to t^+}\frac{f_1f_2(s)-f_1f_2(t)}{g(s)-g(t)}=(f_1)'_g(t)f_2(t)+(f_2)'_g(t)f_1(t),$$

which matches (2.9) since $\Delta^+g(t)=0$. In other words, (2.9) holds in both cases. Hence, if $t\in D_g$ or $g(s)>g(t)$, $s\in[t-\delta,t]$ for some $\delta>0$, then the limit in (2.9) coincides with $(f_1f_2)'_g(t)$ and the proof is complete. Otherwise, $t\notin D_g$ and (2.2) holds. In that case, we obtain from (2.6) that the following limit exists and

$$\lim_{s\to t^-}\frac{f_1f_2(s)-f_1f_2(t)}{g(s)-g(t)}=\frac{(f_1)'_g(t)(f_2(t)+f_2(t^-))+(f_2)'_g(t)(f_1(t)+f_1(t^-))}{2}.$$

However, in that case, Proposition 2.4 ensures that the previous limit equals $(f_1)'_g(t)f_2(t)+(f_2)'_g(t)f_1(t)$, and so f_1f_2 is g-differentiable at t and (2.4) holds.

Now, we show that (2.5) holds. First, observe that the extra hypothesis in (ii) guarantees that $f_2(t)\neq 0$. Furthermore, we also have that $f_2(t)+(f_2)'_g(t)\Delta^+g(t)\neq 0$ which, provided that $t\in D_g$, ensures that $f_2(t^+)\neq 0$, see (2.8).

Assume that (2.3) holds. Since $f_2(t)\neq 0$, it follows from Proposition 2.4 (if $t\notin D_g$) and the definition of limit from the right (if $t\in D_g$) that there exists $\varepsilon>0$ such that f_2 does not vanish in $[t,t+\varepsilon)\cap U_t$. Hence, the following expression is well-defined for any $s\in[t,t+\varepsilon)\cap U_t$,

$$\frac{f_1(s)}{f_2(s)}-\frac{f_1(t)}{f_2(t)}=\frac{f_1(s)f_2(t)-f_1(t)f_2(s)}{f_2(t)f_2(s)}=\frac{(f_1(s)-f_1(t))f_2(t)+f_1(t)(f_2(t)-f_2(s))}{f_2(t)f_2(s)}. \tag{2.10}$$

Taking the corresponding limit from the right, we have that

$$\lim_{s\to t^+}\frac{(f_1/f_2)(s)-(f_1/f_2)(t)}{g(s)-g(t)}=\frac{(f_1)'_g(t)f_2(t)-f_1(t)(f_2)'_g(t)}{f_2(t)f_2(t^+)}. \tag{2.11}$$

Now, if $t\in D_g$, it follows from (2.8) that

$$\lim_{s\to t^+}\frac{(f_1/f_2)(s)-(f_1/f_2)(t)}{g(s)-g(t)}=\frac{(f_1)'_g(t)f_2(t)-f_1(t)(f_2)'_g(t)}{(f_2(t))^2+(f_2)'_g(t)f_2(t)\Delta^+g(t))}. \tag{2.12}$$

On the other hand, if $t\notin D_g$, it follows from Proposition 2.4 and (2.11) that

$$\lim_{s\to t^+}\frac{(f_1/f_2)(s)-(f_1/f_2)(t)}{g(s)-g(t)}=\frac{(f_1)'_g(t)f_2(t)-f_1(t)(f_2)'_g(t)}{(f_2(t))^2},$$

which matches (2.12). That is, (2.12) holds in both cases. Hence, if $t\in D_g$ or $g(s)>g(t)$, $s\in[t-\delta,t]$ for some $\delta>0$, then the limit in (2.12) coincides with $(f_1/f_2)'_g(t)$ and the proof is complete. Otherwise, $t\notin D_g$ and (2.2) holds. In that case, given that $f_2(t)\neq 0$, it follows from Proposition 2.4 that there exists $\varepsilon'>0$ such that f_2 does not vanish in $(t-\varepsilon',t]\cap U_t$. Hence, (2.10) is valid for all $s\in(t-\varepsilon',t]\cap U_t$. As a consequence, we obtain that the following limit exists and

$$\lim_{s\to t^-}\frac{(f_1/f_2)(s)-(f_1/f_2)(t)}{g(s)-g(t)}=\frac{(f_1)'_g(t)f_2(t)-f_1(t)(f_2)'_g(t)}{f_2(t)f_2(t^-)}=\frac{(f_1)'_g(t)f_2(t)-f_1(t)(f_2)'_g(t)}{(f_2(t))^2},$$

where the last equality follows, once again, from Proposition 2.4. This guarantees that f_1/f_2 is g-differentiable at t and (2.5) holds.

Remark 2.6. Observe that the formulas here presented reduce to the usual formulation when $g = \mathrm{Id}$. Furthermore, note that the expressions in Proposition 2.5 do not match those in [10]. Let us illustrate that the formulas there presented are not correct with some examples.

Consider $g, f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} t & \text{if } t \leq 0, \\ 0 & \text{if } 0 < t \leq 1, \\ t & \text{if } t > 1, \end{cases} \qquad f_1(t) = t + 2, \qquad f_2(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ t+2 & \text{if } t > 0. \end{cases}$$

For this choice of functions, we have that $f_1 \cdot f_2, f_1/f_2 : \mathbb{R} \to \mathbb{R}$ are defined as

$$f_1 \cdot f_2(t) = \begin{cases} t+2 & \text{if } t \leq 0, \\ (t+2)^2 & \text{if } t > 0, \end{cases} \qquad \frac{f_1}{f_2}(t) = \begin{cases} t+2 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Observe that, given that $g(t) = g(0)$ for $t \in (0,1)$, the derivatives at 0 are computed as the limit from the left, as pointed out by Remark 2.2. In particular, we have that

$$(f_1)'_g(0) = \lim_{s \to 0^-} \frac{f_1(s) - f_1(0)}{g(s) - g(0)} = \lim_{s \to 0^-} \frac{s + 2 - 2}{s - 0} = 1,$$
$$(f_2)'_g(0) = \lim_{s \to 0^-} \frac{f_2(s) - f_2(0)}{g(s) - g(0)} = \lim_{s \to 0^-} \frac{1 - 1}{s - 0} = 0,$$

and, since $f_1 \cdot f_2 = f_1/f_2 = f_1$ on $(-\infty, 0]$, we have that $(f_1 \cdot f_2)'_g(0) = (f_1/f_2)'_g(0) = 1$. Observe that (2.4) and (2.5) hold at $t = 0$.

