Electronic Journal of Qualitative Theory of Differential Equations
2021, No. **12**, 1–27; https://doi.org/10.14232/ejqtde.2021.1.12 www.math.u-szeged.hu/ejqtde/

General solutions to subclasses of a two-dimensional class of systems of difference equations

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Received 29 September 2020, appeared 26 February 2021
Communicated by Armengol Gasull

Abstract. We show practical solvability of the following two-dimensional systems of difference equations

$$x_{n+1} = \frac{u_{n-2}v_{n-3} + a}{u_{n-2} + v_{n-3}}, \quad y_{n+1} = \frac{w_{n-2}s_{n-3} + a}{w_{n-2} + s_{n-3}}, \quad n \in \mathbb{N}_0,$$

where u_n, v_n, w_n and s_n are x_n or y_n, by presenting closed-form formulas for their solutions in terms of parameter a, initial values, and some sequences for which there are closed-form formulas in terms of index n. This shows that a recently introduced class of systems of difference equations, contains a subclass such that one of the delays in the systems is equal to four, and that they all are practically solvable, which is a bit unexpected fact.

Keywords: system of difference equations, solvable systems, practical solvability.

2020 Mathematics Subject Classification: 39A45.

1 Introduction

Solvability of difference equations is one of the basic topics studied in the area. Presentations of some results in the topic can be found in any book on the equations, for instance, in: [4, 5, 9, 10, 12, 13]. The equations frequently appear in various applications (see, e.g., [4, 5, 7, 8, 11, 12, 23, 25, 41]). There has been also some recent interest in solvability (see, e.g., [2, 22, 28–32, 35, 37–40]). If it is not easy to find solutions to the equations, researchers try to find their invariants, as it was the case in [15–17, 21, 26, 27, 33, 34]. In some cases they can be used also for solving the equations and systems, as it was the case in [33, 34].

Each difference equation can be used for forming systems of difference equations possessing some types of symmetry. A way for forming such systems can be found in [28].

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Papaschinopoulos, Schinas and some of their colleagues proposed studying some systems of this and other types (see, e.g., [6, 14–21, 26, 27]). We have devoted a part of our research to solvable systems of difference equations, unifying the two topics (see, e.g., [2,28–32,35,38–40]).

During the last several years, we have studied, among other things, practical solvability of product-type systems of difference equations. For some of our previous results in the topic see, for instance, [29,30], as well as the related references therein. The systems are theoretically solvable, but only several subclasses are practically solvable, which has been one of the main reasons for our study of the systems.

Quite recently, we have started studying solvability of the, so called, hyperbolic-cotangent-type systems of difference equations. They are given by

$$x_{n+1} = \frac{u_{n-k}v_{n-l} + a}{u_{n-k} + v_{n-l}}, \quad y_{n+1} = \frac{w_{n-k}s_{n-l} + a}{w_{n-k} + s_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where delays k and l are nonnegative integers, parameter a and initial values are complex numbers, whereas each of the four sequences u_n, v_n, w_n and s_n is one of the sequences x_n or y_n for all possible values of index n.

Note that this is a class of nonlinear systems of difference equations which is formed by using the method for forming symmetric types of systems of difference equations described in [28]. For the case of one-dimensional difference equation corresponding to the systems in (1.1) see [24] and [37].

What is interesting is that the systems in (1.1) are connected to product-type ones. As we have mentioned the product-type systems are theoretically solvable, but only few of them are practically solvable. The reason for this lies in impossibility to solve all polynomial equations, as well as the fact that with each product-type system of difference equations is associated a polynomial. The mentioned connection between the systems in (1.1) and product-type ones implies that also only several subclasses of the systems in (1.1) are practically solvable. Moreover, the connection shows that for guaranteeing practical solvability of all the systems in (1.1) values of k and l seems should be small. Note that we may assume $k \le l$. The case $k = 0$ and $l = 1$ was studied in [39] and [40], whereas in [32] was presented another solution to the problem. The case $k = 1$ and $l = 2$ was studied in [31], whereas the case $k = 0$ and $l = 2$ was studied in [35], which finished the study of practical solvability in the case when $\max\{k, l\} \le 2$ and $k \ne l$. The case $k = l \in \mathbb{N}_0$ was solved in [36].

Thus, it is of some interest to see if all the systems in (1.1) are solvable when $l = 3$ and k is such that $0 \le k \le 2$.

One of the cases is obvious. Namely, if $k = 1$, then the systems in (1.1) are with interlacing indices (the notion and some examples can be found in [38]), since each of the systems in (1.1) in this case, reduces to two systems of the exactly same form with $k = 0$ and $l = 1$. Thus, it is of some interest to study the other cases.

Here, we show that the systems of difference equations

$$x_{n+1} = \frac{u_{n-2}v_{n-3} + a}{u_{n-2} + v_{n-3}}, \quad y_{n+1} = \frac{w_{n-2}s_{n-3} + a}{w_{n-2} + s_{n-3}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

are practically solvable, that is, we show the solvability of all sixteen systems in (1.1), in the case $k = 2$ and $l = 3$, which is a bit surprising result. Namely, as we have said, to each system in (1.2) is associated a polynomial, several of which have degree bigger than four (some of them have degree eight). By a well-known theorem of Abel [1], polynomials of degree bigger than four need not be solvable by radicals. However, it turns out that all the associate polynomials to the systems in (1.2) are solvable by radicals, implying practical solvability of the corresponding systems. Using the fact that there is no universal method for showing practical solvability of such systems, as well as the fact that the situation in the case $\max\{k,l\} \geq 5$ is different, shows the importance of studying solvability of the systems in (1.2).

The case $a = 0$ was considered in [32] where it was shown its theoretical solvability. Namely, by using the changes of variables

$$x_n = \frac{1}{\widehat{x}_n}, \quad y_n = \frac{1}{\widehat{y}_n},$$

system (1.2) becomes linear, from which together with a known theorem from the theory of homogeneous linear difference equations with constant coefficients the theoretical solvability of the system follows. Hence, from now on we will consider only the case $a \neq 0$.

2 Connection of (1.2) to product-type systems and a lemma

First, we present above mentioned connection of the systems in (1.2) to some product-type systems.

Some simple calculations yield

$$x_{n+1} \pm \sqrt{a} = \frac{(u_{n-2} \pm \sqrt{a})(v_{n-3} \pm \sqrt{a})}{u_{n-2} + v_{n-3}} \quad \text{and} \quad y_{n+1} \pm \sqrt{a} = \frac{(w_{n-2} \pm \sqrt{a})(s_{n-3} \pm \sqrt{a})}{w_{n-2} + s_{n-3}},$$

for $n \in \mathbb{N}_0$, implying

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{u_{n-2}+\sqrt{a}}{u_{n-2}-\sqrt{a}} \cdot \frac{v_{n-3}+\sqrt{a}}{v_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{w_{n-2}+\sqrt{a}}{w_{n-2}-\sqrt{a}} \cdot \frac{s_{n-3}+\sqrt{a}}{s_{n-3}-\sqrt{a}}, \tag{2.1}$$

for $n \in \mathbb{N}_0$.

System (2.1) written in a compact form, can be written as follows

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.2}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.3}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.4}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}} \cdot \frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.5}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}} \cdot \frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.6}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.7}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.8}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.9}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.10}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.11}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.12}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.13}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.14}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{x_{n-3}+\sqrt{a}}{x_{n-3}-\sqrt{a}}, \tag{2.15}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{x_{n-2}+\sqrt{a}}{x_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.16}$$

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}},\quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}}=\frac{y_{n-2}+\sqrt{a}}{y_{n-2}-\sqrt{a}}\cdot\frac{y_{n-3}+\sqrt{a}}{y_{n-3}-\sqrt{a}}, \tag{2.17}$$

for $n\in\mathbb{N}_0$.

Let

$$\zeta_n=\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\quad\text{and}\quad \eta_n=\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}},$$

then

$$x_n=\sqrt{a}\,\frac{\zeta_n+1}{\zeta_n-1}\quad\text{and}\quad y_n=\sqrt{a}\,\frac{\eta_n+1}{\eta_n-1}, \tag{2.18}$$

so (2.2)–(2.17) become

$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.19}$$
$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.20}$$
$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.21}$$
$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.22}$$
$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.23}$$
$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.24}$$
$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.25}$$
$$\zeta_{n+1} = \eta_{n-2}\zeta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.26}$$
$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.27}$$
$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.28}$$
$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.29}$$
$$\zeta_{n+1} = \zeta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.30}$$
$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \tag{2.31}$$
$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\zeta_{n-3}, \tag{2.32}$$
$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \zeta_{n-2}\eta_{n-3}, \tag{2.33}$$
$$\zeta_{n+1} = \eta_{n-2}\eta_{n-3}, \quad \eta_{n+1} = \eta_{n-2}\eta_{n-3}, \tag{2.34}$$

for $n \in \mathbb{N}_0$.

So, if systems (2.19)–(2.34) are practically solvable, then by using (2.18) the systems (2.2)–(2.17) will be also such. Hence, it should be first proved practical solvability of systems (2.19)–(2.34).

The following auxiliary result is used for several times in the rest of the article. The proof is omitted since it can be found, for example, in [31].

Lemma 2.1. *Assume $R_k(s) = s^k - b_{k-1}s^{k-1} - b_{k-2}s^{k-2} - \cdots - b_0$, $b_0 \neq 0$, is a real polynomial with simple roots s_i, $i = \overline{1,k}$, and a_n, $n \geq l-k$, is defined by*

$$a_n = b_{k-1}a_{n-1} + b_{k-2}a_{n-2} + \cdots + b_0 a_{n-k}, \quad n \geq l,$$

with $a_{j-k} = 0$, $j = \overline{l, l+k-2}$, $a_{l-1} = 1$, and $l \in \mathbb{Z}$. Then

$$a_n = \sum_{i=1}^{k} \frac{s_i^{n+k-l}}{R_k'(s_i)}, \quad n \geq l-k.$$

3 Main results

Here we show that each of the product-type systems of difference equations in (2.19)–(2.34) is practically solvable, and following the analysis of each of the systems, by using the relations in (2.18), we present closed-form formulas for general solutions to systems (2.2)–(2.17).

