Electronic Journal of Qualitative Theory of Differential Equations
2020, No. **73**, 1–23; https://doi.org/10.14232/ejqtde.2020.1.73 www.math.u-szeged.hu/ejqtde/

Influence of singular weights on the asymptotic behavior of positive solutions for classes of quasilinear equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

Maya Chhetri[1], **Pavel Drábek**[✉2] and **Ratnasingham Shivaji**[1]

[1]Department of Mathematics and Statistics, The University of North Carolina at Greensboro, Greensboro, NC 27402, USA
[2]Department of Mathematics and NTIS, University of West Bohemia, Univerzitní 8, CZ-30614 Plzeň, Czech Republic

Received 29 June 2020, appeared 21 December 2020
Communicated by Tibor Krisztin

Abstract. Main objective of this paper is to study positive decaying solutions for a class of quasilinear problems with weights. We consider one dimensional problems on an interval which may be finite or infinite. In particular, when the interval is infinite, unlike the known cases in the history where constant weights force the solution not to decay, we discuss singular weights in the diffusion and reaction terms which produce positive solutions that decay to zero at infinity. We also discuss singular weights that lead to positive solutions not satisfying Hopf's boundary lemma. Further, we apply our results to radially symmetric solutions to classes of problems in higher dimensions, say in an annular domain or in the exterior region of a ball. Finally, we provide examples to illustrate our results.

Keywords: quasilinear problems, singular weights, asymptotic behavior, decaying positive solutions.

2020 Mathematics Subject Classification: 34B18, 34B40, 35B40, 35J60, 35J70.

1 Introduction

We consider the following quasilinear Dirichlet problem with weights

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \sigma(t)f(t,u(t)), \quad t \in (a,b), \\ \lim\limits_{t\to a^+} u(t) = \lim\limits_{t\to b^-} u(t) = 0\,, \end{cases} \tag{1.1}$$

with $p > 1$, $\rho = \rho(t)$ and $\sigma = \sigma(t)$, $t \in (a,b)$ are positive weight functions that are measurable and finite everywhere in (a,b), where $-\infty \le a < b \le \infty$ and $f = f(t,s) : (a,b) \times \mathbb{R} \to \mathbb{R}$

✉Corresponding author. Email: pdrabek@kma.zcu.cz

is continuous. Here we allow the weights ρ and σ to be singular (details are forthcoming in Section 2).

Study of the one dimensional model, such as (1.1), is often helpful to capture the qualitative behavior of the solution in the presence of the weights ρ and σ. Moreover, they provide insights for study of more complex models in higher dimension. Therefore, in this paper we present a careful analysis of the one dimensional problem (1.1), and at the end, also apply the obtained results to study the radially symmetric solutions to a class of problems in the higher dimensional case.

In Section 2, we formulate basic assumptions on weight functions ρ and σ and introduce an appropriate functional setting to study (1.1). In Section 3, we prove a general sub- and supersolution result, Theorem 3.1, using monotone iteration methods. In Section 4 and Section 5, we study two auxiliary problems, solutions of which are used in the construction of sub- and supersolution in order to apply Theorem 3.1. In particular, main results of Section 4 are Theorem 4.3 and Theorem 4.4, and similarly main results of Section 5 are Theorem 5.2 and Theorem 5.3. The asymptotic estimates derived in these theorems are utilized in the construction of a well ordered pair of sub- and supersolution. We obtain rather sharp decay estimates of the first eigenfunction of the p-Laplacian operator with weights in Section 4. These estimates are expressed in terms of the singularity or the degeneracy of the weight ρ, and are of independent interest. In Section 6, we consider the special case $(a, b) = (1, +\infty)$ and weight functions ρ and σ to be of "power type behavior" both near 1 and near $+\infty$. Corollary 6.2 is the special case of Theorem 4.3 and Theorem 4.4, where the asymptotics are expressed in terms of the powers of these weight functions ρ and σ. Similarly, Corollary 6.3 is the special case of Theorem 5.2 and Theorem 5.3. In Section 7, we consider an application of our one dimensional results obtained thus far to a radially symmetric Dirichlet problem for quasilinear PDEs on annular type domains or exterior domains in $\mathbb{R}^N$. In these cases, PDEs transform to special cases of (1.1) with $a > 0$ and $b \leq +\infty$. Therefore, we can reformulate the previous existence result, Corollary 7.2, and asymptotic analysis, Corollaries 7.3–7.6. Two illustrative examples are provided in Section 8. In particular, first we consider a special form of (1.1), and under appropriate assumptions on f, we construct a suitable pair of sub- and supersolution to guarantee the existence of a positive solution with prescribed decay rate at a and b, see Theorem 8.1. Second, we consider an analogous radially symmetric Dirichlet problem for a class of quasilinear PDEs, see Theorem 8.3. When the weights, ρ and σ, have power type behavior, we show that for certain powers, our positive solution cannot satisfy the Hopf maximum principle at the boundary, see Remark 8.5.

2 Notation and functional setting

Let $p > 1$, $p' = \frac{p}{p-1}$ and, $\rho = \rho(t)$ and $\sigma = \sigma(t)$, $t \in (a, b)$ be positive weight functions that are measurable and finite everywhere in (a, b), where $-\infty \leq a < b \leq \infty$. We define the following spaces which will be used throughout the paper. Let

$Y := L^p(a, b; \sigma)$ be the set of all measurable functions $u = u(t)$ in (a, b) satisfying

$$\|u\|_Y := \|u\|_{p,\sigma} = \left(\int_a^b \sigma(t) |u(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} < +\infty;$$

$C_0^\infty(a, b)$ be the set of all smooth functions with a compact support in (a, b);

$X := W_0^{1,p}(a,b;\rho)$ be the closure of $C_0^\infty(a,b)$ with respect to the norm

$$\|u\|_X := \|u\|_{1,p,\rho} = \left(\int_a^b \rho(t)|u'(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} ;$$

$X_L := W_L^{1,p}(a,b;\rho)$ be the set of all functions $u = u(t)$ in (a,b) such that for every compact interval $I \subset (a,b)$, u is absolutely continuous on I, $\lim_{t\to a^+} u(t) = 0$ and $\|u\|_X < \infty$;

$X_R := W_R^{1,p}(a,b;\rho)$ is defined analogously, except requiring $\lim_{t\to b^-} u(t) = 0$.

Properties of function spaces:

If $\sigma \in L^1_{\text{loc}}(a,b)$, then $C_0^\infty(a,b)$ is dense in Y. If $\sigma^{1-p'} \in L^1_{\text{loc}}(a,b)$, then Y is a uniformly convex Banach space. If $\rho^{1-p'} \in L^1_{\text{loc}}(a,b)$, then X, X_L, X_R are uniformly convex Banach spaces, and $\rho \in L^1_{\text{loc}}(a,b)$ implies that $X = X_R \cap X_L$. See [8,11] and [13] for details.

Next two theorems establish sufficient conditions for continuous and compact embeddings between the above defined weighted Sobolev and Lebesgue spaces. The proofs can be found in the book [13, Chapter 1].

Proposition 2.1. *Let*

$$\sup_{a<t<b} \left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} < \infty. \tag{2.1}$$

Then $X_L, X \hookrightarrow Y$ (continuous embedding). Let

$$\sup_{a<t<b} \left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} < \infty. \tag{2.2}$$

Then $X_R, X \hookrightarrow Y$.

Proposition 2.2. *Let*

$$\lim_{\substack{t\to a^+\\ t\to b^-}} \left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0. \tag{2.3}$$

Then $X_L, X \hookrightarrow\hookrightarrow Y$ (compact embedding). Let

$$\lim_{\substack{t\to a^+\\ t\to b^-}} \left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0. \tag{2.4}$$

Then $X_R, X \hookrightarrow\hookrightarrow Y$.

Unless specified otherwise, we always assume that ρ and σ satisfy either (2.3) or (2.4).

For the sake of brevity, we use the same notation for all generic positive constants. In order to avoid confusion, the reader is kindly asked to check the exact meaning of these constants separately in every section.

3 Monotone iterations

A function $u \in X$ is called a weak solution of (1.1) if the integral identity

$$\int_a^b \rho(t)|u'(t)|^{p-2}u'(t)\phi'(t)\mathrm{d}t = \int_a^b \sigma(t)f(t,u(t))\phi(t)\mathrm{d}t \tag{3.1}$$

holds for all test functions $\phi \in X$ with both integrals in (3.1) being finite.

In fact, if ρ and σ are continuous functions in (a,b) then a weak solution $u \in X$ of (1.1) is regular in the following sense (see [9]):

$$\left.\begin{array}{l} u \in C^1(a,b),\ \rho|u'|^{p-2}u' \in C^1(a,b), \text{ the equation (1.1) holds} \\ \text{at every point and the boundary conditions are satisfied} \end{array}\right\}. \tag{3.2}$$

A function $\underline{u} \in X$, such that $\underline{u} \in C^1(a,b)$, $\rho|\underline{u}'|^{p-2}\underline{u}' \in C^1(a,b)$, is called a subsolution of (1.1), if for all $t \in (a,b)$ we have

$$-\left(\rho(t)|\underline{u}'(t)|^{p-2}\underline{u}'(t)\right)' \le \sigma(t)f(t,\underline{u}(t)), \quad t \in (a,b).$$

A supersolution $\overline{u} \in X$ of (1.1) is defined analogously with the reverse inequality. Note that $\underline{u}, \overline{u} \in X$ implies that

$$\lim_{t\to a^+} \underline{u}(t) = \lim_{t\to b^-} \underline{u}(t) = \lim_{t\to a^+} \overline{u}(t) = \lim_{t\to b^-} \overline{u}(t) = 0.$$

We state the following existence theorem.