First, let us show that the formula for the product of two functions in [10],

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t^+) + (f_2)'_g(t) f_1(t^+),$$

is not correct. Indeed, at $t = 0$ we have that

$$(f_1)'_g(0) f_2(0^+) + (f_2)'_g(0) f_1(0^+) = 1 \cdot 2 + 0 \cdot 2 = 2 \neq 1 = (f_1 \cdot f_2)'_g(0).$$

Furthemore, this example also shows that the formula in [14, Lemma 13],

$$(f_1 \cdot f_2)'_g(t) = (f_1)'_g(t) f_2(t^+) + (f_2)'_g(t) f_1(t), \quad t \in D_g, \tag{2.13}$$

cannot be valid for a generic point in $\mathbb{R} \setminus C_g$, as the only difference with respect to the previous formula is that $f_1(0^+)$ is replaced by $f_1(0)$, which has no effect as both terms are multiplied by zero. Nevertheless, observe that (2.4) yields (2.13) for $t \in D_g$ as a consequence of (2.8).

Now, for the quotient formula in [10],

$$\left(\frac{f_1}{f_2}\right)'_g(t) = \frac{(f_1)'_g(t) f_2(t) - (f_2)'_g(t) f_1(t)}{f_2(t) f_2(t^+)}.$$

Once again, this formula fails to be true as

$$\frac{(f_1)'_g(0) f_2(0) - (f_2)'_g(0) f_1(0)}{f_2(0) f_2(0^+)} = \frac{1 \cdot 1 - 0 \cdot 2}{1 \cdot 2} = \frac{1}{2} \neq 1 = \left(\frac{f_1}{f_2}\right)'_g(0).$$

Finally, we include the last pieces of information required for this paper, the two formulations of the Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral. The next result is a reformulation of [10, Theorem 5.4], where we have added the definition of g-absolute continuity, [10, Definition 5.1], to its statement.

Theorem 2.7. *Let $a, b \in \mathbb{R}$, $a < b$, and $F : [a, b] \to \mathbb{R}$. The following conditions are equivalent:*

1. *The function F is g-absolutely continuous on $[a, b]$ according to the following definition: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ verifying*
$$\sum_{n=1}^{m} (g(b_n) - g(a_n)) < \delta,$$
we have that
$$\sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

2. *The function F satisfies the following conditions:*
   (i) *there exists $F'_g(t)$ for g-a.a. $t \in [a, b)$;*
   (ii) *$F'_g \in \mathcal{L}^1_g([a, b), \mathbb{R})$, the set of Lebesgue–Stieltjes integrable functions with respect to μ_g;*
   (iii) *for each $t \in [a, b]$,*
$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, \mathrm{d}\, g(s).$$

Remark 2.8. Observe that in statement 2 (iii) of Theorem 2.7, for $t = a$, we are considering the integral over $[a, a) = \{x \in \mathbb{R} : a \le x < a\} = \emptyset$, which makes the integral null, thus giving the equality.

The other formulation of the Fundamental Theorem of Calculus that we include here is a combination of Theorem 2.4 and Proposition 5.2 in [10] and it reads as follows.

Theorem 2.9. *Let $f \in \mathcal{L}^1_g([a, b), \mathbb{R})$. Then, the function $F : [a, b] \to \mathbb{R}$, defined as*
$$F(t) = \int_{[a,t)} f(s)\, \mathrm{d}\, g(s),$$
is well-defined, g-absolutely continuous on $[a, b]$ and
$$F'_g(t) = f(t), \quad \text{for } g\text{-a.a. } t \in [a, b).$$

In the work that follows, we shall use some known properties for g-absolutely continuous functions, most of which are analogous to those of absolutely continuous functions in the usual sense. For convenience, we refer the reader to [3, 10] for more information on the topic.

3 Linear equation

In this section we focus on the study of the linear equation with Stieltjes derivatives on the real line in its homogeneous and nonhomogeneous formulation. Specifically, given a nondecreasing and left-continuous map, $g : \mathbb{R} \to \mathbb{R}$, we consider the initial value problem
$$x'_g(t) + d(t)x(t) = h(t), \quad t \in [t_0, t_0 + T), \quad x(t_0) = x_0, \tag{3.1}$$
with $x_0 \in \mathbb{R}$ and $d, h : [t_0, t_0 + T) \to \mathbb{R}$. Naturally, (3.1) yields the homogeneous formulation of the problem when $h = 0$. In that case, for simplicity and in order to simplify the connections with [3], we shall write $c(t) = -d(t)$, $t \in [t_0, t_0 + T)$, so that (3.1) reads as
$$x'_g(t) = c(t)x(t), \quad t \in [t_0, t_0 + T), \quad x(t_0) = x_0. \tag{3.2}$$

It is important to note, nevertheless, that [3] is not the only paper available in the study of linear equations in a Stieltjes sense. For example, in [7,16,17] we find linear integral equations in more general settings, for which the different authors were able to obtain the existence and uniqueness of solution. In some cases, an explicit solution is given provided one can find a fundamental matrix for the corresponding problem, which might be hard to obtain. Here, we limit ourselves to a scalar version of the linear differential equation for which we obtain an explicit solution in terms of elemental functions. Interestingly enough, the relations between the different linear problems in the Stieltjes sense arises naturally. For example, condition (6.13) in [17] is a necessary condition for the existence of solution, which yields the condition required in Theorem 3.5 for our solution when both contexts are compatible.