3.1 System (2.19)

The equations in (2.19) immediately imply the following relation

$$\zeta_n = \eta_n, \quad n \in \mathbb{N}. \tag{3.1}$$

The first equation in (2.19) can be written as follows

$$\zeta_n = \zeta_{n-3}\zeta_{n-4} = \zeta_{n-3}^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1}, \tag{3.2}$$

for $n \in \mathbb{N}$, where, of course, the exponents are defined as follows

$$c_1 = d_1 = 1, \quad e_1 = f_1 = 0. \tag{3.3}$$

An application of the first equality in (3.2) into the second one yields

$$\zeta_n = (\zeta_{n-6}\zeta_{n-7})^{c_1}\zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{f_1} = \zeta_{n-4}^{d_1}\zeta_{n-5}^{e_1}\zeta_{n-6}^{c_1+f_1}\zeta_{n-7}^{c_1} = \zeta_{n-4}^{c_2}\zeta_{n-5}^{d_2}\zeta_{n-6}^{e_2}\zeta_{n-7}^{f_2},$$

for $n \geq 4$, where $c_2 := d_1$, $d_2 := e_1$, $e_2 := c_1 + f_1$ and $f_2 := c_1$.

It is natural to assume that the following relations hold

$$\zeta_n = \zeta_{n-k-2}^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k}, \tag{3.4}$$

$$c_k = d_{k-1}, \quad d_k = e_{k-1}, \quad e_k = c_{k-1} + f_{k-1}, \quad f_k = c_{k-1} \tag{3.5}$$

for a $k \geq 2$ and $n \geq k + 2$.

Relations (3.2), (3.4) and (3.5) yield

$$\begin{aligned}\zeta_n &= (\zeta_{n-k-5}\zeta_{n-k-6})^{c_k}\zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{f_k},\\ &= \zeta_{n-k-3}^{d_k}\zeta_{n-k-4}^{e_k}\zeta_{n-k-5}^{c_k+f_k}\zeta_{n-k-6}^{c_k}\\ &= \zeta_{n-k-3}^{c_{k+1}}\zeta_{n-k-4}^{d_{k+1}}\zeta_{n-k-5}^{e_{k+1}}\zeta_{n-k-6}^{f_{k+1}},\end{aligned}$$

where

$$c_{k+1} := d_k, \quad d_{k+1} := e_k, \quad e_{k+1} := c_k + f_k. \quad f_{k+1} := c_k.$$

The inductive argument proves that (3.4) and (3.5) really hold for $2 \leq k \leq n-2$.

It is easy to see that from (3.3) and (3.5), we get

$$c_n = c_{n-3} + c_{n-4}, \tag{3.6}$$

for $n \geq 5$ (in fact, for $n \in \mathbb{Z}$), and

$$c_0 = c_{-1} = 0, \quad c_{-2} = 1, \quad c_{-3} = c_{-4} = c_{-5} = 0, \quad c_{-6} = 1, \quad c_{-7} = -1. \tag{3.7}$$

Choose $k = n - 2$ in relation (3.4). Then (3.5) and (3.6) yield

$$\zeta_n = \zeta_0^{c_{n-2}}\zeta_{-1}^{d_{n-2}}\zeta_{-2}^{e_{n-2}}\zeta_{-3}^{f_{n-2}} = \zeta_0^{c_{n-2}}\zeta_{-1}^{c_{n-1}}\zeta_{-2}^{c_n}\zeta_{-3}^{c_{n-3}}, \tag{3.8}$$

for $n \in \mathbb{N}$. A simple verification shows that (3.8) holds also for $n \geq -3$.

Thus, (3.1) and (3.8) imply

$$\eta_n = \zeta_0^{c_{n-2}}\zeta_{-1}^{c_{n-1}}\zeta_{-2}^{c_n}\zeta_{-3}^{c_{n-3}}, \quad n \in \mathbb{N}. \tag{3.9}$$

Let

$$P_4(\lambda) = \lambda^4 - \lambda - 1 = 0. \tag{3.10}$$

It is the characteristic polynomial associated with (3.6). Its roots λ_j, $j=\overline{1,4}$, are simple and can be found by radicals [3].

Lemma 2.1 shows that the solution to (3.6) satisfying the initial conditions $c_{-5}=c_{-4}=c_{-3}=0$, $c_{-2}=1$, is given by

$$c_n=\sum_{j=1}^{4}\frac{\lambda_j^{n+5}}{P_4'(\lambda_j)},\quad n\in\mathbb{Z}. \tag{3.11}$$

The following theorem follows from (2.18), (3.8) and (3.9).

Theorem 3.1. *If $a\neq 0$, then the general solution to (2.2) is*

$$x_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},\quad n\geq -3,$$

$$y_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},\quad n\in\mathbb{N},$$

where c_n is given by (3.11).

3.2 System (2.20)

Since the first equation in (2.20) is the same as in (2.19), formula (3.8) must hold. Further, we have $\eta_n=\eta_{n-3}\zeta_{n-4}$, $n\in\mathbb{N}$, or equivalently

$$\eta_{3n+i}=\eta_{3(n-1)+i}\zeta_{3(n-1)+i-1},\quad n\in\mathbb{N}, \tag{3.12}$$

for $i=-2,-1,0$, and $n\in\mathbb{N}$.

Relations (3.8) and (3.12), for $i=-2$, yield

$$\begin{aligned}\eta_{3n-2}&=\eta_{-2}\prod_{j=1}^{n}\zeta_{3j-6}\\&=\eta_{-2}\prod_{j=1}^{n}\zeta_0^{c_{3j-8}}\zeta_{-1}^{c_{3j-7}}\zeta_{-2}^{c_{3j-6}}\zeta_{-3}^{c_{3j-9}}\\&=\eta_{-2}\zeta_0^{\sum_{j=1}^{n}c_{3j-8}}\zeta_{-1}^{\sum_{j=1}^{n}c_{3j-7}}\zeta_{-2}^{\sum_{j=1}^{n}c_{3j-6}}\zeta_{-3}^{\sum_{j=1}^{n}c_{3j-9}},\end{aligned} \tag{3.13}$$

for $n\in\mathbb{N}_0$.

From (3.8) and (3.12), for $i=-1$, we obtain

$$\begin{aligned}\eta_{3n-1}&=\eta_{-1}\prod_{j=1}^{n}\zeta_{3j-5}\\&=\eta_{-1}\prod_{j=1}^{n}\zeta_0^{c_{3j-7}}\zeta_{-1}^{c_{3j-6}}\zeta_{-2}^{c_{3j-5}}\zeta_{-3}^{c_{3j-8}}\\&=\eta_{-1}\zeta_0^{\sum_{j=1}^{n}c_{3j-7}}\zeta_{-1}^{\sum_{j=1}^{n}c_{3j-6}}\zeta_{-2}^{\sum_{j=1}^{n}c_{3j-5}}\zeta_{-3}^{\sum_{j=1}^{n}c_{3j-8}},\end{aligned} \tag{3.14}$$

for $n\in\mathbb{N}_0$.

From (3.8) and (3.12), for $i = 0$, it follows that

$$\begin{aligned}\eta_{3n} &= \eta_0 \prod_{j=1}^{n} \zeta_{3j-4} \\ &= \eta_0 \prod_{j=1}^{n} \zeta_0^{c_{3j-6}} \zeta_{-1}^{c_{3j-5}} \zeta_{-2}^{c_{3j-4}} \zeta_{-3}^{c_{3j-7}} \\ &= \eta_0 \zeta_0^{\sum_{j=1}^{n} c_{3j-6}} \zeta_{-1}^{\sum_{j=1}^{n} c_{3j-5}} \zeta_{-2}^{\sum_{j=1}^{n} c_{3j-4}} \zeta_{-3}^{\sum_{j=1}^{n} c_{3j-7}},\end{aligned} \tag{3.15}$$

for $n \in \mathbb{N}_0$.

From (3.6) and (3.7), we have

$$\sum_{j=1}^{n} c_{3j-9} = \sum_{j=1}^{n} (c_{3j-5} - c_{3j-8}) = c_{3n-5}, \tag{3.16}$$

$$\sum_{j=1}^{n} c_{3j-8} = \sum_{j=1}^{n} (c_{3j-4} - c_{3j-7}) = c_{3n-4}, \tag{3.17}$$

$$\sum_{j=1}^{n} c_{3j-7} = \sum_{j=1}^{n} (c_{3j-3} - c_{3j-6}) = c_{3n-3} \tag{3.18}$$

$$\sum_{j=1}^{n} c_{3j-6} = \sum_{j=1}^{n} (c_{3j-2} - c_{3j-5}) = c_{3n-2} - 1, \tag{3.19}$$

$$\sum_{j=1}^{n} c_{3j-5} = \sum_{j=1}^{n} (c_{3j-1} - c_{3j-4}) = c_{3n-1}, \tag{3.20}$$

$$\sum_{j=1}^{n} c_{3j-4} = \sum_{j=1}^{n} (c_{3j} - c_{3j-3}) = c_{3n}, \tag{3.21}$$

for $n \in \mathbb{N}_0$.