Theorem 3.1. *Let $\underline{u}, \overline{u} \in X$ be sub- and supersolutions of (1.1) respectively, and $\underline{u} \le \overline{u}$ in (a,b). Assume that there exist constants $C_0 > 0$ and $\eta > 0$ such that the following hold:*

(H1) *$|f(t,s)| \le C_0|s|^{p-1}$ for all $t \in (a,b)$ and all $s \in \mathbb{R}$;*

(H2) *the function $s \mapsto f(t,s) + \eta|s|^{p-2}s$ is increasing on the interval $[\min_{t\in(a,b)} \underline{u}(t), \max_{t\in(a,b)} \overline{u}(t)]$ for all $t \in (a,b)$.*

Then there exist a minimal weak solution $u_{\min}$ and a maximal weak solution $u_{\max}$ of (1.1) such that

$$\underline{u} \le u_{\min} \le u_{\max} \le \overline{u} \quad \text{in } (a,b).$$

Proof. Let $F(z)(t) := \sigma(t)\left(f(t,z(t)) + \eta|z(t)|^{p-2}z(t)\right)$, $z \in Y$. By (H1), Hölder's inequality and the continuity of the Nemytskii operator, $F : Y \to X^*$ (the dual of X) is a continuous map. For $z \in Y$, consider the following quasilinear Dirichlet problem

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' + \eta\sigma(t)|u(t)|^{p-2}u(t) = F(z)(t), & t \in (a,b), \\ \lim\limits_{t\to a^+} u(t) = \lim\limits_{t\to b^-} u(t) = 0. \end{cases} \tag{3.3}$$

Then (3.3) has a unique weak solution $u \in X$. Indeed, (3.3) understood in the weak sense is equivalent to the operator equation

$$J_\eta(u) = F(z) \tag{3.4}$$

where $J_\eta : X \to X^*$ is strictly monotone, continuous and weakly coercive operator. Therefore (3.4) has a unique solution (see [5, Sec. 12.3]) and hence (3.3) has a unique weak solution.

By [8, Lemma 3.3], $J_\eta^{-1} : X^* \to X$ is continuous. Therefore, $T := J_\eta^{-1} \circ F : Y \to X$ is continuous and by the compact embedding $X \hookrightarrow\hookrightarrow Y$, $T : Y \to Y$ is also compact. It is straight forward to check that $u = T(u)$ if and only if $u \in X$ is a weak solution of problem (1.1).

To complete the proof, we show that T is order preserving (monotone increasing) operator on the order interval $[\underline{u}, \overline{u}] \subset X$, and $\underline{u} \le T(\underline{u})$ and $\overline{u} \ge T(\overline{u})$, i.e., $\underline{u}$ and $\overline{u}$ are sub- and supersolutions of T, respectively, see [10, Section 6.3].

Indeed, let $z_1, z_2 \in Y$ satisfying $\underline{u} \le z_1 \le z_2 \le \overline{u}$, and let $u_i = T(z_i)$, $i = 1, 2$. Then

$$-\Big[\big(\rho(t)|u_2'(t)|^{p-2}u_2'(t)\big)' - \big(\rho(t)|u_1'(t)|^{p-2}u_1'(t)\big)'\Big] + \eta\sigma(t)\left[|u_2(t)|^{p-2}u_2(t) - |u_1(t)|^{p-2}u_1(t)\right]$$
$$= \sigma(t)\left(f(t, z_2(t)) + \eta|z_2(t)|^{p-2}z_2(t)\right) - \sigma(t)\left(f(t, z_1(t)) + \eta|z_1(t)|^{p-2}z_1(t)\right) \ge 0 \quad (3.5)$$

in (a, b), by the assumption (H2). We claim $u_1 \le u_2$ in (a, b). Suppose not. Then by continuity of u_1 and u_2, there is a nonempty open interval $(a_1, b_1) \subseteq (a, b)$ such that $u_2(t) < u_1(t)$, $t \in (a_1, b_1)$, $\lim_{t\to a_1, b_1}(u_2(t) - u_1(t)) = 0$. Now, multiply (3.5) in (a_1, b_1) by $u_2 - u_1$, integrate from a_1 to b_1, perform integration by parts in the first two integrals and use $\lim_{t\to a_1, b_1}(u_2(t) - u_1(t)) = 0$ to get

$$\int_{a_1}^{b_1} \rho(t)\left(|u_2'(t)|^{p-2}u_2'(t) - |u_1'(t)|^{p-2}u_1'(t)\right)'(u_2'(t) - u_1'(t))\mathrm{d}t$$
$$+ \eta\int_{a_1}^{b_1} \sigma(t)\left(|u_2(t)|^{p-2}u_2(t) - |u_1(t)|^{p-2}u_1(t)\right)(u_2(t) - u_1(t))\mathrm{d}t \le 0\,.$$

This contradicts the fact that $s \mapsto |s|^{p-2}s$ is strictly increasing. Hence $u_1 \le u_2$. A similar argument as above yields $\underline{u} \le T(\underline{u})$ and $\overline{u} \ge T(\overline{u})$. Hence Theorem 3.1 holds. □

In the next two sections, we investigate special forms of (1.1) whose solutions are used in the construction of an ordered pair of sub- and supersolution in Section 8.

4 Asymptotic analysis of principal eigenfunction

We consider the following quasilinear eigenvalue problem with weights

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \lambda\sigma(t)|u(t)|^{p-2}u(t), \quad t \in (a, b), \\ \lim\limits_{t\to a^+} u(t) = \lim\limits_{t\to b^-} u(t) = 0\,. \end{cases} \tag{4.1}$$

We define eigenvalues and eigenfunctions associated with (4.1) in the usual way.

Taking advantage of the compact embedding, $X \hookrightarrow\hookrightarrow Y$, from Proposition 2.2, we can construct a sequence of variational eigenvalues and corresponding eigenfunctions of (4.1) using the Lusternik–Schnirelman "inf-sup" argument provided ρ and σ satisfy (2.3) or/and (2.4). In particular, we have the following assertions concerning the principal eigenvalue λ_1 and associated principal eigenfunction $\varphi_1 \in X$.

Proposition 4.1. *Let* (2.3) *or* (2.4) *hold. Then*

$$\lambda_1 := \inf_{\substack{u\ne 0\\ u\in X}} \frac{\int_a^b \rho(t)|u'(t)|^p\mathrm{d}t}{\int_a^b \sigma(t)|u(t)|^p\mathrm{d}t} > 0$$

is the principal eigenvalue of (4.1)*, and the infimum is achieved at a unique* $\varphi_1 \in X, \varphi_1 > 0$ *in* (a,b)*,* $\|\varphi_1\|_Y = 1$*. Moreover, if* ρ *and* σ *are continuous weight functions,* φ_1 *enjoys regularity properties* (3.2).

The proof follows from standard arguments, see for example, [1–3, 8, 12, 14].

Remark 4.2. It follows from Rolle's theorem, from the positivity of φ_1 and from the equation

$$\left(\rho(t)|\varphi_1'(t)|^{p-2}\varphi_1'(t)\right)' = -\lambda_1\sigma(t)\varphi_1^{p-1}(t) \quad (< 0), \quad t \in (a,b), \tag{4.2}$$

that there exist $\tilde{a}, \tilde{b} \in (a,b), \tilde{a} \le \tilde{b}$, such that $\varphi_1'(\tilde{a}) = \varphi_1'(\tilde{b}) = 0$, $\varphi_1'(t) > 0$ for all $t \in (a,\tilde{a})$ and $\varphi_1'(t) < 0$ for all $t \in (\tilde{b}, b)$. Notice that it is possible to have $\tilde{a} = \tilde{b}$. This is the case, when, e.g., $\rho = \sigma = 1$ and $-\infty < a < b < +\infty$.

For certain classes of reaction terms f, the principal eigenfunction φ_1 or its suitable modifications very often serve as positive subsolutions to problem (1.1). To establish the ordering between subsolution and supersolution, behavior of subsolution near the boundary of the domain plays a crucial rule. Therefore, the goal of this section is to study asymptotic properties of $\varphi_1(t)$ as $t \to a^+$ and $t \to b^-$.

Theorem 4.3. *Let* ρ *and* σ *be continuous in* (a,b) *and,* $\tilde{a}$ *be as in* Remark 4.2*. Further, assume*

(i) *there exist* $c > 0$, $\varepsilon \in (0, p-1)$ *such that for all* $t \in (a, \tilde{a})$

$$\left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \le c \tag{4.3}$$

and

(ii)

$$\lim_{t\to b^-}\left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0. \tag{4.4}$$

Then there exist $\overline{a} \in (a,\tilde{a}), c_1, c_2, \tilde{c}_2 > 0$ *such that for all* $t \in (a, \overline{a})$ *we have*

$$c_1\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \le \varphi_1(t) \le c_2\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau, \tag{4.5}$$

and

$$c_1\rho^{1-p'}(t) \le \varphi_1'(t) \le \tilde{c}_2\rho^{1-p'}(t). \tag{4.6}$$

Theorem 4.4. *Let* ρ *and* σ *be continuous in* (a,b) *and,* $\tilde{b}$ *be as in* Remark 4.2*. Further, assume*

(i) *there exist* $d > 0$, $\varepsilon \in (0, p-1)$ *such that for all* $t \in (\tilde{b}, b)$

$$\left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \le d \tag{4.7}$$

and

(ii)

$$\lim_{t\to a^+}\left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0. \tag{4.8}$$

Then there exist $\overline{b} \in (\tilde{b}, b)$, $d_1, d_2, \tilde{d}_2 > 0$ *such that for all* $t \in (\overline{b}, b)$ *we have*

$$d_1 \int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \le \varphi_1(t) \le d_2 \int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \tag{4.9}$$

and

$$d_1\rho^{1-p'}(t) \le -\varphi_1'(t) \le \tilde{d}_2\rho^{1-p'}(t)\,. \tag{4.10}$$

Remark 4.5. Condition (4.3) implies that for any $t \in (a,b)$ we have

$$\sigma \in L^1(t,b) \quad \text{and} \quad \rho^{1-p'} \in L^1(a,t)\,.$$

Similarly, condition (4.7) implies that for any $t \in (a,b)$ we have

$$\sigma \in L^1(a,t) \quad \text{and} \quad \rho^{1-p'} \in L^1(t,b)\,.$$

Remark 4.6. $\varepsilon < p-1$ implies that (4.3) and (4.4) yield

$$\lim_{t\to a^+} \left(\int_t^b \sigma(\tau)\mathrm{d}\tau\right)\left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,. \tag{4.11}$$

Similarly, (4.7) and (4.8) yield

$$\lim_{t\to b^-} \left(\int_a^t \sigma(\tau)\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,. \tag{4.12}$$

Since (4.4) and (4.11) are nothing but (2.3), the assumptions of Theorem 4.3 guarantee that $\varphi_1 \in X$ exists, it is well defined, and satisfies the properties specified in Proposition 4.1. Also, since (4.8) and (4.12) are nothing but (2.4), similar conclusion can be drawn for Theorem 4.4 as well.