Following [3], we start our study of the linear equation studying the homogeneous formulation. A first reasonable guess for a solution for (3.2) would be to consider, under the assumption that $c \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$, the map

$$x(t) = x_0 \exp\left(\int_{[t_0,t)} c(s)\,\mathrm{d}\,g(s)\right), \quad t \in [t_0, t_0 + T], \tag{3.3}$$

as this is the solution for $g = \mathrm{Id}$. Nevertheless, note that this cannot be a solution of (3.2) as for any $t \in [t_0, t_0 + T) \cap D_g$,

$$\begin{aligned} x'_g(t) &= \lim_{s\to t^+} \frac{x(s) - x(t)}{g(s) - g(t)} \\ &= \lim_{s\to t^+} \frac{x(t)\left(\exp\left(\int_{[t,s)} c(r)\,\mathrm{d}\,g(r)\right) - 1\right)}{g(s) - g(t)} = x(t)\frac{\exp\left(\int_{\{t\}} c(r)\,\mathrm{d}\,g(r)\right) - 1}{\Delta^+ g(t)}. \end{aligned}$$

Therefore, we have that

$$x'_g(t) = x(t)\frac{\exp(c(t)\Delta^+ g(t)) - 1}{\Delta^+ g(t)}, \quad t \in [t_0, t_0 + T) \cap D_g,$$

which is not, in general, equal to $x(t)c(t)$. Therefore, the map x in (3.3) cannot be a solution of (3.2). Nevertheless, it is easy to see using the chain rule for the Stieltjes derivative, [10, Theorem 2.3] that x solves the problem in $[t_0, t_0 + T) \setminus D_g$. All this ideas resulted in the modification of the map in (3.3) presented in [3, Definition 6.1]. It is at this point that we encounter the first improvement on the results of [3]. The mentioned modification is subject to a condition regarding the convergence of a series, namely, condition (3.5) in this paper. In the following result we show that such condition is redundant in the considered context.

Lemma 3.1. *Let $c \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$ be such that $1 + c(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + T) \cap D_g$. Then*

$$\sum_{t\in[t_0,t_0+T)\cap D_g} \big|\log|1 + c(t)\Delta^+ g(t)|\big| < +\infty. \tag{3.4}$$

In particular, if $1 + c(t)\Delta^+ g(t) > 0$ for all $t \in [t_0, t_0 + T) \cap D_g$, then

$$\sum_{t\in[t_0,t_0+T)\cap D_g} \big|\log(1 + c(t)\Delta^+ g(t))\big| < +\infty. \tag{3.5}$$

Proof. First, observe that the hypotheses ensure that the logarithms in the corresponding expressions are well-defined and finite for each $t \in [t_0, t_0 + T) \cap D_g$.

Now, elementary calculations show that $\lim_{s\to 0} |\log|1+s|/s| = 1$. Hence, the definition of limit guarantees the existence of some $r > 0$ such that

$$\left|\left|\frac{\log|1+s|}{s}\right| - 1\right| < 1, \quad s \in (-r, r).$$

In particular, this implies that $|\log|1+s|| < 2|s|$ for all $s \in (-r, r)$.

On the other hand, since c is g-integrable on $[t_0, t_0+T)$, we have that

$$\sum_{t\in[t_0,t_0+T)\cap D_g} |c(t)\Delta^+ g(t)| \le \int_{[t_0,t_0+T)} |c(s)|\,\mathrm{d}\,g(s) < +\infty.$$

Therefore, the set $A_r = \{t \in [t_0, t_0+T) \cap D_g : |c(t)\Delta^+ g(t)| \ge r\}$ must be finite. Hence, denoting $B_r = ([t_0, t_0+T) \cap D_g) \setminus A_r$, we have that

$$\begin{aligned}\sum_{t\in[t_0,t_0+T)\cap D_g} |\log|1+c(t)\Delta^+ g(t)|| &= \sum_{t\in A_r} |\log|1+c(t)\Delta^+ g(t)|| + \sum_{t\in B_r} |\log|1+c(t)\Delta^+ g(t)|| \\ &\le \sum_{t\in A_r} |\log|1+c(t)\Delta^+ g(t)|| + 2\sum_{t\in B_r} |c(t)\Delta^+ g(t)| < +\infty.\end{aligned}$$

This shows that (3.4) holds. Now (3.5) follows from the extra hypothesis. □

As a consequence of Lemma 3.1 and the product differentiation rule, we can reformulate Lemmas 6.2 and 6.3 in [3] into the following results.

Theorem 3.2. *Let $c \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ be such that $1 + c(t)\Delta^+ g(t) > 0$ for all $t \in [t_0, t_0+T) \cap D_g$. Then, the map $\widetilde{c} : [t_0, t_0+T) \to \mathbb{R}$, defined as*

$$\widetilde{c}(t) = \begin{cases} c(t) & \text{if } t \in [t_0, t_0+T)\backslash D_g, \\ \dfrac{\log\left(1 + c(t)\Delta^+ g(t)\right)}{\Delta^+ g(t)} & \text{if } t \in [t_0, t_0+T) \cap D_g, \end{cases} \tag{3.6}$$

belongs to $\mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$; the map $e_c(\cdot, t_0) : [t_0, t_0+T] \to (0, +\infty)$,

$$e_c(t, t_0) := \exp\left(\int_{[t_0,t)} \widetilde{c}(s)\,\mathrm{d}\,g(s)\right), \quad t \in [t_0, t_0+T], \tag{3.7}$$

is well-defined and g-absolutely continuous on $[t_0, t_0+T]$; and the map $x : [t_0, t_0+T] \to \mathbb{R}$, given by $x(t) = x_0 e_c(t, t_0)$, $t \in [t_0, t_0+T]$, solves the initial value problem (3.2) g-a.e. in $[t_0, t_0+T)$.