From (3.13)–(3.21), we have

$$\eta_{3n-2} = \eta_{-2} \zeta_0^{c_{3n-4}} \zeta_{-1}^{c_{3n-3}} \zeta_{-2}^{c_{3n-2}-1} \zeta_{-3}^{c_{3n-5}}, \tag{3.22}$$

$$\eta_{3n-1} = \eta_{-1} \zeta_0^{c_{3n-3}} \zeta_{-1}^{c_{3n-2}-1} \zeta_{-2}^{c_{3n-1}} \zeta_{-3}^{c_{3n-4}}, \tag{3.23}$$

$$\eta_{3n} = \eta_0 \zeta_0^{c_{3n-2}-1} \zeta_{-1}^{c_{3n-1}} \zeta_{-2}^{c_{3n}} \zeta_{-3}^{c_{3n-3}}, \tag{3.24}$$

for $n \in \mathbb{N}_0$.

The following theorem follows from (2.18), (3.8), (3.22), (3.23) and (3.24).

Theorem 3.2. *If $a \neq 0$, then the general solution to (2.3) is*

$$x_n = \sqrt{a}\, \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}} - 1}, \quad n \geq -3,$$

$$y_{3n-2} = \sqrt{a}\, \frac{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right) \left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-4}} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-3}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-5}} + 1}{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right) \left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-4}} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-3}} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-5}} - 1}$$

$$y_{3n-1} = \sqrt{a}\, \frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right) \left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-3}} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-1}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-4}} + 1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right) \left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-3}} \left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1} \left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n-1}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-4}} - 1}$$

$$y_{3n}=\sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{3n-3}}-1},$$

for $n\in\mathbb{N}_0$, where c_n is given by (3.11).

3.3 System (2.21)

Since the first equation in (2.21) is the same as in (2.19), formula (3.8) must hold. Further, we have $\eta_n=\zeta_{n-3}\eta_{n-4}$, for $n\in\mathbb{N}$, or equivalently

$$\eta_{4n+i}=\zeta_{4n-3+i}\eta_{4(n-1)+i}, \tag{3.25}$$

for $n\in\mathbb{N}$, $i=-3,-2,-1,0$.

From (3.8) and (3.25), we have

$$\begin{aligned}\eta_{4n-3}&=\eta_{-3}\prod_{j=1}^{n}\zeta_{4j-6}\\&=\eta_{-3}\prod_{j=1}^{n}\zeta_0^{c_{4j-8}}\zeta_{-1}^{c_{4j-7}}\zeta_{-2}^{c_{4j-6}}\zeta_{-3}^{c_{4j-9}}\\&=\eta_{-3}\zeta_0^{\sum_{j=1}^{n}c_{4j-8}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-7}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-6}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-9}},\end{aligned}\tag{3.26}$$

for $n\in\mathbb{N}_0$,

$$\begin{aligned}\eta_{4n-2}&=\eta_{-2}\prod_{j=1}^{n}\zeta_{4j-5}\\&=\eta_{-2}\prod_{j=1}^{n}\zeta_0^{c_{4j-7}}\zeta_{-1}^{c_{4j-6}}\zeta_{-2}^{c_{4j-5}}\zeta_{-3}^{c_{4j-8}}\\&=\eta_{-2}\zeta_0^{\sum_{j=1}^{n}c_{4j-7}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-6}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-5}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-8}},\end{aligned}\tag{3.27}$$

for $n\in\mathbb{N}_0$, and

$$\begin{aligned}\eta_{4n-1}&=\eta_{-1}\prod_{j=1}^{n}\zeta_{4j-4}\\&=\eta_{-1}\prod_{j=1}^{n}\zeta_0^{c_{4j-6}}\zeta_{-1}^{c_{4j-5}}\zeta_{-2}^{c_{4j-4}}\zeta_{-3}^{c_{4j-7}}\\&=\eta_{-1}\zeta_0^{\sum_{j=1}^{n}c_{4j-6}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-5}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-4}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-7}},\end{aligned}\tag{3.28}$$

for $n\in\mathbb{N}_0$,

$$\begin{aligned}\eta_{4n}&=\eta_{0}\prod_{j=1}^{n}\zeta_{4j-3}\\&=\eta_{0}\prod_{j=1}^{n}\zeta_0^{c_{4j-5}}\zeta_{-1}^{c_{4j-4}}\zeta_{-2}^{c_{4j-3}}\zeta_{-3}^{c_{4j-6}}\\&=\eta_{0}\zeta_0^{\sum_{j=1}^{n}c_{4j-5}}\zeta_{-1}^{\sum_{j=1}^{n}c_{4j-4}}\zeta_{-2}^{\sum_{j=1}^{n}c_{4j-3}}\zeta_{-3}^{\sum_{j=1}^{n}c_{4j-6}},\end{aligned}\tag{3.29}$$

for $n\in\mathbb{N}_0$.

Relations (3.6) and (3.7) yield

$$\sum_{j=1}^{n} c_{4j-9} = \sum_{j=1}^{n} (c_{4j-6} - c_{4j-10}) = c_{4n-6} - 1, \tag{3.30}$$

$$\sum_{j=1}^{n} c_{4j-8} = \sum_{j=1}^{n} (c_{4j-5} - c_{4j-9}) = c_{4n-5}, \tag{3.31}$$

$$\sum_{j=1}^{n} c_{4j-7} = \sum_{j=1}^{n} (c_{4j-4} - c_{4j-8}) = c_{4n-4}, \tag{3.32}$$

$$\sum_{j=1}^{n} c_{4j-6} = \sum_{j=1}^{n} (c_{4j-3} - c_{4j-7}) = c_{4n-3}, \tag{3.33}$$

$$\sum_{j=1}^{n} c_{4j-5} = \sum_{j=1}^{n} (c_{4j-2} - c_{4j-6}) = c_{4n-2} - 1, \tag{3.34}$$

$$\sum_{j=1}^{n} c_{4j-4} = \sum_{j=1}^{n} (c_{4j-1} - c_{4j-5}) = c_{4n-1}, \tag{3.35}$$

$$\sum_{j=1}^{n} c_{4j-3} = \sum_{j=1}^{n} (c_{4j} - c_{4j-4}) = c_{4n}, \tag{3.36}$$

for $n \in \mathbb{N}$.

From (3.26)–(3.36), we have

$$\eta_{4n-3} = \eta_{-3}\zeta_0^{c_{4n-5}}\zeta_{-1}^{c_{4n-4}}\zeta_{-2}^{c_{4n-3}}\zeta_{-3}^{c_{4n-6}-1}, \tag{3.37}$$

$$\eta_{4n-2} = \eta_{-2}\zeta_0^{c_{4n-4}}\zeta_{-1}^{c_{4n-3}}\zeta_{-2}^{c_{4n-2}-1}\zeta_{-3}^{c_{4n-5}}, \tag{3.38}$$

$$\eta_{4n-1} = \eta_{-1}\zeta_0^{c_{4n-3}}\zeta_{-1}^{c_{4n-2}-1}\zeta_{-2}^{c_{4n-1}}\zeta_{-3}^{c_{4n-4}}, \tag{3.39}$$

$$\eta_{4n} = \eta_0\zeta_0^{c_{4n-2}-1}\zeta_{-1}^{c_{4n-1}}\zeta_{-2}^{c_{4n}}\zeta_{-3}^{c_{4n-3}}, \tag{3.40}$$

for $n \in \mathbb{N}_0$.

The following theorem follows from (2.18), (3.8), (3.37)–(3.40).

Theorem 3.3. *If $a \neq 0$, then the general solution to (2.4) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \geq -3,$$

$$y_{4n-3} = \sqrt{a}\,\frac{\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}+1}{\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}-1},$$

$$y_{4n-2} = \sqrt{a}\,\frac{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-5}}+1}{\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-5}}-1},$$

$$y_{4n-1} = \sqrt{a}\,\frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-4}}+1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-4}}-1},$$

$$y_{4n}=\sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{4n-3}}-1},$$

for $n\in\mathbb{N}_0$, where sequence c_n is given by (3.11).

3.4 System (2.22)

Since the first equation in (2.22) is the same as in (2.19), formula (3.8) must hold, as well as the following one

$$\eta_n=\eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}},\quad n\geq -3. \tag{3.41}$$

The following theorem follows from (2.18), (3.8) and (3.41).

Theorem 3.4. *If $a\neq 0$, then the general solution to (2.5) is*

$$x_n=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},$$

$$y_n=\sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},$$

for $n\geq -3$, where c_n is given by (3.11).

3.5 System (2.23)

The equations in (2.23) yield the relation

$$\zeta_n=\zeta_{n-4}\zeta_{n-6}\zeta_{n-7},\quad n\geq 4. \tag{3.42}$$

We can write (3.42) as follows

$$\zeta_n=\zeta_{n-4}^{a_1}\zeta_{n-5}^{b_1}\zeta_{n-6}^{c_1}\zeta_{n-7}^{d_1}\zeta_{n-8}^{e_1}\zeta_{n-9}^{f_1}\zeta_{n-10}^{g_1},\quad n\geq 4, \tag{3.43}$$

where, of course, the exponents are defined as follows

$$a_1=1,\quad b_1=0,\quad c_1=d_1=1,\quad e_1=f_1=g_1=0. \tag{3.44}$$

From (3.42) and (3.43), we have

$$\begin{aligned}\zeta_n&=(\zeta_{n-8}\zeta_{n-10}\zeta_{n-11})^{a_1}\zeta_{n-5}^{b_1}\zeta_{n-6}^{c_1}\zeta_{n-7}^{d_1}\zeta_{n-8}^{e_1}\zeta_{n-9}^{f_1}\zeta_{n-10}^{g_1}\\&=\zeta_{n-5}^{b_1}\zeta_{n-6}^{c_1}\zeta_{n-7}^{d_1}\zeta_{n-8}^{a_1+e_1}\zeta_{n-9}^{f_1}\zeta_{n-10}^{a_1+g_1}\zeta_{n-11}^{a_1}\\&=\zeta_{n-5}^{a_2}\zeta_{n-6}^{b_2}\zeta_{n-7}^{c_2}\zeta_{n-8}^{d_2}\zeta_{n-9}^{e_2}\zeta_{n-10}^{f_2}\zeta_{n-11}^{g_2},\end{aligned}$$

for $n\geq 8$, where $a_2:=b_1$, $b_2:=c_1$, $c_2:=d_1$, $d_2:=a_1+e_1$, $e_2:=f_1$, $f_2:=a_1+g_1$ and $g_2:=a_1$.