Remark 4.7. Estimate (4.9) can be found in [7] but its proof contains small gaps. Most gaps are filled in [6] for weights associated with the radial symmetric PDE case, cf. Section 7 of this paper. For completeness, we provide very careful and detailed proof for the general case of weights ρ and σ near the left end point $a \ge -\infty$ of the interval (a,b). The case of the right end point $b \le +\infty$ is similar.

Proof of Theorem 4.3. Let $\varphi_1 \in X$ be the normalized ($\|\varphi_1\|_Y = 1$) and positive principal eigenfunction, the existence of which follows from Proposition 4.1.

We first establish inequalities in (4.6). Integrating (4.2) from $\tau \in (a,\tilde{a})$ to $\tilde{a}$ and using Remark 4.2, we get,

$$\rho(\tau)|\varphi_1'(\tau)|^{p-2}\varphi_1'(\tau) = -\lambda_1 \int_{\tilde{a}}^{\tau} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\,,$$

and hence

$$\varphi_1'(\tau) = \lambda_1^{p'-1}\rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1}. \tag{4.13}$$

Choose $\overline{a} \in (a,\tilde{a})$. Then

$$c_1 := \lambda_1^{p'-1}\left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1} \le \lambda_1^{p'-1}\left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{1}{p(p-1)}}\left(\int_a^b \sigma(\theta)\varphi_1^p(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}} < \infty\,.$$

Thus for $t \in (a, \overline{a})$, we get from (4.13)

$$\varphi_1'(t) \geq \lambda_1^{p'-1} \rho^{1-p'}(t) \left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta) \varphi_1^{p-1}(\theta) \mathrm{d}\theta \right)^{p'-1} = c_1 \rho^{1-p'}(t),$$

establishing the left inequality in (4.6).

We assume for a moment that the right inequality in (4.5) holds and derive from here the right inequality in (4.6). Indeed, using the right inequality from (4.5) in (4.13), for $\tau \in (a, \overline{a})$, we get

$$\begin{aligned}
\varphi_1'(\tau) &\leq c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \right)^{p'-1} \\
&\overset{(4.3)}{\leq} c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_\theta^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{-\frac{p-1}{\varepsilon}} \mathrm{d}\theta \right)^{p'-1} \\
&= \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau)}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left(\int_\tau^{\tilde{a}} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\int_\theta^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{1-\frac{p-1}{\varepsilon}} \mathrm{d}\theta \right)^{p'-1} \\
&= \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'} \rho^{1-p'}(\tau)}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left[\left(\int_{\tilde{a}}^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{1-\frac{p-1}{\varepsilon}} - \left(\int_\tau^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{1-\frac{p-1}{\varepsilon}} \right]^{p'-1} \\
&\leq \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'}}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left(\int_{\tilde{a}}^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{\frac{1}{p-1} - \frac{1}{\varepsilon}} \rho^{1-p'}(\tau) = \tilde{c}_2 \rho^{1-p'}(\tau),
\end{aligned}$$

where

$$\tilde{c}_2 := \frac{c^{\frac{1}{\varepsilon}} c_2 \lambda_1^{1-p'}}{\left(\frac{p-1}{\varepsilon} - 1 \right)^{p'-1}} \left(\int_{\tilde{a}}^b \sigma(\theta_1) \mathrm{d}\theta_1 \right)^{\frac{1}{p-1} - \frac{1}{\varepsilon}} < \infty.$$

The right inequality in (4.6) follows.

Next, we prove the left inequality in (4.5). For $t \in (a, \overline{a})$, we integrate (4.13) from a to t, we get

$$\begin{aligned}
\varphi_1(t) &= \int_a^t \varphi'_1(\tau) \mathrm{d}\tau = \lambda_1^{p'-1} \int_a^t \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}} \sigma(\theta) \varphi_1^{p-1}(\theta) \mathrm{d}\theta \right)^{p'-1} \mathrm{d}\tau \\
&\geq \lambda_1^{p'-1} \left(\int_{\overline{a}}^{\tilde{a}} \sigma(\theta) \varphi_1^{p-1}(\theta) \mathrm{d}\theta \right)^{p'-1} \left(\int_a^t \rho^{1-p'}(\tau) \mathrm{d}\tau \right) = c_1 \int_a^t \rho^{1-p'}(\tau) \mathrm{d}\tau
\end{aligned}$$

and the left inequality of (4.5) follows.

It remains to prove the right inequality in (4.5). This is the most profound part of the proof. We choose $t \in (a, \overline{a})$ and integrate (4.13) from a to t. Then applying Hölder's inequality and using $\left(\int_a^b \sigma(\theta)\varphi_1^p(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}} = \|\varphi_1\|_Y = 1$, we get

$$\begin{aligned}
\varphi_1(t) &= \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1}\mathrm{d}\tau \\
&\le \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p}(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}}\left(\int_\tau^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{p'-1}{p}}\mathrm{d}\tau \\
&\le \lambda_1^{p'-1}\left(\int_a^b \sigma(\theta)\varphi_1^{p}(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{p'-1}{p}}\mathrm{d}\tau \\
&= \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)I_1^{p'-1}(\tau)\mathrm{d}\tau\,,
\end{aligned} \tag{4.14}$$

where

$$I_1(\tau) := \left(\int_\tau^{\tilde{a}} \sigma(\theta)\mathrm{d}\theta\right)^{\frac{1}{p}}.$$

We integrate (4.13) again from a to $t \in (a, \overline{a})$ and use (4.14) to get

$$\begin{aligned}
\varphi_1(t) &= \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\varphi_1^{p-1}(\theta)\mathrm{d}\theta\right)^{p'-1}\mathrm{d}\tau \\
&\le \lambda_1^{p'-1}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^{\tilde{a}} \sigma(\theta)\left(\lambda_1^{p'-1}\int_a^\theta \rho^{1-p'}(\theta_1)I_1^{p'-1}(\theta_1)\mathrm{d}\theta_1\right)^{p-1}\mathrm{d}\theta\right)^{p'-1}\mathrm{d}\tau \\
&= k_2\int_a^t \rho^{1-p'}(\tau)I_2^{p'-1}(\tau)\mathrm{d}\tau\,,
\end{aligned}$$

where $k_2 := \lambda_1^{(p'-1)+(p'-1)^2(p-1)}$ and

$$I_2(\tau) := \int_\tau^{\tilde{a}} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_1)I_1^{p'-1}(\theta_1)\mathrm{d}\theta_1\right)^{p-1}\mathrm{d}\theta\,.$$

By induction, for $n = 3, 4, \ldots$, we get

$$\varphi_1(t) \le k_n\int_a^t \rho^{1-p'}(\tau)I_n^{p'-1}(\tau)\mathrm{d}\tau\,, \tag{4.15}$$

where $k_n := \lambda_1^{(p'-1)+(n-1)(p'-1)^2(p-1)}$ and

$$I_n(\tau) := \int_\tau^{\tilde{a}} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_{n-1})I_{n-1}^{p'-1}(\theta_{n-1})\mathrm{d}\theta_{n-1}\right)^{p-1}\mathrm{d}\theta\,.$$

It suffices to show that there exist $K > 0$ and $n_0 \in \mathbb{N}$, such that for all $\tau \in (a, \overline{a})$ we actually have

$$I_{n_0}(\tau) \le K\,. \tag{4.16}$$

Indeed, once (4.16) is established, then (4.15) and (4.16) would imply the right inequality in (4.5) with $c_2 := k_{n_0}K^{p'-1} > 0$. Therefore, we concentrate on the proof of (4.16) with certain $K > 0$ and $n_0 \in \mathbb{N}$.

We start with the estimate of I_2 (we will denote by $a_1, a_2, \ldots$ the generic positive constants).

$$I_2(\tau) = \int_\tau^{\tilde{a}} \sigma(\theta)\left(\int_a^\theta \rho^{1-p'}(\theta_1)I_1^{p'-1}(\theta_1)\mathrm{d}\theta_1\right)^{p-1}\mathrm{d}\theta$$

$$
\begin{aligned}
&= \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \left(\int_{\theta_1}^{\tilde a} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&\le \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \left(\int_{\theta_1}^{b} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&\overset{(4.3)}{\le} a_1 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_1) \left(\int_a^{\theta_1} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{-\frac{\varepsilon}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&= a_1 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^\theta \frac{\mathrm{d}}{\mathrm{d}\theta_1} \left(\frac{1}{1-\frac{\varepsilon}{p(p-1)}} \right) \left(\int_a^{\theta_1} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{1-\frac{\varepsilon}{p(p-1)}} \mathrm{d}\theta_1 \right)^{p-1} \mathrm{d}\theta \\
&= a_1 \int_\tau^{\tilde a} \sigma(\theta) \left(\frac{1}{1-\frac{\varepsilon}{p(p-1)}} \left(\int_a^{\theta} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{1-\frac{\varepsilon}{p(p-1)}} \right)^{p-1} \mathrm{d}\theta \\
&\le a_2 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_a^{\theta} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{p-1-\frac{\varepsilon}{p}} \mathrm{d}\theta \,.
\end{aligned} \tag{4.17}
$$

Notice that the last inequality holds thanks to $\varepsilon < p(p-1)$. It follows from (4.3) that

$$
\left(\int_a^{\theta} \rho^{1-p'}(\tau_1) \mathrm{d}\tau_1 \right)^{p-1-\frac{\varepsilon}{p}} \le a_3 \left(\int_\theta^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}-\frac{p-1}{\varepsilon}} . \tag{4.18}
$$

Therefore (4.17) and (4.18) yield

$$
\begin{aligned}
I_2(\tau) &\le a_4 \int_\tau^{\tilde a} \sigma(\theta) \left(\int_\theta^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}-\frac{p-1}{\varepsilon}} \mathrm{d}\theta \\
&= a_4 \int_\tau^{\tilde a} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{-1}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \right) \left(\int_\theta^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \mathrm{d}\theta \\
&= \frac{a_4}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \left(\left(\int_\tau^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} - \left(\int_{\tilde a}^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \right) .
\end{aligned} \tag{4.19}
$$

We may assume, without loss of generality, that

$$
\varepsilon \ne \frac{p}{p+1}(p-1) \quad \text{i.e.,} \quad \frac{1}{p} + \frac{\varepsilon-p+1}{\varepsilon} \ne 0 \,.
$$

Therefore, one of the following two cases occurs.