Remark 3.3. Observe that, for any $t \in [t_0, t_0+T) \cap D_g$,

$$\begin{aligned} e_c(t^+, t_0) &= \lim_{s\to t^+} \exp\left(\int_{[t_0,s)} \widetilde{c}(s)\,\mathrm{d}\,g(s)\right) = \lim_{s\to t^+}\left(\exp\left(\int_{[t_0,t)} \widetilde{c}(s)\,\mathrm{d}\,g(s)\right)\exp\left(\int_{[t,s)} \widetilde{c}(s)\,\mathrm{d}\,g(s)\right)\right) \\ &= e_c(t, t_0)\exp\left(\int_{\{t\}} \widetilde{c}(s)\,\mathrm{d}\,g(s)\right) = e_c(t, t_0)(1 + c(t)\Delta^+ g(t)).\end{aligned}$$

Essentially, this shows that the limitations that the map in (3.3) had at the discontinuity points are avoided for $e_c(\cdot, t_0)$.

An analogous improvement to the more general result [3, Lemma 6.5] can be obtained making use of the information in Lemma 3.1 regarding (3.4) instead of (3.5). In that case, we obtain the following result.

Theorem 3.4. *Let $c \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ be such that $1 + c(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0+T) \cap D_g$. Then, the set*

$$T_c^- = \{t \in [t_0, t_0+T) \cap D_g : 1 + c(t)\Delta^+ g(t) < 0\}$$

has finite cardinality. Furthermore, if $T_c^- = \{t_1, \dots, t_k\}$, $t_0 \le t_1 < t_2 < \cdots < t_k < t_{k+1} = t_0 + T$, then the map $\widehat{c} : [t_0, t_0+T) \to \mathbb{R}$, defined as

$$\widehat{c}(t) = \begin{cases} c(t) & \text{if } t \in [t_0, t_0+T) \backslash D_g, \\ \dfrac{\log |1 + c(t)\Delta^+ g(t)|}{\Delta^+ g(t)} & \text{if } t \in [t_0, t_0+T) \cap D_g, \end{cases}$$

belongs to $\mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$; the map $\widehat{e}_c(\cdot, t_0) : [t_0, t_0+T] \to \mathbb{R}\backslash\{0\}$, given by

$$\widehat{e}_c(t, t_0) = \begin{cases} \exp\left(\int_{[t_0,t)} \widehat{c}(s) \, \mathrm{d}\, g(s) \right) & \text{if } t_0 \le t \le t_1, \\ (-1)^j \exp\left(\int_{[t_0,t)} \widehat{c}(s) \, \mathrm{d}\, g(s) \right) & \text{if } t_j < t \le t_{j+1},\ j = 1, \dots, k, \end{cases}$$

is well-defined and g-absolutely continuous on $[t_0, t_0+T]$; and the map $x : [t_0, t_0+T] \to \mathbb{R}$, given by $x(t) = x_0 \widehat{e}_c(t, t_0)$, $t \in [t_0, t_0+T]$, solves the initial value problem (3.2) g-a.e. in $[t_0, t_0+T)$.

Now, we move on to the study of the nonhomogeneous case. The study of this problem was also carried out in [3]. In particular, [3, Proposition 6.8] guarantees the existence of a unique solution of (3.2) under certain hypothesis. Furthermore, although it is not explicitly stated in the result, its proof provides a way to obtain it through the connection with the problem in [3, Proposition 6.7], and they have been made explicit in [2]. However, the proof of [3, Proposition 6.7] relays on the product rule for Stieltjes derivatives which, as it has been pointed out before, was not correct in that paper. Specifically, it is equation (6.16) in [3] that makes use of this property. It is possible to show that such expression remains true with the product formula in Proposition 2.5. Nevertheless, here we will use a different approach to the study of (3.2). Namely, we will recreate the method of variation of constants in this context.

Roughly speaking, the method of variation of constants revolves around the idea that the solution of a nonhomogeneous linear equation can be expressed as the sum of a solution of the homogeneous linear equation plus a particular solution of the nonhomogeneous one. In order to obtain the particular solution, we consider the following family of functions

$$x_C(t) = C x_h(t), \quad t \in [t_0, t_0+T], \quad C \in \mathbb{R},$$

where x_h is a given solution of $x'_g(t) = c(t)x(t)$. Observe that each element of the family x_C, $C \in \mathbb{R}$, also solves the same problem. From there, we make a guess that a particular solution is similar to that one, where we allow the constants to vary, i.e. we consider them as a function. Explicitly, we guess that the solution is of the form

$$x(t) = C(t)x_h(t), \quad t \in [t_0, t_0+T],$$

for some function $C : [t_0, t_0+T] \to \mathbb{R}$. Then, we try our guess on the corresponding nonhomogeneous linear equation. In order to do so, we need to make use of the product rule for Stieltjes derivatives, statement (ii) in Proposition 2.5. Let $t \in [t_0, t_0+T)$ be such that $x'_g(t)$ exists. In that case,

$$\begin{aligned} x'_g(t) &= C'_g(t)x_h(t) + C(t)x_h(t)(-d(t)) + C'_g(t)x_h(t)(-d(t))\Delta^+ g(t) \\ &= x_h(t)(C'_g(t)(1 - d(t)\Delta^+ g(t)) - C(t)d(t)). \end{aligned}$$

Hence, for such $t \in [t_0, t_0+T)$, it follows that $x'_g(t) + d(t)x(t) = x_h(t)C'_g(t)(1 - d(t)\Delta^+ g(t))$. Therefore, if x solves the nonhomogeneous linear equation, we must have that for such $t \in [t_0, t_0+T)$,

$$h(t) = x_h(t)C'_g(t)(1 - d(t)\Delta^+ g(t)).$$

Therefore, if we can find a function C satisfying the equation above, we obtain a particular solution of the nonhomogeneous linear equation and, as a consequence, the general solution of the same problem. Then, imposing the initial condition, we obtain the following result.

Theorem 3.5. *Let $d, h \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ be such that $1 - d(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + T) \cap D_g$. Then the map $x : [t_0, t_0+T] \to \mathbb{R}$, defined as*

$$x(t) = \widehat{e}_{-d}(t, t_0)\left(x_0 + \int_{[t_0,t)} \frac{h(s)}{\widehat{e}_{-d}(s, t_0)(1 - d(s)\Delta^+ g(s))} \, \mathrm{d}\, g(s)\right), \quad t \in [t_0, t_0+T], \tag{3.8}$$

is well-defined, g-absolutely continuous on $[t_0, t_0+T]$ and it solves (3.1) g-a.e. in $[t_0, t_0+T)$.