It is natural to suppose that the following relations hold

$$\zeta_n=\zeta_{n-k-3}^{a_k}\zeta_{n-k-4}^{b_k}\zeta_{n-k-5}^{c_k}\zeta_{n-k-6}^{d_k}\zeta_{n-k-7}^{e_k}\zeta_{n-k-8}^{f_k}\zeta_{n-k-9}^{g_k}, \tag{3.45}$$

$$a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = d_{k-1}, \quad d_k = a_{k-1} + e_{k-1},$$
$$e_k = f_{k-1}, \quad f_k = a_{k-1} + g_{k-1}, \quad g_k = a_{k-1}, \tag{3.46}$$

for a $k \geq 2$ and $n \geq k+6$.

From (3.42) and (3.45), we have

$$\begin{aligned}
\zeta_n &= \zeta_{n-k-3}^{a_k}\zeta_{n-k-4}^{b_k}\zeta_{n-k-5}^{c_k}\zeta_{n-k-6}^{d_k}\zeta_{n-k-7}^{e_k}\zeta_{n-k-8}^{f_k}\zeta_{n-k-9}^{g_k}\\
&= (\zeta_{n-k-7}\zeta_{n-k-9}\zeta_{n-k-10})^{a_k}\zeta_{n-k-4}^{b_k}\zeta_{n-k-5}^{c_k}\zeta_{n-k-6}^{d_k}\zeta_{n-k-7}^{e_k}\zeta_{n-k-8}^{f_k}\zeta_{n-k-9}^{g_k}\\
&= \zeta_{n-k-4}^{b_k}\zeta_{n-k-5}^{c_k}\zeta_{n-k-6}^{d_k}\zeta_{n-k-7}^{a_k+e_k}\zeta_{n-k-8}^{f_k}\zeta_{n-k-9}^{a_k+g_k}\zeta_{n-k-10}^{a_k}\\
&= \zeta_{n-k-4}^{a_{k+1}}\zeta_{n-k-5}^{b_{k+1}}\zeta_{n-k-6}^{c_{k+1}}\zeta_{n-k-7}^{d_{k+1}}\zeta_{n-k-8}^{e_{k+1}}\zeta_{n-k-9}^{f_{k+1}}\zeta_{n-k-10}^{g_{k+1}},
\end{aligned}$$

where

$$a_{k+1} := b_k, \quad b_{k+1} := c_k, \quad c_{k+1} := d_k, \quad d_{k+1} := a_k + e_k,$$
$$e_{k+1} := f_k, \quad f_{k+1} := a_k + g_k, \quad g_{k+1} := a_k,$$

for a $k \geq 2$ and $n \geq k+7$. Thus, (3.45) and (3.46) are true for $2 \leq k \leq n-6$.

Relations (3.44) and (3.46) yield

$$a_n = a_{n-4} + a_{n-6} + a_{n-7}, \tag{3.47}$$

for $n \geq 8$ (in fact, for $n \in \mathbb{Z}$), and

$$a_0 = a_{-1} = a_{-2} = 0, \quad a_{-3} = 1, \quad a_{-j} = 0, \quad j = \overline{4,9}, \quad a_{-10} = 1, \quad a_{-11} = -1.$$

By choosing $k = n-6$ in (3.45), we get

$$\begin{aligned}
\zeta_n &= \zeta_3^{a_{n-6}}\zeta_2^{b_{n-6}}\zeta_1^{c_{n-6}}\zeta_0^{d_{n-6}}\zeta_{-1}^{e_{n-6}}\zeta_{-2}^{f_{n-e}}\zeta_{-3}^{g_{n-6}}\\
&= (\eta_0\zeta_{-1})^{a_{n-6}}(\eta_{-1}\zeta_{-2})^{b_{n-6}}(\eta_{-2}\zeta_{-3})^{c_{n-6}}\zeta_0^{d_{n-6}}\zeta_{-1}^{e_{n-6}}\zeta_{-2}^{f_{n-e}}\zeta_{-3}^{g_{n-6}}\\
&= \zeta_0^{d_{n-6}}\zeta_{-1}^{a_{n-6}+e_{n-6}}\zeta_{-2}^{b_{n-6}+f_{n-6}}\zeta_{-3}^{c_{n-6}+g_{n-6}}\eta_0^{a_{n-6}}\eta_{-1}^{b_{n-6}}\eta_{-2}^{c_{n-6}}\\
&= \zeta_0^{a_{n-3}}\zeta_{-1}^{a_{n-2}}\zeta_{-2}^{a_{n-1}}\zeta_{-3}^{a_{n-4}+a_{n-7}}\eta_0^{a_{n-6}}\eta_{-1}^{a_{n-5}}\eta_{-2}^{a_{n-4}},
\end{aligned} \tag{3.48}$$

for $n \geq -3$.

Relations (2.23) and (3.48) yield

$$\begin{aligned}
\eta_n &= \zeta_{n-3}\zeta_{n-4}\\
&= \zeta_0^{a_{n-6}+a_{n-7}}\zeta_{-1}^{a_{n-5}+a_{n-6}}\zeta_{-2}^{a_{n-4}+a_{n-5}}\zeta_{-3}^{a_{n-4}+a_{n-7}}\eta_0^{a_{n-9}+a_{n-10}}\eta_{-1}^{a_{n-8}+a_{n-9}}\eta_{-2}^{a_{n-7}+a_{n-8}},
\end{aligned} \tag{3.49}$$

for $n \in \mathbb{N}$. A direct check shows that (3.49) also holds for $n = 0$.

Let

$$P_7(\lambda) = \lambda^7 - \lambda^3 - \lambda - 1 = (\lambda^3 + 1)(\lambda^4 - \lambda - 1).$$

Clearly it is the characteristic polynomial of (3.47). Four roots of P_7 are those of (3.10), while $\lambda_{4+j} = e^{i\frac{\pi(2j+1)}{3}}$, $j = \overline{0,2}$. The roots are distinct. Lemma 2.1 shows that

$$a_n = \sum_{j=1}^{7} \frac{\lambda_j^{n+9}}{P_7'(\lambda_j)}, \quad n \in \mathbb{Z}, \tag{3.50}$$

is the solution to (3.47) satisfying the initial conditions $a_{-j} = 0$, $j = \overline{4,9} = 0$, $a_{-3} = 1$.

The following theorem follows from (2.18), (3.48) and (3.49).

Theorem 3.5. *If $a \neq 0$, then the general solution to (2.6) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}-1},$$

for $n \geq -3$, and

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}-1},$$

for $n \in \mathbb{N}_0$, where a_n is given by (3.50) and $b_n = a_n + a_{n-1}$.

3.6 System (2.24)

From the equations in (2.24) we have $\zeta_n = \eta_n$, $n \in \mathbb{N}$. This together with (2.24), implies

$$\zeta_{n+1} = \zeta_{n-2}\zeta_{n-3}, \quad n \geq 3.$$

If we use (3.8), we get

$$\begin{aligned}\zeta_n &= \zeta_4^{c_{n-6}}\zeta_3^{c_{n-5}}\zeta_2^{c_{n-4}}\zeta_1^{c_{n-7}}\\ &= (\eta_{-2}\zeta_0\zeta_{-3})^{c_{n-6}}(\eta_0\zeta_{-1})^{c_{n-5}}(\eta_{-1}\zeta_{-2})^{c_{n-4}}(\eta_{-2}\zeta_{-3})^{c_{n-7}}\\ &= \zeta_0^{c_{n-6}}\zeta_{-1}^{c_{n-5}}\zeta_{-2}^{c_{n-4}}\zeta_{-3}^{c_{n-3}}\eta_0^{c_{n-5}}\eta_{-1}^{c_{n-4}}\eta_{-2}^{c_{n-3}},\end{aligned} \tag{3.51}$$

for $n \in \mathbb{N}_0$, where c_n is given by (3.11).

Therefore

$$\eta_n = \zeta_0^{c_{n-6}}\zeta_{-1}^{c_{n-5}}\zeta_{-2}^{c_{n-4}}\zeta_{-3}^{c_{n-3}}\eta_0^{c_{n-5}}\eta_{-1}^{c_{n-4}}\eta_{-2}^{c_{n-3}}, \quad n \in \mathbb{N}. \tag{3.52}$$

The following theorem follows from (2.18), (3.51) and (3.52).

Theorem 3.6. *If $a \neq 0$, then the general solution to (2.7) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}+1}{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}-1},$$

for $n \in \mathbb{N}_0$, and

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}+1}{\prod_{j=0}^{3}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}-1},$$

for $n \in \mathbb{N}$, where c_n is given by (3.11).