Case 1: $\varepsilon < \frac{p}{p+1}(p-1)$, i.e., $\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon} < 0$. Then it follows from (4.19) that there exists $K > 0$ such that

$$
I_2(\tau) \le - \frac{a_4}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_{\tilde a}^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \le K \,,
$$

i.e., (4.16) holds with $n_0 = 2$ and the proof is complete.

Case 2: $\varepsilon > \frac{p}{p+1}(p-1)$, i.e., $\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon} > 0$. Then it follows from (4.19) that

$$
I_2(\tau) \le \frac{a_4}{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_\tau^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} = a_5 \left(\int_\tau^b \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+\frac{\varepsilon-p+1}{\varepsilon}} . \tag{4.20}
$$

We continue our iterations:

$$
\begin{aligned}
I_3(\tau) &= \int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_2) I_2^{p'-1}(\theta_2) \mathrm{d}\theta_2 \right)^{p-1} \mathrm{d}\theta \\
&\overset{(4.20)}{\leq} a_6 \int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_2) \left(\int_{\theta_2}^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p(p-1)} + \frac{\varepsilon-p+1}{\varepsilon(p-1)}} \mathrm{d}\theta_2 \right)^{p-1} \mathrm{d}\theta \\
&\overset{(4.3)}{\leq} a_7 \int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\theta_2) \left(\int_a^{\theta_2} \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1 \right)^{-\frac{\varepsilon}{p(p-1)} - \frac{\varepsilon-p+1}{p-1}} \mathrm{d}\theta_2 \right)^{p-1} \mathrm{d}\theta \\
&= a_7 \int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_a^\theta \frac{1}{1 - \frac{\varepsilon}{p(p-1)} - \frac{\varepsilon-p+1}{\varepsilon(p-1)}} \frac{\mathrm{d}}{\mathrm{d}\theta_2} \left(\int_a^{\theta_2} \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1 \right)^{1-\frac{\varepsilon}{p(p-1)} - \frac{\varepsilon-p+1}{p-1}} \mathrm{d}\theta_2 \right)^{p-1} \mathrm{d}\theta \\
&= a_7 \int_\tau^{\tilde{a}} \sigma(\theta) \left(\frac{1}{1 - \frac{\varepsilon}{p(p-1)} - \frac{\varepsilon-p+1}{\varepsilon(p-1)}} \left(\int_a^\theta \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1 \right)^{1-\frac{\varepsilon}{p(p-1)} - \frac{\varepsilon-p+1}{p-1}} \right)^{p-1} \mathrm{d}\theta \\
&\leq a_8 \int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_a^\theta \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1 \right)^{2p-2-\frac{\varepsilon}{p}-\varepsilon} \mathrm{d}\theta\,.
\end{aligned}
$$

Notice that $\varepsilon \in (0, p-1)$ and $p > 1$ yield the last inequality thanks to $\varepsilon < \frac{2p}{p+1}(p-1)$, i.e., $1 - \frac{\varepsilon}{p(p-1)} - \frac{\varepsilon-p+1}{p-1} > 0$. It follows from (4.3) that

$$
\left(\int_a^\theta \rho^{1-p'}(\tau_1)\mathrm{d}\tau_1 \right)^{2p-2-\frac{\varepsilon}{p}-\varepsilon} \leq a_9 \left(\int_\theta^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p}+1-\frac{1}{\varepsilon}(2p-2)}.
$$

Therefore,

$$
\begin{aligned}
I_3(\tau) &\leq a_{10} \int_\tau^{\tilde{a}} \sigma(\theta) \left(\int_\theta^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p}+1-\frac{1}{\varepsilon}(2p-2)} \mathrm{d}\theta \\
&= a_{10} \int_\tau^{\tilde{a}} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\frac{-1}{\frac{1}{p} + 2\frac{\varepsilon-p+1}{\varepsilon}} \right) \left(\int_\theta^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}} \mathrm{d}\theta \\
&= \frac{a_{10}}{\frac{1}{p} + 2\frac{\varepsilon-p+1}{\varepsilon}} \left(\left(\int_\tau^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}} - \left(\int_{\tilde{a}}^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}} \right). \qquad (4.21)
\end{aligned}
$$

Without loss of generality, we may assume $\varepsilon \neq \frac{2p}{2p+1}(p-1)$. Therefore, we distinguish between two cases again.

<u>Case 1:</u> $\varepsilon < \frac{2p}{2p+1}(p-1)$ i.e., $\frac{1}{p} + 2\frac{\varepsilon-p+1}{\varepsilon} < 0$. Then it follows from (4.21) that there exists $K > 0$ such that

$$
I_3(\tau) \leq -\frac{a_{10}}{\frac{1}{p} + 2\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_{\tilde{a}}^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}} \leq K,
$$

i.e., (4.16) holds with $n_0 = 3$ and the proof is complete.

<u>Case 2:</u> $\varepsilon > \frac{2p}{2p+1}(p-1)$ i.e., $\frac{1}{p} + 2\frac{\varepsilon-p+1}{\varepsilon} > 0$. Then it follows from (4.21) that

$$
I_3(\tau) \leq \frac{a_{10}}{\frac{1}{p} + 2\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_\tau^b \sigma(\tau_1)\mathrm{d}\tau_1 \right)^{\frac{1}{p}+2\frac{\varepsilon-p+1}{\varepsilon}}
$$

and we continue iterations.

Repeating the argument n times, we may assume without loss of generality, that $\varepsilon \neq \frac{np}{np+1}(p-1)$. We have then two different cases.

Case 1: $\varepsilon < \frac{np}{np+1}(p-1)$ i. e., $\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon} < 0$. Then there exists $K > 0$ such that

$$I_{n+1}(\tau) \leq -\frac{a_{11}}{\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_{\tilde{a}}^{b} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+n\frac{\varepsilon-p+1}{\varepsilon}} \leq K,$$

i.e., (4.16) holds with $n_0 = n$ and the proof is complete.

Case 2: $\varepsilon > \frac{np}{np+1}(p-1)$ i.e., $\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon} > 0$. Then

$$I_{n+1}(\tau) \leq \frac{a_{11}}{\frac{1}{p} + n\frac{\varepsilon-p+1}{\varepsilon}} \left(\int_{\tau}^{b} \sigma(\tau_1) \mathrm{d}\tau_1 \right)^{\frac{1}{p}+n\frac{\varepsilon-p+1}{\varepsilon}}$$

and we continue iterations.

Notice that for a given $\varepsilon \in (0, p-1)$, the second case does not occur after finite number of steps due to $\lim_{n\to\infty} \frac{np}{np+1} = 1$. Therefore the proof is complete after a finite number of iterations. This completes the proof of Theorem 4.3. □

The proof of Theorem 4.4 follows by using analogous arguments.

5 Asymptotic analysis of an auxiliary function

A suitable multiple of the solution $e = e(t)$ of the auxiliary Dirichlet problem

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \sigma(t), & t \in (a,b), \\ \lim\limits_{t\to a^+} u(t) = \lim\limits_{t\to b^-} u(t) = 0 \end{cases} \tag{5.1}$$

with $\sigma \in X^*$ serves as a positive supersolution of the problem (1.1). If we interpret (5.1) in the weak sense, then it is equivalent to the operator equation

$$J(u) = \sigma \tag{5.2}$$

where $J : X \to X^*$ is strictly monotone, continuous and weakly coercive operator. Therefore, there exists unique $e = e(t) \in X$ which is a solution of (5.2) and hence a weak solution of (5.1). Moreover, when $\sigma = \sigma(t)$ and $\rho = \rho(t)$ are continuous in (a,b) then the solution e enjoys regularity properties (3.2) of Section 3.

Moreover, since $\sigma > 0$ in (a,b), it follows from (5.1) that $e(t) > 0$ in (a,b). In addition, there exist $\tilde{a}_e, \tilde{b}_e \in (a,b), \tilde{a}_e \leq \tilde{b}_e$ such that $e'(\tilde{a}_e) = e'(\tilde{b}_e) = 0$, $e'(t) > 0$ for all $t \in (a, \tilde{a}_e)$ and $e'(t) < 0$ for all $t \in (\tilde{b}_e, b)$.

Remark 5.1. Notice that $\sigma \in L^1(a,b)$ is a sufficient condition for $\sigma \in X^*$. Also observe that $\sigma \in L^1(a,b)$ implies that (4.3) and (4.7) hold for an arbitrary $\varepsilon \in (0, p-1)$.

The following assertion is a counterpart of Theorem 4.3.

Theorem 5.2. *Let σ, ρ be continuous in (a,b), $\sigma \in L^1(a,b)$ and $\rho^{1-p'} \in L^1(a,t)$ for any $t \in (a,b)$. Let $\tilde{a}_e$ be associated with $e = e(t)$, and $\varepsilon \in (0, p-1)$. Then there exist $\bar{a}_e \in (a, \tilde{a}_e), c_1, c_2, \tilde{c}_2 > 0$ such that for all $t \in (a, \bar{a}_e)$, we have*

$$c_1 \int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \le e(t) \le c_2 \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}} \tag{5.3}$$

and

$$c_1\rho^{1-p'}(t) \le e'(t) \le \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}}. \tag{5.4}$$

Similarly, the following assertion is a counterpart of Theorem 4.4.