If, in particular, $1 - d(t)\Delta^+ g(t) > 0$ for all $t \in [t_0, t_0+T) \cap D_g$, then

$$x(t) = e_{-d}(t, t_0)\left(x_0 + \int_{[t_0,t)} \frac{h(s)}{e_{-d}(s, t_0)(1 - d(s)\Delta^+ g(s))} \, \mathrm{d}\, g(s)\right), \quad t \in [t_0, t_0+T]. \tag{3.9}$$

Proof. First of all, note that, under the corresponding hypotheses, the maps $\widehat{e}_{-d}(\cdot, t_0)$ and $e_{-d}(\cdot, t_0)$ are well-defined. Let us show that the map x in (3.8) has the stated properties.

Consider the maps $E, H : [t_0, t_0+T) \to \mathbb{R}$ defined as

$$E(t) = \widehat{e}_{-d}(t, t_0)(1 - d(t)\Delta^+ g(t)), \quad H(t) = \frac{h(t)}{E(t)}, \quad t \in [t_0, t_0+T).$$

Since $E(t) = \widehat{e}_{-d}(t, t_0)$ for all $t \in I \setminus D_g$ and D_g is countable, E is g-measurable. Moreover, since $E \neq 0$ by definition, and h and E are g-measurable, it follows that H is g-measurable. Furthermore, H belongs to $\mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$. Indeed, first of all note that for each $t \in [t_0, t_0+T)$,

$$\begin{aligned} |\widehat{e}_{-d}(t, t_0)| &= \exp\left(\int_{[t_0,t)} \widehat{c}(s) \, \mathrm{d}\, g(s)\right) \\ &\geq \exp\left(-\int_{[t_0,t)} |\widehat{c}(s)| \, \mathrm{d}\, g(s)\right) \geq \exp\left(-\int_{[t_0,t_0+T)} |\widehat{c}(s)| \, \mathrm{d}\, g(s)\right). \end{aligned}$$

Observe that $m := \exp\left(-\int_{[t_0,t_0+T)} |\widehat{c}(s)| \, \mathrm{d}\, g(s)\right) > 0$. Hence,

$$|H(t)| \leq \frac{1}{m} \frac{|h(t)|}{|1 - d(t)\Delta^+ g(t)|}, \quad t \in [t_0, t_0+T).$$

Therefore, it is enough to show that the map $\overline{h} : [t_0, t_0+T) \to \mathbb{R}$, defined as

$$\overline{h}(t) = \frac{h(t)}{1 - d(t)\Delta^+ g(t)}, \quad t \in [t_0, t_0+T),$$

is g-integrable to prove that $H \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$. In order to see that $\overline{h}$ is g-integrable, observe that the set $A = \{t \in [t_0, t_0+T) : d(t)\Delta^+ g(t) > 1/2\}$ has finite cardinality as

$$\sum_{t \in A} \frac{1}{2} < \sum_{t \in [t_0,t_0+T) \cap D_g} |d(t)\Delta^+ g(t)| \leq \int_{[t_0,t_0+T)} |d(s)| \, \mathrm{d}\, g(s) < +\infty.$$

As a consequence, we have that $|\overline{h}(t)| \leq 2|h(t)|$ for all $t \in [t_0, t_0 + T) \setminus A$, from which the g-integrability of $\overline{h}$ follows. Hence, $H \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$. Now, Theorem 2.9 yields that x is g-absolutely continuous on $[t_0, t_0 + T]$. Hence, all that is left to do is to check that x solves (3.1).

By definition, we have that $x(t_0) = x_0$. Furthermore, (i) in Proposition 2.5 and Theorems 2.9 and 3.4, ensure that for g-a.a. $t \in [t_0, t_0 + T)$,

$$\begin{aligned} x'_g(t) &= -d(t)\widehat{e}_{-d}(t,t_0)\left(x_0 + \int_{[t_0,t)} H(s)\,\mathrm{d}\,g(s)\right) + \widehat{e}_{-d}(t,t_0)H(t) - \widehat{e}_{-d}(t,t_0)d(t)H(t)\Delta^+ g(t) \\ &= -d(t)x(t) + \widehat{e}_{-d}(t,t_0)H(t)(1 - d(t)\Delta^+ g(t)) = -d(t)x(t) + h(t), \end{aligned}$$

i.e. x solves (3.1).

Now, the expression of x in (3.9) follows from the extra hypothesis and the definition of $\widehat{e}_{-d}(\cdot, t_0)$ and $e_{-d}(\cdot, t_0)$ □

Observe that, unlike [3, Proposition 6.8], Theorem 3.5 does not guarantee the uniqueness of solution of (3.1) but it offers an explicit expression for a solution of the problem under simpler conditions as condition (3.4) is not required. Nevertheless, using the results in the next section, we will be able to show that (3.1) has a unique solution under the assumption that $d \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$.

4 Gronwall's inequality for Lebesgue–Stieltjes integrals

In this section we turn our attention to the Gronwall inequality in the setting of Lebesgue–Stieltjes integrals. Here, following the ideas [5], we obtain an integral inequality involving the solution of the linear problem with Stieltjes derivatives. This argument improves, as we show later, the corresponding results existing in the literature, such as those in [6, 8, 11, 12, 17].

In order to simplify the proof of the main result of this section, Proposition 4.3, we include the following result. By doing this, we can also reflect on the meaning of Proposition 4.1 for the study of the corresponding linear equation in (3.2).