3.7 System (2.25)

It is not difficult to see that in the case of the system the following relation holds

$$\zeta_n = \zeta_{n-4}^2 \zeta_{n-6} \zeta_{n-8}^{-1}, \quad n \geq 5. \tag{3.53}$$

Let

$$\zeta_n^{(i)} = \zeta_{2n+i}, \quad n \geq -1,$$

for $i = -1, 0$, then we have

$$\zeta_n^{(i)} = (\zeta_{n-2}^{(i)})^2 \zeta_{n-3}^{(i)} (\zeta_{n-4}^{(i)})^{-1}, \quad n \geq 3. \tag{3.54}$$

Let further

$$b_1 = 2, \quad c_1 = 1, \quad d_1 = -1, \quad e_1 = 0. \tag{3.55}$$

Then, we have

$$\begin{aligned}
\zeta_n^{(i)} &= (\zeta_{n-2}^{(i)})^{b_1} (\zeta_{n-3}^{(i)})^{c_1} (\zeta_{n-4}^{(i)})^{d_1} (\zeta_{n-5}^{(i)})^{e_1} \\
&= ((\zeta_{n-4}^{(i)})^2 \zeta_{n-5}^{(i)} (\zeta_{n-6}^{(i)})^{-1})^{b_1} (\zeta_{n-3}^{(i)})^{c_1} (\zeta_{n-4}^{(i)})^{d_1} (\zeta_{n-5}^{(i)})^{e_1} \\
&= (\zeta_{n-3}^{(i)})^{c_1} (\zeta_{n-4}^{(i)})^{2b_1+d_1} (\zeta_{n-5}^{(i)})^{b_1+e_1} (\zeta_{n-6}^{(i)})^{-b_1} \\
&= (\zeta_{n-3}^{(i)})^{b_2} (\zeta_{n-4}^{(i)})^{c_2} (\zeta_{n-5}^{(i)})^{d_2} (\zeta_{n-6}^{(i)})^{e_2},
\end{aligned}$$

for $n \geq 5$, where $b_2 := c_1$, $c_2 := 2b_1 + d_1$, $d_2 := b_1 + e_1$ and $e_2 := -b_1$.

It is natural to assume that

$$\zeta_n^{(i)} = (\zeta_{n-k-1}^{(i)})^{b_k} (\zeta_{n-k-2}^{(i)})^{c_k} (\zeta_{n-k-3}^{(i)})^{d_k} (\zeta_{n-k-4}^{(i)})^{e_k}, \tag{3.56}$$

for a $k \geq 2$ and $n \geq k+3$, and

$$b_k = c_{k-1}, \quad c_k = 2b_{k-1} + d_{k-1}, \quad d_k = b_{k-1} + e_{k-1}, \quad e_k = -b_{k-1}. \tag{3.57}$$

From (3.54) and (3.56), we have

$$\begin{aligned}
\zeta_n^{(i)} &= (\zeta_{n-k-1}^{(i)})^{b_k} (\zeta_{n-k-2}^{(i)})^{c_k} (\zeta_{n-k-3}^{(i)})^{d_k} (\zeta_{n-k-4}^{(i)})^{e_k} \\
&= ((\zeta_{n-k-3}^{(i)})^2 \zeta_{n-k-4}^{(i)} (\zeta_{n-k-5}^{(i)})^{-1})^{b_k} (\zeta_{n-k-2}^{(i)})^{c_k} (\zeta_{n-k-3}^{(i)})^{d_k} (\zeta_{n-k-4}^{(i)})^{e_k} \\
&= (\zeta_{n-k-2}^{(i)})^{c_k} (\zeta_{n-k-3}^{(i)})^{2b_k+d_k} (\zeta_{n-k-4}^{(i)})^{b_k+e_k} (\zeta_{n-k-5}^{(i)})^{-b_k} \\
&= (\zeta_{n-k-2}^{(i)})^{b_{k+1}} (\zeta_{n-k-3}^{(i)})^{c_{k+1}} (\zeta_{n-k-4}^{(i)})^{d_{k+1}} (\zeta_{n-k-5}^{(i)})^{e_{k+1}},
\end{aligned}$$

for $n \geq k+4$, where

$$b_{k+1} := c_k, \quad c_{k+1} := 2b_k + d_k, \quad d_{k+1} := b_k + e_k, \quad e_{k+1} := -b_k.$$

So, the method of induction shows that (3.56) and (3.57) hold for $2 \leq k \leq n-3$.

From (3.55) and (3.57) we get

$$b_n = 2b_{n-2} + b_{n-3} - b_{n-4}, \tag{3.58}$$

for $n \geq 5$ (in fact, for $n \in \mathbb{Z}$), and

$$b_0 = 0, \quad b_{-1} = 1, \quad b_{-j} = 0, \ j = \overline{2,4}, \quad b_{-5} = -1.$$

If we choose $k = n - 3$ in (3.56), we get

$$\begin{aligned}\zeta_n^{(i)} &= (\zeta_2^{(i)})^{b_{n-3}}(\zeta_1^{(i)})^{c_{n-3}}(\zeta_0^{(i)})^{d_{n-3}}(\zeta_{-1}^{(i)})^{e_{n-3}}\\ &= (\zeta_2^{(i)})^{b_{n-3}}(\zeta_1^{(i)})^{b_{n-2}}(\zeta_0^{(i)})^{b_{n-1}-2b_{n-3}}(\zeta_{-1}^{(i)})^{-b_{n-4}},\end{aligned}$$

for $n \geq -1$, and $i = -1, 0$.

If $i = 0$, we obtain

$$\begin{aligned}\zeta_{2n} &= \zeta_4^{b_{n-3}}\zeta_2^{b_{n-2}}\zeta_0^{b_{n-1}-2b_{n-3}}\zeta_{-2}^{-b_{n-4}}\\ &= (\zeta_0\zeta_{-2}\eta_{-3})^{b_{n-3}}(\eta_{-1}\zeta_{-2})^{b_{n-2}}\zeta_0^{b_{n-1}-2b_{n-3}}\zeta_{-2}^{-b_{n-4}}\\ &= \zeta_0^{b_{n-1}-b_{n-3}}\zeta_{-2}^{b_n-b_{n-2}}\eta_{-1}^{b_{n-2}}\eta_{-3}^{b_{n-3}},\end{aligned} \tag{3.59}$$

whereas for $i = -1$, we get

$$\begin{aligned}\zeta_{2n-1} &= \zeta_3^{b_{n-3}}\zeta_1^{b_{n-2}}\zeta_{-1}^{b_{n-1}-2b_{n-3}}\zeta_{-3}^{-b_{n-4}}\\ &= (\eta_0\zeta_{-1})^{b_{n-3}}(\eta_{-2}\zeta_{-3})^{b_{n-2}}\zeta_{-1}^{b_{n-1}-2b_{n-3}}\zeta_{-3}^{-b_{n-4}}\\ &= \zeta_{-1}^{b_{n-1}-b_{n-3}}\zeta_{-3}^{b_{n-2}-b_{n-4}}\eta_0^{b_{n-3}}\eta_{-2}^{b_{n-2}},\end{aligned} \tag{3.60}$$

for $n \geq -1$.

Since (2.25) is symmetric, we get

$$\eta_{2n} = \eta_0^{b_{n-1}-b_{n-3}}\eta_{-2}^{b_n-b_{n-2}}\zeta_{-1}^{b_{n-2}}\zeta_{-3}^{b_{n-3}}, \tag{3.61}$$

$$\eta_{2n-1} = \eta_{-1}^{b_{n-1}-b_{n-3}}\eta_{-3}^{b_{n-2}-b_{n-4}}\zeta_0^{b_{n-3}}\zeta_{-2}^{b_{n-2}}, \tag{3.62}$$

for $n \geq -1$.

Let

$$\widehat{P}_4(\lambda) = \lambda^4 - 2\lambda^2 - \lambda + 1,$$

It is the characteristic polynomial of (3.58). Its zeros $\widehat{\lambda}_j$, $j = \overline{1,4}$, are distinct.

Therefore

$$b_n = \sum_{j=1}^{4}\frac{\widehat{\lambda}_j^{n+4}}{\widehat{P}_4'(\widehat{\lambda}_j)}, \quad n \in \mathbb{Z}, \tag{3.63}$$

is the solution to (3.58) satisfying the initial conditions $b_{-j} = 0$, $k = \overline{2,4}$, $b_{-1} = 1$.

The following theorem follows from (2.18), (3.59)–(3.62).

Theorem 3.7. *If $a \neq 0$, then the general solution to (2.8) is*

$$x_{2n} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}}-1},$$

$$x_{2n-1} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_{n-2}-1}},$$

$$y_{2n} = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)^{b_{n-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}}-1},$$

$$y_{2n-1}=\sqrt{a}\,\frac{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}}+1}{\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{b_{n-1}-b_{n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_n-b_{n-2}}\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{b_{n-3}}\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)^{b_{n-2}-1}},$$

for $n\ge -1$, where the sequence b_n is given by (3.63).

3.8 System (2.26)

By interchanging letters ζ and η, system (2.26) is obtained from (2.21). Hence

$$\zeta_{4n-3}=\zeta_{-3}\eta_0^{c_{4n-5}}\eta_{-1}^{c_{4n-4}}\eta_{-2}^{c_{4n-3}}\eta_{-3}^{c_{4n-6}-1},\tag{3.64}$$

$$\zeta_{4n-2}=\zeta_{-2}\eta_0^{c_{4n-4}}\eta_{-1}^{c_{4n-3}}\eta_{-2}^{c_{4n-2}-1}\eta_{-3}^{c_{4n-5}},\tag{3.65}$$

$$\zeta_{4n-1}=\zeta_{-1}\eta_0^{c_{4n-3}}\eta_{-1}^{c_{4n-2}-1}\eta_{-2}^{c_{4n-1}}\eta_{-3}^{c_{4n-4}},\tag{3.66}$$

$$\zeta_{4n}=\zeta_0\eta_0^{c_{4n-2}-1}\eta_{-1}^{c_{4n-1}}\eta_{-2}^{c_{4n}}\eta_{-3}^{c_{4n-3}},\tag{3.67}$$

for $n\in\mathbb{N}_0$,

$$\eta_n=\eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}},\quad n\ge -3.\tag{3.68}$$

The following theorem follows from (2.18), (3.64)–(3.68).