Theorem 5.3. *Let σ, ρ be continuous in (a,b), $\sigma \in L^1(a,b)$ and $\rho^{1-p'} \in L^1(t,b)$ for any $t \in (a,b)$. Let $\tilde{b}_e$ be associated with $e = e(t)$, and $\varepsilon \in (0, p-1)$. Then there exist $\bar{b}_e \in (\tilde{b}_e, b), d_1, d_2, \tilde{d}_2 > 0$ such that for all $t \in (\bar{b}_e, b)$, we have*

$$d_1 \int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \le e(t) \le d_2 \left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}} \tag{5.5}$$

and

$$d_1\rho^{1-p'}(t) \le -e'(t) \le \tilde{d}_2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_t^b \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}}. \tag{5.6}$$

Proof of Theorem 5.2. It follows by directly integrating the equation in (5.1) from $\tilde{a}_e$ to $t \in (a, \bar{a}_e)$ with $\bar{a}_e < \tilde{a}_e$ that

$$e'(t) = \rho^{1-p'}(t) \left(\int_t^{\tilde{a}_e} \sigma(\tau)\mathrm{d}\tau \right)^{p'-1} \ge c_1\rho^{1-p'}(t), \tag{5.7}$$

with $c_1 := \left(\int_{\bar{a}_e}^{\tilde{a}_e} \sigma(\tau)\mathrm{d}\tau \right)^{p'-1}$, i.e., the left inequality in (5.4) holds. Now, integrating the equality in (5.7) from a to $t \in (a, \bar{a}_e)$ yields

$$\begin{aligned} e(t) = \int_a^t e'(\tau)\mathrm{d}\tau &= \int_a^t \rho^{1-p'}(\tau) \left(\int_\tau^{\tilde{a}_e} \sigma(\theta)\mathrm{d}\theta \right)^{p'-1} \mathrm{d}\tau \\ &\ge \left(\int_{\bar{a}_e}^{\tilde{a}_e} \sigma(\theta)\mathrm{d}\theta \right)^{p'-1} \int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau = c_1 \int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \end{aligned}$$

and the left inequality in (5.3) follows.

In view of Remark 5.1, the condition (4.3) is satisfied for any $\varepsilon \in (0, p-1)$. For $\varepsilon \in (0, p-1)$ arbitrary, and for $t \in (a, \bar{a}_e)$, we have

$$\begin{aligned} e'(t) &= \rho^{1-p'}(t) \left(\int_t^{\tilde{a}_e} \sigma(\tau)\mathrm{d}\tau \right)^{p'-1} \le \rho^{1-p'}(t) \left(\int_t^b \sigma(\tau)\mathrm{d}\tau \right)^{p'-1} \\ &\overset{(4.3)}{\le} c^{p'-1}\rho^{1-p'}(t) \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{-\varepsilon(p'-1)} = \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}}, \end{aligned} \tag{5.8}$$

where $\tilde{c}_2 := \frac{c^{p'-1}}{1-\frac{\varepsilon}{p-1}}$. Thus the right inequality in (5.4) holds. Finally, integrating (5.8) from a to $t \in (a, \bar{a}_e)$, we establish the right inequality in (5.3). Indeed,

$$e(t) = \int_a^t e'(\tau)\mathrm{d}\tau \le \int_a^t \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\int_a^\tau \rho^{1-p'}(\theta)\mathrm{d}\theta \right)^{1-\frac{\varepsilon}{p-1}} \mathrm{d}\tau = c_2 \left(\int_a^t \rho^{1-p'}(\tau)\mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}},$$

where $c_2 := c^{p'-1}$. The proof of Theorem 5.2 is complete. □

The proof of Theorem 5.3 is similar.

6 Weight functions of special type

Here we consider the case $a = 1$, $b = +\infty$ and the following pair of continuous weight functions ρ and σ defined on $(1, +\infty)$:

$$\rho(t) = \begin{cases} (t-1)^{\alpha_1}, & t \in (1,2)\,, \\ 1, & t \in [2,10]\,, \\ \left(\frac{10}{t}\right)^{\alpha_\infty}, & t \in (10,+\infty)\,, \end{cases} \quad \text{and} \quad \sigma(t) = \begin{cases} (t-1)^{\beta_1}, & t \in (1,2)\,, \\ 1, & t \in [2,10]\,, \\ \left(\frac{10}{t}\right)^{\beta_\infty}, & t \in (10,+\infty)\,. \end{cases} \tag{6.1}$$

The weight functions ρ and σ have "power type behavior" prescribed by α_1 and β_1 near $a = 1$ and by α_∞ and β_∞ near $b = +\infty$. The following assertion is an immediate consequence of (6.1), (2.3) and (2.4).

Lemma 6.1. *Condition* (2.3) *holds if and only if*

$$\alpha_1 < \min\{\beta_1 + p, p-1\} \quad \textit{and} \quad \beta_\infty > \max\{\alpha_\infty + p, 1\}\,. \tag{6.2}$$

Condition (2.4) *holds if and only if*

$$\beta_1 > \max\{\alpha_1 - p, -1\} \quad \textit{and} \quad \alpha_\infty < \min\{\beta_\infty - p, 1-p\}\,. \tag{6.3}$$

In particular,

$$(6.2) \Rightarrow X_L, X \hookrightarrow\hookrightarrow Y \quad \textit{and} \quad (6.3) \Rightarrow X_R, X \hookrightarrow\hookrightarrow Y\,.$$

In this section, we discuss an application of Theorems 4.3, 4.4, 5.2 and 5.3. At first, we concentrate on assumptions (4.3) and (4.7) and interpret an asymptotic behavior of φ_1 given by (4.5), (4.6), (4.9) and (4.10) in terms of $\alpha_1, \alpha_\infty, \beta_1$ and β_∞.

Corollary 6.2. *Let us assume that* (6.2) *holds and* $\varphi_1 \in X$ *be the principal eigenfunction of* (4.1) *with* ρ *and* σ *given by* (6.1). *Then there exist* $\bar{a} > 1, c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ *such that for all* $t \in (1, \bar{a})$ *we have*

$$c_1(t-1)^{1-\frac{\alpha_1}{p-1}} \leq \varphi_1(t) \leq c_2(t-1)^{1-\frac{\alpha_1}{p-1}}$$

and

$$\tilde{c}_1(t-1)^{-\frac{\alpha_1}{p-1}} \leq \varphi_1'(t) \leq \tilde{c}_2(t-1)^{-\frac{\alpha_1}{p-1}}\,.$$

Similarly, assume that (6.3) *holds. Then there exist* $\bar{b} > 1, d_1, \tilde{d}_1, d_2, \tilde{d}_2 > 0$ *such that for all* $t \in (\bar{b}, +\infty)$ *we have*

$$d_1 t^{1+\frac{\alpha_\infty}{p-1}} \leq \varphi_1(t) \leq d_2 t^{1+\frac{\alpha_\infty}{p-1}}$$

and

$$\tilde{d}_1 t^{\frac{\alpha_\infty}{p-1}} \leq -\varphi_1'(t) \leq \tilde{d}_2 t^{\frac{\alpha_\infty}{p-1}}\,.$$

Proof. The proof consists of verifying the assumptions of Theorem 4.3 and Theorem 4.4 in the case of the weight functions, ρ and σ, given by (6.1). Indeed, if we assume (6.2) then we distinguish between two cases. In the case $\beta_1 \geq -1$ the condition (4.3) holds with arbitrary $\varepsilon \in (0, p-1)$, and in the case $\beta_1 < -1$ we can take any $\varepsilon \in \left(\frac{(p-1)(\beta_1+1)}{\alpha_1 - p + 1}, p-1\right)$. Similarly, if we assume (6.3), condition (4.7) holds with arbitrary $\varepsilon \in (0, p-1)$ in the case $\beta_\infty \geq 1$, and any $\varepsilon \in \left(\frac{(p-1)(1-\beta_\infty)}{1-p-\alpha_\infty}, p-1\right)$ in the case $\beta_\infty < 1$. □

Secondly, we discuss the asymptotic behavior of solution e of (5.1). Notice that in order to guarantee $\sigma \in L^1(1,+\infty)$, we must assume $\beta_1 > -1$ and $\beta_\infty > 1$. Then condition (6.2) reduces to

$$\alpha_1 < p-1 \quad \text{and} \quad \beta_\infty > \max\{\alpha_\infty + p, 1\} \tag{6.4}$$

and condition (6.3) reduces to

$$\beta_1 > \max\{\alpha_1 - p, -1\} \quad \text{and} \quad \alpha_\infty < 1-p\,. \tag{6.5}$$

Corollary 6.3. *Let us assume that* (6.4) *holds and $e \in X$ is a weak solution of* (5.1) *with ρ and σ given by* (6.1)*. Let $\varepsilon \in (0, p-1)$ be arbitrary. Then there exist $\overline{a}_e > 1$, $c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ such that for all $t \in (1, \overline{a}_e)$ we have*

$$c_1(t-1)^{1-\frac{\alpha_1}{p-1}} \le e(t) \le c_2(t-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}$$

and

$$\tilde{c}_1(t-1)^{-\frac{\alpha_1}{p-1}} \le e'(t) \le \tilde{c}_2(t-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Similarly, assume that (6.5) *holds, and $\varepsilon \in (0, p-1)$ is arbitrary. Then there exist $\overline{b}_e > 1$, d_1, $\tilde{d}_1$, d_2, $\tilde{d}_2 > 0$ such that for all $t \in (\overline{b}_e, +\infty)$ we have*

$$d_1 t^{1+\frac{\alpha_\infty}{p-1}} \le e(t) \le d_2 t^{\left(1+\frac{\alpha_\infty}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}$$

and

$$\tilde{d}_1 t^{\frac{\alpha_\infty}{p-1}} \le -e'(t) \le \tilde{d}_2 t^{\left(1+\frac{\alpha_\infty}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Remark 6.4. With obvious modifications we can derive analogous assertions if $a, b \in \mathbb{R}$ (i.e., (a,b) is a bounded interval), $a = -\infty, b \in \mathbb{R}$ (i.e., $(a,b) = (-\infty, b)$ is bounded above) and $a = -\infty, b = +\infty$ (i.e., $(a,b) = \mathbb{R}$).