Proposition 4.1. *Let $c \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$ be such that $1 + c(t)\Delta^+ g(t) > 0$ for all $t \in [t_0, t_0 + T) \cap D_g$. Then the map $h : [t_0, t_0 + T] \to \mathbb{R}$, defined as*

$$h(t) = (e_c(t, t_0))^{-1}, \quad t \in [t_0, t_0 + T], \tag{4.1}$$

is well-defined, g-absolutely continuous on $[t_0, t_0 + T]$ and

$$h'_g(t) = \frac{-c(t)}{e_c(t,t_0)(1 + c(t)\Delta^+ g(t))}, \quad g\text{-a.a. } t \in [t_0, t_0 + T). \tag{4.2}$$

Proof. Define $h_1(t) = e_c(t, t_0)$, $t \in [t_0, t_0 + T]$. Since h_1 is g-absolutely continuous on $[t_0, t_0 + T]$, it has bounded variation on that interval (see [10, Proposition 5.3]) and thus, it is bounded on $[t_0, t_0 + T]$. In particular, if we take

$$m := \exp\left(-\int_{[t_0,t_0+T)} |\widetilde{c}(s)|\,\mathrm{d}\,g(s)\right), \quad M := \exp\left(\int_{[t_0,t_0+T)} |\widetilde{c}(s)|\,\mathrm{d}\,g(s)\right),$$

where $\widetilde{c}$ is the modified function in (3.6), we have that $0 < m \leq h(t) \leq M < +\infty$, $t \in [t_0, t_0 + T]$. Hence, taking $h_2(t) = 1/t$, $t \in [m, M]$, we can rewrite h as $h(t) = h_2 \circ h_1$, which shows that it is well-defined. Now, as in the classical setting, this is enough to ensure that h is g-absolutely continuous on $[t_0, t_0 + T]$, see [3, Proposition 5.3].

Let $t \in [t_0, t_0 + T)$ be such that $h'_g(t)$ exists. If $t \notin D_g$, then by the chain rule, [10, Theorem 2.3],

$$h'_g(t) = h'_2(h_1(t))(h_2)'_g(t) = \frac{-1}{(e_c(t,t_0))^2} e_c(t,t_0)c(t) = \frac{-c(t)}{e_c(t,t_0)},$$

which coincides with (4.2) since $\Delta^+ g(t) = 0$. On the other hand, if $t \in D_g$, using Remarks 2.3 and 3.3 we have that

$$h'_g(t) = \frac{(e_c(t^+,t_0))^{-1} - (e_c(t,t_0))^{-1}}{\Delta^+ g(t)} = \frac{(1 + c(t)\Delta^+ g(t))^{-1} - 1}{e_c(t,t_0)\Delta^+ g(t)} = \frac{-c(t)}{e_c(t,t_0)(1 + c(t)\Delta^+ g(t))}$$

which concludes the proof. □

Remark 4.2. Observe that Proposition 4.1 shows that, under the corresponding hypotheses, $(e_c(t,t_0))^{-1}$ solves the Stieltjes differential equation $x'_g(t) = -c(t)x(t)$ except at the discontinuity points of the derivator, presenting the limitations that the map in (3.3) had. In order to obtain an equality at those points, one would have to modify the map c in an analogous way to (3.6), which would lead to Theorem 3.2 under the corresponding hypotheses for $-c$.

As we mentioned before, Proposition 4.1 allows us to derive a version of Gronwall's inequality in the context of Lebesgue–Stieltjes integrals. Naturally, in this context, the exponential map involved in the inequality is the one in (3.7). However, as we will see later, we can obtain a different version of Gronwall's inequality involving the usual exponential map. Let us state and prove our first version of Gronwall's inequality for the Lebesgue–Stieltjes integral.

Proposition 4.3. *Let $u, K, L : [t_0, t_0 + T) \to [0, +\infty)$ be such that $L, K \cdot L, u \cdot L \in \mathcal{L}^1_g([t_0, t_0 + T), [0, +\infty))$. If*

$$u(t) \le K(t) + \int_{[t_0,t)} L(s)u(s)\, \mathrm{d}\, g(s), \quad t \in [t_0, t_0 + T), \tag{4.3}$$

then

$$u(t) \le K(t) + \int_{[t_0,t)} K(s)L(s) \exp\left(\int_{[s,t)} \widetilde{L}(r)\, \mathrm{d}\, g(r) \right) \mathrm{d}\, g(s), \quad t \in [t_0, t_0 + T), \tag{4.4}$$

where $\widetilde{L}$ is the modified function in (3.6). Moreover, if the map $\varphi : [t_0, t_0 + T) \to \mathbb{R}$, defined as $\varphi(t) = K(t)(1 + L(t)\Delta^+ g(t))$, is nondecreasing, then

$$u(t) \le \varphi(t) e_L(t, t_0), \quad t \in [t_0, t_0 + T). \tag{4.5}$$

Proof. First, observe that $1 + L(t)\Delta^+ g(t) > 0$ for all $t \in [t_0, t_0 + T) \cap D_g$. Therefore, the maps $\widetilde{L}$ and $e_L(\cdot, t_0)$ are well-defined.

Define $U(t) = \int_{[t_0,t)} L(s)u(s)\, \mathrm{d}\, g(s)$, $t \in [t_0, t_0 + T]$. It follows from the hypotheses and Theorem 2.9 that U is well-defined, g-absolutely continuous on $[t_0, t_0 + T]$ and

$$U'_g(t) = L(t)u(t), \quad g\text{-a.a. } t \in [t_0, t_0 + T).$$

Let $h : [t_0, t_0 + T] \to \mathbb{R}$ be as in (4.1) for $c = L$ and define $v(t) = U(t)h(t)$, $t \in [t_0, t_0 + T]$. This is enough to ensure that v is g-absolutely continuous on $[t_0, t_0 + T]$, which guarantees that $v'_g(t)$ exists g-almost everywhere in $[t_0, t_0 + T)$.