Theorem 3.8. *If $a\neq 0$, then the general solution to* (2.9) *is*

$$y_n=\sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1},\quad n\ge -3,$$

$$x_{4n-3}=\sqrt{a}\,\frac{\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}+1}{\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-5}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-6}-1}-1},$$

$$x_{4n-2}=\sqrt{a}\,\frac{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-5}}+1}{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-5}}-1},$$

$$x_{4n-1}=\sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-4}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-4}}-1},$$

$$x_{4n}=\sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{4n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{4n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{4n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{4n-3}}-1},$$

for $n\in\mathbb{N}_0$, where sequence c_n is given by (3.11).

3.9 System (2.27)

It is easy to see that the following relation holds

$$\zeta_n=\zeta_{n-3}\zeta_{n-7}\zeta_{n-8},\quad n\ge 5.\tag{3.69}$$

We can write (3.69) as follows

$$\zeta_n = \zeta_{n-3}^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}, \quad n \geq 5, \tag{3.70}$$

where, of course, the exponents are defined as follows

$$a_1 = 1, \quad b_1 = c_1 = d_1 = 0, \quad e_1 = f_1 = 1, \quad g_1 = h_1 = 0. \tag{3.71}$$

Employing (3.69) in (3.70), we have

$$\begin{aligned}\zeta_n &= (\zeta_{n-6}\zeta_{n-10}\zeta_{n-11})^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}\\ &= \zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{a_1+d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{a_1+h_1}\zeta_{n-11}^{a_1}\\ &= \zeta_{n-4}^{a_2}\zeta_{n-5}^{b_2}\zeta_{n-6}^{c_2}\zeta_{n-7}^{d_2}\zeta_{n-8}^{e_2}\zeta_{n-9}^{f_2}\zeta_{n-10}^{g_2}\zeta_{n-11}^{h_2},\end{aligned}$$

for $n \geq 8$, where $a_2 := b_1$, $b_2 := c_1$, $c_2 := a_1 + d_1$, $d_2 := e_1$, $e_2 := f_1$, $f_2 := g_1$, $g_2 := a_1 + h_1$ and $h_2 := a_1$.

As in the case of equation (3.53) is obtained

$$\zeta_n = \zeta_{n-k-2}^{a_k}\zeta_{n-k-3}^{b_k}\zeta_{n-k-4}^{c_k}\zeta_{n-k-5}^{d_k}\zeta_{n-k-6}^{e_k}\zeta_{n-k-7}^{f_k}\zeta_{n-k-8}^{g_k}\zeta_{n-k-9}^{h_k}, \tag{3.72}$$

for a $k \geq 2$ and $n \geq k+6$, and

$$\begin{aligned}&a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = a_{k-1} + d_{k-1}, \quad d_k = e_{k-1}\\ &e_k = f_{k-1}, \quad f_k = g_{k-1}, \quad g_k = a_{k-1} + h_{k-1}, \quad h_k = a_{k-1}.\end{aligned} \tag{3.73}$$

Relations (3.71) and (3.73) imply

$$a_n = a_{n-3} + a_{n-7} + a_{n-8}, \tag{3.74}$$

for $n \geq 9$ (in fact, for $n \in \mathbb{Z}$), and

$$a_0 = a_{-1} = 0, \quad a_{-2} = 1, \quad a_{-j} = 0, \ j = \overline{3,9}, \quad a_{-10} = 1, \quad a_{-11} = -1.$$

By choosing $k = n-6$ in (3.72), it follows that

$$\begin{aligned}\zeta_n &= \zeta_4^{a_{n-6}}\zeta_3^{b_{n-6}}\zeta_2^{c_{n-6}}\zeta_1^{d_{n-6}}\zeta_0^{e_{n-6}}\zeta_{-1}^{f_{n-6}}\zeta_{-2}^{g_{n-6}}\zeta_{-3}^{h_{n-6}}\\ &= (\zeta_{-2}\eta_0\eta_{-3})^{a_{n-6}}(\zeta_0\eta_{-1})^{b_{n-6}}(\zeta_{-1}\eta_{-2})^{c_{n-6}}(\zeta_{-2}\eta_{-3})^{d_{n-6}}\zeta_0^{e_{n-6}}\zeta_{-1}^{f_{n-6}}\zeta_{-2}^{g_{n-6}}\zeta_{-3}^{h_{n-6}}\\ &= \zeta_0^{b_{n-6}+e_{n-6}}\zeta_{-1}^{c_{n-6}+f_{n-6}}\zeta_{-2}^{a_{n-6}+d_{n-6}+g_{n-6}}\zeta_{-3}^{h_{n-6}}\eta_0^{a_{n-6}}\eta_{-1}^{b_{n-6}}\eta_{-2}^{c_{n-6}}\eta_{-3}^{a_{n-6}+d_{n-6}}\\ &= \zeta_0^{a_{n-2}}\zeta_{-1}^{a_{n-1}}\zeta_{-2}^{a_n}\zeta_{-3}^{a_{n-7}}\eta_0^{a_{n-6}}\eta_{-1}^{a_{n-5}}\eta_{-2}^{a_{n-4}}\eta_{-3}^{a_{n-3}},\end{aligned} \tag{3.75}$$

for $n \geq -3$.

Relations (3.75) and (2.27) yield

$$\begin{aligned}\eta_n = \zeta_{n-3}\zeta_{n-4} = &\ \zeta_0^{a_{n-5}+a_{n-6}}\zeta_{-1}^{a_{n-4}+a_{n-5}}\zeta_{-2}^{a_{n-3}+a_{n-4}}\zeta_{-3}^{a_{n-10}+a_{n-11}}\\ &\times\eta_0^{a_{n-9}+a_{n-10}}\eta_{-1}^{a_{n-8}+a_{n-9}}\eta_{-2}^{a_{n-7}+a_{n-8}}\eta_{-3}^{a_{n-6}+a_{n-7}},\end{aligned} \tag{3.76}$$

for $n \geq -3$.

Let

$$\widetilde{P}_8(t) = t^8 - t^5 - t - 1 = (t^4 - t - 1)(t^4 + 1).$$

It is the characteristic polynomial associated to (3.74). Its roots are those of (3.10) and $t_{j+4} = e^{i\frac{\pi(2j+1)}{4}}$, $j = \overline{0,3}$. It is not difficult to see that they are distinct.

Lemma 2.1 shows that

$$a_n = \sum_{j=1}^{8} \frac{t_j^{n+9}}{\widetilde{P}_8'(t_j)}, \quad n \in \mathbb{Z}, \tag{3.77}$$

is the solution to (3.74) satisfying the initial conditions $a_{-j} = 0$, $j = \overline{3,9}$, $a_{-2} = 1$.

The following theorem follows from (2.18), (3.75) and (3.76).

Theorem 3.9. *If $a \neq 0$, then the general solution to (2.10) is*

$$x_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}-1},$$

$$y_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{y_{-l}+\sqrt{a}}{y_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}-1},$$

for $n \geq -3$, where the sequence a_n is given by (3.77) and $b_n = a_n + a_{n-1}$.

3.10 System (2.28)

It is easy to see that the following relation holds

$$\zeta_n = \zeta_{n-3}^2\zeta_{n-6}^{-1}\zeta_{n-8}, \quad n \geq 5, \tag{3.78}$$

which can be written as

$$\zeta_n = \zeta_{n-3}^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}, \tag{3.79}$$

for $n \geq 5$, where, of course, the exponents are given by

$$a_1 = 2, \quad b_1 = c_1 = 0, \quad d_1 = -1, \quad e_1 = 0, \quad f_1 = 1, \quad g_1 = h_1 = 0. \tag{3.80}$$

Relations (3.78) and (3.79) yield

$$\begin{aligned}\zeta_n &= \zeta_{n-3}^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}\\ &= (\zeta_{n-6}^2\zeta_{n-9}^{-1}\zeta_{n-11})^{a_1}\zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{g_1}\zeta_{n-10}^{h_1}\\ &= \zeta_{n-4}^{b_1}\zeta_{n-5}^{c_1}\zeta_{n-6}^{2a_1+d_1}\zeta_{n-7}^{e_1}\zeta_{n-8}^{f_1}\zeta_{n-9}^{-a_1+g_1}\zeta_{n-10}^{h_1}\zeta_{n-11}^{a_1}\\ &= \zeta_{n-4}^{a_2}\zeta_{n-5}^{b_2}\zeta_{n-6}^{c_2}\zeta_{n-7}^{d_2}\zeta_{n-8}^{e_2}\zeta_{n-9}^{f_2}\zeta_{n-10}^{g_2}\zeta_{n-11}^{h_2},\end{aligned}$$

for $n \geq 8$, where $a_2 := b_1$, $b_2 := c_1$, $c_2 := 2a_1 + d_1$, $d_2 := e_1$, $e_2 := f_1$, $f_2 := -a_1 + g_1$, $g_2 := h_1$ and $h_2 := a_1$.

As in (3.53) we obtain

$$\zeta_n = \zeta_{n-k-2}^{a_k}\zeta_{n-k-3}^{b_k}\zeta_{n-k-4}^{c_k}\zeta_{n-k-5}^{d_k}\zeta_{n-k-6}^{e_k}\zeta_{n-k-7}^{f_k}\zeta_{n-k-8}^{g_k}\zeta_{n-k-9}^{h_k}, \tag{3.81}$$

and

$$\begin{aligned} &a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = 2a_{k-1} + d_{k-1}, \quad d_k = e_{k-1}, \\ &e_k = f_{k-1}, \quad f_k = -a_{k-1} + g_{k-1}, \quad g_k = h_{k-1}, \quad h_k = a_{k-1}, \end{aligned} \tag{3.82}$$

for $2 \le k \le n-6$.