7 Application to partial differential equations

In this section, we will apply the one dimensional results obtained thus far to study the radially symmetric solutions to a class of quasilinear PDEs satisfying Dirichlet boundary conditions. Our results in this section are valid in various domains in $\mathbb{R}^N$ with $N \ge 2$ such as $B_R := \left\{x \in \mathbb{R}^N : |x| < R\right\} \subset \mathbb{R}^N$ where B_R is a ball if $R < +\infty$ and entire $\mathbb{R}^N$ if $R = +\infty$, or

$$A_{R_1}^{R_2} := \left\{x \in \mathbb{R}^N : R_1 < |x| < R_2\right\} \quad \text{for } 0 < R_1 < R_2 \le +\infty$$

where $A_{R_1}^{R_2}$ is an annular domain if $R_2 < +\infty$ and an exterior domain if $R_2 = +\infty$.

Here we focus on radially symmetric solutions to the boundary value problem:

$$\begin{cases} -\operatorname{div}\left(v(|x|)|\nabla u(|x|)|^{p-2}\nabla u(|x|)\right) = w(|x|)f(|x|, u(|x|)), & x \in A_{R_1}^{R_2} \\ u(x) = 0, \quad x \in \partial A_{R_1}^{R_2}, \end{cases} \tag{7.1}$$

where v and w are positive continuous weight functions. After substitution $r = |x|$, the above problem transforms to

$$\begin{cases} -\left(r^{N-1}v(r)|u'(r)|^{p-2}u'(r)\right)' = r^{N-1}w(r)f(r,u(r)), & r \in (R_1, R_2), \\ \lim\limits_{r\to R_1} u(r) = \lim\limits_{r\to R_2} u(r) = 0\,, \end{cases} \tag{7.2}$$

where $f : (R_1, R_2) \times \mathbb{R} \to \mathbb{R}$ is as in Section 3. The problem (7.2) corresponds to (1.1) with the following change of notation:

$$t = r, \quad a = R_1, \quad b = R_2, \quad \rho(t) = r^{N-1}v(r), \quad \sigma(t) = r^{N-1}w(r)\,.$$

We say that a radially symmetric function $u = u(|x|), x \in A_{R_1}^{R_2}$, is a weak solution of problem (7.1) if the function $u = u(r)$, $r \in (R_1, R_2)$, is a weak solution of problem (7.2) in the sense mentioned at the beginning of Section 3. Similarly, using corresponding notions from Section 3, we can define radially symmetric sub- and supersolutions to (7.1).

Natural spaces to study the radially symmetric solutions to problem (7.1) are Sobolev and Lebesgue spaces X and Y of all radially symmetric functions with norms depending on v and w, respectively. More precisely, let $v^{1-p'}, w^{1-p'}, v, w \in L^1_{\mathrm{loc}}(A_{R_1}^{R_2})$. Then X and Y are uniformly convex Banach spaces and $C_0^\infty(A_{R_1}^{R_2})$ is dense in both X and Y. A radial function $u \in Y$ if and only if $u = u(r)$ is a measurable function in (R_1, R_2) satisfying

$$\|u\|_Y = \left(\int_{R_1}^{R_2} r^{N-1} w(r) |u(r)|^p \mathrm{d}r \right)^{\frac{1}{p}} < \infty\,.$$

Similarly, a radial function $u \in X$ if and only if $u = u(r)$ is absolutely continuous on every compact subinterval of (R_1, R_2), $\lim\limits_{r \to R_1} u(r) = \lim\limits_{r \to R_2} u(r) = 0$ and

$$\|u\|_X = \left(\int_{R_1}^{R_2} r^{N-1} v(r) |u'(r)|^p \mathrm{d}r \right)^{\frac{1}{p}} < \infty\,.$$

Obvious change of the notation in (2.1)–(2.4) leads to the following sufficient conditions for continuous and compact embeddings $X \hookrightarrow Y$ and $X \hookrightarrow\hookrightarrow Y$, respectively.

Proposition 7.1.

(A) Let either

$$\sup_{R_1 < r < R_2} \left(\int_r^{R_2} \tau^{N-1} w(\tau) \mathrm{d}\tau \right) \left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau) \mathrm{d}\tau \right)^{p-1} < \infty$$

or

$$\sup_{R_1 < r < R_2} \left(\int_{R_1}^r \tau^{N-1} w(\tau) \mathrm{d}\tau \right) \left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau) \mathrm{d}\tau \right)^{p-1} < \infty$$

hold. Then $X \hookrightarrow Y$.

(B) Let either

$$\lim_{r \to R_1, R_2} \left(\int_r^{R_2} \tau^{N-1} w(\tau) \mathrm{d}\tau \right) \left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau) \mathrm{d}\tau \right)^{p-1} = 0 \tag{7.3}$$

or

$$\lim_{r \to R_1, R_2} \left(\int_{R_1}^r \tau^{N-1} w(\tau) \mathrm{d}\tau \right) \left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau) \mathrm{d}\tau \right)^{p-1} = 0 \tag{7.4}$$

hold. Then $X \hookrightarrow\hookrightarrow Y$.

As a consequence of this compact embedding, the following result follows from Theorem 3.1.

Corollary 7.2. *Let $\underline{u} \in X$ and $\overline{u} \in X$ be subsolution and supersolution of (7.2), respectively, and $\underline{u} \le \overline{u}$ in (R_1, R_2). Let $f : (R_1, R_2) \times \mathbb{R} \to \mathbb{R}$ be as in Section 3. Then there exist a minimal weak solution $u_{\min} \in X$ and a maximal weak solution $u_{\max} \in X$ of (7.2) which satisfy $\underline{u} \le u_{\min} \le u_{\max} \le \overline{u}$ in (R_1, R_2).*

Next, let us consider the eigenvalue problem

$$\begin{cases} -\left(r^{N-1}v(r)|u'(r)|^{p-2}u'(r)\right)' = \lambda r^{N-1}w(r)|u(r)|^{p-2}u(r), & r \in (R_1, R_2), \\ \lim\limits_{r\to R_1} u(r) = \lim\limits_{r\to R_2} u(r) = 0\,. \end{cases} \tag{7.5}$$

Under the assumption (7.3) or (7.4) the principal eigenvalue of (7.5),

$$\lambda_1 := \inf_{\substack{u\neq 0\\ u\in X}} \frac{\int_{R_1}^{R_2} r^{N-1}v(r)|u'(r)|^p \mathrm{d}r}{\int_{R_1}^{R_2} r^{N-1}w(r)|u(r)|^p \mathrm{d}r} > 0$$

is achieved at a unique $\varphi_1 \in X, \varphi_1 > 0$ in (R_1, R_2) and $\|\varphi_1\|_Y = 1$. Asymptotic estimates of φ_1 for $r \to R_1$ and $r \to R_2$ follow from Theorem 4.3 and Theorem 4.4. Indeed, Let $R_1 < \tilde{R}_1 \le \tilde{R}_2 < R_2$ be such that $\varphi_1'(\tilde{R}_1) = \varphi_1'(\tilde{R}_2) = 0$ and $\varphi_1'(r) > 0$ in $(R_1, \tilde{R}_1)$ and $\varphi_1'(r) < 0$ in $(\tilde{R}_2, R_2)$. The existence of $\tilde{R}_1$ and $\tilde{R}_2$ are explained in Remark 4.2. Then due to Theorem 4.3 and Theorem 4.4, we have:

Corollary 7.3. *Let $c > 0$, $\varepsilon \in (0, p-1)$ be such that for all $r \in (R_1, \tilde{R}_1)$*

$$\left(\int_r^{R_2} \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \le c$$

and

$$\lim_{r\to R_2}\left(\int_r^{R_2} \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,.$$

Then there exist $\overline{R}_1 \in (R_1, \tilde{R}_1), c_1, c_2, \tilde{c}_2 > 0$ such that for all $r \in (R_1, \overline{R}_1)$ we have

$$c_1 \int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau \le \varphi_1(r) \le c_2 \int_{R_1}^r \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau$$

and

$$c_1 r^{\frac{1-N}{p-1}}v^{1-p'}(r) \le \varphi_1'(r) \le \tilde{c}_2 r^{\frac{1-N}{p-1}}v^{1-p'}(r)\,.$$

Let $d > 0$, $\varepsilon \in (0, p-1)$ be such that for all $r \in (\tilde{R}_2, R_2)$

$$\left(\int_{R_1}^r \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{\varepsilon} \le d$$

and

$$\lim_{r\to R_1}\left(\int_{R_1}^r \tau^{N-1}w(\tau)\mathrm{d}\tau\right)\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau\right)^{p-1} = 0\,.$$

Then there exist $\overline{R}_2 \in (\tilde{R}_2, R_2), d_1, d_2, \tilde{d}_2 > 0$ such that for all $r \in (\overline{R}_2, R_2)$ we have

$$d_1 \int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau \le \varphi_1(r) \le d_2 \int_r^{R_2} \tau^{\frac{1-N}{p-1}}v^{1-p'}(\tau)\mathrm{d}\tau$$

and

$$d_1 r^{\frac{1-N}{p-1}}v^{1-p'}(r) \le -\varphi_1'(r) \le \tilde{d}_2 r^{\frac{1-N}{p-1}}v^{1-p'}(r)\,.$$

Next, we consider the case $R_1 = 1$, $R_2 = +\infty$, i.e., $A_{R_1}^{R_2} = \overline{B}_1^c$ is exterior of unit ball centered at the origin. Let us consider continuous radial weights v and w defined on $(1, +\infty)$ as follows:

$$v(r) = \begin{cases} (r-1)^{\alpha_1}, & r \in (1,2), \\ 1, & r \in [2,10], \\ \left(\frac{10}{r}\right)^{\alpha_\infty}, & r \in (10,+\infty); \end{cases} \qquad w(r) = \begin{cases} (r-1)^{\beta_1}, & r \in (1,2), \\ 1, & r \in [2,10], \\ \left(\frac{10}{r}\right)^{\beta_\infty}, & r \in (10,+\infty). \end{cases} \tag{7.6}$$

Similarly to Section 6, we can now reformulate the sufficient conditions in Proposition 7.1. We also express conditions stated in Corollary 7.3 in terms of $\alpha_1, \alpha_\infty, \beta_1$ and β_∞. Clearly, now also the dimension $N \geq 2$ will be involved in these conditions. Indeed, condition (7.3) holds if and only if

$$\alpha_1 < \min\{\beta_1 + p, p-1\} \text{ and } \beta_\infty > \max\{\alpha_\infty + p, N\} \tag{7.7}$$

and condition (7.4) holds if and only if

$$\beta_1 > \max\{\alpha_1 - p, -1\} \text{ and } \alpha_\infty < \min\{\beta_\infty - p, N-p\}. \tag{7.8}$$

In particular, the compact embedding $X \hookrightarrow\hookrightarrow Y$ holds if either (7.7) or (7.8) holds. Since v and w are continuous, $\varphi_1(r)$ is regular in the sense of (3.2) from Section 3.