Given $t \in [t_0, t_0+T)$ such that $v_g'(t)$ exists, Propositions 2.5 and 4.1 yield

$$\begin{aligned}
v_g'(t) &= U_g'(t)(h(t) + h_g'(t)\Delta^+ g(t)) + h_g'(t)U(t) \\
&= u(t)L(t)\left(h(t) - \frac{L(t)h(t)}{1+L(t)\Delta^+ g(t)}\Delta^+ g(t)\right) - \frac{L(t)h(t)}{1+L(t)\Delta^+ g(t)}U(t) \\
&= u(t)L(t)h(t)\frac{1}{1+L(t)\Delta^+ g(t)} - \frac{L(t)h(t)U(t)}{1+L(t)\Delta^+ g(t)} \\
&= \frac{L(t)h(t)}{1+L(t)\Delta^+ g(t)}\left(u(t) - \int_{[t_0,t)} L(s)u(s)\,\mathrm{d}\,g(s)\right).
\end{aligned}$$

Thus, inequality (4.3) and the fact that $1 + L(t)\Delta^+ g(t) \geq 1$ for all $t \in [t_0, t_0+T)$, ensure that

$$v_g'(t) \leq \frac{K(t)L(t)h(t)}{1+L(t)\Delta^+ g(t)} \leq K(t)L(t)h(t), \quad g\text{-a.a. } t \in [t_0, t_0+T).$$

Therefore, it follows from Fundamental Theorem of Calculus for the Lebesgue–Stieltjes integral, Theorem 2.7, that

$$v(t) = v(t_0) + \int_{[t_0,t)} v_g'(s)\,\mathrm{d}\,g(s) \leq \int_{[t_0,t)} K(s)L(s)h(s)\,\mathrm{d}\,g(s), \quad t \in [t_0, t_0+T]$$

and, as a consequence, for all $t \in [t_0, t_0+T]$ we have

$$\begin{aligned}
\int_{[t_0,t)} L(s)u(s)\,\mathrm{d}\,g(s) &= e_L(t,t_0)v(t) \\
&\leq e_L(t,t_0)\int_{[t_0,t)} K(s)L(s)h(s)\,\mathrm{d}\,g(s) \\
&= e_L(t,t_0)\int_{[t_0,t)} K(s)L(s)\left(e_L(s,t_0)\right)^{-1}\mathrm{d}\,g(s) \\
&= \int_{[t_0,t)} K(s)L(s)\exp\left(\int_{[s,t)} \widetilde{L}(r)\,\mathrm{d}\,g(r)\right)\mathrm{d}\,g(s).
\end{aligned}$$

Thus, it follows from 4.3 that

$$u(t) \leq K(t) + \int_{[t_0,t)} K(s)L(s)\exp\left(\int_{[s,t)} \widetilde{L}(r)\,\mathrm{d}\,g(r)\right)\mathrm{d}\,g(s), \quad t \in [t_0, t_0+T);$$

that is, (4.4) holds.

To prove (4.5), for each $t \in [t_0, t_0+T)$, define

$$\psi_t(s) = \exp\left(\int_{[s,t)} \widetilde{L}(r)\,\mathrm{d}\,g(r)\right) = \frac{e_L(t,t_0)}{e_L(s,t_0)}, \quad s \in [t_0, t].$$

Then, it follows from (4.4) that for all $t \in [t_0, t_0+T)$,

$$\begin{aligned}
u(t) &\leq K(t) + \int_{[t_0,t)} K(s)L(s)\psi_t(s)\,\mathrm{d}\,g(s) \\
&\leq K(t)(1+L(t)\Delta^+ g(t)) + \int_{[t_0,t)} K(s)(1+L(s)\Delta^+ g(s))\frac{L(s)\psi_t(s)}{1+L(s)\Delta^+ g(s)}\,\mathrm{d}\,g(s).
\end{aligned}$$

Now, since $\varphi(t) = K(t)(1+L(t)\Delta^+ g(t))$ is nondecreasing, we have that

$$u(t) \leq \varphi(t)\left(1 + \int_{[t_0,t)} \frac{L(s)\psi_t(s)}{1+L(s)\Delta g(s)}\,\mathrm{d}\,g(s)\right), \quad t \in [t_0, t_0+T].$$

On the other hand, Proposition 4.1 ensures that for all $t \in [t_0, t_0 + T)$, the map ψ_t is g-absolutely continuous on $[t_0, t]$ and

$$(\psi_t)'_g(s) = \frac{-L(s)}{1 + L(s)\Delta g(s)}\psi_t(s) \quad g\text{-a.a. } s \in [t_0, t).$$

This fact, together with the Fundamental Theorem of Calculus, Theorem 2.7, yields that

$$u(t) \le \varphi(t)\left(1 - \int_{[t_0,t)} (\psi_t)'_g(s) \, \mathrm{d}\, g(s)\right) = \varphi(t)\left(1 - (\psi_t(t) - \psi_t(t_0))\right) = \varphi(t)\psi_t(t_0),$$

for all $t \in [t_0, t_0 + T)$, from which the result follows. □

Remark 4.4. The bound (4.4), under the corresponding hypotheses, is sharp. Indeed, let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function, $K : [t_0, t_0 + T) \to [0, +\infty)$ be constant and L be g-integrable on $[t_0, t_0 + T)$. The map $x(t) = Ke_L(t, t_0)$, $t \in [t_0, t_0 + T]$, is g-absolutely continuous on $[t_0, t_0 + T]$. As a consequence, and with the aid of Theorems 2.7 and 3.2, we have that

$$x(t) = K + \int_{[t_0,t)} L(s)Ke_L(s, t_0) \, \mathrm{d}\, g(s), \quad t \in [t_0, t_0 + T],$$

that is, (4.3) holds. Furthermore, that same expression shows that (4.4) also holds with the equality.

This type of inequalities for Stieltjes integrals already exist in the literature, see for example [6, 8, 11, 12, 17]. Let us briefly discuss the relations between the mentioned references and Proposition 4.3. First, in [12, Theorem 7.5.3], the authors worked in the more general context of the Kurzweil–Stieltjes integral. Nevertheless, the results can be compared in the context of the Lebesgue–Stieltjes integral as the integrability in this sense implies the integrability in the Kurzweil–Stieltjes sense. In that case, we can see that the hypotheses required there are stronger than the ones in Proposition 4.3. Furthermore, it is possible to deduce through our next result, Corollary 4.5, that (4.5) gives a sharper bound than the one [12, Theorem 7.5.3]. A similar argument can be done for [8, Chapter 22], where the authors imposes some condition regarding the length of the jumps that the map g presents to arrive to a similar inequality that is not as sharp as the one provided in Proposition 4.3. The same thing happens when we consider the generalized version of the Gronwall inequality in [17, Theorem 1.40]. For the particular setting in which we recover the usual Gronwall inequality (namely, when $\omega(r) = r$) then we obtain the same inequality as in [12, Theorem 7.5.3], which we have already discussed. Now, for [6, 11], the authors obtained a Gronwall type inequality in the context of a certain family of linear operators. The operators there considered can be the Lebesgue–Stieltjes integrals in this paper. In that case, the authors impose some conditions on the discontinuities of the map g, and moreover, the inequality is expressed using an unknown function introduced in [6], called Gronwall majorant. Hence, in the context of our work, the inequality in Proposition 4.3 provides more information.