Relations (3.80) and (3.82) yield

$$a_n = 2a_{n-3} - a_{n-6} + a_{n-8}, \tag{3.83}$$

for $n \ge 9$ (in fact, for $n \in \mathbb{Z}$), and

$$a_0 = a_{-1} = 0, \quad a_{-2} = 1, \quad a_{-j} = 0, \ j = \overline{3,9}, \quad a_{-10} = 1, \quad a_{-11} = 0.$$

By choosing $k = n-6$ in (3.81), we obtain

$$\begin{aligned} \zeta_n &= \zeta_4^{a_{n-6}} \zeta_3^{b_{n-6}} \zeta_2^{c_{n-6}} \zeta_1^{d_{n-6}} \zeta_0^{e_{n-6}} \zeta_{-1}^{f_{n-6}} \zeta_{-2}^{g_{n-6}} \zeta_{-3}^{h_{n-6}} \\ &= (\zeta_{-2}\eta_0\eta_{-3})^{a_{n-6}} (\zeta_0\eta_{-1})^{b_{n-6}} (\zeta_{-1}\eta_{-2})^{c_{n-6}} (\zeta_{-2}\eta_{-3})^{d_{n-6}} \zeta_0^{e_{n-6}} \zeta_{-1}^{f_{n-6}} \zeta_{-2}^{g_{n-6}} \zeta_{-3}^{h_{n-6}} \\ &= \zeta_0^{b_{n-6}+e_{n-6}} \zeta_{-1}^{c_{n-6}+f_{n-6}} \zeta_{-2}^{a_{n-6}+d_{n-6}+g_{n-6}} \zeta_{-3}^{h_{n-6}} \eta_0^{a_{n-6}} \eta_{-1}^{b_{n-6}} \eta_{-2}^{c_{n-6}} \eta_{-3}^{a_{n-6}+d_{n-6}} \\ &= \zeta_0^{a_{n-2}-a_{n-5}} \zeta_{-1}^{a_{n-1}-a_{n-4}} \zeta_{-2}^{a_n-a_{n-3}} \zeta_{-3}^{a_{n-7}} \eta_0^{a_{n-6}} \eta_{-1}^{a_{n-5}} \eta_{-2}^{a_{n-4}} \eta_{-3}^{a_{n-3}-a_{n-6}}, \end{aligned} \tag{3.84}$$

for $n \ge -3$. System (2.28) is symmetric implying that

$$\eta_n = \eta_0^{a_{n-2}-a_{n-5}} \eta_{-1}^{a_{n-1}-a_{n-4}} \eta_{-2}^{a_n-a_{n-3}} \eta_{-3}^{a_{n-7}} \zeta_0^{a_{n-6}} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \zeta_{-3}^{a_{n-3}-a_{n-6}}, \tag{3.85}$$

for $n \ge -3$.

Let

$$\widehat{P}_8(t) = t^8 - 2t^5 + t^2 - 1 = (t^4 - t - 1)(t^4 - t + 1).$$

It is the characteristic polynomial of (3.83). Let $\widetilde{t}_j$, $j = \overline{1,8}$, be the roots of $\widehat{P}_8$. They are simple. So, the solution to (3.83) such that $a_{-j} = 0$, $j = \overline{3,9}$, and $a_{-2} = 1$, is

$$a_n = \sum_{j=1}^{8} \frac{\widetilde{t}_j^{\,n+9}}{\widehat{P}_8'(\widetilde{t}_j)}, \quad n \in \mathbb{Z}. \tag{3.86}$$

The following theorem follows from (2.18), (3.84) and (3.85).

Theorem 3.10. *If $a \neq 0$, then the general solution to (2.11) is*

$$x_n = \sqrt{a}\, \frac{\prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}} + 1}{\prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-3}} - 1},$$

$$y_n = \sqrt{a}\, \frac{\prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}} + 1}{\prod_{j=0}^{2} \left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-2}} \left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}} \prod_{j=0}^{2} \left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}} \left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-3}} - 1},$$

for $n \ge -3$, where the sequence a_n is given by (3.86) and $b_n = a_n - a_{n-3}$.

3.11 System (2.29)

We have $\zeta_n = \eta_n$, $n \in \mathbb{N}$, and consequently

$$\zeta_n = \zeta_{n-3}\zeta_{n-4},$$

for $n \geq 5$.

From (3.8), we obtain

$$\begin{aligned}\zeta_n &= \zeta_4^{c_{n-6}}\zeta_3^{c_{n-5}}\zeta_2^{c_{n-4}}\zeta_1^{c_{n-7}}\\ &= (\zeta_{-2}\eta_0\eta_{-3})^{c_{n-6}}(\zeta_0\eta_{-1})^{c_{n-5}}(\zeta_{-1}\eta_{-2})^{c_{n-4}}(\zeta_{-2}\eta_{-3})^{c_{n-7}}\\ &= \zeta_0^{c_{n-5}}\zeta_{-1}^{c_{n-4}}\zeta_{-2}^{c_{n-3}}\eta_0^{c_{n-6}}\eta_{-1}^{c_{n-5}}\eta_{-2}^{c_{n-4}}\eta_{-3}^{c_{n-3}},\end{aligned} \tag{3.87}$$

for $n \in \mathbb{N}$, where c_n is given by (3.11). Thus

$$\eta_n = \zeta_0^{c_{n-5}}\zeta_{-1}^{c_{n-4}}\zeta_{-2}^{c_{n-3}}\eta_0^{c_{n-6}}\eta_{-1}^{c_{n-5}}\eta_{-2}^{c_{n-4}}\eta_{-3}^{c_{n-3}}, \tag{3.88}$$

for $n \in \mathbb{N}_0$.

The following theorem follows from (2.18), (3.87) and (3.88).

Theorem 3.11. *If $a \neq 0$, then the general solution to (2.12) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}-1},$$

for $n \in \mathbb{N}$, and

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{c_{n+j-5}}\prod_{j=0}^{3}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{c_{n+j-6}}-1},$$

for $n \in \mathbb{N}_0$, where c_n is given by (3.11).

3.12 System (2.30)

By interchanging letters ζ and η, system (2.30) is got from (2.20). Hence

$$\zeta_{3n-2} = \zeta_{-2}\eta_0^{c_{3n-4}}\eta_{-1}^{c_{3n-3}}\eta_{-2}^{c_{3n-2}-1}\eta_{-3}^{c_{3n-5}}, \tag{3.89}$$

$$\zeta_{3n-1} = \zeta_{-1}\eta_0^{c_{3n-3}}\eta_{-1}^{c_{3n-2}-1}\eta_{-2}^{c_{3n-1}}\eta_{-3}^{c_{3n-4}}, \tag{3.90}$$

$$\zeta_{3n} = \zeta_0\eta_0^{c_{3n-2}-1}\eta_{-1}^{c_{3n-1}}\eta_{-2}^{c_{3n}}\eta_{-3}^{c_{3n-3}}, \tag{3.91}$$

for $n \in \mathbb{N}_0$, and

$$\eta_n = \eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}}, \tag{3.92}$$

for $n \geq -3$.

The following theorem follows (2.18), (3.89)–(3.92).

Theorem 3.12. *If $a \neq 0$, then the general solution to (2.13) is*

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \geq -3,$$

$$x_{3n-2} = \sqrt{a}\,\frac{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-5}}+1}{\left(\frac{x_{-2}+\sqrt{a}}{x_{-2}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-4}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-5}}-1}$$

$$x_{3n-1} = \sqrt{a}\,\frac{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-4}}+1}{\left(\frac{x_{-1}+\sqrt{a}}{x_{-1}-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-3}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-4}}-1}$$

$$x_{3n} = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-3}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{3n-2}-1}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{3n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_{3n}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{3n-3}}-1},$$

for $n \in \mathbb{N}$, where c_n is given by (3.11).

3.13 System (2.31)

It is easy to see that the following relation holds

$$\zeta_n = \zeta_{n-6}\zeta_{n-7}^2\zeta_{n-8}, \quad n \geq 5, \tag{3.93}$$

which can be written as follows

$$\zeta_n = \zeta_{n-6}^{a_1}\zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{g_1}\zeta_{n-13}^{h_1}, \tag{3.94}$$

for $n \geq 5$, where, of course, the exponents are given by

$$a_1 = 1, \quad b_1 = 2, \quad c_1 = 1, \quad d_1 = e_1 = f_1 = g_1 = h_1 = 0. \tag{3.95}$$

Relations (3.93) in (3.94) yield

$$\begin{aligned}
\zeta_n &= \zeta_{n-6}^{a_1}\zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{g_1}\zeta_{n-13}^{h_1}\\
&= (\zeta_{n-12}\zeta_{n-13}^2\zeta_{n-14})^{a_1}\zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{g_1}\zeta_{n-13}^{h_1}\\
&= \zeta_{n-7}^{b_1}\zeta_{n-8}^{c_1}\zeta_{n-9}^{d_1}\zeta_{n-10}^{e_1}\zeta_{n-11}^{f_1}\zeta_{n-12}^{a_1+g_1}\zeta_{n-13}^{2a_1+h_1}\zeta_{n-14}^{a_1}\\
&= \zeta_{n-7}^{a_2}\zeta_{n-8}^{b_2}\zeta_{n-9}^{c_2}\zeta_{n-10}^{d_2}\zeta_{n-11}^{e_2}\zeta_{n-12}^{f_2}\zeta_{n-13}^{g_2}\zeta_{n-14}^{h_2},
\end{aligned}$$

for $n \geq 11$, where $a_2 := b_1$, $b_2 := c_1$, $c_2 := d_1$, $d_2 := e_1$, $e_2 := f_1$, $f_2 := a_1 + g_1$, $g_2 := 2a_1 + h_1$ and $h_2 := a_1$.