Next, we formulate asymptotic behavior of φ_1, see Corollary 7.3, in the language of powers $\alpha_1, \alpha_\infty, \beta_1$ and β_∞.

Corollary 7.4. *If (7.7) holds, then there exist $\overline{R}_1 > 1, c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ such that for all $r \in (1, \overline{R}_1)$ we have*

$$c_1 (r-1)^{1-\frac{\alpha_1}{p-1}} \leq \varphi_1(r) \leq c_2 (r-1)^{1-\frac{\alpha_1}{p-1}}$$

and

$$\tilde{c}_1 (r-1)^{-\frac{\alpha_1}{p-1}} \leq \varphi_1'(r) \leq \tilde{c}_2 (r-1)^{-\frac{\alpha_1}{p-1}}.$$

If (7.8) holds, then there exist $\overline{R}_2 > 1, d_1, \tilde{d}_1, d_2, \tilde{d}_2 > 0$ such that for all $r \in (\overline{R}_2, +\infty)$ we have

$$d_1 r^{1+\frac{\alpha_\infty + 1 - N}{p-1}} \leq \varphi_1(r) \leq d_2 r^{1+\frac{\alpha_\infty+1-N}{p-1}}$$

and

$$\tilde{d}_1 r^{\frac{\alpha_\infty+1-N}{p-1}} \leq -\varphi_1'(r) \leq \tilde{d}_2 r^{\frac{\alpha_\infty+1-N}{p-1}}.$$

While the asymptotics near 1 corresponds to the asymptotics in the first part of Corollary 6.2, the asymptotics near $+\infty$ is affected by an additional term "r^{N-1}".

Similarly, we can study the asymptotic properties of the weak solution $e(r)$ to the following auxiliary problem

$$\begin{cases} -\left(r^{N-1} v(r) |u'(r)|^{p-2} u'(r)\right)' = r^{N-1} w(r), & r \in (R_1, R_2), \\ \lim\limits_{r \to R_1} u(r) = \lim\limits_{r \to R_2} u(r) = 0. \end{cases} \tag{7.9}$$

In fact, we can formulate an analogue of Theorem 5.2 and Theorem 5.3.

Corollary 7.5. *Let $r^{N-1} w(r) \in L^1(R_1, R_2)$. Given $\varepsilon \in (0, p-1)$ arbitrary, there exist $\overline{R}_1^e \in (R_1, R_2)$, c_1, c_2, $\tilde{c}_2 > 0$ such that for all $r \in (R_1, \overline{R}_1^e)$ we have*

$$c_1 \int_{R_1}^r \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau) \mathrm{d}\tau \leq e(r) \leq c_2 \left(\int_{R_1}^r \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau) \mathrm{d}\tau \right)^{1-\frac{\varepsilon}{p-1}}$$

and

$$c_1 r^{\frac{1-N}{p-1}} v^{1-p'}(r) \le e'(r) \le \tilde{c}_2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_{R_1}^{r} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau\right)^{1-\frac{\varepsilon}{p-1}}.$$

Similarly, given $\varepsilon \in (0, p-1)$ *arbitrary, there exist* $\overline{R}_2^e \in (R_1, R_2)$, d_1, d_2, $\tilde{d}_2 > 0$ *such that for all* $r \in (\overline{R}_2^e, R_2)$ *we have*

$$d_1 \int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau \le e(r) \le d_2 \left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau\right)^{1-\frac{\varepsilon}{p-1}}$$

and

$$d_1 r^{\frac{1-N}{p-1}} v^{1-p'}(r) \le -e'(r) \le \tilde{d}_2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_r^{R_2} \tau^{\frac{1-N}{p-1}} v^{1-p'}(\tau)\mathrm{d}\tau\right)^{1-\frac{\varepsilon}{p-1}}.$$

If v and w are given by (7.6) then $w \in L^1(\overline{B}_1^c)$ requires $\beta_1 > -1$ and $\beta_\infty > N$. In particular, (7.7) reduces to

$$\alpha_1 < p-1 \quad \text{and} \quad \beta_\infty > \max\{\alpha_\infty + p, N\} \tag{7.10}$$

and (7.8) reduces to

$$\beta_1 > \max\{\alpha_1 - p, -1\} \quad \text{and} \quad \alpha_\infty < N - p. \tag{7.11}$$

Note that (4.3) holds for arbitrary $\varepsilon \in (0, p-1)$ in this special case. Then the asymptotic estimates for e and e' read as follows.

Corollary 7.6. *Given* $\varepsilon \in (0, p-1)$ *arbitrary, there exist* $\overline{R}_1^e > 1, c_1, \tilde{c}_1, c_2, \tilde{c}_2 > 0$ *such that for all* $r \in (1, \overline{R}_1^e)$ *we have*

$$c_1 (r-1)^{1-\frac{\alpha_1}{p-1}} \le e(r) \le c_2 (r-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}$$

and

$$\tilde{c}_1 (r-1)^{-\frac{\alpha_1}{p-1}} \le e'(r) \le \tilde{c}_2 (r-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Similarly, given $\varepsilon \in (0, p-1)$ *arbitrary, there exist* $\overline{R}_2^e > 1$, $d_1, \tilde{d}_1, d_2, \tilde{d}_2 > 0$ *such that for all* $r \in (\overline{R}_2^e, +\infty)$ *we have*

$$d_1 r^{1+\frac{\alpha_\infty+1-N}{p-1}} \le e(r) \le d_2 r^{\left(1+\frac{\alpha_\infty+1-N}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)} \tag{7.12}$$

and

$$\tilde{d}_1 r^{\frac{\alpha_\infty+1-N}{p-1}} \le -e'(r) \le \tilde{d}_2 r^{\left(1+\frac{\alpha_\infty+1-N}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)-1}.$$

Remark 7.7. Let us emphasize the importance of asymptotic estimates presented above. We will utilize them later for constructing an ordered pair of sub- and supersolution for problem (7.1). Since modifications of φ_1 and e will serve as a subsolution and a supersolution, respectively, the estimates above will allow to compare the resulting subsolution and a supersolution near the finite boundary and near infinity.

Remark 7.8. We compare our results for (7.9) with $R_2 = +\infty$ and corresponding results of Bidaut-Véron and Pohozaev [4, Prop. 2.6, (ii)]. Let $N > p$. Consider v and w as given in (7.6) with $\alpha_1 = \alpha_\infty = \beta_1 = 0$ and $\beta_\infty > N$. Then the left inequality in (7.12) coincides with the lower estimate from [4], the first inequality in (2.34). Let $N \le p$. The second inequality in (2.34) from [4] implies that any possible nonnegative weak solution of equation in (7.9) cannot decay to zero as $r \to +\infty$, i.e., (7.9) does not have a weak solution. On the other hand, choosing now $\alpha_1 = \beta_1 = 0$, $\alpha_\infty < N - p$, $\beta_\infty > N$, problem (7.9) has a positive weak solution satisfying decay asymptotic estimates presented above. This says that a sufficiently singular diffusion coefficient $v(r)$ could guarantee the existence of a weak solution having prescribed decay at infinity.

8 Examples

We will discuss some examples to demonstrate our general existence result from Theorem 3.1 and the use of asymptotics obtained for the eigenfunction φ_1 and the auxiliary function e in Section 4 and Section 5, respectively. For simplicity, we consider $f(t,s) = f(s)$, where $f : [0,+\infty) \to \mathbb{R}$ is C^1 and satisfies the following additional assumptions:

(H3) there exists a constant $K > 0$ such that $\lim_{s\to 0} \frac{f(s)}{s^{p-1}} = K$;

(H4) there exists $r_0 > 0$ such that $f(s)(r_0 - s) > 0$ for all $s > 0$, $s \neq r_0$.

We observe that since f is C^1, (H3)–(H4) imply that f satisfies (H1)–(H2).

We consider the following one dimensional quasilinear problem

$$\begin{cases} -\left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' = \lambda\sigma(t)f(u(t)), & t \in (1,+\infty), \\ \lim\limits_{t\to 1^+} u(t) = \lim\limits_{t\to+\infty} u(t) = 0 \end{cases} \tag{8.1}$$

where $\lambda > 0$ is a parameter.

Then we prove the following result.