Note that (4.5) in Proposition 4.3 becomes the usual Gronwall's inequality when the derivator g is the identity map. Furthermore, as we mentioned before, we can obtain a different Gronwall type inequality involving the usual exponential map, i.e. not involving the modified map in (3.6). However, the bound in Proposition 4.3 is sharper than the one in the following result.

Corollary 4.5. *Let $u, K, L : [t_0, t_0+T) \to [0,+\infty)$ be such that $L, K\cdot L, u\cdot L \in \mathcal{L}^1_g([t_0,t_0+T), [0,+\infty))$. If (4.3) holds, then*

$$u(t) \leq K(t) + \int_{[t_0,t)} K(s)L(s)\exp\left(\int_{[s,t)} L(r)\,\mathrm{d}g(r)\right)\mathrm{d}g(s), \quad t\in[t_0,t_0+T).$$

Moreover, if the map $\varphi(t) = K(t)(1+L(t)\Delta g(t))$ is nondecreasing, then

$$u(t) \leq \varphi(t)\exp\left(\int_{[t_0,t)} L(r)\,\mathrm{d}g(r)\right), \quad t\in[t_0,t_0+T).$$

Proof. Given the inequalities in Proposition 4.3, it is enough to show that $\widetilde{L} \leq L$ on $[t_0,t_0+T)$. Observe that $\widetilde{L} = L$ on $[t_0,t_0+T)\setminus D_g$. Thus, we only need to show the inequality for $[t_0,t_0+T)\cap D_g$.

For $t\in[t_0,t_0+T)\cap D_g$, we have that $1+L(t)\Delta g(t) > 0$. Now, since $\log(1+s)\leq s$ for $s\in(-1,+\infty)$, it follows that

$$\widetilde{L}(t) = \frac{\log\left(1+L(t)\Delta g(t)\right)}{\Delta g(t)} \leq \frac{L(t)\Delta g(t)}{\Delta g(t)} = L(t),$$

which concludes the proof. □

As in the classical setting, Gronwall's inequality allows us to obtain a uniqueness result for a general initial value problem under the assumption that the map defining the problem satisfies a Lipschitz condition. We present this information in the following result.

Theorem 4.6. *Let $X\subset\mathbb{R}^n$, $x_0\in X$ and $f:[t_0,t_0+T)\times X\to\mathbb{R}^n$. If there exists $\tau\in(0,T]$ and $L\in\mathcal{L}^1_g([t_0,t_0+\tau),[0,+\infty))$ such that*

$$\|f(t,x)-f(t,y)\| \leq L(t)\|x-y\|, \quad g\text{-a.a. } t\in[t_0,t_0+\tau), \quad x,y\in X,$$

then the initial value problem

$$x_g'(t) = f(t,x(t)), \quad g\text{-a.a. } t\in[t_0,t_0+T), \quad x(t_0)=x_0, \tag{4.6}$$

has at most one g-absolutely continuous solution on $[t_0,t_0+\tau)$.

Proof. Suppose that $x_1,x_2\in\mathcal{AC}_g([t_0,t_0+\tau],\mathbb{R}^n)$ are two solutions of (4.6) on $[t_0,t_0+\tau)$. It follows from Theorem 2.7 that $f(\cdot,x_i(\cdot))\in\mathcal{L}^1_g([t_0,t_0+\tau),\mathbb{R}^n)$, $i=1,2$. As a consequence, we have that the map $\|f(\cdot,x_1(\cdot))-f(\cdot,x_2(\cdot))\|$ is g-integrable over $[t_0,t_0+\tau)$.

Define $u(t)=\|x_1(t)-x_2(t)\|$, $t\in[t_0,t_0+\tau]$. Clearly, u is nonnegative and bounded on $[t_0,t_0+\tau]$ as x_1 and x_2 are bounded, see [10, Proposition 5.3]. Hence, it follows that $u, u\cdot L\in\mathcal{L}^1_g([t_0,t_0+\tau),[0,+\infty))$. Furthermore, the Fundamental Theorem of Calculus yields that for $t\in[t_0,t_0+\tau]$,

$$\begin{aligned} u(t) &= \left\|\int_{[t_0,t)} f(s,x_1(s))\,\mathrm{d}g(s) - \int_{[t_0,t)} f(s,x_2(s))\,\mathrm{d}g(s)\right\| \\ &\leq \int_{[t_0,t)} \|f(s,x_1(s))-f(s,x_2(s))\|\,\mathrm{d}g(s) \leq \int_{[t_0,t)} L(s)u(s)\,\mathrm{d}g(s). \end{aligned}$$

Hence, (4.3) holds with $K=0$. As a consequence, (4.4) holds for $K=0$, which implies that $u=0$ on $[t_0,t_0+\tau)$, or equivalently, $x_1=x_2$ on that interval. □

We can now combine Theorems 3.5 and 4.6 to obtain the following result which is, to some extend, a revision of [3, Proposition 6.8].

Theorem 4.7. *Let $d, h \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$ be such that $1 - d(t)\Delta g(t) \neq 0$ for all $t \in [t_0, t_0 + T) \cap D_g$. Then the unique g-absolutely continuous solution of* (3.1) *is given by the map in* (3.8)*. If, in particular, $1 - d(t)\Delta g(t) > 0$ for all $t \in [t_0, t_0 + T) \cap D_g$, then the unique g-absolutely continuous solution of* (3.1) *matches* (3.9)*.*

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