As in (3.2) are obtained the folowing relations

$$\zeta_n = \zeta_{n-k-5}^{a_k}\zeta_{n-k-6}^{b_k}\zeta_{n-k-7}^{c_k}\zeta_{n-k-8}^{d_k}\zeta_{n-k-9}^{e_k}\zeta_{n-k-10}^{f_k}\zeta_{n-k-11}^{g_k}\zeta_{n-k-12}^{h_k}, \tag{3.96}$$

$$\begin{aligned}
&a_k = b_{k-1}, \quad b_k = c_{k-1}, \quad c_k = d_{k-1}, \quad d_k = e_{k-1},\\
&e_k = f_{k-1}, \quad f_k = a_{k-1} + g_{k-1}, \quad g_k = 2a_{k-1} + h_{k-1}, \quad h_k = a_{k-1}.
\end{aligned} \tag{3.97}$$

for a $k \geq 2$ and $n \geq k+9$.

Relations (3.95) and (3.97) yield

$$a_k = a_{k-6} + 2a_{k-7} + a_{k-8}, \tag{3.98}$$

and

$$a_{-l} = 0, \quad l = \overline{0,4}, \quad a_{-5} = 1, \quad a_{-j} = 0, \quad j = \overline{6,12}.$$

By choosing $k = n - 9$ in (3.96), we get

$$\begin{aligned} \zeta_n &= \zeta_4^{a_{n-9}} \zeta_3^{b_{n-9}} \zeta_2^{c_{n-9}} \zeta_1^{d_{n-9}} \zeta_0^{e_{n-9}} \zeta_{-1}^{f_{n-9}} \zeta_{-2}^{g_{n-9}} \zeta_{-3}^{h_{n-9}} \\ &= (\zeta_{-2}\zeta_{-3}\eta_0)^{a_{n-9}} (\eta_0\eta_{-1})^{b_{n-9}} (\eta_{-1}\eta_{-2})^{c_{n-9}} (\eta_{-2}\eta_{-3})^{d_{n-9}} \zeta_0^{e_{n-9}} \zeta_{-1}^{f_{n-9}} \zeta_{-2}^{g_{n-9}} \zeta_{-3}^{h_{n-9}} \\ &= \zeta_0^{e_{n-9}} \zeta_{-1}^{f_{n-9}} \zeta_{-2}^{a_{n-9}+g_{n-9}} \zeta_{-3}^{a_{n-9}+h_{n-9}} \eta_0^{a_{n-9}+b_{n-9}} \eta_{-1}^{b_{n-9}+c_{n-9}} \eta_{-2}^{c_{n-9}+d_{n-9}} \eta_{-3}^{d_{n-9}} \\ &= \zeta_0^{a_{n-5}} \zeta_{-1}^{a_{n-4}} \zeta_{-2}^{a_{n-3}} \zeta_{-3}^{a_{n-9}+a_{n-10}} \eta_0^{a_{n-8}+a_{n-9}} \eta_{-1}^{a_{n-7}+a_{n-8}} \eta_{-2}^{a_{n-6}+a_{n-7}} \eta_{-3}^{a_{n-6}}, \end{aligned} \tag{3.99}$$

for $n \ge -3$.

System (2.31) is symmetric implying that

$$\eta_n = \eta_0^{a_{n-5}} \eta_{-1}^{a_{n-4}} \eta_{-2}^{a_{n-3}} \eta_{-3}^{a_{n-9}+a_{n-10}} \zeta_0^{a_{n-8}+a_{n-9}} \zeta_{-1}^{a_{n-7}+a_{n-8}} \zeta_{-2}^{a_{n-6}+a_{n-7}} \zeta_{-3}^{a_{n-6}} \tag{3.100}$$

for $n \ge -3$.

Let

$$\widehat{P}_8(t) = t^8 - t^2 - 2t - 1 = (t^4 - t - 1)(t^4 + t + 1).$$

It is the characteristic polynomial associated to (3.98). Let t_j, $j = \overline{1,8}$, be its roots. It is not difficult to see that they are simple. Then, the solution to (3.98) such that $a_{-j} = 0$, $j = \overline{6,12}$, and $a_{-5} = 1$, is given by

$$a_n = \sum_{j=1}^{8} \frac{t_j^{n+12}}{\widehat{P}_8'(t_j)}, \quad n \in \mathbb{Z}. \tag{3.101}$$

The following theorem follows from (2.18), (3.99) and (3.100).

Theorem 3.13. *If $a \neq 0$, then the general solution to (2.14) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-5}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-9}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-8}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-6}}+1}{\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-5}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{b_{n-9}}\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-8}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-6}}-1},$$

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-5}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-9}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-8}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-6}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-5}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-9}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-8}}\left(\frac{x_{-3}+\sqrt{a}}{x_{-3}-\sqrt{a}}\right)^{a_{n-6}}-1},$$

for $n \ge -3$, where the sequence a_n is given by (3.101) and $b_n = a_n + a_{n-1}$.

3.14 System (2.32)

By interchanging letters ζ and η, (2.32) is got from (2.27). Hence

$$\zeta_n = \eta_0^{a_{n-5}+a_{n-6}} \eta_{-1}^{a_{n-4}+a_{n-5}} \eta_{-2}^{a_{n-3}+a_{n-4}} \eta_{-3}^{a_{n-10}+a_{n-11}} \zeta_0^{a_{n-9}+a_{n-10}} \zeta_{-1}^{a_{n-8}+a_{n-9}} \zeta_{-2}^{a_{n-7}+a_{n-8}} \zeta_{-3}^{a_{n-6}+a_{n-7}}, \tag{3.102}$$

$$\eta_n = \eta_0^{a_{n-2}} \eta_{-1}^{a_{n-1}} \eta_{-2}^{a_n} \eta_{-3}^{a_{n-7}} \zeta_0^{a_{n-6}} \zeta_{-1}^{a_{n-5}} \zeta_{-2}^{a_{n-4}} \zeta_{-3}^{a_{n-3}}, \tag{3.103}$$

for $n \ge -3$.

The following theorem follows from (2.18), (3.102) and (3.103).

Theorem 3.14. *If $a \neq 0$, then the general solution to (2.15) is*

$$x_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-5}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{b_{n-10}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{b_{n+l-9}}-1},$$

$$y_n = \sqrt{a}\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-2}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-7}}\prod_{l=0}^{3}\left(\frac{x_{-l}+\sqrt{a}}{x_{-l}-\sqrt{a}}\right)^{a_{n+l-6}}-1},$$

for $n \geq -3$, where the sequence a_n is given by (3.77) and $b_n = a_n + a_{n-1}$.

3.15 System (2.33)

By interchanging letters ζ and η, (2.33) is got from (2.23). Hence

$$\zeta_n = \eta_0^{a_{n-6}+a_{n-7}}\eta_{-1}^{a_{n-5}+a_{n-6}}\eta_{-2}^{a_{n-4}+a_{n-5}}\eta_{-3}^{a_{n-4}+a_{n-7}}\zeta_0^{a_{n-9}+a_{n-10}}\zeta_{-1}^{a_{n-8}+a_{n-9}}\zeta_{-2}^{a_{n-7}+a_{n-8}}, \tag{3.104}$$
$$\eta_n = \eta_0^{a_{n-3}}\eta_{-1}^{a_{n-2}}\eta_{-2}^{a_{n-1}}\eta_{-3}^{a_{n-4}+a_{n-7}}\zeta_0^{a_{n-6}}\zeta_{-1}^{a_{n-5}}\zeta_{-2}^{a_{n-4}}, \tag{3.105}$$

for $n \in \mathbb{N}$.

The following theorem follows from (2.18), (3.104) and (3.105).

Theorem 3.15. *If $a \neq 0$, then the general solution to (2.16) is*

$$x_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{b_{n+j-6}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{b_{n+j-9}}-1},$$

for $n \in \mathbb{N}_0$,

$$y_n = \sqrt{a}\,\frac{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}+1}{\prod_{j=0}^{2}\left(\frac{y_{-j}+\sqrt{a}}{y_{-j}-\sqrt{a}}\right)^{a_{n+j-3}}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{a_{n-4}+a_{n-7}}\prod_{j=0}^{2}\left(\frac{x_{-j}+\sqrt{a}}{x_{-j}-\sqrt{a}}\right)^{a_{n+j-6}}-1},$$

for $n \geq -3$, and where a_n is given by (3.50) and $b_n = a_n + a_{n-1}$.

3.16 System (2.34)

By interchanging letters ζ and η, (2.34) is got from (2.19). Hence

$$\zeta_n = \eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}}, \quad n \in \mathbb{N}, \tag{3.106}$$

and

$$\eta_n = \eta_0^{c_{n-2}}\eta_{-1}^{c_{n-1}}\eta_{-2}^{c_n}\eta_{-3}^{c_{n-3}}, \quad n \geq -3. \tag{3.107}$$

The following theorem follows from (2.18), (3.106) and (3.107).

Theorem 3.16. *If $a \neq 0$, then the general solution to* (2.17) *is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \in \mathbb{N},$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{c_{n-2}}\left(\frac{y_{-1}+\sqrt{a}}{y_{-1}-\sqrt{a}}\right)^{c_{n-1}}\left(\frac{y_{-2}+\sqrt{a}}{y_{-2}-\sqrt{a}}\right)^{c_n}\left(\frac{y_{-3}+\sqrt{a}}{y_{-3}-\sqrt{a}}\right)^{c_{n-3}}-1}, \quad n \geq -3,$$

where c_n is given by (3.11).

Remark 3.17. From (2.18) we see that a solution to a system in (1.2) is well defined if and only if $\zeta_n \neq 1$ and $\eta_n \neq 1$ for every n belonging to the domain of the system. Using this fact, as well as above presented expressions for the sequences ζ_n and η_n, can be described the sets of not well defined solutions for each of the systems. We leave it to the reader.

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