Theorem 8.1. *Let the weight functions ρ and σ be as in (6.1) with $\alpha_1, \beta_1, \alpha_\infty, \beta_\infty$ satisfying (6.4) and (6.5). Let $p > 1$ and (H3)–(H4) hold. Then for any $\lambda > \frac{\lambda_1}{K}$, there exist a minimal weak solution $u_{\min}$ and a maximal weak solution $u_{\max}$ of (8.1). Moreover, given $\varepsilon \in (0, p-1)$, there exist constants $C > 1$, $C_1, C_2 > 0$ such that for all $t \in (1, C)$ we have*

$$C_1(t-1)^{1-\frac{\alpha_1}{p-1}} \le u_{\min}(t) \le u_{\max}(t) \le C_2(t-1)^{\left(1-\frac{\alpha_1}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}.$$

Similarly, given $\varepsilon \in (0, p-1)$, there exist constants $D > 1$, $D_1, D_2 > 0$ such that for all $t \in (D,+\infty)$ we have

$$D_1 t^{1+\frac{\alpha_\infty}{p-1}} \le u_{\min}(t) \le u_{\max}(t) \le D_2 t^{\left(1+\frac{\alpha_\infty}{p-1}\right)\left(1-\frac{\varepsilon}{p-1}\right)}.$$

Proof. In order to apply Theorem 3.1, we construct a suitable pair of well ordered sub- and supersolution of (8.1). We will first construct a positive supersolution of (8.1) with the help of the auxiliary function $e > 0$, weak solution of (5.1). Let $\overline{u} := (\lambda A_0)^{\frac{1}{p-1}} e$, where $A_0 := \sup_{s\ge 0} f(s) > 0$. Then

$$-\left(\rho(t)|\overline{u}'(t)|^{p-2}\overline{u}'(t)\right)' = \lambda A_0 \sigma(t) \ge \lambda\sigma(t) f(\overline{u}).$$

Now we construct a positive subsolution of (8.1) using the eigenfunction $\varphi_1 > 0$ corresponding to the principal eigenvalue λ_1 of (4.1). Note that continuity, and decay properties, (4.5) and (4.9), of the eigenfunction φ_1 imply that $\|\varphi_1\|_\infty < +\infty$. First, we consider a function

$$G(s) := \lambda_1 s^{p-1} - \lambda f(s) \quad \text{for } s \ge 0.$$

Using hypothesis (H3), we see that $G(s) = \lambda_1 s^{p-1} - \lambda K s^{p-1} - o(s^{p-1})$. Let $\lambda > \frac{\lambda_1}{K}$ be fixed. Then there exists $s_\lambda > 0$ such that for any $s \in (0, s_\lambda)$, we have $G(s) < 0$. For $m \leq \frac{s_\lambda}{\|\varphi_1\|_\infty}$, we show that $\underline{u} := m\varphi_1$ is a subsolution of (8.1). Indeed, it follows from the discussion above and the fact that $\sigma(t) > 0$ in $(1, +\infty)$

$$-\left(\rho(t)|\underline{u}'(t)|^{p-2}\underline{u}'(t)\right)' = \lambda_1 \sigma(t) m^{p-1} \varphi_1^{p-1} \leq \lambda \sigma(t) f(m\varphi_1) = \lambda \sigma(t) f(\underline{u}) .$$

Now using the decay estimates in Corollary 6.2 of the eigenfunction φ_1 and Corollary 6.3 of the auxiliary function e at the end points of the interval $(1, +\infty)$, we can adjust the constant $m \approx 0$ so that $\underline{u} \leq \overline{u}$ in $(1, +\infty)$. Then by Theorem 3.1, there exist a minimal weak solution u_{min} and a maximal weak solution $u_{\max}$ of (8.1) such that

$$0 < \underline{u} \leq u_{\min} \leq u_{\max} \leq \overline{u} \quad \text{in } (1, +\infty) ,$$

and enjoy the regularity properties (3.2). This completes the proof. □

Remark 8.2. We observe that the rates of decay of positive weak solutions obtained in Theorem 8.1 are independent of the nonlinearity f.

Next, we consider radially symmetric positive solutions of the following PDE in dimension $N > 1$

$$\begin{cases} -\text{div}\left(v(|x|)|\nabla u(|x|)|^{p-2}\nabla u(|x|)\right) = \lambda w(|x|) f(u(|x|)), & x \in A_1^{+\infty} \subset \mathbb{R}^N, \\ u(x) = 0, \quad x \in \partial A_1^{+\infty}, \end{cases} \tag{8.2}$$

where $\lambda > 0$ is a parameter, f is as above, and $A_1^{+\infty} = \overline{B}_1^c$ is the exterior of a unit ball. We obtain the counterpart of Theorem 8.1 below.

Theorem 8.3. *Let the weight functions v and w be as in (7.6) with $\alpha_1, \beta_1, \alpha_\infty, \beta_\infty$ satisfying (7.10) and (7.11). Let $p > 1$ and (H3)–(H4) hold. Then for any $\lambda > \frac{\lambda_1}{K}$, there exist a minimal weak solution $u_{\min}$ and a maximal weak solution $u_{\max}$ of (8.2). Moreover, given $\varepsilon \in (0, p-1)$, there exist constants $C > 1$, $C_1, C_2 > 0$ such that for all $|x| \in (1, C)$ we have*

$$C_1 \left(|x| - 1\right)^{1 - \frac{\alpha_1}{p-1}} \leq u_{\min}(|x|) \leq u_{\max}(|x|) \leq C_2 \left(|x| - 1\right)^{\left(1 - \frac{\alpha_1}{p-1}\right)\left(1 - \frac{\varepsilon}{p-1}\right)} . \tag{8.3}$$

Similarly, given $\varepsilon \in (0, p-1)$, there exist constants $D > 1$, $D_1, D_2 > 0$ such that for all $|x| \in (D, +\infty)$ we have

$$D_1 |x|^{1 + \frac{\alpha_\infty + 1 - N}{p-1}} \leq u_{\min}(|x|) \leq u_{\max}(|x|) \leq D_2 |x|^{\left(1 + \frac{\alpha_\infty + 1 - N}{p-1}\right)\left(1 - \frac{\varepsilon}{p-1}\right)} .$$

Proof. Substituting $r = |x|$, (8.2) transforms to

$$\begin{cases} -\left(r^{N-1} v(r)|u'(r)|^{p-2}u'(r)\right)' = \lambda r^{N-1} w(r) f(u(r)), & r \in (1, +\infty), \\ \lim\limits_{r \to 1^+} u(r) = \lim\limits_{r \to +\infty} u(r) = 0 . \end{cases} \tag{8.4}$$

Observe that (8.4) is a special case of (8.1) with $\rho(t) = t^{N-1} v(t)$ and $\sigma(t) = t^{N-1} w(t)$ for $t \in (1, +\infty)$. Then the proof follows by repeating the constructions in the proof of Theorem 8.1. □

Remark 8.4. We observe again that the rates of decay of positive weak solutions obtained in Theorem 8.3 are independent of the nonlinearity f. However, the decay rate at infinity depends on the dimension $N > 1$.

Remark 8.5. Notice that it follows from (7.10) that $\alpha_1 < p-1$. If $\alpha_1 \in (0, p-1)$ then $1 - \frac{\alpha_1}{p-1} < 1$ and hence the left inequality in (8.3) yields that $\frac{\partial u}{\partial \vec{n}} = +\infty$ on ∂B_1, where $\vec{n}$ denotes the outer unit normal vector of ∂B_1. On the other hand, if $\alpha_1 < 0$ then we can choose $\varepsilon \in (0, p-1)$ so that $(1 - \frac{\alpha_1}{p-1})(1 - \frac{\varepsilon}{p-1}) > 1$ and then the right inequality in (8.3) yields that $\frac{\partial u}{\partial \vec{n}} = 0$ on ∂B_1. Therefore, if $\alpha_1 \in (-\infty, 0) \cup (0, p-1)$, any weak solution u of (8.2) violates the Hopf maximum principle on ∂B_1, cf. [15, Thm. 5].

Acknowledgements

The second author was supported by the Grant Agency of the Czech Republic Project No. 18-03253S.

References

[1] A. Anane, Simplicité et isolation de la première valeur propre du p-laplacien avec poids, *C. R. Acad. Sci. Paris Sér. I Math.* **305**(1987), No. 16, 725–728. MR920052

[2] G. Barles, Remarks on uniqueness results of the first eigenvalue of the p-Laplacian, *Ann. Fac. Sci. Toulouse Math. (5)* **9**(1988), No. 1, 65–75. MR971814

[3] T. Bhattacharya, Radial symmetry of the first eigenfunction for the p-Laplacian in the ball, *Proc. Amer. Math. Soc.* **104**(1988), No. 1, 169–174. https://doi.org/10.2307/2047480; MR958061

[4] M.-F. Bidaut-Véron, S. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, *J. Anal. Math.* **84**(2001), 1–49. https://doi.org/10.1007/BF02788105; MR1849197

[5] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985. https://doi.org/10.1007/978-3-662-00547-7; MR787404

[6] P. Drábek, K. Ho, A. Sarkar, On the eigenvalue problem involving the weighted p-Laplacian in radially symmetric domains, *J. Math. Anal. Appl.* **468**(2018), 716–756. https://doi.org/10.1016/j.jmaa.2018.08.046; MR3852550

[7] P. Drábek, A. Kufner, K. Kuliev, Half-linear Sturm–Liouville problem with weights: asymptotic behavior of eigenfunctions, *Proc. Steklov Inst. Math.* **284**(2014), No. 1, 148–154. https://doi.org/10.1134/S008154381401009X; MR3479970

[8] P. Drábek, A. Kufner, F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 5, Walter de Gruyter & Co., Berlin, 1997. https://doi.org/10.1515/9783110804775; MR1460729

[9] P. Drábek, K. Kuliev, Half-linear Sturm–Liouville problem with weights. *Bull. Belg. Math. Soc. Simon Stevin* **19**(2012), No. 1, 107–119. MR2952799

[10] P. Drábek, J. Milota, *Methods of nonlinear analysis. Applications to differential equations*, 2nd edition, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser/Springer Basel AG, Basel, 2013. https://doi.org/10.1007/978-3-0348-0387-8; MR3025694

[11] A. Kufner, B. Opic, How to define reasonably weighted Sobolev spaces, *Comment. Math. Univ. Carolin.* **25**(1984), No. 3, 537–554. MR1069756.

[12] P. Lindqvist, On the equation div $(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$. *Proc. Amer. Math. Soc.* **109**(1990), No. 1, 157–164. https://doi.org/10.2307/2048375; MR1007505

[13] B. Opic, A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, Vol. 219, Longman Scientific & Technical, Harlow, 1990. MR1069756

[14] M. Otani, T. Teshima, On the first eigenvalue of some quasilinear elliptic equations, *Proc. Japan Acad. Ser. A Math. Sci.* **64**(1988), No. 1, 8–10. MR953752

[15] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**(1984), No. 3, 191–202. https://doi.org/10.1007/BF01449041; MR768629